Journal of Computational and Applied Mathematics
Manuscript Draft

Manuscript Number: CAM-D-08-00325

Title: Subdivision Depth Computation for N-ary Subdivision Curves/Surfaces

Article Type: Research Paper

Section/Category: All Other 65Cxx

Keywords: Subdivision curve, subdivision surfaces, subdivision depth, error bound, control polygon, forward differences

Manuscript Region of Origin:

Abstract:

The editor,
Journal of Computational and Applied mathematics

Subject: Submission of manuscript for publication

Please here find the manuscript titled: **“Subdivision Depth Computation for N-ary Subdivision Curves/Surfaces”** for publication in your journal. Please do the needful and oblige.

Regards,

Mustafa

Subdivision Depth Computation for N-ary Subdivision Curves/Surfaces

Ghulam Mustafa*, Sadiq Hashmi†
Department of Mathematics
The Islamia University of Bahawal pur Pakistan

Abstract

This paper deals with subdivision depth computation technique for n-ary subdivision curves/surfaces. This technique also includes error bound evaluation technique for n-ary subdivision curves/surfaces with their control polygon. Both techniques provide error control tools in subdivision schemes.

Keywords: Subdivision curve, subdivision surfaces, subdivision depth, error bound, control polygon, forward differences
AMS Subject Classifications: 65D17, 65D07, 65D05

1 Introduction

Computer Aided Geometric Design (CAGD) is a branch of applied Mathematics concerned with algorithms for the design of smooth curves/ surfaces. One common approach to the design of curves/surfaces which related to CAGD is the subdivision schemes. It is an algorithm to generate smooth curves and surfaces as a sequence of successively refined control polygons. At each refinement level, new points are added into the existing polygon and the original points remain existed or discarded in all subsequent sequences of control polygons. The number of points inserted at level $k+1$ between two consecutive points from level k is called arity of the scheme. In the case when number of points inserted are 2, 3,..., n the subdivision schemes are called binary, ternary,..., n-ary respectively. For more details on n-ary subdivision schemes, we may refer to the thesis of N. Aspert [1] and Kwan [7].

*Corresponding author: mustafa_rakib@yahoo.com, Cell No. +92-3006855221
†sadiq.hashmi@gmail.com, Cell No. +92-3216804886
This work is supported by the Indigenous PhD Scholarship Scheme of Higher Education Commission (HEC) Pakistan.

Although subdivision schemes have become important in recent years because they provide a precise and efficient way to describe smooth curves/surfaces. However, the little have been done in the area of error control for n-ary subdivision curves/surfaces. For example, given an error tolerance, how many levels of subdivision should be performed on the initial control polygon so that the error/distance between the resulting control polygon and the limit curve/ surface would be less than the error tolerance? This error control technique is called subdivision depth computation.
A subdivision depth and error bound are based on forward differences of control points have been presented by [2], [3], [4], [5], [6] and [11], while the methods [8], [9] and [10] are based on eigenanalysis. But nothing in this area has been done for more general n-ary subdivision curves/surafces yet. In this paper we will present a subdivision depth computation technique based on error bounds for n-ary subdivision curves/surfaces. The paper is arranged as follows:
Section 2 is devoted for basic definitions and notations. In Section 3 and 4 we have computed subdivision depth for n-ary subdivision curves and n-ary subdivision surfaces respectively. Conclusion and future research directions are given in Section 5. The typical mathematical proofs are placed in appendices for transparent presentation of the paper.

2 Definitions and notations

N-ary subdivision curve:
Given a sequence of control points $p_i^k \in \mathbb{R}^N$, $i \in \mathbb{Z}$, $N \geqslant 1$, where the upper index $k \geqslant 0$ indicates the subdivision level. An n-ary subdivision curve [1] is defined by

$$p_{ni+\alpha}^{k+1} = \sum_{j=0}^{m} a_{\alpha,j} p_{i+j}^k, \quad \alpha = 0, 1, \dots, n-1, \tag{2.1}$$

where $m > 0$ and

$$\sum_{j=0}^{m} a_{\alpha,j} = 1, \quad \alpha = 0, 1, \dots, n-1. \tag{2.2}$$

The set of coefficients $\{a_{\alpha,j},\ \alpha = 0, 1, \dots, n-1\}_{j=0}^m$ is called subdivision mask. Given initial values $p_i^0 \in \mathbb{R}^N, i \in \mathbb{Z}$. Then in the limit $k \to \infty$, the process (2.1) defines an infinite set of points in $\mathbb{R}^N$. The sequence of control points $\{p_i^k\}$ is related, in a natural way, with the diadic mesh points $t_i^k = i/n^k,\ i \in \mathbb{Z}$. The process then defines a scheme whereby p_{ni}^{k+1} & p_{ni+n}^{k+1} replace the values p_i^k & p_{i+1}^k at the mesh points $t_{ni}^{k+1} = t_i^k$ & $t_{ni+n}^{k+1} = t_{i+1}^k$ respectively, while $p_{ni+\alpha}^{k+1}$ are inserted at the new mesh points $t_{ni+\alpha}^{k+1} = \frac{1}{n}((n-\alpha)t_i^k + \alpha t_{i+1}^k)$ for $\alpha = 1, 2, \dots, n-1$.

N-ary subdivision surface:
Given a sequence of control points $p^k_{i,j} \in \mathbb{R}^N$, $i,j \in \mathbb{Z}$, $N \geqslant 2$, where the upper index $k \geqslant 0$ indicates the subdivision level. N-ary subdivision surface is tensor product of (2.1) defined by

$$p^{k+1}_{ni+\alpha,nj+\beta} = \sum_{r=0}^{m}\sum_{s=0}^{m} a_{\alpha,r}a_{\beta,s}p^k_{i+r,j+s}, \quad \alpha,\beta = 0,1,\ldots,n-1, \tag{2.3}$$

where $a_{\alpha,r}$ satisfies (2.2). Given initial values $p^0_{i,j} \in \mathbb{R}^N, i,j \in \mathbb{Z}$, then in the limit $k \to \infty$, the process (2.3) defines an infinite set of points in $\mathbb{R}^{\mathbb{N}}$. The sequence of values $\{p^k_{i,j}\}$ is related, in a natural way, with the diadic mesh points $(\frac{i}{n^k}, \frac{j}{n^k})$, $i,j \in \mathbb{Z}$. The process then defines a scheme whereby $p^{k+1}_{ni+\alpha,nj+\beta}$ replaces the value $p^k_{i+\alpha/n,j+\beta/n}$ at the mesh point $(\frac{i+\alpha/n}{n^k}, \frac{j+\beta/n}{n^k})$ for $\alpha,\beta \in \{0,n\}$, while the values $p^{k+1}_{ni+\alpha,nj+\beta}$ are inserted at the new mesh points $(\frac{ni+\alpha}{n^{k+1}}, \frac{nj+\beta}{n^{k+1}})$ for $\alpha,\beta = 0,1,\ldots,n-1$ (where α & β are not zero at the same time).

Subdivision depth:
Given control polygon of n-ary subdivision curve/surface and an error tolerance ϵ, if we subdivide control polygon k times so that the error between resulting polygon and subdivision curve/surface is smaller than ϵ, then k is called subdivision depth of subdivision curve/surface with respect to ϵ.

Notations:
Here we settle some notations for fair reading of this paper. Assume

$$\chi = \max_i \left\| p^0_{i+1} - p^0_i \right\|, \tag{2.4}$$

$$\delta_1 = \max_\beta \left\{ \left| \sum_{j=0}^{m} b_{\beta,j} \right|, \ \beta = 0,1,\ldots,n-1 \right\}, \tag{2.5}$$

$$\delta_2 = \max_{\alpha,\beta} \left\{ \left| \sum_{s=0}^{m} a_{\alpha,s} \sum_{r=0}^{m} b_{\beta,r} \right|, \ \alpha,\beta = 0,1,\ldots,n-1 \right\}, \tag{2.6}$$

where

$$\begin{cases} b_{\beta,j} = \sum\limits_{t=0}^{j}(a_{\beta,t} - a_{\beta+1,t}), & \beta = 0,1,\ldots,n-2, \\ b_{n-1,j} = a_{0,j} - \sum\limits_{\beta=0}^{n-2} b_{\beta,j}. \end{cases} \tag{2.7}$$

Also

$$\gamma = \max_\alpha \left\{ \left| \sum_{j=0}^{m-1} \tilde{a}_{\alpha,j} \right|, \ \alpha = 0,1,\ldots,n-1 \right\}, \tag{2.8}$$

$$\eta^1_{\alpha,\beta} = \left| a_{\beta,0}\sum_{t=1}^{m} a_{\alpha,t} - \frac{\alpha(n-\beta)}{n^2}\right| + \left| a_{\beta,0}\sum_{s=1}^{m-1} \tilde{a}_{\alpha,s}\right|, \tag{2.9}$$

$$\eta^2_{\alpha,\beta} = \left| \sum_{t=1}^{m} a_{\beta,t} - \frac{\beta}{n}\right| + \left| \sum_{r=0}^{m} a_{\alpha,r}\sum_{s=1}^{m-1} \tilde{a}_{\beta,s}\right|, \tag{2.10}$$

$$\eta^3_{\alpha,\beta} = \left| \sum_{t=1}^{m} a_{\alpha,t}\sum_{t=1}^{m} a_{\beta,t} - \frac{\alpha\beta}{n^2}\right| + \left| \sum_{t=1}^{m} a_{\beta,t}\sum_{s=1}^{m-1} \tilde{a}_{\alpha,s}\right|, \tag{2.11}$$

where

$$\begin{cases} \tilde{a}_{\alpha,0} = \sum\limits_{t=1}^{m} a_{\alpha,t} - \frac{\alpha}{n}, \\ \tilde{a}_{\alpha,j} = \sum\limits_{t=j+1}^{m} a_{\alpha,t},\, j \geqslant 1, \end{cases} \quad \alpha = 0,1,\ldots,n-1. \tag{2.12}$$

Suppose further for $\alpha, \beta = 0, 1, \ldots, n-1$,

$$N^k_\alpha = \left\| p^{k+1}_{ni+\alpha} - \frac{1}{n}\left((n-\alpha)p^k_i + \alpha p^k_{i+1}\right)\right\|, \tag{2.13}$$

$$\begin{aligned} M^k_{\alpha,\beta} = \Big\| p^{k+1}_{ni+\alpha,nj+\beta} - \frac{1}{n^2}\{&(n-\alpha)(n-\beta)p^k_{i,j} \\ &+\alpha(n-\beta)p^k_{i+1,j} + (n-\alpha)\beta p^k_{i,j+1} + \alpha\beta p^k_{i+1,j+1}\}\Big\|, \end{aligned} \tag{2.14}$$

Assume

$$\begin{cases} \Delta^k_{i,j,1} = p^k_{i+1,j} - p^k_{i,j}, \\ \Delta^k_{i,j,2} = p^k_{i,j+1} - p^k_{i,j}, \\ \Delta^k_{i,j,3} = p^k_{i+1,j+1} - p^k_{i,j+1}, \end{cases} \quad \chi_t = \max_{i,j}\left\|\Delta^0_{i,j,t}\right\|,\ t = 1,2,3. \tag{2.15}$$

Further more suppose

$$\vartheta = \max_{\alpha,\beta}\left\{ \sum_{t=1}^{3} (\chi_t)(\eta^t_{\alpha,\beta}),\ \alpha,\beta = 0,1,\ldots,n-1 \right\}. \tag{2.16}$$

3 Depth for n-ary subdivision curves

In this section we find subdivision depth for n-ary subdivision curves. Moreover we prove that error bounds for binary and ternary subdivision curves [4] & [5] are special cases of our bounds.

Lemma 3.1. *Given an initial control polygon $p^0_i = p_i$, $i \in \mathbb{Z}$, let the values p^k_i, $k \geqslant 1$ be defined recursively by subdivision process* (2.1) *together with* (2.2) *then*

$$N^k_\alpha \leqslant \gamma\chi(\delta^k_1), \tag{3.1}$$

where χ, δ_1, N^k_α and γ are defined by (2.4), (2.5), (2.8) & (2.15) respectively.

Proof. Proof is given in Appendix A. $\square$

Lemma 3.2. *Given an initial control polygon $p_i^0 = p_i$, $i \in \mathbb{Z}$, let the values p_i^k, $k \geqslant 1$ be defined recursively by subdivision process* (2.1) *together with* (2.2). *Suppose P^k be the piecewise linear interpolant to the values p_i^k and P^∞ be the limit curve of the process. If $\delta_1 < 1$ then the error bound between limit curve and its control polygon after k-fold subdivision is*

$$\left\|P^k - P^\infty\right\|_\infty \leqslant \gamma\chi\left(\frac{(\delta_1)^k}{1-\delta_1}\right), \tag{3.2}$$

where χ, δ_1 and γ are defined by (2.4), (2.5) & (2.8) respectively.

Proof. Let $\|.\|_\infty$ denote the maximum norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)$th mesh, we have

$$\left\|P^{k+1} - P^k\right\|_\infty \leqslant \max_\alpha\{N_\alpha^k,\ \alpha = 0, 1, \ldots, n-1\}, \tag{3.3}$$

where N_α^k is defined by (2.13). From (3.1) and (3.3) we get

$$\left\|P^{k+1} - P^k\right\|_\infty \leqslant \gamma\chi(\delta_1^k),$$

where χ, δ_1 and γ are defined by (2.4), (2.5) & (2.8) respectively.
Triangle inequality yields (3.2). This completes the proof. $\square$

Remark 3.1. Here we mention that for $n = 2$ & 3 Lemma 3.2 reduces to Theorem 1 [4] and Theorem 2.1 [5] respectively.

Now we offer the computational formula of subdivision depth for n-ary subdivision curves.

Theorem 3.3. *Let k be subdivision depth and let d^k be the error bound between n-ary subdivision curve P^∞ and its k-level control polygon P^k. For arbitrary $\epsilon > 0$, if*

$$k \geqslant \log_{\delta_1^{-1}}\left(\frac{\gamma\chi}{\epsilon(1-\delta_1)}\right),$$

then

$$d^k \leqslant \epsilon.$$

Proof. From (3.2), we have

$$d^k = \left\|P^k - P^\infty\right\|_\infty \leqslant \gamma\chi\left(\frac{(\delta_1)^k}{1-\delta_1}\right),$$

This implies, for arbitrary given $\epsilon > 0$, when subdivision depth k satisfy the following inequality

$$k \;\geqslant\; \log_{\delta_1{}^{-1}}\left(\frac{\gamma\chi}{\epsilon(1-\delta_1)}\right),$$

then

$$d^k \leqslant \epsilon.$$

This completes the proof. □

4 Depth for n-ary subdivision surfaces

In this paragraph we compute subdivision depth for n-ary subdivision surfaces. Moreover, we show that results of error bounds for binary and ternary subdivision surfaces [4] & [5] are special cases of our result. Here we need following lemmas for Theorem 4.4. The proof of first two lemmas are shown in Appendices B & C respectively.

Lemma 4.1. *Given an initial control polygon $p^0_{i,j} = p_{i,j}$, $i, j \in \mathbb{Z}$, let the values $p^k_{i,j}$, $k \geqslant 1$ be defined recursively by subdivision process* (2.3) *together with* (2.2) *then*

$$\max_{i,j}\left\|\Delta^k_{i,j,t}\right\| \leqslant (\delta_2)^k \max_{i,j}\left\|\Delta^0_{i,j,t}\right\|, \tag{4.1}$$

where δ_2, $\Delta^k_{i,j,t}$, $t = 1, 2, 3$ are defined by (2.6) & (2.15) respectively.

Lemma 4.2. *Given an initial control polygon $p^0_{i,j} = p_{i,j}$, $i, j \in \mathbb{Z}$, let the values $p^k_{i,j}$, $k \geqslant 1$ be defined recursively by subdivision process (2.3) together with* (2.2) *then*

$$M^k_{\alpha,\beta} \leqslant (\delta_2)^k \sum_{t=1}^{3}(\chi_t)(\eta^t_{\alpha,\beta}), \tag{4.2}$$

where $\delta_2, \eta^t_{\alpha,\beta}, M^k_{\alpha,\beta}, \chi_t$, $\alpha, \beta = 0, 1, \ldots, n-1$ are defined by (2.6),(2.9)–(2.11),(2.14) & (2.15).

Lemma 4.3. *Given an initial control polygon $p^0_{i,j} = p_{i,j}$, $i, j \in \mathbb{Z}$, let the values $p^k_{i,j}$, $k \geqslant 1$ be defined recursively by subdivision process* (2.3) *together with* (2.2). *Suppose P^k be the piecewise linear interpolant to the values $p^k_{i,j}$ and P^∞ be the limit surface of the subdivision process* (2.3). *If $\delta_2 < 1$, then the error bound between the limit surface and its control polygon after k-fold subdivision is*

$$\left\|P^k - P^\infty\right\|_\infty \leqslant \vartheta\left(\frac{(\delta_2)^k}{1-\delta_2}\right), \tag{4.3}$$

where δ_2 and ϑ are defined by (2.6) & (2.16) respectively.

Proof. Let $\|.\|_\infty$ denote the uniform norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)$th mesh, we have

$$\left\|P^{k+1} - P^k\right\|_\infty \leqslant \max_{\alpha,\beta} \left\{M^k_{\alpha,\beta},\ \alpha,\beta = 0,1,\ldots,n-1\right\}, \tag{4.4}$$

where $M^k_{\alpha,\beta}$ is defined by (2.14). Using (4.2) & (4.4) we get

$$\left\|P^{k+1} - P^k\right\|_\infty \leqslant \vartheta(\delta_2)^k,$$

where δ_2 and ϑ are defined by (2.6) & (2.16) respectively.
By triangle inequality we get (4.3). This completes the proof. □

Remark 4.1. Here we point out that for $n = 2$ & 3 Lemma 4.3 reduces to Theorem 7 [4] and Theorem 3.2 [5] respectively.

Here we suggest the computational formula of subdivision depth for n-ary subdivision surfaces.

Theorem 4.4. *Let k be subdivision depth and let d^k be the error bound between n-ary subdivision surface P^∞ and its k-level control polygon P^k. For arbitrary $\epsilon > 0$, if*

$$k \geqslant \log_{\delta_2{}^{-1}}\left(\frac{\vartheta}{\epsilon(1-\delta_2)}\right),$$

then

$$d^k \leqslant \epsilon.$$

Proof. From (4.3), we have

$$d^k = \left\|P^k - P^\infty\right\|_\infty \leqslant \vartheta\left(\frac{(\delta_2)^k}{1-\delta_2}\right),$$

This implies, for arbitrary given $\epsilon > 0$, when subdivision depth k satisfy the following inequality

$$k \geqslant \log_{\delta_2{}^{-1}}\left(\frac{\vartheta}{\epsilon(1-\delta_2)}\right),$$

then

$$d^k \leqslant \epsilon.$$

This completes the proof. □

5 Conclusion and Future work

We have computed subdivision depth based on error bounds for more general n-ary subdivision schemes. Furthermore, we have shown that error bounds for binary and ternary subdivision schemes [4] & [5] are special cases of our bounds. The authors are looking, as a future work, to extend the computational techniques of subdivision depth for n-ary subdivision schemes over volumetric models.

References

[1] N. Aspert (2003), Non-linear subdivision of univariate signals and discrete surfaces, EPFL thesis

[2] F. Cheng and J. H. Yong (2006), Subdivision depth computation for Catmull-Clark subdivision surface, Computer Aided Geometric Design and Applications, 3(1-4): 485-494

[3] F. Cheng, G. Chen and J. Yong, Subdivision depth computation for extraordinary Catmull-Clark subdivision surface patches (complete version), www.cs.uky.edu/∼cheng/PUBL/sub_depth_2.pdf

[4] Ghulam Mustafa, Chen Falai, Jiansong Deng (2006), Estimating error bounds for binary subdivision curves/surfaces, J. Comp. Appl. Math., 193: 596-613

[5] Ghulam Mustafa, Jiangsong Deng (2007), Estimating error bounds for ternary subdivision curve/surfaces, J. Comput. Math., 24(4): 473-484

[6] Ghulam Mustafa, Sadiq Hashmi, Nusrat Anjum Noshi (2006), Estimating error bounds for tensor product binary subdivision volumetric model, International J. Computer Math., 12(83): 879–903

[7] Kwan Pyo Ko (2007), A study on subdivision scheme-draft, Dongseo University Busan South Korea, http://kowon.dongseo.ac.kr/∼kpko/ publication/ 2004book.pdf

[8] Wang Huawei, K. H. Qin (2004), Estimating subdivision depth of Catmull-Clark surfaces, J. Comp. Sci. Technol., 19(5): 657-664

[9] Wang Huawei, Guan Youjiang and Qin Kaihuai (2002), Error estimate for Doo-Sabin surfaces, Progress in Natural Sci., 12(9): 697-700

[10] Wang Huawei, Hanqiu Sun, Kaihuai Qin (2004), Estimating recursion depth for Loop subdivision, International Journal of CAD/CAM, 4(1): 11-18

[11] Xiao-Ming Zeng, X. J. Chen (2006), Computational formula of depth for Catmull-Clark subdivion surfaces, J. Comp. Appl. Math., 195(1-2): 252-262

6 Appendix A: Proof of Lemma 3.1

Proof. From (2.1) and (2.2) for $\alpha = 0, 1, \ldots, n-1$ we obtain

$$p^{k+1}_{ni+\alpha} - \frac{1}{n}\left((n-\alpha)p^k_i + \alpha p^k_{i+1}\right) = \sum_{j=0}^{m-1} \tilde{a}_{\alpha,j}(p^k_{i+j+\alpha+1} - p^k_{i+j+\alpha}), \tag{6.1}$$

where $\tilde{a}_{\alpha,j}$ is defined by (2.12).
By (2.1), (2.2) and induction on m for $\beta = 0, 1, \ldots, n-1$ we get

$$p^k_{ni+\beta+1} - p^k_{ni+\beta} = \sum_{j=0}^{m} b_{\beta,j}(p^{k-1}_{i+j+1} - p^{k-1}_{i+j}), \tag{6.2}$$

where $b_{\beta,j}$ is defined by (2.7). It follows from (2.12), (2.13) and (6.1) that

$$N^k_\alpha \leqslant \max_\alpha \left|\sum_{j=0}^{m-1} \tilde{a}_{\alpha,j}\right| \max_i \left\|p^k_{i+1} - p^k_i\right\|. \tag{6.3}$$

Using (6.2) recursively gives

$$\max_i \left\|p^k_{i+1} - p^k_i\right\| \leqslant \left(\max_\beta \left|\sum_{j=0}^{m} b_{\beta,j}\right|\right)^k \max_i \left\|p^0_{i+1} - p^0_i\right\|. \tag{6.4}$$

By (6.3) and (6.4) we get (3.1). This completes the proof. □

7 Appendix B: Proof of Lemma 4.1

Proof. From (2.2), (2.3) and using similar approach as we did for (6.2), we obtain

$$p^k_{ni+\alpha+1,nj+\beta} - p^k_{ni+\alpha,nj+\beta} = \sum_{s=0}^{m} a_{\beta,s}\left(\sum_{r=0}^{m} b_{\alpha,r}\left(p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}\right)\right), \tag{7.1}$$

$$p^k_{ni+\alpha+1,nj+n} - p^k_{ni+\alpha,nj+n} = \sum_{s=0}^{m} a_{0,s}\left(\sum_{r=0}^{m} b_{\alpha,r}\left(p^{k-1}_{i+r+1,j+s+1} - p^{k-1}_{i+r,j+s+1}\right)\right), \tag{7.2}$$

$$p^k_{ni+\alpha,nj+\beta+1} - p^k_{ni+\alpha,nj+\beta} = \sum_{r=0}^{m} a_{\alpha,r}\left(\sum_{s=0}^{m} b_{\beta,s}\left(p^{k-1}_{i+r,j+s+1} - p^{k-1}_{i+r,j+s}\right)\right), \tag{7.3}$$

$$p^k_{ni+n,nj+\beta+1} - p^k_{ni+n,nj+\beta} = \sum_{r=0}^{m} a_{0,r}\left(\sum_{s=0}^{m} b_{\beta,s}\left(p^{k-1}_{i+r+1,j+s+1} - p^{k-1}_{i+r+1,j+s}\right)\right), \tag{7.4}$$

where $b_{\beta,r}$ is defined by (2.7) and $\alpha, \beta = 0, 1, \ldots, n-1$.
Now using (7.1) recursively together with notations defined by (2.15), we get

$$\max_{i,j} \left\| \Delta^k_{i,j,1} \right\| \leqslant \left(\max_{\alpha,\beta} \left| \sum_{s=0}^{m} a_{\alpha,s} \sum_{r=0}^{m} b_{\beta,r} \right| \right)^k \max_{i,j} \left\| \Delta^0_{i,j,1} \right\|.$$

From (2.6) and above inequality we get

$$\max_{i,j} \left\| \Delta^k_{i,j,1} \right\| \leqslant (\delta_2)^k \max_{i,j} \left\| \Delta^0_{i,j,1} \right\|.$$

Again using (7.2) recursively and by utilizing (2.6) & (2.15) we have

$$\max_{i,j} \left\| \Delta^k_{i,j,3} \right\| \leqslant (\delta_2)^k \max_{i,j} \left\| \Delta^0_{i,j,3} \right\|.$$

Similarly, using (7.3) & (7.4) recursively together with (2.6) & (2.15)

$$\max_{i,j} \left\| \Delta^k_{i,j,2} \right\| \leqslant (\delta_2)^k \max_{i,j} \left\| \Delta^0_{i,j,2} \right\|.$$

This completes the proof. □

8 Appendix C: Proof of Lemma 4.2

Proof. From (2.2) and (2.3) we get

$$p^{k+1}_{ni,nj} - p^k_{i,j} = \sum_{r=0}^{m} a_{0,r} \left(\sum_{s=0}^{m} a_{0,s} (p^k_{i+r,j+s} - p^k_{i,j}) \right). \tag{8.1}$$

Since

$$\begin{aligned}
&\sum_{s=0}^{m} a_{0,s} (p^k_{i+r,j+s} - p^k_{i,j}) \\
&\quad = a_{0,0}(p^k_{i+r,j} - p^k_{i,j}) + a_{0,1}(p^k_{i+r,j+1} - p^k_{i,j}) \\
&\quad\quad + a_{0,2}(p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}) \\
&\quad\quad + a_{0,3}(p^k_{i+r,j+3} - p^k_{i+r,j+2} + p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}) + \ldots \\
&\quad\quad + a_{0,m}(p^k_{i+r,j+m} - p^k_{i+r,j+m-1} + p^k_{i+r,j+m-1} - \ldots - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}),
\end{aligned}$$

therefore

$$\begin{aligned}
&\sum_{s=0}^{m} a_{0,s} (p^k_{i+r,j+s} - p^k_{i,j}) \\
&\quad = a_{0,0}(p^k_{i+r,j} - p^k_{i,j}) + \sum_{t=1}^{m} a_{0,t}(p^k_{i+r,j+1} - p^k_{i,j}) + \sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s}),
\end{aligned}$$

where $\tilde{a}_{0,s}$ is defined by (2.12). Taking summation on both side of above equation we get

$$\sum_{r=0}^{m} a_{0,r}\left(\sum_{s=0}^{m} a_{0,s}(p^k_{i+r,j+s} - p^k_{i,j})\right) = a_{0,0}\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j} - p^k_{i,j}) +$$
$$\sum_{t=1}^{m} a_{0,t}\left(\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j+1} - p^k_{i,j})\right) + \sum_{r=0}^{m} a_{0,r}\left(\sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right).$$

Since

$$\begin{aligned}\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j} - p^k_{i,j}) &= a_{0,1}(p^k_{i+1,j} - p^k_{i,j}) + a_{0,2}(p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j}) \\ &+ a_{0,3}(p^k_{i+3,j} - p^k_{i+2,j} + p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j}) + \dots \\ &+ a_{0,m}(p^k_{i+m,j} - p^k_{i+m-1,j} + p^k_{i+m-1,j} - \dots + p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j}),\end{aligned}$$

therefore

$$\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j} - p^k_{i,j}) = \sum_{t=1}^{m} a_{0,t}(p^k_{i+1,j} - p^k_{i,j}) + \sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j} - p^k_{i+s,j}).$$

Similarly

$$\begin{aligned}\sum_{r=0}^{m} a_{0,r}(p^k_{i+r,j+1} - p^k_{i,j}) = a_{0,0}(p^k_{i,j+1} - p^k_{i,j}) &+ \sum_{t=1}^{m} a_{0,t}(p^k_{i+1,j+1} - p^k_{i,j}) \\ &+ \sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j+1} - p^k_{i+s,j+1}).\end{aligned}$$

Substituting these summations into (8.1) then by (2.14) we obtain

$$\begin{aligned}M^k_{0,0} = \left(a_{0,0}\sum_{t=1}^{m} a_{0,t}\right)(p^k_{i+1,j} - p^k_{i,j}) &+ \left(\sum_{t=1}^{m} a_{0,t}\right)^2 (p^k_{i+1,j+1} - p^k_{i,j+1}) \\ &+ \sum_{t=1}^{m} a_{0,t}(p^k_{i,j+1} - p^k_{i,j}) + a_{0,0}\sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j} - p^k_{i+s,j}) \\ &+ \sum_{t=1}^{m} a_{0,t}\sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+s+1,j+1} - p^k_{i+s,j+1}) \\ &+ \sum_{r=0}^{m} a_{0,r}\left(\sum_{s=1}^{m-1} \tilde{a}_{0,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right).\end{aligned} \tag{8.2}$$

Similarly from (2.2), (2.3) and (2.14) for $\alpha, \beta = 0, 1, \ldots, n-1$ (where α & β are not zero at the same time) we obtain

$$
\begin{aligned}
M^k_{\alpha,\beta} = &\left(a_{\beta,0}\sum_{t=1}^{m} a_{\alpha,t} - \frac{\alpha(n-\beta)}{n^2}\right)(p^k_{i+1,j} - p^k_{i,j}) \\
&+ \left(\sum_{t=1}^{m} a_{\alpha,t}\sum_{t=1}^{m} a_{\beta,t} - \frac{\alpha\beta}{n^2}\right)(p^k_{i+1,j+1} - p^k_{i,j+1}) \\
&+ \left(\sum_{t=1}^{m} a_{\beta,t} - \frac{\beta}{n}\right)(p^k_{i,j+1} - p^k_{i,j}) + a_{\beta,0}\sum_{s=1}^{m-1}\tilde{a}_{\alpha,s}(p^k_{i+s+1,j} - p^k_{i+s,j}) \\
&+ \sum_{t=1}^{m} a_{\beta,t}\sum_{s=1}^{m-1}\tilde{a}_{\alpha,s}(p^k_{i+s+1,j+1} - p^k_{i+s,j+1}) \\
&+ \sum_{r=0}^{m} a_{\alpha,r}\left(\sum_{s=1}^{m-1}\tilde{a}_{\beta,s}(p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right),
\end{aligned}
\tag{8.3}
$$

where $\tilde{a}_{\alpha,s}$ is defined by (2.12).
Using (2.6), (4.1), (8.2) and (8.3) for $\alpha, \beta = 0, 1, \ldots, n-1$ we get

$$
\begin{aligned}
M^k_{\alpha,\beta} \leqslant (\delta_2)^k \Bigg\{ &\left(\left|a_{\beta,0}\sum_{t=1}^{m} a_{\alpha,t} - \frac{\alpha(n-\beta)}{n^2}\right| + \left|a_{\beta,0}\sum_{s=1}^{m-1}\tilde{a}_{\alpha,s}\right|\right)\max_{i,j}\left\|\Delta^0_{i,j,1}\right\| \\
&+ \left(\left|\sum_{t=1}^{m} a_{\beta,t} - \frac{\beta}{n}\right| + \left|\sum_{r=0}^{m} a_{\alpha,r}\sum_{s=1}^{m-1}\tilde{a}_{\beta,s}\right|\right)\max_{i,j}\left\|\Delta^0_{i,j,2}\right\| \\
&+ \left(\left|\sum_{t=1}^{m} a_{\alpha,t}\sum_{t=1}^{m} a_{\beta,t} - \frac{\alpha\beta}{n^2}\right| + \left|\sum_{t=1}^{m} a_{\beta,t}\sum_{s=1}^{m-1}\tilde{a}_{\alpha,s}\right|\right)\max_{i,j}\left\|\Delta^0_{i,j,3}\right\|\Bigg\}.
\end{aligned}
$$

Utilizing notations (2.9)–(2.11), (2.14) & (2.15) we get (4.2).
This completes the proof. □