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Analysis of a collocation method for integrating rapidly oscillatory functions

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Abstract

A collocation method for approximating integrals of rapidly oscillatory functions is analyzed. The method is efficient for integrals involving Bessel functions $J_\nu(rx)$ with a large oscillation frequency parameter r, as well as for many other one- and multi-dimensional integrals of functions with rapid irregular oscillations. The analysis provides a convergence rate and it shows that the relative error of the method is even decreasing as the frequency of the oscillations increases.

Keywords: Oscillatory integrals; Collocation analysis

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1. Introduction

In [2] a new method for the numerical integration of rapidly oscillatory integrals is presented and tested. The integrals are of the form

$$I = \int_a^b g(x)S(rx)\,\mathrm{d}x, \tag{1.1}$$

where S is an oscillatory function and r is a large parameter and a and b are real and finite. Approximating I by usual numerical integration algorithms requires many function evaluations of g and S. Sometimes the evaluation of g is very expensive, and the computation becomes highly time consuming for very large values of r. The method in [2] is an extension of the collocation method presented in [1] for integrals of the type

$$I = \int_a^b g(x)\mathrm{e}^{\mathrm{i}q(x)}\,\mathrm{d}x, \tag{1.2}$$

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with $\max_{x\in[a,b]}\{|q'(x)|\} \gg (b-a)^{-1}$. The collocation method presented in [2] is applicable to a wide class of oscillatory integrals with weight functions S satisfying certain differential conditions. For example, it is appropriate for computing integrals of the form

$$\int_a^b g(x)\cos(r_1x)J_\nu(r_2x)\,dx, \tag{1.3}$$

for large r_1 and r_2 and integrals involving $S(x) = J_\nu^2(rx)$. It is demonstrated in [2] by numerical examples that the efficiency of the method does not deteriorate as the parameters r, r_1 or r_2 increase. Simple classification rules for a large class of oscillatory functions S which satisfy the required conditions for the application of the method are also presented in [2].

An analysis of the collocation method presented in [1], for integrals of the form (1.2), is presented in [3]. In the present paper we extend the analysis in [3] to the more general framework of [2]. In the next section we review the collocation method of [2]. It is presented as an nth-order collocation method, where n is the number of collocation points, or equivalently the dimension of the approximation space. In the present work we analyze an h-method version of that method. The interval $[a,b]$ is subdivided into subintervals of length h, on each of which an nth-order collocation method is applied and the resulting approximations are aggregated. The error analysis presented in Section 3 deals with the case of a *fixed* order n and $h \to 0$. The important result obtained here is that the relative error in the collocation approximation is even decreasing as the frequency of the oscillations increases.

2. The collocation scheme

The method presented in [2] is designed to handle a wide class of rapidly oscillatory integrals of the form

$$I = \int_a^b w^t(x)f(x)\,dx \equiv \int_a^b \langle w,f\rangle(x)\,dx, \tag{2.1}$$

where $f(x) = (f_1(x),\dots,f_m(x))^t$ is an m-vector of nonrapidly oscillatory functions and $w(x) = (w_1(x),\dots,w_m(x))^t$ is an m-vector of linearly independent rapidly oscillatory functions. It is assumed that $\{w_i\}_{i=1}^m$ satisfy a system of ordinary differential equation of the form

$$w'(x) = A(x)w(x), \tag{2.2}$$

where $A(x)$ is an $m\times m$ matrix of nonrapidly oscillatory functions. The integrals considered in [1] satisfy (2.1)–(2.2) with $m=1$, while integrals involving Bessel functions, as $\int_a^b g(x)J_\nu(rx)\,dx$ with large r, satisfy (2.1)–(2.2) with $m=2$. For the second case we take $w(x) = (J_{\nu-1}(rx), J_\nu(rx))^t$, which satisfies (2.2) with

$$A(x) = \begin{pmatrix} (\nu-1)/x & -r \\ r & -\nu/x \end{pmatrix}. \tag{2.3}$$

The idea in [2] is to represent the integrand in (2.1) as a derivative of a known function. Namely, let $v(x) = (v_1(x),\dots,v_m(x))^t$ be an m-vector function such that

$$\langle w,v\rangle' = \langle w,f\rangle, \tag{2.4}$$

then the integral I can be expressed as

$$I = \int_a^b \langle w, v\rangle'(x)\,\mathrm{d}x = w^{\mathrm{t}}(b)v(b) - w^{\mathrm{t}}(a)v(a). \tag{2.5}$$

It follows from (2.4) that v should satisfy a system of ordinary differential equations

$$Lv \equiv v' + A^{\mathrm{t}}v = f. \tag{2.6}$$

The problem of evaluating I is thus transformed into the problem of finding an approximation to **a** solution of (2.6), with no boundary conditions prescribed.

By the above assumptions, f and A are not rapidly oscillatory. Therefore, as we are going to prove here, the system (2.6) has a particular solution which is not rapidly oscillatory. It is suggested in [2] to find an approximation to this particular solution by collocation with 'nice' functions, e.g., polynomials, as follows.

For $i = 1,\dots,m$ let $\{u_k^{[i]}\}_{k=1}^n$ be some linearly independent basis functions on $[a,b]$. An n-point (nth-order) collocation approximation to the solution of (2.6) is defined as $\tilde{v}(x) = (\tilde{v}_1(x),\dots,\tilde{v}_m(x))^{\mathrm{t}}$, where

$$\tilde{v}_i(x) = \sum_{k-1}^{n} c_{i,k}u_k^{[i]}(x), \quad i = 1,\dots,m, \tag{2.7}$$

where the coefficients $\{c_{i,k}\}_{i=1,k=1}^{m,n}$ are determined by the collocation conditions

$$L\tilde{v}(x_j) = f(x_j), \quad j = 1,\dots,n, \tag{2.8}$$

where $\{x_j\}_{j=1}^n$ are regularly distributed in $[a,b]$. Following (2.5), the corresponding n-point approximation to the integral (2.1) is given by

$$I_n \equiv (w(b))^{\mathrm{t}}\tilde{v}(b) - (w(a))^{\mathrm{t}}\tilde{v}(a). \tag{2.9}$$

We remark that the order n is going to be fixed throughout the paper. Therefore, to reduce notation complexity, we avoid indexing the nth-order approximation to the solution of (2.6) by n. It follows from (2.5) that the error $I - I_n$ is determined by the error in the approximate solution of (2.6) at the endpoints of the interval, and this matter is pursued in the next section. Numerical examples demonstrating the application of the collocation method are given in [2]. The examples include $S(x) = J_\nu(rx)$, $S(x) = J_\nu^2(rx)$ and $S(x) = \cos(r_1x)J_\nu(r_2x)$, and it is shown that the collocation method is easily applicable for all these cases, and is very efficient.

3. Error analysis

In the present work we fix the order n of the collocation and consider an h-method version as follows: The interval $[a,b]$ is subdivided into ℓ equal subintervals $[t_{j-1},t_j]$, $1\leqslant j\leqslant \ell$, $t_j = a + jh$, of length $h = (b-a)/\ell$. On each subinterval we choose n collocation points

$$s_i^{[j]} = t_{j-1} + hs_i, \quad 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant \ell,$$

where $0 = s_1 < s_2 < \cdots < s_n = 1$ are reference points. The above nth-order collocation method is performed separately on each subinterval, using the monomial basis functions, namely, $\{u_k^{[i]}(x) =$

$x^{k-1}\}_{k=1}^{n}$. Let $v^{[j]}$ denote the resulting nth-order collocation approximation on $[t_{j-1}, t_j]$, $1 \leqslant j \leqslant \ell$. Summing up the ℓ approximations to the integrals on the individual intervals, we thus obtain an approximation of order n to I, which we denote as $I_{n,h}$:

$$I_{n,h} = \sum_{j=1}^{\ell} [(w(t_j))^{t} v^{[j]}(t_j) - (w(t_{j-1}))^{t} v^{[j]}(t_{j-1})]. \tag{3.1}$$

The error analysis is carried out for a class of integrals of the form (2.1), where $w(x)$ satisfies the o.d.e. system

$$w'(x) = \frac{1}{\varepsilon}\tilde{A}(x)w(x). \tag{3.2}$$

The major assumptions we make here are that $B(x) \equiv (\tilde{A}(x))^{-1}$ exists, is in $C^{2n+1}[a,b]$, and that w and the $2n+1$ derivatives of B are bounded for $0 < \varepsilon < \alpha < 1$. For example, for the case of integrals involving Bessel functions, we can write the matrix function $A(x)$ defined in (2.3) as

$$A(x) = \frac{1}{\varepsilon}\tilde{A}(x),$$

where $\varepsilon = 1/r$. For $a > 0$ and a large enough r the above conditions hold.

In analyzing a numerical method for computing integrals of rapidly oscillatory functions, it is important to note that in many cases the value of the integral I itself tends to zero as the oscillations' frequency increases. It is important to have an error estimate which accounts for this fact. Under the conditions specified in Theorem 3.1 below, it is shown that we have

$$I = \int_a^b w^{t}(x) f(x)\,dx = O(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

The main parameters governing the collocation method are h, the subintervals' length, n, the number of collocation points at each subinterval, and the oscillation parameter ε (or the 'frequency' $1/\varepsilon$). The main result of this paper is the following theorem which gives the convergence rate, for a *fixed* order n, as a function of ε and h.

Theorem 3.1. *Let $f \in C^{2n+1}[a,b]$, $B(x) \equiv (\varepsilon A(x))^{-1}$ exists, $B \in C^{2n+1}[a,b]$, and its $2n+1$ derivatives are bounded, uniformly in ε, for $\varepsilon \in (0,\alpha)$. Then*

$$|I - I_{n,h}| < c\varepsilon^2 h^{n-1}, \tag{3.4}$$

for $\varepsilon \in (0,\beta]$, $\beta > 0$, where c does not depend on ε and h.

The proof of Theorem 3.1 follows from the following two lemmas. The first lemma is concerned with the existence of a 'nonoscillatory' solution of (2.6), namely, a solution which has sufficiently many derivatives bounded uniformly for $\varepsilon \in (0,\alpha)$. The second lemma deals with the error in the collocation approximation to that solution.

Lemma 3.2. *Let f and A satisfy the conditions of Theorem* 3.1. *Then there exists a solution $v(x)$ of* (2.6),

$$Lv = v' + A^{\mathrm{t}}v = f,$$

which satisfies

$$\|D^i v(x)\| < c\varepsilon \quad \text{on } [a,b], \quad i = 0,\dots,n, \tag{3.5}$$

where $D^i v(x) \equiv (\mathrm{d}^i/\mathrm{d}x^i)v(x)$.

Proof. Rewriting the equation as

$$v = \varepsilon B^{\mathrm{t}}(f - v'),$$

let us define a sequence of successive approximations to the solution

$$u^{[k+1]}(x) = \varepsilon(B(x))^{\mathrm{t}}(f(x) - Du^{[k]}(x)), \quad k \geqslant 0, \tag{3.6}$$

starting with $u^{[0]} \equiv 0$. Differentiating (3.6) and using the assumption that $2n+1$ derivatives of B are bounded uniformly in ε, we find out that

$$\|D^i u^{[k]}(x)\| = \mathrm{O}(\varepsilon) \quad \text{on } [a,b], \quad i = 0,\dots,n, \ k = 1,\dots,n+1. \tag{3.7}$$

Therefore, $u^{[n+1]}$ satisfies the desired condition, but it is not a solution of (2.6). We observe that $u^{[n+1]}$ satisfies $Lu^{[n+1]} = f + D(u^{[n+1]} - u^{[n]})$. From the relation

$$u^{[k+1]}(x) - u^{[k]}(x) = \varepsilon(B(x))^{\mathrm{t}}(D(u^{[k]} - u^{[k-1]})(x)), \quad k \geqslant 0,$$

it follows that

$$\|D(u^{[n+1]} - u^{[n]})(x)\| = \mathrm{O}(\varepsilon^{n+1}).$$

Now we define $v(x)$ to be the solution of the o.d.e system (2.6) with the initial conditions $v(a) = u^{[n+1]}(a)$. The difference $d(x) = v(x) - u^{[n+1]}(x)$ satisfies

$$Ld(x) = D(u^{[n]} - u^{[n+1]})(x) \equiv g^{[0]}(x), \tag{3.8}$$

with zero initial conditions, where $g^{[0]} \in C^n[a,b]$. By standard o.d.e. theory, the solution of $Lq = g$ in $[a,b]$, with $g \in C^0[a,b]$ and initial condition $q(a) = 0$, satisfies

$$\|q(x)\| \leqslant c\|g\|_\infty, \tag{3.9}$$

for any $\varepsilon \in (0,\alpha)$, where $\|g\|_\infty \equiv \max_{x\in[a,b]} \|g(x)\|$. Here we have $\|g^{[0]}\|_\infty = \mathrm{O}(\varepsilon^{n+1})$, hence $\|d(x)\| = \mathrm{O}(\varepsilon^{n+1})$. Differentiating (3.8) it follows that d' satisfies $Ld' = g^{[1]}$ with $g^{[1]} = Dg^{[0]} - 1/\varepsilon\widetilde{A}'d$. We have $g^{[1]} \in C^{n-1}[a,b]$ and $\|g^{[1]}\|_\infty = \mathrm{O}(\varepsilon^n)$, which by (3.3) implies that $\|d'(x)\| = \mathrm{O}(\varepsilon^n)$. Repeating the differentiation process we get

$$\|D^i d(x)\| = \mathrm{O}(\varepsilon^{n-i+1}), \quad i = 0,\dots,n. \tag{3.10}$$

The result (3.4) now follows from (3.7) and (3.10). □

The estimate (3.3) results from Lemma 3.2 by representing the integral (2.1) as

$$I = \int_a^b \langle w, v\rangle'(x)\,\mathrm{d}x = w^{\mathrm{t}}(b)v(b) - w^{\mathrm{t}}(a)v(a). \tag{3.11}$$

The result of Lemma 3.2 certainly holds for any subinterval $[t_{j-1}, t_j]$, $1 \leqslant j \leqslant \ell$. To examine the error in the nth-order approximation to the integral we express it in terms of that special solution v found in Lemma 3.2:

$$I - I_{n,h} = \sum_{j-1}^{\ell} w(t_j)^{\mathrm{t}}(v(t_j) - (v^{[j]}(t_j))) - w(t_{j-1})^{\mathrm{t}}(v(t_{j-1}) - v^{[j]}(t_{j-1})). \tag{3.12}$$

Therefore, it is enough to bound the error in the local collocation approximation $v^{[j]}$ to v at the end points of each subinterval. The next lemma establishes the error in the local collocation approximation $v^{[j]}$ at the collocation points in $[t_{j-1}, t_j]$, which include the endpoints.

Lemma 3.3. *Under the assumptions of Theorem* 3.1, *let $v^{[j]}$ denote the nth-order collocation approximation on $[t_{j-1}, t_j]$, $1 \leqslant j \leqslant \ell$, satisfying*

$$Lv^{[j]}(s_i^{[j]}) = f(s_i^{[j]}), \quad i = 1, \ldots, n \tag{3.13}$$

and let v be the solution of (2.6) *ensured by Lemma* 3.2. *Then*

$$\|v(s_i^{[j]}) - v^{[j]}(s_i^{[j]})\| \leqslant c\varepsilon^2 h^{n-1}, \tag{3.14}$$

where c does not depend on ε and h.

Proof. Let $\{\lambda_k\}_{k=1}^n$ be the Lagrange basic functions for interpolation by a polynomial of degree $n-1$ at the points $\{s_i^{[j]}\}_{i=1}^n$. With these we can represent the collocation polynomial approximation on the interval $[t_{j-1}, t_j]$ as $v^{[j]}(x) = \sum_{k=1}^n v^{[j]}(s_k^{[j]})\lambda_k(x)$. Since $\lambda_k(s_i^{[j]}) = \delta_{i,k}$, the collocation equations defining $v^{[j]}$ take the form

$$Dv^{[j]}(s_i^{[j]}) + \frac{1}{\varepsilon}(\tilde{A}(s_i^{[j]}))^{\mathrm{t}} v^{[j]}(s_i^{[j]}) = f(s_i^{[j]}), \quad i = 1, \ldots, n. \tag{3.15}$$

This is a system for the vector of unknowns

$$\hat{v}^{[j]} = (v^{[j]}(s_1^{[j]}), \ldots, v^{[j]}(s_n^{[j]}))^{\mathrm{t}}$$

which can be written in the form

$$\left(L + \frac{1}{\varepsilon}\Delta\right)\hat{v}^{[j]} = \hat{f}, \tag{3.16}$$

where $\hat{f} = (f(s_1^{[j]}), \ldots, f(s_n^{[j]}))^{\mathrm{t}}$. The matrices L and Δ are of size $nm \times nm$, which can be written as block matrices:

$$L = \{L_{i,k}\}_{i,k=1}^m, \quad L_{i,k} = \lambda_k'(s_i^{[j]})I_{n\times n},$$

$$\Delta = \operatorname{diag}\{\Delta_i\}_{i=1}^m, \quad \Delta_i = (\tilde{A}(s_i^{[j]}))^{\mathrm{t}}.$$

Now we consider the particular solution v of (2.6) ensured by Lemma 3.2. Let q be the polynomial of degree $n-1$ interpolating v at $\{s_i^{[j]}\}_{i=1}^n$, then the vector of values $\widehat{v} = (v(s_1^{[j]}), \ldots, v(s_n^{[j]}))^t$ satisfies the system of equations:

$$\left(L + \frac{1}{\varepsilon}\Delta\right)\widehat{v} = \widehat{f} + \widehat{r}, \tag{3.17}$$

where $\widehat{r} = (r_1, \ldots, r_n)^t$, where

$$r_i = v'(s_i^{[j]}) - q'(s_i^{[j]}), \quad 1 \leqslant i \leqslant n.$$

From the theory of polynomial interpolation we have that

$$\|r_i\| < ch^{n-1}\|D^n v\|_\infty. \tag{3.18}$$

The vector $\widehat{e} = \widehat{v} - \widehat{v}^{[j]}$ satisfies the system

$$(\varepsilon L + \Delta)\widehat{e} = \varepsilon\widehat{r}. \tag{3.19}$$

Since Δ is block diagonal, with invertible blocks, it follows that for an ε small enough we can write $\widehat{e} = \varepsilon(\varepsilon L + \Delta)^{-1}\widehat{r}$. By the conditions on the invertibility of $\widetilde{A}$ we conclude that

$$\|\widehat{e}\| \leqslant c\varepsilon \max_{1 \leqslant i \leqslant n} \|r_i\|. \tag{3.20}$$

The result (3.14) now results from (3.20), (3.18) and (3.5). $\square$

Proof of Theorem 3.1. The proof is derived by expressing the error $I - I_{n,h}$ in the form (3.12) and using the result of Lemma 3.3. The parameter β in Theorem 3.1 is the maximal $\beta < \alpha$ so that $(\varepsilon L + \Delta)^{-1}$ exists for $\varepsilon \in (0, \beta]$. $\square$

Remark 3.4. The choice of the end points of each subinterval as collocation points is not just a technical necessity for the proof. Due to this choice we obtain an $O(\varepsilon^2)$ term in (3.4) instead of just an $O(\varepsilon)$.

Remark 3.5. The factor c in the error estimate (3.4) depends on the order n. This dependence is influenced by the bounds on the derivatives of f and Bf and by the bound on the derivatives in polynomial approximation. In case the derivatives of f and Bf are uniformly bounded in n, and the collocation points are equidistant, we get $c = O(n^{-(n-1)})$ in (3.4).

Remark 3.6. In the numerical examples in [2], the collocation points are equidistant. Better approximation results were obtained by choosing the collocation points as the extended Chebyshev points in each subinterval.

Remark 3.7. For ε small enough, and in the case that the derivatives of f and B are available, or not too expensive to compute, the proof of Lemma 3.2 suggests another approximation approach.

Using the recursive relation (3.6) we can construct an approximation $u^{[k]}$ to the solution of (2.6), and approximate I as

$$I \approx \tilde{I}_k \equiv \int_a^b \langle w, u^{[k]} \rangle'(x)\,\mathrm{d}x = w^{\mathrm{t}}(b)u^{[k]}(b) - w^{\mathrm{t}}(a)u^{[k]}(a). \tag{3.21}$$

It follows that

$$|I - \tilde{I}_k| = \mathrm{O}(\varepsilon^{k+1}). \tag{3.22}$$

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