Experimental computation with oscillatory integrals

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ABSTRACT. A previous study by one of the present authors, together with D. Borwein and I. E. Leonard [8], studied the asymptotic behavior of the p-norm of the sinc function: $\mathrm{sinc}(x) = (\sin x)/x$ and along the way looked at closed forms for integer values of p. In this study we address these integrals with the tools of experimental mathematics, namely by computing their numerical values to high precision, both as a challenge in itself, and also in an attempt to recognize the numerical values as closed-form constants. With this approach, we are able to reproduce several of the results of [8] and to find new results, both numeric and analytic, that go beyond the previous study.

1. Introduction

A previous work by one of the present authors, together with D. Borwein and I. Leonard [8], studied the behavior of the p-norm of the sinc function: $\mathrm{sinc}(x) = (\sin x)/x$. In particular, these authors considered the function $I(p)$ defined by:

$$(1) \qquad I(p) \quad := \quad \sqrt{p} \int_0^\infty \left| \frac{\sin t}{t} \right|^p \, \mathrm{d}t.$$

Plots of $I(p)$ over $(0, 10)$ and $(0, 100)$ are shown in Figures 1 and 2. In this study we wish to further explore this function, both numerically and analytically. Indeed, in [8] one finds proofs of the following composite result.

THEOREM 1. *For all $p > 1$ one has*

$$(2) \qquad I(p) \quad > \quad \sqrt{\frac{3\pi}{2}} \frac{2p}{2p+1} \quad > \quad \sqrt{\frac{3\pi}{2}} \left(1 - \frac{1}{2p} \right).$$

Moreover

$$(3) \qquad \lim_{p \to \infty} I(p) \quad = \quad \sqrt{\frac{3\pi}{2}},$$
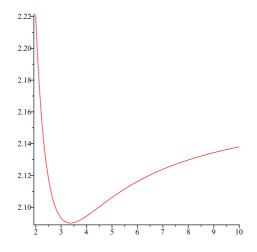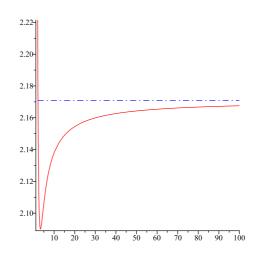
FIGURE 1.   The function I on $[2, 10]$

FIGURE 2.   The function I and its limiting value on $[2, 100]$

and there are real constants c_s such that

$$I(p) \sim \frac{1}{2}\sqrt{p} \int_{-\pi}^{\pi} \left| \frac{\sin(x)}{x} \right|^p \, dx$$

$$(4) \qquad \sim \sqrt{\frac{3\pi}{2}} - \frac{3}{20}\sqrt{\frac{3\pi}{2}}\frac{1}{p} + \sum_{s=2}^{\infty} c_s \frac{1}{p^s} + \cdots .$$

A first challenge we set ourselves was to compute the limit value in (3) sufficiently well to recognize the limit from numerical computation. Given the slow rate of convergence in (4) this is a very hard task, the lower bound being of the correct

order. The following related conjecture is made in [**8**] and is well supported by the plots in Figures 1 and 2.

Conjecture. *I is increasing for p above the conjectured global minimum near* 3.36 *and concave for p above an inflection point near* 4.469.

Note that (4) shows that $I(p)$ is concave increasing for sufficiently large p. While we can not fully resolve this conjecture, we are able—inter alia—to resolve both the critical point and inflection point to very high precision. We could not thusly identify a closed form but someone else may be able to determine a closed form for one or both of these quantities.

2. Quadrature for highly oscillatory integrals

In a previous study of quadrature algorithms for very high precision, one of the present authors, together with X. S. Li and K. Jeyabalan, found that Gaussian quadrature is generally the most efficient scheme for regular functions (even at endpoints), and for moderately high precision levels (up to several hundred digits). For functions with singularities or other difficulties at the endpoints, or for any function at very high precision (beyond about 500 digits), these authors found that the tanh-sinh quadrature algorithm of Takahasi and Mori is generally the most effective [**5**] [**26**].

The tanh-sinh scheme is simply the observation that if $x = g(t) = \tanh(\pi/2 \sinh t)$, then

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} f(g(t))g'(t)\,\mathrm{d}t \;\approx\; h \sum_{j=-N}^{N} w_j f(x_j),$$

where $x_j = g(hj)$, $w_j = g(hj)$ and N is chosen large enough that the terms of the summation are smaller than the "epsilon" of the numeric precision being used for $j > N$. Note that the resulting quadrature rule is a simple sum of weighted function values at precalculated abscissas x_j, with weights w_j — very similar to Gaussian quadrature in this regard. Because of the nature of the function $g(t)$, the transformed integrand $f(g(t))g'(t)$ rapidly goes to zero (together with all higher derivatives), even in (most) cases where the original integrand function $f(x)$ has blow-up singularities at the endpoints of the interval $[-1, 1]$. Thus tanh-sinh quadrature exhibits "exponential" or "quadratic" convergence for a wide range of integrands— dividing the interval length h by two (or, equivalently, taking twice as many abscissa points) typically doubles the number of correct digits.

Even armed with advanced quadrature techniques, numerical evaluation of (1) to high precision (say several hundred digits) presents numerous challenges. To begin with, it is first necessary to rewrite this integral as one or more integrals on finite intervals. For most integrands this is handled by a simple transformation such as:

$$\int_{0}^{\infty} f(t)\,\mathrm{d}t \;=\; \int_{0}^{1} f(t)\,\mathrm{d}t + \int_{0}^{1} \frac{f(1/t)\,\mathrm{d}t}{t^2}$$

Unfortunately, in the case of integrals such as (1), this transformation yields, as the second integrand, the function $\sin(1/t)/t^2$, which oscillates wildly near the origin. Partly for such reasons, this integral (with $p = 1$, and without the absolute value) was selected to be one of 15 test integrals in the above-mentioned study of quadrature algorithms [**5**].

As it turns out, when p is an even integer (so that the absolute value signs can be replaced by simple parentheses), such integrals can be efficiently computed by applying a clever technique recently introduced by Ooura and Mori [**23**]. Let $x = g(t) = Mt/(1 - \exp(-2\pi \sinh t))$. Then in the case of $p = 2$, for instance,

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x \; = \; \int_{-\infty}^\infty \left(\frac{\sin g(t)}{g(t)}\right)^2 \cdot g'(t)\, \mathrm{d}t \; \approx \; h \sum_{k=-N}^N \left(\frac{\sin g(hk)}{g(hk)}\right)^2 \cdot g'(hk)$$

Now note that if one chooses $M = \pi/h$, then for large k, the $g(hk)$ values are all very close to $k\pi$, so the $\sin(g(hk))$ values are all very close to zero. Thus the sum can be truncated after a modest number of terms, as in tanh-sinh quadrature. In practice, this scheme is very effective for oscillatory integrands such as this.

When p is not an even integer, neither than tanh-sinh nor the Ooura-Mori scheme can be used, because the condition of regularity except at endpoints is not met due to the absolute value signs in (1). What happens is that convergence is very slow, and even with high precision arithmetic and many abscissa-weight pairs, accuracy is typically only a few digits. One response is to write, for integer N and $2N - 1 > |\theta|$,

$$\int_0^\infty \left(\left|\frac{\sin(t)}{t}\right|\right)^{2N+\theta} \mathrm{d}t$$

(5)
$$= \sum_{n=0}^\infty \frac{\theta^n}{n!\, 2^n} \int_0^\infty \left(\frac{\sin(t)}{t}\right)^{2N} \left(\log\left(\frac{\sin(t)}{t}\right)^2\right)^n \mathrm{d}t$$

on expanding the θ-power as an exponential series. This eliminates the absolute value signs, but unfortunately the log sin term results in singularities at every integer multiple of π, and thus, sadly, is also not suitable for high-precision quadrature computation as it stands. We shall return to this type of integral in Section 4.

An alternate strategy is to employ the *Hurwitz zeta* function, which is defined as:

$$\zeta(p, x) \; := \; \sum_{n=0}^\infty \frac{1}{(n+x)^p}$$

With this definition, we can derive:

(6) $$I(p) \; := \; \sqrt{p} \int_0^\infty \left|\frac{\sin(x)}{x}\right|^p \mathrm{d}x \; = \; \frac{\sqrt{p}}{\pi^{p-1}} \int_0^1 \sin^p(\pi x)\, \zeta(p, x)\, \mathrm{d}x.$$

This can be obtained by breaking the integral in (5) on the intervals $[n\pi, (n+1)\pi]$ and gathering up the translations.

We recall that the Bateman project [**15**, (7), p. 24] records

$$\zeta(s, v) \; = \; \frac{1}{\Gamma(s)} \int_0^1 \frac{x^{v-1}(-\ln(x))^{s-1}}{1-x} \mathrm{d}x.$$

An even more efficient scheme is given by the following formula supplied by Richard Crandall [**3**]:

LEMMA 1 (Crandall). *The complete analytic continuation of* $\zeta(s,a)$ *for* $a \in (0,1), s \neq 1 + 0i,$ *is given by*

$$\zeta(s, a) \; = \; \frac{1}{\Gamma(s)} \sum_{n\geq 0} \frac{\Gamma(s, \lambda(n+a))}{(n+a)^s} + \frac{1}{\Gamma(s)} \sum_{m\geq 0} \frac{(-1)^m B_m(a)}{m!} \frac{\lambda^{m+s-1}}{m+s-1},$$

with the following interpretations: $\Gamma(s, \cdot)$ is the standard incomplete gamma function, B_n is the standard Bernoulli polynomial, λ is a free parameter with $|\lambda| < 2\pi$. For any case of integer $s = -n \leq 0$, the $\Gamma(s)$ divergence cancels a divergent m-summand, and so $\zeta(-n, a) = -B_{n+1}(a)/(n+1)$.

We chose not to use this formula because it requires access to the incomplete gamma function. Instead, for our purposes here it suffices to use the Euler-Maclaurin summation formula [**2**, pg. 180]. Let $m \geq 0$ and $n \geq 1$ be integers, and define $h = (b - a)/n$ and $x_j = a + jh$ for $0 \leq j \leq n$. Further assume that the function $f(x)$ is at least $(2m + 2)$-times continuously differentiable on $[a, b]$. Then the Euler-Maclaurin summation formula is:

$$h \sum_{j=0}^{n} f(x_j) = \int_a^b f(x) \mathrm{d}x + \frac{h}{2} \left(f(a) + f(b)\right)$$

$$+ \sum_{i=1}^{m} \frac{h^{2i} B_{2i}}{(2i)!} \left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right) + E,$$

where B_{2i} denote the Bernoulli numbers, and

$$E = \frac{h^{2m+2}(b - a)B_{2m+2}f^{2m+2}(\xi)}{(2m + 2)!}$$

for some $\xi \in (a, b)$.

As written, this formula is not particularly effective to numerically evaluate series. The strategy is instead to evaluate a series manually for, say, $N = 1000$ terms, then to use the Euler-Maclaurin formula to evaluate only the tail of the series, namely $\sum_{j>1000} f(x_j)$. The Bernoulli numbers B_{2k}, which are required here, can be easily computed by recalling that for $k > 0$ [**1**, pg. 807],

$$\zeta(2k) = \frac{(2\pi)^{2k}|B_{2k}|}{2(2k)!}.$$

This can be rewritten as

$$\frac{B_{2k}}{(2k)!} = \frac{2(-1)^{k+1}\zeta(2k)}{(2\pi)^{2k}}.$$

The Riemann zeta function at integer arguments can, in turn, be computed using the formula due to Peter Borwein [**10**].

With a scheme to compute the Hurwitz zeta in hand, formula (6) can be used, in conjunction with a high-precision quadrature routine, to compute $I(p)$.

One question here is whether to use Gaussian quadrature or tanh-sinh. Because the integrand function $\sin^p(\pi x)\zeta(p, x)$ for various p seems quite well-behaved, our first inclination was to use Gaussian quadrature. This works quite well for $p = 3$, for instance, but then we found that it gave very poor results for $p = 3.5$ — even with a very small h and a correspondingly large number of evaluation points, the accuracy was only a few digits. We naturally suspected a programming "bug," but after observing the same behavior whether we coded the algorithm in our own ARPREC software [**4**] or in *Mathematica*, we looked elsewhere.

As it turns out, this strange behavior is due to the fact that when $p = 3.5$, while the integrand function and its derivatives are well-behaved up to the third derivative, the fourth derivative exhibits severe blow-up singularities at 0 and 1. This is illustrated in Figure 3, which shows the function $\sin^p(\pi x)\zeta(p, x)$ for $p = 3$
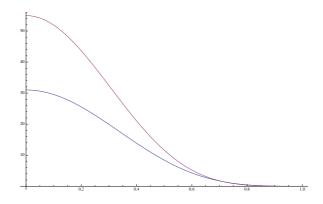
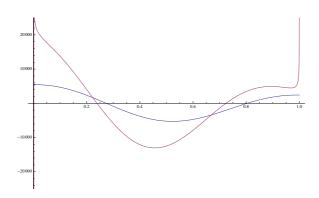FIGURE 3. Integrand function for $p = 3$ (blue) and $p = 3.5$ (red).

FIGURE 4. Fourth derivative of integrand function for $p = 3$ (blue)
and $p = 3.5$ (red).

(blue) and $p = 3.5$ (red), and Figure 4, which shows the fourth derivatives of these
functions with the same color scheme. Because of these singularities in the higher-
order derivatives, Gaussian quadrature gives poor results. The tanh-sinh scheme,
in contrast, is not bothered by these singularities and gives excellent exponential
convergence to high-precision values.

We should add, more generally, that if $p > 0$ is not an integer, then the func-
tion $\sin^p(\pi t)\zeta(p, t)$ will exhibit singular behavior on $[0, 1]$ beginning with the k-th
derivative, where $k = \lfloor p \rfloor + 1$.

Our first goal with this computer program was to find the minimum of the
function $I(p)$ near $p = 3.36$ and also the point of inflection near $p = 4.46$. We did
this by starting with a "comb" of four equispaced abscissa values (five in the case
of the inflection point) near 3.36 (4.46 for the inflection point). We evaluated $I(p)$
for these values of p, then used polynomial regression to fit these $(p, I(p))$ pairs to
a quartic curve (quintic for the inflection point). We then used Newton iteration
to calculate a new estimate for the zero of the derivative of this function (zero of
the second derivative, for the inflection point). This new estimate for the p at the
critical point was then taken as a new element of the "comb," and the process was
repeated, until successive new elements differed by an amount roughly equal to the

square root of the "epsilon" of the numeric precision being used, which is as good as can be expected.

This scheme appeared to work quite well. We obtained the results below and believe them to be accurate to the precision shown:

- **p at critical point (conjectured minimum):**
  3.3635487602245153281633430155354110698234097301 0200
  9339302427452685362432280882211178063052274354 6839
  6516854667296148546282707784684178641121861308 9950
  8745727158152731
- **$I(p)$ at critical point (conjectured minimum):**
  2.0900286026918041225495649155078117735383497494 9186
  7516155894611577041927127462449177641134431475 8189
  9346130671184603074736322373502311886888801790 2470
  2980223273478188838606173485063108224384639425 7215
  3851191162210810094581882751317041088948108059 3453
  36438830185161897153124688334006896341 9076
- **p at inflection point:**
  4.4698778865856457891778082067498869317159691986 7299
  1163425397552598383794145970545164697950992842 4279
  4233718363336416486397093

With regards to the Hurwitz zeta function, we mention here a related result (communicated to us by M. Coffey) that may be of use both in computation and analysis:

THEOREM 2. *Given an integer $M > 0$, the following holds for all $\operatorname{Re}(p) > -M$ and uniformly for all $0 < t \leq 1$:*

$$\zeta(p,t) \quad = \quad t^{-p} + (t+1)^{-p} + \sum_{0}^{M} \binom{-p}{m}(\zeta(p+m) - 1)t^m + O(t^{M+1})$$

Proof. This follows from

$$\zeta(p,t) \quad = \quad t^{-p} + (t+1)^{-p} + \sum_{j=2}^{\infty} j^{-p}(1 + t/j)^{-p}$$

$$= \quad t^{-p} + (t+1)^{-p} + \sum_{m=0}^{\infty} \binom{p}{m}(\zeta(p+m) - 1)t^m.$$

Therefore, we also have, for example,

$$\lim_{p \to \infty} I(p) \quad = \quad \lim_{p \to \infty} \sqrt{p}\,\pi^{1-p} \int_{0}^{1} \frac{\sin^p(\pi t)}{t^p}\, \mathrm{d}t.$$

One other related asymptotic of interest can be deduced from a result in [**21**]. Define

$$Q_n(0) \quad := \quad \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin x}{x}\right)^n \mathrm{d}x.$$

Then we can write for even n that

$$
\begin{aligned}
(7) \qquad I(n) &= \frac{\pi\sqrt{n}}{2} Q_n(0) \\
&= \sqrt{\frac{3\pi}{2}} \left[1 - \frac{3}{20n} - \frac{13}{1120n^2} + \frac{27}{3200n^3} + \frac{52791}{3942400n^4} \right. \\
&\qquad \left. + \frac{482427}{66560000n^5} - \frac{124996631}{10035200000n^6} + \cdots \right)\Bigg]
\end{aligned}
$$

which, for integer n, is formally a more precise version of the asymptotic in (4) which was developed from general critical point analysis [**22**, Theorem 7.1, p. 127]. We should note that [**21**] makes no claim as to the rigorous properties of this estimation which comes from the Bernoulli number expansion

$$
\log\left(\frac{\sin x}{x}\right) = -\sum_{k=1}^{\infty} \frac{B_k}{(2k)(2k)!} (2x)^{2k}.
$$

3. The sinc norm at integer values

For integer $M \geq 1$, on expressing $\zeta(M, x)$ in terms of $\Psi^{(M-1)}(x)$ and integrating by parts M-times we have also the representation

$$
\begin{aligned}
I(M) &= \frac{\sqrt{M}}{\pi^{M-1}} \int_0^1 \sin^M(\pi x)\, \zeta(M, x)\, dx \\
&= \frac{\sqrt{M}}{\Gamma(M)\pi^{M-1}} \int_0^1 \log\Gamma(x)\, \frac{\partial^M}{\partial x^M} \sin^M(\pi x)\, dx.
\end{aligned}
$$

Note also that when M is even we may substitute $x = 1 - x$ and average to get

$$
\begin{aligned}
I(2N) &= \frac{\sqrt{2N}}{\pi^{2N-1}} \int_0^1 \sin^{2N}(\pi x)\, \zeta(2N, x)\, dx \\
&= -\frac{\sqrt{N/2}}{\Gamma(2N)\pi^{2N-1}} \int_0^1 \log\left(\frac{\sin(\pi x)}{\pi}\right) \frac{\partial^{2N}}{\partial x^{2N}} \sin^{2N}(\pi x)\, dx.
\end{aligned}
$$

We can to use these last two formulae—via their Fourier expansions and Parseval's formula—to recapture the known closed form for even integers and to provide something new for odd integers. Indeed, we have

$$
(8) \qquad -\ln\left(\frac{\sin(\pi t)}{\pi}\right) = \log(2\pi) + \sum_{n=1}^{\infty} \frac{\cos(2n\pi t)}{n},
$$

$$
(9) \qquad \frac{1}{2} - t = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\pi t)}{n},
$$

$$
\begin{aligned}
\log\Gamma(t) &= -\frac{1}{2}\log\left(\frac{\sin(\pi t)}{\pi}\right) + (\gamma + \ln(2\pi))\left(\frac{1}{2} - t\right) \\
(10) &\qquad + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\log(n)}{n} \sin(2n\pi t)
\end{aligned}
$$

where the final composite Fourier series is originally due to Kummer [**19**]. Also, it is known [**6**] and easy to show inductively that

$$(11) \qquad \sin^{2N-1}(\pi t) = \frac{1}{2^{2N-2}} \sum_{n=1}^{N} (-1)^{n+1} \binom{2N-1}{N-n} \sin\left((2n-1)\pi t\right)$$

and

$$(12) \quad \sin^{2N}(\pi t) = \frac{1}{2^{2N-1}} \left(\frac{1}{2}\binom{2N}{N} + \sum_{n=1}^{N} (-1)^n \binom{2N}{N-n} \cos(2n\pi t) \right).$$

Finally, for completeness, we recall that for $0 \le t \le 1$,

$$(13) \qquad \sum_{n=1}^{\infty} \frac{\sin(2\pi nt)}{n^{2N+1}} = \frac{(-1)^{N-1}}{2} (2\pi)^{2N+1} \phi_{2N+1}(t)$$

and

$$(14) \qquad \sum_{n=1}^{\infty} \frac{\cos(2\pi nt)}{n^{2N}} = \frac{(-1)^{N-1}}{2} (2\pi)^{2N} \phi_{2N}(t),$$

where $\phi_N(x)$ is the Nth *Bernoulli polynomial*, normalized so that the highest-order coefficient is $1/N!$, see [**25**, p. 430]. Now it transpires that the first two terms of Kummer's formula (8) and (9) are orthogonal to (11) and hence

$$
\begin{aligned}
I(2N-1) \;=\;\; & \frac{\sqrt{2N-1}}{\Gamma(2N-1)(2\pi)^{2N-2}} \\
& \times \sum_{n=1}^{\infty} \int_{0}^{1} \frac{\log(n)}{n} \sin(2n\pi t) \frac{\partial^{2N-1}}{\partial x^{2N-1}} \sin^{2N-1}(\pi x) \, dx \\
=\;\; & \frac{1}{\pi} \frac{\sqrt{2N-1}}{\Gamma(2N-1)(2\pi)^{2N-2}} \sum_{n=1}^{\infty} \sum_{m=1}^{N} \frac{(-1)^{m+1}\binom{2N-1}{N-m}\log(n)}{n} \\
& \times \int_{0}^{1} \sin(2n\pi t) \frac{\partial^{2N-1}}{\partial x^{2N-1}} \sin\left((2m-1)\pi t\right) dx \\
=\;\; & \frac{(-1)^N \sqrt{2N-1}}{\Gamma(2N-1)\, 4^{N-1}} \sum_{n=1}^{\infty} \sum_{m=1}^{N} \frac{(-1)^m \binom{2N-1}{N-m}(2m-1)^{2N-1}\log(n)}{n} \\
& \times \int_{0}^{1} \sin(2n\pi t) \cos\left((2m-1)\pi t\right) dx \\
=\;\; & \frac{1}{\pi} \frac{(-1)^N \sqrt{2N-1}}{\Gamma(2N-1)} \\
& \times \sum_{n=1}^{\infty} \sum_{m=1}^{N} (-1)^m \binom{2N-1}{N-m} \left(m - \frac{1}{2}\right)^{2N-1} \frac{\log(n)}{n^2 - \left(m-\frac{1}{2}\right)^2} \\
=\;\; & \frac{1}{\pi} \frac{(-1)^N \sqrt{2N-1}}{\Gamma(2N-1)} \\
& \times \sum_{m=1}^{N} (-1)^m \binom{2N-1}{N-m} \left(m - \frac{1}{2}\right)^{2N-1} \sum_{n=1}^{\infty} \frac{\log(n)}{n^2 - \left(m-\frac{1}{2}\right)^2}.
\end{aligned}
$$

Let us set

$$\mathcal{L}(x) \quad := \quad \sum_{n=2}^{\infty} \frac{\log(n)}{n^2 - x^2} = -\sum_{s=1}^{\infty} \zeta'(2s)\, x^{2s-2},$$

where the power series has radius of convergence two, and the derivative (here and elsewhere) is with respect to the first variable. For general x, we may similarly write

$$(15) \qquad \mathcal{L}(x) \quad = \quad \sum_{n=2}^{\lfloor x \rfloor} \frac{\log(n)}{n^2 - x^2} - \sum_{s=1}^{\infty} \zeta'(2s, \lceil x \rceil)\, x^{2s-2},$$

with radius of convergence $\lceil x \rceil$. Thence we obtain

$$I(2N-1) \quad = \quad \sqrt{2N-1} \cdot \frac{(-1)^N}{(2N-2)!}$$

$$(16) \qquad \times \sum_{m=1}^{N} (-1)^m \binom{2N-1}{N-m} \left(m - \frac{1}{2}\right)^{2N-1} \frac{\mathcal{L}\left(m - \frac{1}{2}\right)}{\pi}.$$

In particular $I(1) = \infty$ and

$$(17)\quad I(3) \quad = \quad \frac{3\sqrt{3}}{2\pi} \sum_{s=1}^{\infty} \frac{9^s - 1}{4^s} \zeta'(2s)$$

$$= \quad -2.0930867689497938424321336535746596878058055573140\ldots.$$

$$(18)\quad I(5) \quad = \quad -\frac{5\sqrt{5}}{64\pi} \left(\frac{1}{3} \mathcal{L}\left(\frac{1}{2}\right) - \frac{81}{2} \mathcal{L}\left(\frac{3}{2}\right) + \frac{625}{6} \mathcal{L}\left(\frac{5}{2}\right) \right)$$

$$= \quad 2.1061252846080842088918826986669843292200478501493\ldots,$$

since

$$-\mathcal{L}\left(\frac{5}{2}\right) \quad = \quad \frac{4}{9} \log(2) + \sum_{s=1}^{\infty} \left(\zeta'(2s) + \frac{\log(2)}{2^{2s}} \right) \left(\frac{5}{2}\right)^{2s-2}$$

while $\mathcal{L}\left(\frac{1}{2}\right)$ and $\mathcal{L}\left(\frac{3}{2}\right)$ are as already engaged in (17). Such computations can be done quite efficiently to much higher precision in either *Maple* or *Mathematica*. Richard Crandall has noted the following pretty formula

$$I(3) \quad = \quad \frac{3\sqrt{3}}{8} \pi + 6\sqrt{3} \int_0^{\infty} \frac{t^2}{(t^2 + 1)(t^2 + 9)(e^{\pi t} - 1)} \, dt.$$

Herein, the integral term only accounts for 2.5% of the value.

For even integers the computation is much simpler as the Fourier series have only cosine terms and so are orthogonal on $[0, 1]$. We arrive at:

$$I(2N) \quad = \quad \sqrt{2N} \int_0^{\infty} \left(\frac{\sin x}{x}\right)^{2N} dx$$

$$(19) \qquad = \quad \sqrt{2N} \cdot \frac{(-1)^N}{(2N-1)!} \cdot \sum_{m=0}^{N} (-1)^m \binom{2N}{N-m} m^{2N-1} \frac{\pi}{2}.$$

Hence $I(2) = \pi/2, I(4) = 2\pi/3 = 2.0943951023932\ldots$ and $I(6) = 11\sqrt{6}\pi/40 = 2.1162072197671\ldots$. Result (19) can be found in Bromwich [**11**, Exercise 22, p. 518], where it is attributed to Wolstenholme, and in many other places—including

two relatively recent articles on integrals of more general products of sinc functions [**7, 9**]. We recapitulate with a theorem.

THEOREM 3. *For integer $N \geq 1$ one has*

$$I(2N-1) = \sqrt{2N-1} \cdot \frac{(-1)^N}{(2N-2)!}$$

$$(20) \qquad \times \sum_{m=1}^{N}(-1)^m \binom{2N-1}{N-m}\left(m-\frac{1}{2}\right)^{2N-1}\frac{\mathcal{L}\left(m-\frac{1}{2}\right)}{\pi},$$

with $\mathcal{L}$ given by (15), while

$$I(2N) = \sqrt{2N}\int_0^\infty \left(\frac{\sin x}{x}\right)^{2N}\mathrm{d}x$$

$$(21) \qquad = \sqrt{2N}\cdot\frac{(-1)^N}{(2N-1)!}\cdot\sum_{m=0}^{N}(-1)^m\binom{2N}{N-m}m^{2N-1}\frac{\pi}{2}.$$

For the limit of the $I(p)$, as $p \to \infty$ the best we could obtain purely numerically is **2.1708037**537.... This required computing $I(p)$ to high precision, using the Hurwitz integral representation (6), for $p = 4, 16, \ldots,$ up to $p = 4^{12}$, and then using Richardson extrapolation. This is consistent with the actual closed form $\sqrt{3\pi/2} = \mathbf{2.1708037}636748029781\ldots$ which was given an elementary proof in [**8**]; but it certainly does not provide compelling evidence. A discussion of the problems entailed in using formula (21) directly is given in [**6**].

3.1. Another direct proof of the limit $\sqrt{3\pi/2}$. A referee of this paper was able to formulate an alternative delightful and direct—if non-elementary—proof of this limit, as follows. The expression (21) for $p = 2N$ can be rewritten by employing the *Eulerian numbers* (found by Euler in 1755), which may be defined by

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k+1}(-1)^j\binom{n+1}{j}(k-j+1)^n.$$

Note that

$$(-1)^N\sum_{m=0}^{N}(-1)^m\binom{2N}{N-m}m^{2N-1} = \sum_{m=0}^{N}(-1)^m 2Nm(N-m)^{2N-1}$$

$$= \left\langle {2N-1 \atop N-1} \right\rangle = A(2N-1,N)$$

in the A notation of [**13**, pg. 243], namely $A(n,k) = \left\langle {n \atop k-1} \right\rangle$. Moreover, $A(n,m)$ counts the number of permutations of the numbers 1 to n in which exactly m elements are greater than the previous element. This leads to the recursion

$$(22) \qquad A(n,m) = (n-m+1)A(n-1,m-1) + mA(n-1,m).$$

Then we have the following beautiful formula:

$$\frac{I(2N)}{\sqrt{2N}} = \int_0^\infty \left(\frac{\sin x}{x}\right)^{2N}\mathrm{d}x$$

$$(23) \qquad = \frac{\pi}{2(2N-1)!}\left\langle {2N-1 \atop N-1} \right\rangle = \frac{\pi A(2N-1,N)}{2(2N-1)!}.$$

With the result (23) in hand, we can apply known asymptotic results for A [**17**, 6.16], such as

$$A\left(n, \frac{n-1}{2}\right) \quad \sim_n \quad \frac{2\sqrt{3}}{e}\left(\frac{n+1}{e}\right)^n,$$

together with Stirling's formula for $\Gamma(2N)$, to again deduce that

$$\lim_{N\to\infty} I(2N) = \sqrt{\frac{3\pi}{2}}.$$

Note that $A(2N-1, N)$ can be computed "in-place" by starting with a $(2N-1)$-long array $\bar{A} = (1, 0, 0, 0, \cdots, 0)$, and then applying the recursion

$$\bar{A}(m) := (n - m + 1)\bar{A}(m - 1) + m\bar{A}(m)$$

for $m = n$ to 2 in reverse order, repeating for $n = 2$ to $2N-1$. In this way, with $N = 10,000$, we calculate $A(2N - 1, N) = \bar{A}(N) = 8.8886211119258922301\ldots \times 10^{77330}$, so that $I(N) = 2.1707874825835856332\ldots$, a value that differs from $\sqrt{3\pi/2}$ by 1.62811×10^{-5}. This calculation, using 56-digit arithmetic, required 341 seconds run time on a single processor core.

4. More general oscillatory integrals

More complex integrands of the form studied in [**7**] can be handled by variations of (6). For example

$$(24) \quad \int_{-\pi}^{\infty} \left|\frac{\sin(x)}{x}\right|^p \left|\frac{\sin(x/2)}{x/2}\right|^q \, \mathrm{d}x$$

$$= \frac{2}{(2\pi)^{p+q-1}} \int_{-1}^{1} \sin^p(\pi x)\sin^q\left(\pi\frac{x}{2}\right)\zeta\left(p+q, \frac{x}{2}\right)\, \mathrm{d}x$$

$$+ \int_0^{\pi} \left(\frac{\sin(x)}{x}\right)^p \left(\frac{\sin(x/2)}{x/2}\right)^q \, \mathrm{d}x.$$

Likewise, logarithmic integrals of the form in (5) can be treated as follows. For $p > 1, q \geq 0$ and q integer we write

$$
\begin{aligned}
I(p, q) \quad &:= \quad \int_0^{\infty} \left|\frac{\sin(t)}{t}\right|^p \left(\log\left|\frac{\sin(t)}{t}\right|\right)^q \, \mathrm{d}t \\
&= \quad \frac{1}{\pi^{p-1}} \int_0^{\infty} \left|\frac{\sin(\pi t)}{t}\right|^p \left(\log\left|\frac{\sin(\pi t)}{\pi}\right| - \log|t|\right)^q \, \mathrm{d}t \\
&= \quad \frac{1}{\pi^{p-1}} \sum_{r=0}^{q} \binom{q}{r}(-1)^r \int_0^{\infty} \left|\frac{\sin(\pi t)}{t}\right|^p \left(\log\left|\frac{\sin(\pi t)}{\pi}\right|\right)^{q-r} (\log t)^r \, \mathrm{d}t \\
&= \quad \frac{1}{\pi^{p-1}} \sum_{r=0}^{q} \binom{q}{r}(-1)^r \int_0^{\infty} |\sin(\pi t)|^p \left(\log\left|\frac{\sin(\pi t)}{\pi}\right|\right)^{q-r} \frac{(\log t)^r}{t^p} \, \mathrm{d}t \\
&= \quad \frac{1}{\pi^{p-1}} \sum_{r=0}^{q} \binom{q}{r}(-1)^r \sum_{n=0}^{\infty} \int_0^1 \sin^p(\pi t)\log^{q-r}\left(\frac{\sin(\pi t)}{\pi}\right) \frac{\log^r(n+t)}{(n+t)^p} \, \mathrm{d}t,
\end{aligned}
$$

where this last expression has again broken up the integral. On exchanging the integral and sum, we we arrive at our final theorem:

THEOREM 4. *For all real numbers $p > 1$ and integers $q \geq 0$ one has*

$$\int_0^\infty \left| \frac{\sin(t)}{t} \right|^p \log^q \left| \frac{\sin(t)}{t} \right| \, \mathrm{d}t$$

$$(25) \qquad = \frac{1}{\pi^{p-1}} \sum_{r=0}^q \binom{q}{r} \int_0^1 \sin^p(\pi t) \log^{q-r} \left(\frac{\sin(\pi t)}{\pi} \right) \zeta^{(r)}(p, t) \, \mathrm{d}t$$

where derivatives of the Hurwitz zeta function are taken with respect to the first variable.

Observe that for computational purposes it is preferable to write

$$\int_0^\infty \left| \frac{\sin(t)}{t} \right|^p \log^q \left| \frac{\sin(t)}{t} \right| \, \mathrm{d}t$$

$$(26) \qquad = \frac{1}{\pi^{p-1}} \int_0^1 \sin^p(\pi t) \sum_{r=0}^q \binom{q}{r} \log^{q-r} \left(\frac{\sin(\pi t)}{\pi} \right) \zeta^{(r)}(p, t) \, \mathrm{d}t.$$

Note that formula (25) recaptures (6). Observe also that the Fourier series of $\log \left(\frac{\sin(\pi t)}{\pi} \right)$ was recorded in equation (8).

To illustrate the virtues of Theorem 3, with $p = 3$ and $q = 1, 2, 3$ we computed in *Maple* that

$$
\begin{aligned}
I(3,1) = \quad - \quad & 0.21269444360025161284678764161199575472915948205259008888 \\
& 37953040469514339043760331248955363745973320 \\
I(3,2) = \quad + \quad & 0.15046077234624478828956577555259908739964416463107895870 \\
& 09495322070870213832829012135200745982056446 \\
I(3,3) = \quad - \quad & 0.21673166110791925450854193518248175083146189217945509989 \\
& 94616313036889771963481594421547345585858421
\end{aligned}
$$

Each hundred digit computation took roughly three minutes on a modern laptop.

We can also handle quite general sinc products of the sort analyzed in [7]. Let $0 < n_1 \leq n_2 \leq \cdots \leq n_N$ be natural numbers. Set $L := \mathrm{LCM}(n_1, n_2, \ldots, n_N)$ and set $M := n_1 \cdot n_2 \cdot \ldots \cdot n_N$. Then

$$(27) \qquad \int_0^\infty \prod_{k=1}^N \mathrm{sinc}\left(\frac{x}{n_k} \right) \, \mathrm{d}x \quad = \quad \frac{M/L}{(2L\pi)^{N-1}} \int_0^1 \prod_{k=1}^N \sin\left(\frac{2L}{n_k} \pi x \right) \zeta(N, x) \, \mathrm{d}x.$$

For example,

$$\frac{1}{51840000\,\pi^6} \int_0^1 \sin^4(10\,\pi\,x) \sin^3(6\,\pi\,x)\, \Psi(6, x)\ dx$$

$$= -2.81369873784840103\ldots = -\frac{286601}{320000} \pi.$$

Equation (27) is most efficacious when the number of distinct integers n_k, and hence L, is small. The factor of 2 is not needed when each integer occurs an even number of times. To conclude, we illustrate in Figure 5 the effect of the Hurwitz kernel for $\sin^2(x)/x^2$. The increases in the values of the function on $[0, 1]$ exactly balance the amount needed to compensate for the values on $[1, \infty]$.
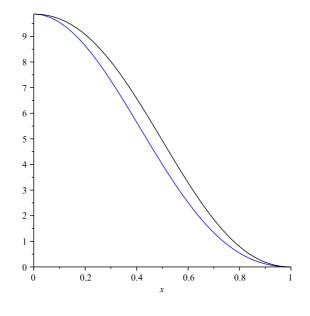
FIGURE 5. $\frac{\sin^2(\pi x)}{x^2}$ (blue) and $\sin^2(\pi x)\zeta(2,x)$ (black).

5. Concluding remarks

We found this study stimulating for several reasons.

(1) Such numerical computations to high or extreme precision are quite difficult and worthwhile challenges in themselves.
(2) In the integer cases for which we have a closed form we obtain excellent test integrals for different high-precision numerical methods.
(3) The numerical tools presented new analytic opportunities such as the discovery of the formula (16) which represented an unexpected windfall. It is probably possible to emulate that result for integer p, q in (24).
(4) One other unexpected discovery in this study is the fact that the function $\sin^p(\pi t)\zeta(p, t)$ (when $p = 3.5$) is a striking example of a function that appears entirely smooth and well-behaved, but which in fact cannot be integrated using Gaussian quadrature to more than a few digits, due to bad behavior of a higher derivative (the *fourth* derivative in this case). And for the same reason it is yet one more example of the power of the tanh-sinh quadrature algorithm, since the tanh-sinh scheme integrates this function without difficulty to very high precision.
(5) Even partial failures, like the limited success in numerically computing $\lim_{p \to \infty} I(p)$ from the integral definition, have left us with a much better understanding of the issues involved with such experimental techniques.
(6) Finally, our inability to determine closed forms for the critical point or the inflection point has still resulted in high-precision data sets that can be revisited if and when inspiration strikes.

6. Acknowledgements

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