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Research article

Homogenization of nonlinear nonlocal diffusion equation with periodic and stationary structure

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Abstract: This paper is devoted to the homogenization of a class of nonlinear nonlocal parabolic equations with time dependent coefficients in a periodic and stationary structure. In the first part, we consider the homogenization problem with a periodic structure. Inspired by the idea of Akagi and Oka for local nonlinear homogenization, by a change of unknown function, we transform the nonlinear nonlocal term in space into a linear nonlocal scaled diffusive term, while the corresponding linear time derivative term becomes a nonlinear one. By constructing some corrector functions, for different time scales r and the nonlinear parameter p, we obtain that the limit equation is a local nonlinear diffusion equation with coefficients depending on r and p. In addition, we also consider the homogenization of the nonlocal porous medium equation with non negative initial values and get similar homogenization results. In the second part, we consider the previous problem in a stationary environment and get some similar homogenization results. The novelty of this paper is two folds. First, for the determination equation with a periodic structure, our study complements the results in literature for $r = 2$ and $p = 1$. Second, we consider the corresponding equation with a stationary structure.

Keywords: nonlocal diffusion; time dependent coefficient; scale parameter; convolutional kernel; stochastic homogenization

1. Introduction

The homogenization theory is to establish the macroscopic behavior of a system which is microscopically heterogeneous in order to describe some characteristics of the heterogeneous medium [1]. In recent years many papers have been faced with the problem of how to get an effective behavior as a scale parameter $\varepsilon \to 0^+$. Nguetseng [2] and Allaire [3] first proposed two-scale convergence, and in 1997, Holmbom [4] proved the homogenization result of the parabolic equation that the main operator depends on time t by using two-scale convergence.

Recently, Akagi and Oka [5] considered a space-time homogenization problem for nonlinear diffusion equations with periodically oscillating space and time coefficients:

$$\begin{cases} \partial_t u^\varepsilon(x,t) - \operatorname{div}\Big(A(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\nabla(|u^\varepsilon|^{m-2}u^\varepsilon)\Big)(x,t) = f(x,t), & (x,t) \in \Omega \times I, \\ u^\varepsilon(x,0) = u_0(x), & x \in \Omega, \\ |u^\varepsilon|^{m-2}u^\varepsilon(x,t) = 0, & x \in \partial\Omega \times I, \end{cases} \tag{1.1}$$

their main results are based on the two-scale convergence theory for space-time homogenization.

Akagi and Oka [6] also considered space-time homogenization problems for porous medium equations with nonnegative initial data. These are important developments of the homogenization of local second-order parabolic equations where the operator depends on the time t. Geng and Shen [7] and Niu and Xu [8] discussed the convergence rates in periodic homogenization of a second-order parabolic system depending on time t. There are many qualitative and quantitative studies on the homogenization theory of parabolic equations with periodic and stationary coefficients [9–12].

The nonlocal operator homogenization theory is based on the regular convolutional kernel and the singularity kernel corresponding to the fractional Laplace equation. Piatnitski and Zhizhina [13] gave a scaling operator:

$$\mathscr{L}^\varepsilon u(x) = \int_{\mathbb{R}^d} \frac{1}{\varepsilon^{d+2}} J(\frac{x-y}{\varepsilon})\lambda(\frac{x}{\varepsilon})\mu(\frac{y}{\varepsilon})\Big(u(y) - u(x)\Big)dy, \tag{1.2}$$

where there are two natural length scales, one being the macroscopic scale of order 1 and the other being the microscopic pore scale of order $\varepsilon > 0$; the scale parameter ε measures the oscillation. The bounded 1−periodic functions $\lambda(\xi), \mu(\eta)$ describe the periodic structure. As $\varepsilon \to 0^+$, the limit of operators $\{\mathscr{L}^\varepsilon\}_{\varepsilon>0}$ is a second order elliptic differential operator L corresponding to the macroscopic scale. Piatnitski and Zhizhina [14] dealt with the homogenization of parabolic problems for integral convolutional type operators with a non-symmetric jump kernel in a periodic elliptic medium

$$\mathscr{L}^\varepsilon u(x) = \int_{\mathbb{R}^d} J(\frac{x-y}{\varepsilon})\mu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})\Big(u(y) - u(x)\Big)dy, \tag{1.3}$$

where $\mu(\xi, \eta)$ is a positive periodic function in ξ and η. Kassmann, Piatnitski and Zhizhina [15] considered the homogenization of a Lévy-type operator.

Karch, Kassmann and Krupski [16] discussed the existence of the Cauchy problem

$$\begin{cases} \partial_t u(x,t) = \displaystyle\int_{\mathbb{R}^d} \rho\Big(u(x,t), u(y,t); x, y\Big)\Big(u(y,t) - u(x,t)\Big)dy, \\ u(x,0) = u_0(x), \end{cases} \tag{1.4}$$

for $(x,t) \in \mathbb{R}^d \times [0,\infty)$ with a given homogeneous jump kernel ρ. Their models contain both integrable and non-integrable kernels.

Next, we introduce some examples about the nonlocal evolution of porous medium equations and fast diffusion equations. Cortazar et al. [17] considered the rescaled problem

$$\partial_t u^\varepsilon(x,t) = \frac{1}{\varepsilon^2}\left(\int_R J\left(\frac{x-y}{\varepsilon u^\varepsilon(y,t)}\right)\frac{dy}{\varepsilon} - u^\varepsilon(x,t)\right), (x,t) \in \mathbb{R} \times [0,\infty) \tag{1.5}$$

with a fixed initial condition $u_0(x)$ and they proved that the limit $\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = u(x,t)$ is a solution to the porous medium equation $u_t = D(u^3)_{xx}$ for a suitable constant D, where J is a smooth non-negative even function supported in $[-1,1]$.

Andreu et al. [18, Chapter 5] discussed a class of nonlinear nonlocal evolution equations with the Neumann boundary condition

$$\begin{cases} \partial_t z(t,x) = \int_\Omega J(x-y)(u(t,y)-u(t,x))dy, & x\in\Omega, t>0, \\ z(t,x)\in\theta(u(t,x)), & x\in\Omega, t>0, \\ z(0,x)=z_0(x), & x\in\Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded domain. If the maximal monotone function $\theta(r) = |r|^{p-1}r$, under the suitable rescale model problem (1.6) corresponds to the nonlocal version of the porous medium equation if $0 < p < 1$, or to the fast diffusion equation if $p > 1$.

Nonlocal porous medium equations with a non-integrable kernel is an important example of nonlinear and nonlocal diffusion equations; the different properties of solutions to the fractional porous medium equation

$$\partial_t u(x,t) + (-\Delta)^{\frac{\alpha}{2}}(|u|^{p-1}u) = 0 \tag{1.7}$$

have been studied from various viewpoints [19–24].

The two types of equations studied in this paper have recently been widely researched in the following form:

$$\partial_t u(x,t) = \mathfrak{L}(|u|^{p-1}u), \tag{1.8}$$

where $p > 0$, $\mathfrak{L}$ is a linear, symmetric, and nonnegative operator ($p \geq 1$) and sub-Markovian operator ($0 < p < 1$). More details can be seen in [25, 26].

Inspired by the thought of local nonlinear homogenization described by Akagi and Oka [5, 6], the goal of this paper is to investigate the homogenization theory of nonlocal nonlinear parabolic equations in a periodic environment with the following nonlocal scaling operator:

$$\mathscr{L}^\varepsilon u(x,t) = \int_{\mathbb{R}^d} \frac{J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})}{\varepsilon^{d+2}} \nu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})\Big(|u(y,t)|^{p-1}u(y,t) - |u(x,t)|^{p-1}u(x,t)\Big)dy, \tag{1.9}$$

which means that we take the jump kernel in Eq (1.4) as follows

$$\rho\Big(u(x), u(y); x, y\Big) = \frac{|u(y)|^{p-1}u(y) - |u(x)|^{p-1}u(x)}{u(y)-u(x)} \frac{J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})}{\varepsilon^{d+2}} \nu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}). \tag{1.10}$$

The difference between the kernels in Eq (1.10) and in the equation

$$\partial_t u(x,t) = \int_{\mathbb{R}^d} \frac{C_{\alpha,d}}{|x-y|^{d+\alpha}}\Big(|u(y)|^{p-1}u(y) - |u(x)|^{p-1}u(x)\Big)dy, \tag{1.11}$$

where $C_{\alpha,d}$ is a constant and $\alpha \in (0,1)$, can be seen in the work of Karch, Kassmann and Krupski [16]. For more literatures about time-dependent regular kernels (integrable) and Lévy kernels (non-integrable), see, for instance, [27, 28] for more details.

This paper is mainly divided into two parts.

The first part is the homogenization problem under the periodic framework. Our goal is to characterize the limit operator by homogenizing the nonlocal operators $\{\mathscr{L}^{\varepsilon}\}_{\varepsilon>0}$ as the scale parameter $\varepsilon \to 0^{+}$. The paper is organized as follows. The first step, in the case of $0 < p \leq 2$, we transfer the spatial nonlinearity to the time derivative term through a kirchhoff transform, which can simplify the difficulty of nonlinearity in the nonlocal operator. In the second step, we construct axillary functions that work on the operator and then divide the operator in Eq (1.9) into three parts we then deal with it part by part separately according to the parameters r and p. We prove that the first part is zero. For the second part, we get that the limit is a nonlinear diffusion operator. Finally, from the third part we get an error function ϕ_{ε} and we can prove that it tends to zero in $L^2((0,T), L^2(\mathbb{R}^d))$ as $\varepsilon \to 0^{+}$. We also consider the homogenization of the nonlocal porous medium equation $(1 \leq p < +\infty)$ with non negative initial values and get similar homogenization results.

The second part is the homogenization problem under the stationary framework. The idea of the proof is divided into the following steps. The first step is to construct an approximation sequence equation when $p = 1$; by approximation we obtain a random corrector function. The second step is to prove the existence and uniqueness of the corrector functions, as well as the properties of sub-linear growth and stationarity. The third step is to prove the convergent limit equation. There will be some additional stationary matrix-field $F(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)$ with zero average and the non-stationary term Υ^{ε} during the process of solving the coefficients of the limit equation. It is necessary to prove that there are some functions v_2 and v_3 to cancel the additional part, and also to prove the positive definiteness of the matrix Θ and the existence of the limit equation. The fourth step focuses on the effects of nonlinearity and give some key proofs of our results.

The novelty of this paper is two folds. First, for the determination equation with a periodic structure, our study complements the results in literature for $r = 2$ and $p = 1$. Second, we consider the corresponding equation with a stationary structure.

It is worth noting that we need to require that $0 < p \leq 2$ become $|u^{\varepsilon}|^{p-1}u^{\varepsilon} \in L^2((0,T) \times L^2_{loc}(\mathbb{R}^d))$. So far it is not actually clear how to solve the case of $p > 2$. The local equation in the case that $p \in (0,2)$ $u^{\varepsilon} \in L^{\infty}((0,T) \times L^{3-p}(\Omega)) \cap L^2((0,T) \times H^1_0(\Omega))$ does not hold when $p = 2$; the specific proof is in [5, Lemma 4.1].

2. Preliminaries

In order to deal with the homogenization of nonlinear nonlocal operators, we first introduce some results on nonlinear functional analysis, semigroups and the nonlocal diffusion of knowledge; the main references are [29, 30].

Notation. $X = L^2(\mathbb{R}^d, \varrho)$, $I = (0,T), E = (0,1)$ and $Y = \mathbb{T}^d = [0,1]^d$. We have that $x_0 \in \mathbb{R}, Q_R(x_0) = x_0 + (-\frac{R}{2}, \frac{R}{2})^d$ and $B_r(x_0)$ is the open ball in $\mathbb{R}^d$ centered at x_0 and radius r. Moreover, $Q_R = Q_R(0), B_r = B_r(0)$, and $\widetilde{Q}_R = Q_R \times I_R = (-\frac{R}{2}, \frac{R}{2})^{d+1} \subset \mathbb{R}^{d+1}$, while $\widetilde{Q}$ and $\widehat{Q}$ are used for any cube in $\mathbb{R}^{d+1}$. Additionally, $a \lesssim_{\alpha} b$ means that there exists a constant $C = C(\alpha) > 0$ such that $a \leq Cb$. We write $a = b$ if $a \lesssim_{\alpha} b$ and $b \lesssim_{\alpha} a$.

Assume that the kernel J is a nonnegative symmetric function that satisfies the time periodicity that

$$J(z, s+1) = J(z, s), \quad \forall s \in I, \tag{2.1}$$

and that $J(\cdot,\cdot)$ is compactly supported in the set $\{(x,t) \in \mathbb{R}^{d+1} : t \geq 0\}$. In addition,

$$\begin{cases} J(z,s) \in L^{\infty}\Big((0,T), C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\Big) \\ J(x,t) \geq J(y,t) \text{ if } |x| \leq |y|, \forall\, t > 0. \quad \exists j_1 > 0 \text{ and } J_1(z) \geq j_1 \text{ such that} \\ \displaystyle\int_{\mathbb{R}^d} J_1(z)|z|dz = j_0,\ \|J(z,\cdot)\|_{L^{\infty}(0,T)} \leq J_1(z), \forall z \in \mathbf{U}, \end{cases} \tag{2.2}$$

where $\mathbf{U}$ is any tube in $\mathbb{R}^d$.

We also assume that the bounded periodic function $\nu(x,y)$ satisfies

$$0 < \alpha_1 \leq \nu(x,y) \leq \alpha_2 < +\infty, \tag{2.3}$$

where α_1 and α_2 are positive constants. Here ν contains the case that $\nu(x,y) = \lambda(x)\mu(y)$.

Definition 2.1. (Monotone operator) Let X and X^* be a Banach space and its dual space. A set-valued operator $A : X \to 2^{X^*}$ is said to be monotone, if it holds that

$$< u - v, \xi - \eta > \geq 0 \text{ for all } [u,\xi], [v,\eta] \in G(A), \tag{2.4}$$

where $G(A)$ denotes the graph of A, i.e., $G(A) = \Big\{[u,\xi] \in X \times X^* : \xi \in Au\Big\}$.

Finally, let us recall the notion of subdifferentials for convex functionals.

Definition 2.2. (Subdifferential operator) Let X and X^* be a Banach space and its dual space, respectively. Let $\phi : X \to (-\infty, +\infty]$ be a proper (i.e., $D(\phi) \neq \emptyset$) lower semicontinuous and convex functional with the effective domain $D(\phi) := \{u \in X : \phi(u) < +\infty\}$. The subdifferential operator $\partial\phi : X \to 2^{X^*}$ of ϕ is defined by

$$\partial\phi(u) = \Big\{\xi \in X^* : \phi(v) - \phi(u) \geq \langle \xi, v - u\rangle_X \text{ for all } v \in D(\phi)\Big\}$$

with domain $D(\partial\phi) := \Big\{u \in D(\phi) : \partial\phi(u) \neq \emptyset\Big\}$. Subdifferential operators form a subclass of maximal monotone operators.

Theorem 2.3. *(Minty) Every subdifferential operator is maximal monotone.*

Lemma 2.1. *[29, Prop. 6.19, Poincaré-type inequality] For $q \geq 1$, assume that $J(x) \geq J(y)$ if $|x| \leq |y|$ and Ω is a bounded domain in $\mathbb{R}^d$; the quantity*

$$\beta_{q-1} := \beta_{q-1}(J,\Omega,q) = \inf_{\substack{u \in L^q(\Omega), \\ \int_\Omega u dx = 0}} \frac{\int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^q dy dx}{2\int_\Omega |u(x)|^q dx} \tag{2.5}$$

is strictly positive. Consequently, for every $u \in L^q(\Omega)$,

$$\beta_{q-1} \int_\Omega \left| u(x) - \frac{1}{|\Omega|}\int_\Omega u(x) dx \right|^q dx \leq \frac{1}{2}\int_\Omega \int_\Omega J(x-y) \Big| u(y) - u(x) \Big|^q dy dx. \tag{2.6}$$

Theorem 2.4. *[30, Prop. 32D] Let $V \subseteq H \subseteq V^*$ be an evolution triple; and $X = L^p(0, T; V)$, where $1 < p < \infty$ and $0 < T < \infty$. Suppose that the operator $A : X \to X^*$ is pseudomonotone, coercive and bounded. Then, for each $b \in X^*$ and the operators*

$$\begin{cases} L_1 u = u', & D(L_1) = \left\{u \in W^{1,p}(0, T; V, H) : u(0) = 0\right\}, \\ L_2 u = u', & D(L_2) = \left\{u \in W^{1,p}(0, T; V, H) : u(0) = u(T)\right\}, \end{cases} \tag{2.7}$$

the equations

$$\begin{cases} L_1 u + Au = b, & u \in D(L_1), \\ L_2 u + Au = b, & u \in D(L_2) \end{cases} \tag{2.8}$$

have respective solutions. In addition, if A is strictly monotone, then the corresponding solutions are unique.

3. Statement of the problem and the main results

We consider the following nonlocal scaling operator

$$\mathscr{L}^\varepsilon u(x, t) = \int_{\mathbb{R}^d} \frac{J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})}{\varepsilon^{d+2}} \nu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})\Big(|u(y, t)|^{p-1} u(y, t) - |u(x, t)|^{p-1} u(x, t)\Big) dy, \tag{3.1}$$

and the corresponding Cauchy problem

$$\begin{cases} \partial_t u^\varepsilon(x, t) - \mathscr{L}^\varepsilon u^\varepsilon(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, T), \\ u^\varepsilon(x, 0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.2}$$

with

$$\varphi(x) \in L^{[1,\infty]}(\mathbb{R}^d) := L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{3.3}$$

As the scale parameter $\varepsilon \to 0^+$, we will prove that the effective Cauchy problem for Eq (3.2) is

$$\begin{cases} \partial_t u^0(x, t) - \mathscr{L}^0 u^0(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, T), \\ u^0(x, 0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.4}$$

where

$$\mathscr{L}^0 u^0(x, t) = \Theta \cdot \nabla\nabla\Big(|u^0|^{p-1} u^0\Big) = \sum_{i,j=1}^{d} \Theta^{ij} \frac{\partial^2}{\partial x_i \partial x_j}(|u^0|^{p-1} u^0), \tag{3.5}$$

and the positive definite constant matrix $\Theta = (\Theta^{ij})$ will be given below. For writing convenience, we omit Σ in Eq (3.5).

Remark 1. According to [31, 32], for $1 < p < \infty$, the Cauchy problem of porous medium equations admits a solution when $\varphi(x) \in L^1_{loc}(\mathbb{R}^d)$, but the corresponding result for the Cauchy problem to fast diffusion equations was only established for $d \geq 3$, $\frac{d-2}{d} < p < 1$ and for $d = \{1, 2\}, 0 < p < 1$ when $\varphi(x) \in L^1_{loc}(\mathbb{R}^d)$. Therefore, the index conditions in the critical situation are also satisfied here. For $p = 1$, the operator $\mathscr{L}^\varepsilon$ in Eq (3.1) is linear, the Cauchy problems of parabolic Eqs (3.2) and (3.4) have solutions u^ε and $u^0 \in L^\infty((0, T), L^2(\mathbb{R}^d))$ respectively. But the existence of solutions is not obvious for $0 < p < 1$ and $p > 1$, so we need to prove it before going to investigate the limit behavior.

3.1. Existence and uniqueness of nonlocal porous medium equation and fast diffusion equation

We apply the space of functions of bounded variation, following [16, 33]. Suppose that for $u \in L^1(\mathbb{R}^d)$, there exist finite signed Radon measures $\lambda_i (i = 1, 2, \cdots, d)$ such that

$$\int_{\mathbb{R}^d} u\partial_{x_i}\phi dx = -\int_{\mathbb{R}^d} \phi d\lambda_i, \ \forall \phi \in C_c^\infty(\mathbb{R}^d),$$
$$|Du|(\mathbb{R}^d) = \sum_{i=1}^d \sup\left\{\int_{\mathbb{R}^d} \Phi_i d\lambda_i : \Phi \in C_0(\mathbb{R}^d, \mathbb{R}^d), \|\Phi\|_{C_0(\mathbb{R}^d,\mathbb{R}^d)} < 1\right\}.$$

Then we say that $u \in BV(\mathbb{R}^d)$ if the norm $\|u\|_{BV} = 2\|u\|_1 + |Du|(\mathbb{R}^d) < \infty$.

For every $\delta \in (0, 1]$, we consider a function $h_\delta \in C^\infty([0, \infty)), 0 \le h_\delta(x) \le 1$ which is nondecreasing and satisfies that $h_\delta(x) = 0$ for $x \le \frac{\delta}{2}$ and $h_\delta(x) = 1$ for $x \ge \delta$. Denote

$$\Gamma_0\Big(u(x), u(y), x, y, t\Big) = \frac{|u(y)|^{p-1}u(y) - |u(x)|^{p-1}u(x)}{u(y) - u(x)},$$
$$\Gamma\Big(u(x), u(y), x, y, t\Big) = J(x - y, t)\nu(x, y)\Gamma_0\Big(u(x), u(y), x, y, t\Big),$$
$$\Gamma^\delta(a, b; x, y) = h_\delta(|a - b|)\mathbb{1}_{|x-y|\ge\delta}(x, y)\Gamma(a, b; x, y),$$
$$\mathcal{L}_v^{t,\delta}u(x) = \int_{\mathbb{R}^d} \Gamma^\delta\Big(v(x), v(y), x, y, t\Big)\Big(u(y) - u(x)\Big)dy.$$

Lemma 3.1. *For $1 \le p < +\infty$, the operator $\mathfrak{B}^\delta(u) := \mathcal{L}_u^{t,\delta}u$ is locally Lipschitz as a mapping $\mathfrak{B}^\delta : L^{[1,\infty]}(\mathbb{R}^d) \to L^{[1,\infty]}(\mathbb{R}^d)$ for a.e. t.*

Proof. See Appendix A for a detailed proof.

Lemma 3.2. *For every initial data point $u_0^\delta(x) \in L^{[1,\infty]}(\mathbb{R}^d), 1 < p < +\infty$ and $\forall\ T > 0$, the problem (3.2) admits a unique global classical solution*

$$u^\delta(x, t) \in C^1([0, T], L^{[1,\infty]}(\mathbb{R}^d)).$$

Proof. For $v \in C^1([0, T], L^{[1,\infty]}(\mathbb{R}^d))$, consider an integral operator

$$\begin{cases} \mathfrak{B}^\delta : X = C([0, T], L^{[1,\infty]}(\mathbb{R}^d)) \to C^1([0, T], L^{[1,\infty]}(\mathbb{R}^d)), \\ (\mathfrak{B}^\delta v)(t) = \displaystyle\int_0^t \mathfrak{B}^\delta(v(s))ds, \ v \in X. \end{cases} \tag{3.6}$$

From Lemma 3.1, the operator $\mathfrak{B}^\delta(u) = \mathcal{L}_u^{t,\delta}u$ is locally Lipschitz. Fix $T \in (0, \infty)$; for $v_1, v_2 \in X$, we have

$$\begin{aligned} |||\mathfrak{B}^\delta v_1 - \mathfrak{B}^\delta v_2|||_X &\le \int_0^T \|\mathfrak{B}^\delta(v_1(s)) - \mathfrak{B}^\delta(v_2(s))\|_{[1,\infty]}ds \\ &\le T \max_{0\le t\le T} \|\mathfrak{B}^\delta v_1 - \mathfrak{B}^\delta v_2\|_{[1,\infty]} \le M(p, \alpha_2, \Lambda)T|||v_1 - v_2|||_X, \end{aligned} \tag{3.7}$$

and

$$|||\partial_t\mathfrak{B}^\delta v_1 - \partial_t\mathfrak{B}^\delta v_2|||_X \le \max_{0\le t\le T} \|\mathfrak{B}^\delta v_1 - \mathfrak{B}^\delta v_2\|_{[1,\infty]} \le M(p, \alpha_2, \Lambda)|||v_1 - v_2|||_X,$$

where $M(p, \alpha_2, \Lambda)$ is introduced in Appendix A. For a small enough T such that $CT < 1$, the Banach contraction mapping principle implies that the problem (3.2) admits a unique local classical solution $u^\delta \in C^1([0,T], L^{[1,\infty]}(\mathbb{R}^d))$.

This local classical solution u^δ is actually global. According to [16, Lemma 3.5], we have

$$\|u(t)\|_{[1,\infty]} \leq C\|\varphi\|_{[1,\infty]}, \; t \in [0,T]. \tag{3.8}$$

Taking a ball $B(\varphi, C\|\varphi\|_{[1,\infty]}) \subset X$, then $M(p, \alpha_2, \Lambda)$ only depends on $\|\varphi\|_{[1,\infty]}$. Therefore, the problem (3.2) admits a global solution.

Theorem 3.1. *(Existence of strong solutions) For $1 < p < +\infty$ and the initial condition $\varphi \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the problem (3.2) has a strong solution (and still denotes u)*

$$u \in L^\infty\left([0,\infty), BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right) \cap C\left([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}^d)\right). \tag{3.9}$$

Proof. For an arbitrarily fixed $T > 0$, by applying the Aubin-Lions-Simon lemma [34, Theorem 1] in the space $L^\infty\left([0,T], L^1(\Omega)\right)$, we can get the convergent subsequence in the usual way. So there exist a subsequence $\left\{u^{\delta_j}\right\}$ and a function u such that $u^{\delta_j} \to u$ in $C([0,T], L^1_{loc}(\mathbb{R}^d))$. The specific proof process can be found in [16,25].

The case $0 < p < 1$ can be obtained in [35], and the existence of a doubly nonlinear equation is consistent with our equation. Noticed that more studies focus on fractional nonlocal fast diffusion equations, e.g. [36,37]. The general framework was recently studied in [16]. We now describe our main results on $u^\varepsilon(x,t), u^0(x,t)$ corresponding to the Cauchy problems (3.2) and (3.4), respectively.

3.2. Main results on homogenization

Theorem 3.2. *Assume that the functions $J(z,s)$ and $\nu(x,y)$ satisfy the conditions (2.1)–(2.3). Let $u^\varepsilon(x,t)$ be the solution of the Cauchy problem (3.2) and $u^0(x,t)$ be the solution of the effective Cauchy problem (3.4). Then there exist a vector $\varpi \in \mathbb{R}^d$ ($\varpi = 0$ for $p \neq 1$) and a positive definite matrix Θ such that for any $T > 0$, we have*

$$\left\|u^\varepsilon(x + \frac{\varpi}{\varepsilon}t, t) - u^0(x,t)\right\|_{L^1((0,T), L^1_{loc}(\mathbb{R}^d))} \to 0 \text{ as } \varepsilon \to 0^+. \tag{3.10}$$

Theorem 3.2 implies that the homogenization of the nonlocal operator in Eq (3.1) is a local porous medium operator. The Cauchy problem of porous medium equations has been extensively studied in [31,32,38,39].

The homogenized flux $\Theta(x,t)$ can be characterized as follows.

Case I. For $0 < r < 2$ and $0 < p \leq 2$, Θ is a constant $d \times d$ matrix given by

$$\begin{aligned} \Theta &= \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{1}{2}(\xi - q)(\xi - q) J(\xi - q, s) \nu(\xi, q) m(\xi) dq d\xi ds \\ &- \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} J(\xi - q, s) \nu(\xi, q)(\xi - q) \chi_1(q,s) dq d\xi ds \\ &+ \varpi \int_0^1 \int_{\mathbb{T}^d} \frac{1}{p} |u^0|^{1-p} \chi_1(\xi, s) \mu(\xi, s) d\xi ds, \end{aligned} \tag{3.11}$$

where the periodic function $\chi_1(\xi, s)$ ($(\xi, s) \in \mathbb{T}^d \times \mathbb{T}$) solves the cell-problem

$$\begin{cases} \int_{\mathbb{R}^d} J(\xi - q, s)\nu(\xi, q)\Big(q - \xi + \chi_1(q, s) - \chi_1(\xi, s)\Big)dq = -\frac{1}{p}|u^0|^{1-p}\varpi, \\ \chi_1(y, 0) = \chi_1(y, 1), \ y \in \mathbb{T}^d, \end{cases} \tag{3.12}$$

$u^0(x, t)$ is the solution of Eq (3.4) and m will be defined in Eq (4.40).

Remark 2. For $p \neq 1, \chi_1$ does not depend on x and t, so $\varpi = 0$, $w^\varepsilon(x, t) = u(x, t) + \varepsilon u_1(x, t) + \varepsilon^2 u_2(x, t)$ and the pair (u, u_1) is uniquely determined. Moreover, the function $u_1(x, t, y, s)$ can be written as

$$u_1(x, t, y, s) = \sum_{k=1}^{d} \partial_{x_k} \left(\left|u^0\right|^{p-1} u^0\right)(x, t) \cdot \chi_1^k(y, s). \tag{3.13}$$

Case II. For $r = 2$ and $p \in (0, 1]$, the homogenized matrix function $\Theta(x, t)$ is characterized by

$$\begin{aligned} \Theta(x, t) &= \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{1}{2}(\xi - q)(\xi - q)J(\xi - q)\nu(\xi, q)m(\xi)dqd\xi ds \\ &- \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} J(\xi - q, s)\nu(\xi, q)m(\xi)(\xi - q)\chi_1(x, t, q, s)dqd\xi ds \\ &+ \varpi \int_0^1 \int_{\mathbb{T}^d} \frac{1}{p}|u^0|^{1-p}\chi_1(x, t, \xi, s)\nu(\xi, q)m(\xi)d\xi ds, \end{aligned} \tag{3.14}$$

where $\chi_1^k = \chi_1^k(x, t, y, s) \in L^\infty\left(\mathbb{R}^d \times (0, T); L^2\left(E; L^2_{per}(Y)/\mathbb{R}\right)\right)$ solves the cell problem

$$\begin{cases} \int_{\mathbb{R}^d} J(\xi - q, s)\nu(\xi, q)(q - \xi + \chi_1(x, t, q, s) - \chi_1(x, t, \xi, s))dq \\ \qquad = \frac{1}{p}|u^0|^{1-p}(\partial_s\chi_1(x, t, \xi, s) - \varpi), \ (\xi, s) \in \mathbb{T}^d \times \mathbb{T}, \\ \chi_1(x, t, y, 0) = \chi_1(x, t, y, 1), \ y \in \mathbb{T}^d, \end{cases} \tag{3.15}$$

such that, for each $(x, t) \in \mathbb{R}^d \times (0, T)$,

$$|u^0|^{1-p}\chi_1^k \in L^\infty\left(\mathbb{R}^d \times (0, T); L^2\left(E; \left[L^2_{per}(Y)/\mathbb{R}\right]^*\right)\right), \tag{3.16}$$

$$|u^0|^{\frac{1-p}{2}}\chi_1^k \in L^\infty\left(\mathbb{R}^d \times (0, T); C\left(\bar{E}; L^2(Y)/\mathbb{R}\right)\right). \tag{3.17}$$

Case III. For $r = 2$ and $p \in (1, 2]$, the homogenized matrix function $\Theta(x, t)$ is characterized by Eq (3.14), where

$$\chi_1^k(x, t, y, s) = \begin{cases} p|u^0|^{p-1}\natural_1^k(x, t, y, s) & \text{if} \quad u^0(x, t) \neq 0, \\ 0 & \text{if} \quad u^0(x, t) = 0, \end{cases} \tag{3.18}$$

and $\natural_1^k = \natural_1^k(x, t, y, s) \in L^\infty\left(\left[u^0 \neq 0\right]; H^1\left(E; \left[L^2_{\text{per}}(Y)/\mathbb{R}\right]^*\right)\right)$ solves the cell problem for each $(x, t) \in [u_0 \neq 0]$:

$$\begin{cases} \partial_s\natural_1(x, t, \xi, s) = \int_{\mathbb{R}^d} J(\xi - q, s)\nu(\xi, q)\Big(q - \xi \\ \qquad + p|u^0|^{p-1}\natural_1(x, t, q, s) - p|u^0|^{p-1}\natural_1(x, t, \xi, s)\Big)dq, \\ \natural_1(x, t, y, 0) = \natural_1(x, t, y, 1), \ y \in \mathbb{T}^d, \end{cases} \tag{3.19}$$

such that

$$|u^0|^{p-1}\natural_1^k \in L^\infty\left(\left[u^0 \neq 0\right]; L^2\left(E; L^2_{\text{per}}(Y)/\mathbb{R}\right)\right), \tag{3.20}$$

$$|u^0|^{\frac{p-1}{2}}\natural_1^k \in L^\infty\left(\left[u^0 \neq 0\right]; C\left(\bar{E}; L^2(Y)/\mathbb{R}\right)\right), \tag{3.21}$$

and the measurable set $\left[u^0 \neq 0\right] := \left\{(x,t) \in \mathbb{R}^d \times (0,T) : u^0(x,t) \neq 0\right\}$.

Case IV. For $2 < r < +\infty$ and $0 < p \leq 1$, Θ is a constant $d \times d$ matrix given by

$$\begin{aligned}
\Theta &= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \frac{1}{2}(\xi - q)(\xi - q)\Big[\int_0^1 J(\xi - q, s)ds\Big]\nu(\xi, q)m(\xi)dqd\xi \\
&- \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\Big[\int_0^1 J(\xi - q, s)ds\Big]\nu(\xi, q)m(\xi)(\xi - q)\chi_1(x, t, q, s)dqd\xi \\
&+ \varpi\int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{p}|u^0|^{1-p}\chi_1(x, t, \xi, s)\nu(\xi, q)m(\xi)dqd\xi ds,
\end{aligned} \tag{3.22}$$

where χ_1 satisfies the following problem with $(\xi, s) \in \mathbb{T}^d \times \mathbb{T}$:

$$\int_{\mathbb{T}^d}\int_0^1 J(\xi - q, s)ds\nu(\xi, q)\Big(q - \xi + \chi_1(q) - \chi_1(\xi)\Big)dq = -\frac{1}{p}|u^0|^{1-p}\varpi, \tag{3.23}$$

χ_1 does not include s because $\varpi = 0$; we found that χ_1 does not include x and t.

Case V. For $2 < r < +\infty$, $1 < p \leq 2$, $\varpi = 0$ and Θ is a constant $d \times d$ matrix given by

$$\begin{aligned}
\Theta &= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \frac{1}{2}(\xi - q)(\xi - q)\Big[\int_0^1 J(\xi - q, s)ds\Big]\nu(\xi, q)m(\xi)dqd\xi \\
&- \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\Big[\int_0^1 J(\xi - q, s)ds\Big]\nu(\xi, q)m(\xi)(\xi - q)\chi_1(q)dqd\xi,
\end{aligned} \tag{3.24}$$

where χ_1 also satisfies Eq (3.23).

Now for the operator given by Eq (3.1), we consider the following nonlocal scaling operator

$$L^\varepsilon v(x,t) = \frac{1}{\varepsilon^{d+2}}\int_{\mathbb{R}^d} J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})\nu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})\Big(v(y,t) - v(x,t)\Big)dy; \tag{3.25}$$

thus for $v(x,t) = |u(x,t)|^{p-1}u(x,t)$, we have that

$$L^\varepsilon v(x,t) = L^\varepsilon(|u|^{p-1}u) = \mathscr{L}^\varepsilon u(x,t).$$

Therefore we transform the problems (3.2) and (3.4) into the following Cauchy problems

$$\begin{cases} \partial_t v^\varepsilon(x,t)^{\frac{1}{p}} - L^\varepsilon v^\varepsilon(x,t) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ v^\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.26}$$

and

$$\begin{cases} \partial_t v(x,t)^{\frac{1}{p}} - \Theta \cdot \nabla\nabla v(x,t) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ v(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.27}$$

respectively. We first study the existence and uniqueness of solutions to the Cauchy problems (3.26) and (3.27) with the nonlocal operator (3.25), where $-L^{\varepsilon}$ is a non-positive and self-adjoint operator in the space $L^2(\mathbb{R}^d, m)$ for $\nu(x, y)$. In fact, for any $u, v \in L^2(\mathbb{R}^d, m)$,

$$\begin{aligned}&(L^{\varepsilon}u(x), u(x))_{L^2(\mathbb{R},m)}\\&= -\frac{1}{2\varepsilon^{d+2}}\iint_{\mathbb{R}^{2d}} J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})\nu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})m(\frac{x}{\varepsilon})|u(y)-u(x)|^2 dydx \le 0.\end{aligned} \tag{3.28}$$

We directly give the following theorem.

Theorem 3.3. *The hypotheses given above are satisfied. If there is a homogenized solution denoted as v of the problem (3.26). Then we have*

$$v \in L^{\infty}((0,T); BV(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)) \cap H^1((0,T); L^2_{loc}(\mathbb{R}^d)). \tag{3.29}$$

Proof. The proof of existence is similar to Theorem 3.1 so we do not show it here. What needs to be emphasized here is that, in the three cases of $0 < p < 1$, $p = 1$ and $p > 1$, the method of proof of existence could be different. These will not affect our subsequent homogenization proof.

4. Auxiliary cell problems and existence of first corrector χ_1 and drift ϖ

Due to the classical method of asymptotic expansion, we first construct some auxiliary functions to prove Theorem 3.2, i.e., our main results on homogenization of nonlinear nonlocal equations with a periodic structure. Denote $x^{\varepsilon} = x - \frac{\varpi t}{\varepsilon}$ and $y = x - \varepsilon z$, we first give a chain-rule formula.

Lemma 4.1. *(Chain-rule formula) If $v^{\varepsilon}(x,t)^{\frac{1}{p}}$ is bounded in $H^1(I; L^2(\mathbb{R}^d))$ $(0 < p \le 2)$, then for a.e. $(x,t) \in \mathbb{R}^d \times I$, we have*

$$\frac{\partial v^{\varepsilon}(x,t)^{\frac{1}{p}}}{\partial t} = \frac{1}{p}|v^{\varepsilon}(x,t)|^{\frac{1-p}{p}}\frac{\partial v^{\varepsilon}(x,t)}{\partial t},\ p \in (0,1), \tag{4.1}$$

$$\frac{\partial v^{\varepsilon}(x,t)}{\partial t} = p|v^{\varepsilon}(x,t)|^{\frac{p-1}{p}}\frac{\partial v^{\varepsilon}(x,t)^{\frac{1}{p}}}{\partial t},\ p \in (1,2]. \tag{4.2}$$

For a given $v \in C^{\infty}((0,T), \mathcal{S}(\mathbb{R}^d))$, we introduce some auxiliary functions:

$$w^{\varepsilon}(x,t) = v(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t). \tag{4.3}$$

For different cases of r, p, we construct the corresponding auxiliary functions $w_i^{\varepsilon}, i = 0, 1, 2$.
(i) $r = 2, 0 < p < 1$

$$w_0^{\varepsilon}(x,t) = v(x,t) + \varepsilon\chi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla v(x,t) + \varepsilon^2\chi_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla\nabla v(x,t). \tag{4.4}$$

(ii) $r = 2, 1 < p \le 2$

$$\begin{aligned}w_1^{\varepsilon}(x,t) = v(x,t) \quad &+ \quad \varepsilon p|v(x,t)|^{\frac{p-1}{p}}\natural_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla v(x,t)\\&+ \quad \varepsilon^2 p|v(x,t)|^{\frac{p-1}{p}}\natural_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla\nabla v(x,t).\end{aligned} \tag{4.5}$$

(iii) $r = 2, p = 1$

$$\begin{aligned} w_2^\varepsilon(x,t) &= v(x - \frac{\varpi t}{\varepsilon}, t) + \varepsilon\chi_1(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\nabla v(x - \frac{\varpi t}{\varepsilon}, t) \\ &+ \varepsilon^2\chi_2(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\nabla\nabla v(x - \frac{\varpi t}{\varepsilon}, t). \end{aligned} \tag{4.6}$$

(iv) $r \neq 2, \chi_i$ and $\natural_i$ do not depend on x and t.

Lemma 4.2. *For a given $v \in C^\infty((0,T), \mathcal{S}(\mathbb{R}^d))$, w_i^ε $(i = 0, 1, 2)$ is defined by Eqs (4.4)–(4.6). Then there exist two functions*

$$\chi_1 \in (L^\infty((0,T), (L^2(\mathbb{T}^d \times \mathbb{T}))^d, \quad \chi_2 \in (L^\infty((0,T), (L^2(\mathbb{T}^d \times \mathbb{T}))^{d\times d}, \tag{4.7}$$

a vector $\varpi \in \mathbb{R}^d (p = 1)$ and a positive definite matrix Θ such that

$$\begin{aligned} \mathrm{H}^\varepsilon w_i^\varepsilon(x,t) &:= \frac{\partial w_i^\varepsilon(x,t)^{\frac{1}{p}}}{\partial t} - L^\varepsilon w_i^\varepsilon \\ &= \Big(\frac{1}{p}|w_i^\varepsilon|^{\frac{1-p}{p}}\frac{\partial v}{\partial t}(x^\varepsilon, t) - \Theta\cdot\nabla\nabla v(x^\varepsilon, t) + \phi_\varepsilon(x,t)\Big)\Big|_{x^\varepsilon = x - \frac{\varpi}{\varepsilon}t}, \end{aligned} \tag{4.8}$$

where $\phi_\varepsilon \to 0$ in $L^2((0,T), L^2(\mathbb{R}^d))$ as $\varepsilon \to 0$.

Remark 3. For $p = 1$, the homogenization takes place in the moving coordinates $X_t = x - \frac{\varpi}{\varepsilon}t$ with an appropriate constant vector ϖ. But it does not work in the nonlinear situation ($p \neq 1$) when $\varpi \in \mathbb{R}^d$.

Proof. Substitute the expressions on the right-hand side of Eqs (4.4)–(4.6) into Eq (4.8) and using the notation $x^\varepsilon = x - \frac{\varpi}{\varepsilon}t$ and $\partial_t\chi(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}) = \partial_t\chi + \frac{1}{\varepsilon^r}\partial_s\chi$, where the symbol $\otimes$ stands for the tensor product:

$$\begin{aligned} &z\otimes z = (z^i z^j)_{d\times d},\ z\otimes z\cdot\nabla\nabla v = z^i z^j\frac{\partial^2 v}{\partial x^i\partial x^j}, \\ &z\otimes z\otimes z\cdot\nabla\nabla\nabla v = z^i z^j z^k\frac{\partial^3 v}{\partial x^i\partial x^j\partial x^k}, \\ &\chi_2(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\otimes(-\frac{\varpi}{\varepsilon})\cdot\nabla\nabla\nabla v = \chi_2^{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})(-\frac{x^k}{\varepsilon})\partial_{x^i}\partial_{x^j}\partial_{x^k}v. \end{aligned}$$

Case 1. For $p \in (0, 1)$,

$$\begin{aligned} \frac{\partial w_0^\varepsilon(x,t)}{\partial t} &= \frac{1}{p}|w_0^\varepsilon|^{\frac{1-p}{p}}\Big[\frac{\partial v}{\partial t}(x,t) + (\varepsilon^{1-r}\frac{\partial\chi_1}{\partial s})\cdot\nabla v(x,t) \\ &+ \varepsilon^{2-r}\frac{\partial\chi_2}{\partial s}\cdot\nabla\nabla v(x,t) + \phi_\varepsilon^{(time)}(x,t)\Big] \end{aligned}$$

with

$$\begin{aligned} &\phi_\varepsilon^{(time)}(x,t) = \varepsilon\frac{\partial\chi_1}{\partial t}\cdot\nabla v(x,t) + \varepsilon^2\frac{\partial\chi_2}{\partial t}\cdot\nabla\nabla v(x,t) \\ &+\varepsilon\chi_1(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\cdot\nabla\frac{\partial v}{\partial t}(x,t) + \varepsilon^2\chi_2(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\cdot\nabla\nabla\frac{\partial v}{\partial t}(x,t). \end{aligned} \tag{4.9}$$

Set $z = \frac{x-y}{\varepsilon}$; we get

$$
\begin{aligned}
&(L^\varepsilon w_0^\varepsilon)(x,t)\\
&= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} J(z,s)v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\Big\{v(x-\varepsilon z,t)+\varepsilon\chi_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\cdot\nabla v(x-\varepsilon z,t)\\
&+\varepsilon^2\chi_2(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\cdot\nabla\nabla v(x-\varepsilon z,t)-v(x,t)-\varepsilon\chi_1(y,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\cdot\nabla v(x,t)\\
&-\varepsilon^2\chi_2(y,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\cdot\nabla\nabla v(x,t)\Big\}dz. \qquad (4.10)
\end{aligned}
$$

Using the Taylor expansions

$$
\begin{aligned}
v(y) &= v(x)+\int_0^1 \frac{d}{d\theta}v(x+(y-x)\theta)d\theta\\
&= v(x)+\int_0^1 \nabla v(x+(y-x)\theta)(y-x)d\theta, \qquad (4.11)\\
v(y) &= v(x)+\nabla v(x)(y-x)+\int_0^1 \nabla\nabla v(x+(y-x)\theta)(y-x)^2(1-\theta)d\theta, \qquad (4.12)
\end{aligned}
$$

we have

$$
\begin{aligned}
(L^\varepsilon w_0^\varepsilon)(x,t) &= \int_{\mathbb{R}^d}\frac{1}{\varepsilon^2}J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\Big[v(x,t)-\varepsilon z\nabla v(x,t)\\
&+ \varepsilon^2\int_0^1 \nabla\nabla v(x-\varepsilon z\theta,t)z^2(1-\theta)d\theta+\varepsilon\chi_1(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\nabla v(x,t)\\
&- \varepsilon^2\chi_1(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})z\nabla\nabla v(x)\\
&+ \varepsilon^3\chi_1(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\int_0^1 \nabla\nabla\nabla v(x-\varepsilon z\theta,t)z^2(1-\theta)d\theta\\
&+ \varepsilon^2\chi_2(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\nabla\nabla v(x-\varepsilon z)-v(x)-\varepsilon\chi_1(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla v(x,t)\\
&- \varepsilon^2\chi_2(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla\nabla v(x,t)\Big]dz, \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
&H^\varepsilon w_0^\varepsilon(x,t)\\
&= \frac{\partial w_0^\varepsilon(x,t)}{\partial t}-\frac{1}{\varepsilon^{d+2}}\int_{\mathbb{R}^d} J(\frac{x-y}{\varepsilon},\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{y}{\varepsilon})(w_0^\varepsilon(y,t)-w_0^\varepsilon(x,t))dy\\
&= \frac{1}{p}|w_0^\varepsilon|^{\frac{1-p}{p}}\frac{\partial v}{\partial t}(x,t)+\frac{1}{\varepsilon}\mathcal{M}_0(x,t)+\mathcal{M}_\varepsilon(x,t)+\phi_\varepsilon(x,t) \quad \text{as } \varepsilon\to 0^+, \qquad (4.14)
\end{aligned}
$$

where

$$
\phi_\varepsilon(x,t) = \frac{1}{p}|w_0^\varepsilon(x,t)|^{\frac{1-p}{p}}\phi_\varepsilon^{(time)}(x,t)-\phi_\varepsilon^{(space)}(x,t), \qquad (4.15)
$$

and

$$
\phi_\varepsilon^{(space)} = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} dzJ(z,\frac{t}{\varepsilon^r})v\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z\right)
$$

$$
\begin{aligned}
&\cdot\Big\{\varepsilon^2\int_0^1 \nabla\nabla v\,(x-\varepsilon zq,t)\cdot z\otimes z(1-q)dq-\frac{\varepsilon^2}{2}\nabla\nabla v\,(x,t)\cdot z\otimes z\\
&+\ \varepsilon^3\varkappa_1\Big(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}\Big)\cdot\int_0^1 \nabla\nabla\nabla v\,(x-\varepsilon zq,t)\,z\otimes z(1-q)dq\\
&-\ \varepsilon^3\varkappa_2\Big(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}\Big)\cdot\int_0^1 \nabla\nabla\nabla v\,(x-\varepsilon zq,t)\,zdq\Big\},
\end{aligned}\tag{4.16}
$$

$$
\begin{aligned}
\mathcal{M}_0(x,t)\ &=\ \varepsilon^{2-r}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial\chi_1}{\partial s}\nabla v(x,t)-\nabla v(x,t)\Big[\int_{\mathbb{R}^d}J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\
&\cdot\Big(-z+\chi_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})-\chi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\Big)dz\Big],
\end{aligned}\tag{4.17}
$$

$$
\begin{aligned}
\mathcal{M}_\varepsilon(x,t)\ &=\ \varepsilon^{2-r}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial\chi_2}{\partial s}\nabla\nabla v(x,t)-\nabla\nabla v(x,t)\Big[\int_{\mathbb{R}^d}J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\
&\cdot\Big(-z\chi_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})+\chi_2(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})-\chi_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})+\frac{1}{2}z^2\Big)dz\Big].
\end{aligned}\tag{4.18}
$$

Case 2. For $p\in(1,2]$,

$$
\begin{aligned}
\frac{\partial w_1^\varepsilon(x,t)}{\partial t}\ &=\ \frac{1}{p}|w_1^\varepsilon|^{\frac{1-p}{p}}\Big[\frac{\partial v}{\partial t}(x,t)+\varepsilon^{1-r}p|v|^{\frac{p-1}{p}}\frac{\partial\natural_1}{\partial s}\cdot\nabla v(x,t)\\
&+\ \varepsilon^{2-r}p|v|^{\frac{p-1}{p}}\frac{\partial\natural_2}{\partial s}\cdot\nabla\nabla v(x,t)+\phi_\varepsilon^{(time)}(x,t)\Big],
\end{aligned}
$$

with

$$
\begin{aligned}
&\phi_\varepsilon^{(time)}=\varepsilon(p-1)|v|^{-(1+\frac{1}{p})}v\frac{\partial v}{\partial t}\natural_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla v(x,t)\\
&+\varepsilon^2(p-1)|v|^{-(1+\frac{1}{p})}v\frac{\partial v}{\partial t}\natural_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla\nabla v(x,t)+\varepsilon p|v|^{\frac{p-1}{p}}\frac{\partial\natural_1}{\partial t}\cdot\nabla v(x,t)\\
&+\varepsilon^2p|v|^{\frac{p-1}{p}}\frac{\partial\natural_2}{\partial t}\cdot\nabla\nabla v(x,t)+\varepsilon p|v|^{\frac{p-1}{p}}\natural_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\cdot\nabla\frac{\partial v}{\partial t}(x,t)\\
&+\varepsilon^2p|v|^{\frac{p-1}{p}}\natural_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\cdot\nabla\nabla\frac{\partial v}{\partial t}(x,t).
\end{aligned}\tag{4.19}
$$

Set $z=\frac{x-y}{\varepsilon}$; we get

$$
\begin{aligned}
(L^\varepsilon w_1^\varepsilon)(x,t)\ &=\ \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}J(z,s)v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\Big\{v(x-\varepsilon z,t)\\
&+\ \varepsilon p|v(y,t)|^{\frac{p-1}{p}}\natural_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\cdot\nabla v(x-\varepsilon z,t)\\
&+\ \varepsilon^2p|v(y,t)|^{\frac{p-1}{p}}\natural_2(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\cdot\nabla\nabla v(x-\varepsilon z,t)\\
&-\ v(x,t)-\varepsilon p|v(x,t)|^{\frac{p-1}{p}}\natural_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\cdot\nabla v(x,t)\\
&-\ \varepsilon^2p|v(x,t)|^{\frac{p-1}{p}}\natural_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\cdot\nabla\nabla v(x,t)\Big\}dz.
\end{aligned}\tag{4.20}
$$

Using the Taylor expansions again,

$$
H^\varepsilon w_1^\varepsilon(x,t)\ =\ \frac{1}{p}|w_1^\varepsilon|^{\frac{1-p}{p}}\frac{\partial v}{\partial t}+\frac{1}{\varepsilon}\mathcal{M}_0(x,t)+\mathcal{M}_\varepsilon(x,t)
$$

$$+\quad \frac{1}{p}|w_1^\varepsilon|^{\frac{1-p}{p}}\phi_\varepsilon^{(time)}+\phi_\varepsilon^{(space)} \text{ as } \varepsilon\to 0^+, \tag{4.21}$$

where

$$\begin{aligned}\mathcal{M}_0(x,t) = \varepsilon^{2-r}\Big|\frac{v}{w_1^\varepsilon}\Big|^{\frac{p-1}{p}}\frac{\partial\natural_1}{\partial s}\nabla v(x,t)-\nabla v(x,t)\Big[\int_{\mathbb{R}^d}J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ \cdot\Big(-z+p|v(y,t)|^{\frac{p-1}{p}}\natural_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})-p|v(x,t)|^{\frac{p-1}{p}}\natural_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\Big)dz\Big],\end{aligned} \tag{4.22}$$

$$\begin{aligned}\mathcal{M}_\varepsilon(x,t) &= \varepsilon^{2-r}\Big|\frac{v}{w_1^\varepsilon}\Big|^{\frac{p-1}{p}}\frac{\partial\natural_2}{\partial s}\nabla\nabla v(x,t)-\nabla\nabla v(x,t)\Big[\int_{\mathbb{R}^d}J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ &\cdot\Big(-zp|v(y,t)|^{\frac{p-1}{p}}\natural_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})+p|v(y,t)|^{\frac{p-1}{p}}\natural_2(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\\ &-p|v(x,t)|^{\frac{p-1}{p}}\natural_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})+\frac{1}{2}z^2\Big)dz\Big],\end{aligned} \tag{4.23}$$

and $\phi_\varepsilon^{(space)}$ is similar to that in the case that $0<p<1$.

Case 3. For $p=1$, similar to the derivations in [13, 14], we only need to notice that the correctors are the functions $\chi_i(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})$ $(i=1,2)$. Therefore, substituting the expression on the right-hand side of H^ε for w_2^ε in Eq (4.6) and using the notation $x^\varepsilon = x-\frac{\varpi}{\varepsilon}t$ we get

$$H^\varepsilon w_2^\varepsilon(x,t)=\frac{\partial v}{\partial t}(x^\varepsilon,t)+\frac{1}{\varepsilon}\mathcal{M}_0(x,t)+\mathcal{M}_\varepsilon(x,t)+\phi_\varepsilon(x,t) \quad \text{as } \varepsilon\to 0^+, \tag{4.24}$$

where

$$\phi_\varepsilon(x,t)=\phi_\varepsilon^{(time)}(x,t)-\phi_\varepsilon^{(space)}(x,t), \tag{4.25}$$

and

$$\begin{aligned}\mathcal{M}_0(x,t) &= \varepsilon^{2-r}\frac{\partial\chi_1}{\partial s}\nabla v(x^\varepsilon,t)-\nabla v(x^\varepsilon,t)\Big[\int_{\mathbb{R}^d}J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ &\cdot\Big(-z+\chi_1(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})-\chi_1(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\Big)dz+\varpi\Big],\end{aligned} \tag{4.26}$$

$$\begin{aligned}\mathcal{M}_\varepsilon(x,t) &= \varepsilon^{2-r}\frac{\partial\chi_2}{\partial s}-\nabla\nabla v(x^\varepsilon,t)\Big[\int_{\mathbb{R}^d}J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ &\cdot\Big(-z+\chi_1(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})+\chi_2(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})-\chi_2(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})+\frac{1}{2}z^2\Big)dz+\varpi\chi_1\Big].\end{aligned} \tag{4.27}$$

Due to the order of ε, we put the terms with $O(\varepsilon)$ and the higher-order terms with $o(\varepsilon)$ into the remainder as the fourth part. For the given functions χ_1 and χ_2, it is easy to show that the fourth part is an infinitesimal $O(\varepsilon)$ as $\varepsilon\to 0^+$.

This completes the proof of Lemma 4.2.

We now consider the asymptotic decomposition of $(L^\varepsilon w^\varepsilon)(x,t)$ in ε, deal with the last three parts $\mathcal{M}_0(x,t)$, $\mathcal{M}_\varepsilon(x,t)$ and $\phi_\varepsilon(x,t)$ and get more precisely asymptotic behavior.

1. Constructing auxiliary functions to guarantee the first part $\mathcal{M}_0(x,t)$ of L^ε satisfies that $0<r\le 2, \mathcal{M}_0=0$ and $r>2, \varepsilon^{r-2}\mathcal{M}_0=0$.

2. From the second part $\mathcal{M}_\varepsilon(x,t)$ of L^ε we can get a second order differential operator L^0 such that $L^0 v(x,t) = \Theta \cdot \nabla\nabla v$ as $\varepsilon \to 0^+$.

3. The third part ϕ_ε satisfies that

$$\lim_{\varepsilon\to 0^+} \|\phi_\varepsilon\|_{L^2((0,T),L^2(\mathbb{R}^d))} = 0. \tag{4.28}$$

Finishing the above three steps, for $w^\varepsilon(x,t) = v(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t)$, we can prove that the operator L^ε has the following asymptotic representation

$$(L^\varepsilon w^\varepsilon)(x) = \Theta \cdot \nabla\nabla v + \phi_\varepsilon(x,t) \quad \text{as } \varepsilon \to 0^+. \tag{4.29}$$

We now construct an auxiliary function in order to prove that $\mathcal{M}_0 = 0$, where $\mathcal{M}_0(x,t)$ is defined by Eq (4.17). Because $v(x,t)$ and its derivatives are in $C^\infty((0,T)\times \mathcal{S}(\mathbb{R}^d))$, we need not to deal with this part and only solve the theorem as follows.

Theorem 4.1. *Assume that there exists a function $\chi_1(x,t,\xi,s) \in L^\infty(\mathbb{R}^d \times I; L^2_{per}(\mathbb{T}\times\mathbb{Y}))$ and $\varpi \in \mathbb{R}^d$ such that $\mathcal{M}_0 = 0$.*

Proof. We need to consider the solvability of the following equations due to the time scale r and given $p \in (0,2]$. For $r = 2$ and $0 < p \le 1$, it is straightforward to see that $\chi_1(x,t,\xi,s) = \chi_1(\xi,s)$ when $p = 1$. In case that $p \neq 1$, we sometimes omit x and t for simplicity to write $\chi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})$ as $\chi_1(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})$. For any $\varepsilon > 0$, we have

$$\begin{aligned}\varepsilon^{2-r}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial\chi_1}{\partial s} - \frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\varpi - \Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ \cdot\Big(-z+\chi_1(x-\varepsilon z,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}) - \chi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})dz\Big] = 0.\end{aligned} \tag{4.30}$$

Denote $\xi = \frac{x}{\varepsilon}$ and $s = \frac{t}{\varepsilon^r}$, which is a variable with the period $\xi, s \in \mathbb{T}^d = [0,1]^d$, also $v(\xi), \chi_1(\xi,s)$ and $\chi_2(\xi,s)$ are functions on $\mathbb{T}^d$. We solve Eq (4.30) for the functions $\chi_1(\xi,s)$ and $\chi_2(\xi,s)$ on the torus. Let

$$\psi_\varepsilon(x,t) = \varepsilon^{2-r}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}.$$

For $(x,t) \in \mathbb{R}^d \times I$, $\psi_\varepsilon(x,t) \to 0$ as $\varepsilon \to 0$, and from

$$\begin{aligned}&\psi_\varepsilon\frac{\partial\chi_1}{\partial s} - \varepsilon^{r-2}\psi_\varepsilon\varpi\\ &= \int_{\mathbb{R}^d} J(z,s)\mu(\xi,\xi-z)\Big(-z+\chi_1(\varepsilon\xi-\varepsilon z,t,\xi-z,s) - \chi_1(\varepsilon\xi,t,\xi,s)\Big)dz,\end{aligned} \tag{4.31}$$

we have

$$\Xi(y,s) = \int_{\mathbb{R}^d} J(\eta-y,s)v(y,q)(y-q)dq + \varepsilon^{r-2}\psi_\varepsilon\varpi = \emptyset(y,s) + \varepsilon^{r-2}\psi_\varepsilon\varpi. \tag{4.32}$$

We consider that $X = L^2((0,1)\times\mathbb{T}^d)$. Let $\mathrm{L} : X \supseteq D(L) \to X^*$ be defined by $\mathrm{L}u = u'$, where u' is understood in the sense of distributions, i.e.,

$$\int_0^1 u'(t)\psi(t)dt = -\int_0^1 u(t)\psi'(t)dt, \psi \in C_0^\infty(0,1)$$

with domain $Dom(L) = \left\{u \in X : u' \in X^*, u(0) = u(1)\right\}$. We can see that

$$\langle \mathrm{L}u, \rho\rangle = \int_0^1 \langle u'(t), \rho(t)\rangle_V \, dt, u \in Dom(\mathrm{L}), \rho \in X.$$

It is easy to obtain that $\mathrm{L} : X \supseteq D(\mathrm{L}) \to X^*$ is densely defined maximal monotone. For details about the operator L, the reader is referred to Zeidler [30, Prop. 32.10]. Let $\tilde{N} : X \to X^*$ be defined by

$$\tilde{N}\Pi = \int_{\mathbb{R}^d} J(\eta - \zeta, s)\nu(\eta, \zeta)(\Pi(\varepsilon\zeta, \zeta) - \Pi(\varepsilon\eta, \eta))d\zeta = (G - K)\Pi,$$

$$\begin{aligned}
&\langle \tilde{N}\Pi, \Pi\rangle_m \\
&= \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} J(\eta - \zeta, s)\nu(\eta, \zeta)m(\eta)(\Pi(\varepsilon\zeta, \zeta) - \Pi(\varepsilon\eta, \eta))\Pi(\varepsilon\eta, \eta)d\zeta d\eta ds \\
&= -\frac{1}{2}\int_0^1 \int_{\mathbb{T}^d} \mu(\eta, s) \int_{\mathbb{R}^d} J(\eta - \zeta, s)\nu(\eta, \zeta)m(\eta)(\Pi(\varepsilon\zeta, \zeta) - \Pi(\varepsilon\eta, \eta))^2 d\zeta d\eta ds \\
&\leq 0;
\end{aligned}$$

then, $-\tilde{N}$ is a monotone operator in X.

$$\aleph_1\Pi = \int_{\mathbb{R}^d} J(\eta - \zeta, s)\nu(\eta, \zeta)\Pi(\zeta)d\zeta, \tag{4.33}$$

$$\aleph_2\Pi = \int_{\mathbb{R}^d} J(\eta - \zeta, s)\nu(\eta, \zeta)d\zeta\Pi(\eta); \tag{4.34}$$

we know that $\aleph_1$ is bounded in $X \to X^*$ and $\aleph_2$ is a positive and invertible operator. Denote

$$\kappa\pi(\xi) = \int_{\mathbb{R}^d} J(\xi - q, s)\nu(\xi, q)\Pi(q)dq, \ \pi \in L^2(\mathrm{T}^d). \tag{4.35}$$

We first introduce a proposition.

Proposition 1. *[6] For $\check{J}(\eta) = \sum_{k\in\mathbb{Z}^d} J(\eta + k), \eta \in \mathbb{T}^d$, the operator*

$$\kappa\varphi(\xi) = \int_{\mathbb{R}^d} J(\xi - q)\nu(\xi, q)\varphi(q)dq = \int_{\mathbb{T}^d} \check{J}(\xi - \eta)\nu(\xi, \eta)\varphi(\eta)d\eta, \ \varphi \in L^2(\mathbb{T}^d) \tag{4.36}$$

is a compact operator in $L^2(\mathbb{T}^d)$.

From Proposition 1 and Lemma 2.1, we have

$$\begin{aligned}
(-\tilde{N}\chi, \chi) &\geq \frac{1}{2}\alpha_1^2 \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} J(x - y, s)|\chi(y, s) - \chi(x, s)|^2 dy dx ds \\
&\geq c\beta_1 \int_0^1 \int_{\mathbb{T}^d} \left|\chi - \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} \chi\right|^2 dy ds = c\beta_1 \|\chi\|^2_{L^2(\mathbb{T}\times\mathbb{T}^d)};
\end{aligned} \tag{4.37}$$

we know the $-\tilde{N}$ is coercive.

Lemma 4.3. *There exists a function* $\chi_1^\varepsilon(x,t,\xi,s)$ *on* $\mathbb{R}^d \times \mathbb{R} \times [0,1]^d \times [0,1]$ *such that Eq (4.30) holds true.*

Proof. We first rewrite Eq (4.31) as follows

$$\psi_\varepsilon(x,t)\mathrm{L}\chi_1^\varepsilon(x,t,y,s) - \tilde{N}\chi_1^\varepsilon(x,t,y,s) = \Xi, \tag{4.38}$$

where the operator $\psi_\varepsilon(x,t), \mathrm{L}$ and $\tilde{N}$ are defined above, $L : X \supseteq D(\mathrm{L}) \to X^*$ is a densely defined maximal monotone operator in X and $\tilde{N}$ is bounded pseudomonotone and coercive in $X \to X^*$ from the inequality (4.37); due to Eq (4.30) we fix an arbitrary $(x,t) \in \mathbb{R}^d \times I$. By applying Theorem 2.4, Eq (4.38) has a solution, that is, there exists a function $\chi_1^\varepsilon(x,t,\xi,s)$ on the torus $\mathbb{T}^d \times \mathbb{T}$ such that Eq (4.38) holds true. The proof is completed.

For $p > 1$ and $\tilde{\psi}_\varepsilon = \varepsilon^{2-r}|\frac{v}{w_1^\varepsilon}|^{\frac{p-1}{p}}$, we have

$$\begin{aligned}\mathcal{M}_0(x,t) = \psi_\varepsilon \frac{\partial \natural_1^\varepsilon}{\partial s}\nabla v(x,t) - \nabla v(x,t)\Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\Big(-z \\ +p|v(y,t)|^{\frac{p-1}{p}}\natural_1^\varepsilon(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}) - p|v(x,t)|^{\frac{p-1}{p}}\natural_1^\varepsilon(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\Big)dz\Big] = 0.\end{aligned} \tag{4.39}$$

Then we have the following lemma.

Lemma 4.4. *Fix* $\varepsilon > 0$; *there exists a function* $\natural_1^\varepsilon$ *on* $\mathbb{R}^d \times \mathbb{R} \times [0,1]^d \times [0,1]$ *such that Eq (4.39) holds true.*

The proof is similar to the case of $0 < p < 1$, so we omit the details.

For $p = 1, 0 < r \leq 2$ and $\psi_\varepsilon = \varepsilon^{2-r}$, so χ_1 does not include x and t because time derivative term tends to zero when $\varepsilon \to 0$. We obtain the existence of χ_1. Next, we also need to determine ϖ. Then, the solvability condition for Eq (4.31) is that $-\tilde{N}$ is the sum of a positive invertible operator K and a compact operator $-G$. In [14] it shows that the dimension of space $Ker(K-G)^*$ is one and that

$$Ker(K-G)^* = K^{-1}(\xi)\pi_0(\xi) := m(\xi), \tag{4.40}$$

where $\pi_0(\xi)$ and $m(\xi)$ are positive and bounded.

According to the Fredholm theory, $dim(G-K) = dim(G-K)^*$, thus there exists $m(\xi) \in Ker(G-K)$ that satisfies $(G-K)m(\xi) = 0$ such that

$$\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} J(\xi-q,s)v(\xi,q)(\xi-q)dq m(\xi)d\xi ds - \varpi\int_0^1\int_{\mathbb{T}^d} m(\xi)d\xi ds = 0.$$

Taking the normalized $m(\xi)$ with $\int_{\mathbb{T}^d} m(\xi)d\xi = 1$, and choosing ϖ as

$$\varpi = \int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} J(\xi-q,s)v(\xi,q)(\xi-q)dq m(\xi)d\xi ds.$$

We also need the following lemma in order to use symmetry of the integral; it is obviously right when the nonlocal structure is symmetric.

Lemma 4.5. *The compact operator $(K^{-1}G)^*$ has a simple eigenvalue at $\lambda = 1$. The corresponding eigenfunction η_0 satisfies the equation*

$$(K^{-1}G)^*\eta_0 = \eta_0,$$

and there exists a unique (up to an additive constant) function $m \in L^2\left(\mathbb{T}^d\right)$ satisfying

$$(K^{-1}G)^*m = m, \int_{\mathbb{R}^d} J(q-\xi)\nu(q,\xi)m(q)dq = m(\xi)\int_{\mathbb{R}^d} J(\xi-q)\nu(\xi,q)dq,$$

i.e., Span(m)=Ker$(K-G)^*$. This function obeys the following lower and upper bounds:

$$0 < \kappa_1 \le \eta_0(\xi) \le \kappa_2 < \infty, 0 < \tilde{\kappa}_1 \le m(\xi) \le \tilde{\kappa}_2 < \infty, \forall\, \xi \in \mathbb{T}^d,$$

where $\kappa_1, \kappa_2, \tilde{\kappa}_1, \tilde{\kappa}_2$ are positive constants.

We can obtain from the Krein-Rutman theorem [40] that the operator $(K^{-1}G)^*$ has the maximal eigenvalue equal to 1.

Since we find that χ_1^ε is related to ε, we also need to discuss strong measurability in (x,t). This section is devoted to discussing the existence, uniqueness and regularity of solutions to cell problems at the critical ratio $r = 2$. The cases $r < 2$ and $r > 2$ are similar to the case of $r = 2$ and E and $\mathbb{T}^d$ correspond to the cell area of time and space respectively. We simply write $w(y,s)$ for the functions $w = w(x,t,y,s)$ by omitting the variables x and t, unless any confusion may arise. We first explain $V = L^2_{per}(Y)$; the $\{\chi_1^k\}_{k=1}^d$ in this section refers to χ_1.

4.1. $r = 2$ and $p \neq 1$

Case I. For $r = 2$ and $0 < p < 1$. For each $(x,t) \in \mathbb{R}^d \times (0,T)$, the cell problem reads that

$$\begin{cases} \frac{1}{p}\,|\nu|^{(1-p)/p}\,\partial_s\chi_1(\xi,s) = \int_{\mathbb{T}^d} J(\xi-q,s)\nu(\xi,q)\Big(q-\xi+\chi_1(q,s)-\chi_1(\xi,s)\Big)dqd\xi, \\ \chi^k(y,0) = \chi^k(y,1), y \in \mathbb{T}^d, \end{cases} \tag{4.41}$$

such that $M_Y(\chi_1^k(\cdot,s)_y) = 0$ for $s \in \mathbb{T}$. It can be regarded as a constant to discuss the existence, uniqueness and regularity of solutions to Eq (4.41) in view of $\nu = \nu(x,t)$ depending only on (x,t) for each (x,t) that is fixed. In case that $\nu(x,t) \neq 0$, assuming

$$\Xi + \frac{1}{p}\,|\nu|^{(1-p)/p}\,\varpi \in \left[L^2\left(J; L^2_{per}(Y)\right)\right]^d,$$

one can construct a unique weak solution $\chi_1^k(x,t,\cdot,\cdot) \in L^2(E;V) \cap H^1\left(E; L^2(V^*)\right)$, where $V = L^2_{per}(Y)/\mathbb{R}, k \in \mathbb{N}$.

Lemma 4.6. *(Strong measurability in (x,t)) Assume that $r = 2$ and $p \in (0,1)$. For $k \in d$, the function:*

$$(x,t) \mapsto \chi_1^k(x,t,\cdot,\cdot)\ (resp.,\ |\nu(x,t)|^{(1-p)/p}\chi_1^k(x,t,\cdot,\cdot))$$

is strongly measurable in $\mathbb{R}^d \times (0,T)$ with values in $L^2(E;V)$. Moreover,

$$\chi_1^k \in L^\infty(\mathbb{R}^d \times I; L^2(E;V)),\ |\nu|^{(1-p)/p}\chi_1^k \in L^\infty(\mathbb{R}^d \times I; H^1(E; L^2(V))).$$

Proof. Since $|v|^{(1-p)/p}$ lies in $L^{(p+1)/(1-p)}(\mathbb{R}^d \times I)$, one can take a sequence (ψ_ε) of step functions from $\mathbb{R}^d \times I$ into $\mathbb{R}$ such that $\psi_\varepsilon(x,t) \to (1/p)|v(x,t)|^{(1-p)/p}$ for $(x,t) \in M_0$, where M_0 is a measurable set in $\mathbb{R}^d \times I$ satisfying $|(\mathbb{R}^d \times I)\backslash M_0| = 0$, as $\varepsilon \to +\infty$. Fix $(x,t) \in M_0$ and let $\chi_1^\varepsilon(x,t,\cdot,\cdot) \in L^2(E;V)$ be the unique solution to

$$\begin{cases} \frac{1}{p}\,|w_\varepsilon|^{(1-p)/p}\,\partial_s\chi_1^\varepsilon(\xi,s) = \int_{\mathbb{R}^d} J(\xi-q,s)v(\xi,q)\Big(q-\xi+\chi_1^\varepsilon(q,s)-\chi_1^\varepsilon(\xi,s)\Big)dq, \\ \chi_1^\varepsilon(y,0) = \chi_1^\varepsilon(y,1), y \in \mathbb{T}^d \end{cases} \tag{4.42}$$

such that $M_Y(\chi_1^\varepsilon(\cdot,s)_y) = 0$. Moreover, we note that the vector-valued function $(x,t) \mapsto \chi^\varepsilon(x,t,\cdot,\cdot)$ is defined over $\mathbb{R}^d \times I$. Test Eq (4.42) by using χ_1^ε and with respect to the ξ integral in Y. We observe by using the nonlocal Poincaré inequality that

$$\begin{aligned} \frac{\psi_\varepsilon}{2}\frac{d}{ds}\left\|\chi_1^\varepsilon(y,s)\right\|^2_{L^2(Y)} + \beta_1\left\|\chi_1^\varepsilon(y,s)\right\|^2_{L^2(Y)} &\le \int_Y |\Xi|\cdot\chi_1^\varepsilon(y,s)dy \\ &\le \frac{\beta_1}{2}\left\|\chi_1^\varepsilon(y,s)\right\|^2_{L^2(Y)} + C\,\|\Xi\|^2_{L^2(Y)}. \end{aligned}$$

Integrate both sides over $(0,1)$ and employ the periodicity $\chi^\varepsilon(\cdot,0) = \chi^\varepsilon(\cdot,1)$ in $\mathbb{T}^d$. It then follows that

$$\frac{\beta_1}{2}\int_0^1 \left\|\chi_1^\varepsilon(y,s)\right\|^2_{L^2(Y)} ds \le C\int_0^1 \|\Xi\|^2_{L^2(Y)} ds;$$

then, one can also get

$$\frac{\psi_\varepsilon}{2}\int_0^1 \left\|\partial_s\chi_1^\varepsilon(y,s)\right\|^2_{L^2(Y)} ds \le C\int_0^1 \|\Xi\|^2_{L^2(Y)} ds.$$

Therefore we can select a subsequence and still note χ_1^ε as a limit $\chi_1(x,t,\cdot,\cdot) \in L^2(E;V)$ such that

$$|v(x,t)|^{(1-p)/p}\chi(x,t,\cdot,\cdot) \in H^1\left(E;L^2(V)\right)$$

and

$$\begin{aligned} &\chi_1^\varepsilon(x,t,\cdot,\cdot) \rightharpoonup \chi_1(x,t,\cdot,\cdot) && \text{weakly in } L^2(E;V). \\ &\psi_\varepsilon(x,t)\chi_1^\varepsilon(x,t,\cdot,\cdot) \rightharpoonup \tfrac{1}{p}|v(x,t)|^{\frac{1}{p}-1}\chi_1(x,t,\cdot,\cdot) && \text{weakly in } H^1\left(E;L^2(V)\right). \end{aligned}$$

Hence, $(x,t) \mapsto \chi_1(x,t,\cdot,\cdot)$ is weakly measurable in $\mathbb{R}^d \times I$, with values in $L^2(E;V)$; therefore, due to Pettis' theorem, it is also strongly measurable. Moreover, the fact that the convergence $\psi_\varepsilon(x,t) \to (1/p)\,|v(x,t)|^{(1-p)/p}$ a.e. in $\mathbb{R}^d \times I$ as $\varepsilon \to 0$, it can be verified that the unique solution χ_1 solves Eq (4.41) for a.e. $(x,t) \in \mathbb{R}^d \times I$.

Finally, it is easy to check that

$$\chi_1,\ |v(x,t)|^{(1-p)/p}\chi_1 \in L^\infty\left(\mathbb{R}^d \times (0,T);H^1\left(E;L^2(V)\right)\right).$$

In case that $0 < p < 1$, for a.e. $(x,t) \in \mathbb{R}^d \times I$ and all $l \in V$ and $l_1 \in C^\infty_{per}(E)$, we observe that

$$\int_0^1\int_{\mathbb{T}^d}\Big[\frac{1}{p}\,|v(x,t)|^{(1-p)/p}\chi_1(\xi,s)l(\xi)\partial_s l_1(s) - \int_{\mathbb{R}^d} J(\xi-q,s)v(\xi,q)$$

$$\cdot\Big(q-\xi+\chi_1(q,s)-\chi_1(\xi,s)\Big)dql(\xi)l_1(s)\Big]d\xi ds=0. \tag{4.43}$$

Next we will show that

$$\frac{1}{p}|v(x,t)|^{(1-p)/p}\chi_1\in L^2\left(\mathbb{R}^d\times I;H^1(E;V^*)\right). \tag{4.44}$$

Actually let us define $\xi(x,t,\cdot,\cdot)\in L^2(E;V^*)$ by

$$\int_0^1\langle\xi(x,t,\cdot,s),\varsigma(\cdot,s)\rangle_V ds=\int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}J(\xi-q,s)v(\xi,q)$$
$$\cdot\Big(q-\xi+\chi_1(q,s)-\chi_1(\xi,s)\Big)dq\varsigma(\xi,s)d\xi ds \tag{4.45}$$

for $\varsigma\in L^2(J;V)$. Then $\xi:\mathbb{R}^d\times I\to L^2(E;V^*)$ is weakly measurable, and actually it is strongly measurable by Pettis' theorem.

Since $\chi_1\in L^2(\mathbb{R}^d\times I\times\mathbb{T}^d\times E)$, one can verify that $\xi\in L^2\left(\mathbb{R}^d\times I;L^2(E;V^*)\right)$. Furthermore, we deduce by Eq (4.45) that

$$\int_0^1\frac{1}{p}|v|^{(1-p)/p}\chi_1(x,t,\cdot,s)\partial_s t_1(s)ds=\int_0^1\xi(x,t,\cdot,s)t_1(s)ds\ in\ V^*,$$

which along with the arbitrariness of $t_1\in C^\infty_{\rm per}(E)$ in the distributional sense for a.e. $(x,t,s)\in\mathbb{R}^d\times I\times E$ implies that

$$\frac{1}{p}|v|^{(1-p)/p}\partial_s\chi_1(x,t,\xi,s)=-\xi(x,t,\cdot,s)\ in\ V^*$$

This yields Eq (4.44). It is easy to check that

$$\frac{1}{p}|v(x,t)|^{(1-p)/p}\chi(x,t,\cdot,1)=\frac{1}{p}|v(x,t)|^{(1-p)/p}\chi(x,t,\cdot,0)\text{ in }V^*$$

for a.e. $(x,t)\in\mathbb{R}^d\times I$. Case I is proved.

Case II. $r=2$ and $1<p\le 2$. It is enough to consider the case that $v(x,t)\neq 0$ only. For each $(x,t)\in[v\neq 0]:=\left\{(x,t)\in\mathbb{R}^d\times(0,T):v(x,t)\neq 0\right\}$, the existence and uniqueness of a weak solution $\natural_1^k(x,t,\cdot,\cdot)\in L^\infty[v\neq 0]\cap L^2(E;V)$ to the cell problem can be verified

$$\begin{cases}\partial_s\natural_1(\xi,s)=\displaystyle\int_{\mathbb{T}^d}J(\xi-q,s)v(\xi,q)\Big(q-\xi+p|v|^{(p-1)/p}\natural_1(\xi,s)-p|v|^{(p-1)/p}\natural_1(q,s)\Big)dq,\\ \natural_1^k(y,0)=\natural_1^k(y,1),\ y\in\mathbb{T}^d,\end{cases} \tag{4.46}$$

such that $M_Y(\natural^k(\cdot,s)_y)=0$ for $s\in\mathbb{T}$.

Moreover, we claim that

$$|u|^{(p-1)/p}\natural_1^k\in L^\infty\left([v\neq 0];L^2(E;V)\right),\ \natural_1^k\in L^\infty\left([v\neq 0];L^2(E;V)\right),$$

which implies that $\natural^k\in L^\infty\left([v\neq 0];L^2(E;V^*)\right)$.

The proof is similar to the case for $0<p<1$.

4.2. $r \neq 2$ and $0 < p \leq 2$

Case III. $0 < r < 2$ and $0 < p \leq 1$. Let $\psi_\varepsilon(x,t) = \varepsilon^{2-r}\psi_\varepsilon \to 0$ for $(x,t) \in \mathbb{R}^d \times I$ as $\varepsilon \to 0$, χ_1 satisfies the following equation

$$\begin{cases} \int_{\mathbb{R}^d} J(\xi - q, s)\nu(\xi, q)\Big(q - \xi + \chi_1(q,s) - \chi_1(\xi, s)\Big)dq = 0, \\ \chi_1(y,0) = \chi_1(y,1), \ \ y \in \mathbb{T}^d \end{cases} \tag{4.47}$$

as $\varpi = 0$; we found that χ_1 does not include x and t, and that $\chi_1 \in L^2(E \times V)$; then, we have

$$u_1(x,t,y,s) = \sum_{k=1}^{d} \partial_{x_k}(|u|^{p-1}\, u(x,t)) \cdot \chi_1^k(y,s). \tag{4.48}$$

Case IV. $0 < r < 2$ and $1 < p \leq 2$. It is enough to just consider the case that $v(x,t) \neq 0$. For each $(x,t) \in [v \neq 0] := \Big\{(x,t) \in \mathbb{R}^d \times (0,T) : v(x,t) \neq 0\Big\}$, we can verify the existence and uniqueness of the solution $\natural_1^k(\cdot,\cdot) \in L^2(E;V) \cap H^1\Big(E; L^2(V)\Big)$ to the cell problem

$$\begin{cases} \int_{\mathbb{T}^d} J(\xi - q, s)\nu(\xi, q) \\ \qquad \cdot\Big(q - \xi + p|v|^{(p-1)/p}\natural_1(\xi, s) - p|v|^{(p-1)/p}\natural_1(q,s)\Big)dq = 0, \\ \natural_1^k(y,0) = \natural_1^k(y,1), \ \ y \in \mathbb{T}^d, \end{cases} \tag{4.49}$$

where, actually, $\natural_1 = \frac{1}{p}|v|^{(1-p)/p}\chi_1$, the situation is similar to Case III.

For $r > 2$ and $0 < p \leq 2$, we consider two cases.

Case V. $r > 2$ and $0 < p \leq 1$. For any $\varepsilon > 0$, we have

$$\begin{aligned} &\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial \chi_1}{\partial s} - \varepsilon^{r-2}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\varpi \\ &= \varepsilon^{r-2}\Big[\int_{\mathbb{R}} J(z, \frac{t}{\varepsilon^r})\mu(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} - z)\Big(- z + \chi_1(\frac{x}{\varepsilon} - z, \frac{t}{\varepsilon^r}) - \chi_1(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\Big)dz\Big]; \end{aligned} \tag{4.50}$$

let $\varepsilon \to 0$; we get

$$\frac{1}{p}|v|^{\frac{1-p}{p}}\frac{\partial \chi_1}{\partial s} = 0, \tag{4.51}$$

which implies that χ_1 is in fact independent of s and satisfies

$$\frac{1}{p}|v|^{\frac{1-p}{p}}\varpi = \int_{\mathbb{R}}\int_0^1 J(z,s)ds\mu(\xi, \xi - z)\Big(- z + \chi_1(\xi - z) - \chi_1(\xi)\Big)dz \ in \ \mathbb{T}^d. \tag{4.52}$$

Case VI. For $r > 2$ and $1 < p \leq 2$. For any $\varepsilon > 0$, we have

$$\begin{aligned} &\Big|\frac{v}{w_1^\varepsilon}\Big|^{\frac{p-1}{p}}\frac{\partial \Phi_1}{\partial s} - \varepsilon^{r-2}\Big[\int_{\mathbb{R}^d} J(z, \frac{t}{\varepsilon^r})\mu(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} - z)\Big(- z \\ &+ p|v|^{\frac{p-1}{p}}\Phi_1(\frac{x}{\varepsilon} - z, \frac{t}{\varepsilon^r}) - p|v|^{\frac{p-1}{p}}\Phi_1(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\Big)dz\Big] = 0, \end{aligned} \tag{4.53}$$

let $\varepsilon \to 0$; we get

$$\frac{\partial \Phi_1}{\partial s} = 0, \tag{4.54}$$

which means that $\Phi_1(y,s)$ does not depend on s and satisfies Eq (4.53).

5. Existence of χ_2 and positive definiteness of Θ

Actually the existence of χ_2 is similar to χ_1; the proof steps are the same as before; next, we prove that the symmetric part of the matrix Θ defined in Theorem 3.2 is positive definite.

Case I. $r = 2$ and $0 < p \le 1$:

From Eq (4.18), we have

$$\begin{aligned}\mathcal{M}_\varepsilon(x,t) = \varepsilon^{2-r}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial\chi_2}{\partial s} - \frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\varpi\chi_1 - [\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^r})\nu(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ \cdot(-z\chi_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}) + \chi_2(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}) - \chi_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}) + \frac{1}{2}z^2)dz].\end{aligned} \tag{5.1}$$

For $r = 2$,

$$\varepsilon^{2-r}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial\chi_2}{\partial s} \to \frac{1}{p}|v|^{\frac{1-p}{p}}\frac{\partial\chi_2}{\partial s} \quad \text{as } \varepsilon \to 0. \tag{5.2}$$

Next, using the time periodicity of J, we consider

$$\begin{aligned}\Theta(x,t) = \int_0^1\int_Y[\int_{\mathbb{R}^d} J(z,s)\mu(y,y-z)(-z\chi_1(x,t,y-z,s) + \chi_2(x,t,y-z,s)\\ -\chi_2(x,t,y,s) + \frac{1}{2}z^2)dz - \frac{1}{p}|v|^{\frac{1-p}{p}}(\frac{\partial\chi_2}{\partial s} - \varpi\chi_1)]dyds,\end{aligned} \tag{5.3}$$

$$\begin{aligned}\Theta^{ij}(x,t) &= \Theta(x,t)\int_0^1\int_{\mathbb{T}^d} m(\xi)d\xi ds\\ &= \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{2}(\xi-q)^i(\xi-q)^jJ(\xi-q,s)\nu(\xi,q)m(\xi)dqd\xi ds\\ &- \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} J(\xi-q,s)\nu(\xi,q)m(\xi)(\xi-q)^i\chi_1^j(x,t,q,s)dqd\xi ds\\ &+ \varpi^i\int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{p}|v|^{\frac{1-p}{p}}\chi_1^j(x,t,\xi,s)\nu(\xi,q)m(\xi)dqd\xi ds\\ &- \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{p}|v|^{\frac{1-p}{p}}\frac{\partial\chi_2}{\partial s}\nu(\xi,q)m(\xi)dqd\xi ds,\end{aligned} \tag{5.4}$$

the last formula on the right side of Eq (5.4) is zero by using the periodicity of χ_2.

Our aim is to show that the symmetric part of the right-hand side of Eq (5.4) is equal to $\mathfrak{B}$ such that

$$\begin{aligned}\mathfrak{B}^{ij} &= \tilde{\Theta}^{ij} + \tilde{\Theta}^{ji}\\ &= \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(\xi-q)^i(\xi-q)^jJ(\xi-q,s)\nu(\xi,q)m(\xi)dqd\xi ds\\ &- \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} J(\xi-q,s)\nu(\xi,q)m(\xi)\left((\xi-q)^i\chi_1^j + (\xi-q)^j\chi_1^i\right)dqd\xi ds\\ &+ \varpi^i\int_0^1\int_{\mathbb{T}^d}\frac{1}{p}|v|^{\frac{1-p}{p}}\chi_1^j(x,t,\xi,s)\nu(\xi,q)m(\xi)d\xi ds\end{aligned}$$

$$+\quad \varpi^j \int_0^1 \int_{\mathbb{T}^d} \frac{1}{p}|v|^{\frac{1-p}{p}} \chi_1^i(x,t,\xi,s)v(\xi,q)m(\xi)d\xi ds. \tag{5.5}$$

We also want to prove $\mathfrak{B}^{ij}$ is positive definite. For brevity, we write $\chi_1(x,t,\xi,s) = \chi_1(\xi,s)$. From Eq (5.5), we have

$$\mathfrak{B}^{ij} = \mathfrak{B}_1^{ij} + \mathfrak{B}_2^{ij} + \mathfrak{B}_3^{ij}, \tag{5.6}$$

where

$$\mathfrak{B}_1^{ij} = \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\xi-q)^i(\xi-q)^j J(\xi-q,s)v(\xi,q)m(\xi)dqd\xi ds, \tag{5.7}$$

$$\begin{aligned}\mathfrak{B}_2^{ij} = &\int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} J(\xi-q,s)v(\xi,q)m(\xi)\left(\chi_1(\xi,s)-\chi_1(q,s)\right)^i \\ &\cdot \left(\chi_1(\xi,s)-\chi_1(q,s)\right)^j dqd\xi ds,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\mathfrak{B}_3^{ij} = &\int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} J(\xi-q,s)v(\xi,q)m(\xi)\Big((\xi-q)^i\left(\chi_1(\xi,s)-\chi_1(q,s)\right)^j \\ &+(\xi-q)^j\left(\chi_1(\xi,s)-\chi_1(q,s)\right)^i\Big)dqd\xi ds.\end{aligned} \tag{5.9}$$

Obviously, we find that $\mathfrak{B}_1^{ij}$ is the first integral of Eq (5.5). Let us rearrange the integral in $\mathfrak{B}_3^{ij}$ as follows:

$$\begin{aligned}&\mathfrak{B}_3^{ij} \\ =&\int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} J(\xi-q,s)v(\xi,q)m(\xi)((\xi-q)^i\chi_1^j(\xi,s)+(\xi-q)^j\chi_1^i(\xi,s))dqd\xi ds \\ -&\int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} J(\xi-q,s)v(\xi,q)m(\xi)((\xi-q)^i\chi_1^j(q,s)+(\xi-q)^j\chi_1^i(q,s))dqd\xi ds \\ =&\breve{\iota}_2^{ij} + \iota_2^{ij}.\end{aligned} \tag{5.10}$$

Then, ι_2^{ij} coincides with the second integral in Eq (5.5). Further, we rearrange the integral $\breve{\iota}_2^{ij}$ and recall the definition of the function $\emptyset$ in Eq (4.32):

$$\begin{aligned}\breve{\iota}_2^{ij} =& \int_0^1 \int_{\mathbb{T}^d} \emptyset^i(\xi,s)\chi_1^j(\xi,s)v(\xi,q)m(\xi)d\xi ds \\ +&\int_0^1 \int_{\mathbb{T}^d} \emptyset^j(\xi,s)\chi_1^i(\xi,s)v(\xi,q)m(\xi)d\xi ds \\ =&\int_0^1 \int_{\mathbb{T}^d} \chi_1^j(\xi,s)v(\xi,q)m(\xi)(\frac{1}{p}|v|^{\frac{1-p}{p}}\varpi^i + A\chi_1^i(\xi,s) - \frac{1}{p}|v|^{\frac{1-p}{p}}\frac{\partial \chi_1^i}{\partial s})d\xi ds \\ +&\int_0^1 \int_{\mathbb{T}^d} \chi_1^i(\xi,s)v(\xi,q)m(\xi)(\frac{1}{p}|v|^{\frac{1-p}{p}}\varpi^j + A\chi_1^j(\xi,s) - \frac{1}{p}|v|^{\frac{1-p}{p}}\frac{\partial \chi_1^j}{\partial s})d\xi ds \\ =&\int_0^1 \int_{\mathbb{T}^d} \frac{1}{p}|v|^{\frac{1-p}{p}}\Big(\varpi^i\chi_1^j(\xi,s)+\varpi^j\chi_1^i(\xi,s)\Big)v(\xi,q)m(\xi)d\xi ds \\ +&\int_0^1 \int_{\mathbb{T}^d} v(\xi,q)m(\xi)\Big(\chi_1^j(\xi,s)A\chi_1^i(\xi,s)+\chi_1^i(\xi,s)A\chi_1^j(\xi,s)\Big)d\xi ds\end{aligned}$$

$$-\int_0^1\int_{\mathbb{T}^d}\nu(\xi,q)m(\xi)\frac{1}{p}|\nu|^{\frac{1-p}{p}}\left(\chi_1^j(\xi,s)\frac{\partial\chi_1^i}{\partial s}+\chi_1^i(\xi,s)\frac{\partial\chi_1^j}{\partial s}\right)d\xi ds.$$

The last two formulas on the right side of the above equation are zero by using the periodicity of χ_1. Denote

$$\begin{aligned}\mathfrak{I}_2^{ij} &= \varpi^i\int_0^1\int_{\mathbb{T}^d}\frac{1}{p}|\nu|^{\frac{1-p}{p}}\chi_1^j(\xi,s)\nu(\xi,q)m(\xi)d\xi ds\\ &+ \varpi^j\int_0^1\int_{\mathbb{T}^d}\frac{1}{p}|\nu|^{\frac{1-p}{p}}\chi_1^i(\xi,s)\nu(\xi,q)m(\xi)d\xi ds,\end{aligned}$$

$$\begin{aligned}\tilde{\mathfrak{I}}_2^{ij} &= \int_0^1\int_{\mathbb{T}^d}\chi_1^j(\xi,s)\nu(\xi,q)m(\xi)\left(A\chi_1^i(\xi,s)\right)d\xi ds\\ &+ \int_0^1\int_{\mathbb{T}^d}\chi_1^i(\xi,s)\nu(\xi,q)m(\xi)\left(A\chi_1^j(\xi,s)\right)d\xi ds.\end{aligned}$$

Then, $\mathfrak{I}_2^{ij}$ coincides with the third integral in Eq (5.5). We only need to prove that $\mathfrak{B}_2^{ij}=-\tilde{\mathfrak{I}}_2^{ij}$. We have

$$\begin{aligned}\mathfrak{B}_2^{ij} &= \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}J(\xi-q,s)\nu(\xi,q)m(\xi)\left(\chi_1(\xi,s)-\chi_1(q,s)\right)^i\chi_1^j(\xi,s)dqd\xi ds\\ &- \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}J(\xi-q,s)\nu(\xi,q)m(\xi)\left(\chi_1(\xi,s)-\chi_1(q,s)\right)^i\chi_1^j(q,s)dqd\xi ds\\ &= -\int_0^1\int_{\mathbb{T}^d}A\chi_1^i(\xi,s)\chi_1^j(\xi,s)\nu(\xi,q)m(\xi)d\xi ds+\iota_3^{ij}.\end{aligned}$$

We rearrange ι_3^{ij}:

$$\begin{aligned}\iota_3^{ij} &= \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}J(\xi-q,s)\nu(\xi,q)m(\xi)\left(\chi_1(q,s)-\chi_1(\xi,s)\right)^i\chi_1^j(q,s)dqd\xi ds\\ &= \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}J(\xi-q,s)\nu(\xi,q)m(\xi)\left(\chi_1(\xi,s)-\chi_1(q,s)\right)^i\chi_1^j(\xi,s)dqd\xi ds\\ &= \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}J(\xi-q,s)\nu(\xi,q)m(\xi)\chi_1^i(\xi,s)\chi_1^j(\xi,s)dqd\xi ds\\ &- \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}J(\xi-q,s)\nu(\xi,q)m(\xi)\chi_1^j(q,s)\chi_1^i(\xi,s)dqd\xi ds\\ &= -\int_0^1\int_{\mathbb{T}^d}A\chi_1^j(\xi,s)\chi_1^i(\xi,s)\nu(\xi,q)m(\xi)d\xi ds.\end{aligned}\tag{5.11}$$

Thus, $\mathfrak{B}_2^{ij}=-\tilde{\mathfrak{I}}_2^{ij}$ and the proof of Eq (5.4) is done by this relation. The structure of Eq (5.4) means that $(Ir,r)\geq 0$, for any $q\in\mathbb{R}^d$; moreover, $(Ir,r)>0$ since $m>0$ and $\chi_1(q,s)$ is the periodic function while q is a linear function. Consequently, $\lfloor((\xi-q)+(\chi_1(\xi,s)-\chi_1(q,s)))\cdot r\rfloor^2$ will not be identical to 0 if $r\neq 0$.

Case II. $r>2$ and $0<p\leq 1$:

For $r > 2$, χ_1 does not include s, and

$$\begin{aligned}\Theta &= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{2}(\xi-q)(\xi-q)\Big[\int_0^1 J(\xi-q,s)ds\Big]v(\xi,q)m(\xi)dqd\xi \\ &- \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\Big[\int_0^1 J(\xi-q,s)ds\Big]v(\xi,q)m(\xi)(\xi-q)\chi_1(q)dqd\xi \\ &+ \varpi\int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{p}|v|^{\frac{1-p}{p}}\chi_1(\xi)v(\xi,q)m(\xi)dqd\xi ds. \end{aligned} \tag{5.12}$$

Case III. $r < 2$ and $0 < p \le 1$:

$$\varepsilon^{2-r}\frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial\chi_2}{\partial s} \to 0 \text{ as } \varepsilon \to 0; \tag{5.13}$$

there is one less item here than Case I and χ does not contain x and t.

Case IV. $0 < r < \infty$ and $1 < p \le 2$:

The proof is similar to that in Case I, so it will not be described here.

6. Estimation of the remainder and a priori estimates

The estimation of error is similar to the linear equation, so we do not provide a detailed description here; this gives the result of $p = 1$, and other situations can be obtained by analogous argument.

Proposition 2. *Let $v \in C^\infty\left((0,T), \mathcal{S}\left(\mathbb{R}^d\right)\right)$. For the functions $\phi_\varepsilon^{(time)}$ and $\phi_\varepsilon^{(space)}$:*

$$\begin{aligned}\phi_\varepsilon^{(time)}(x,t) &= \varepsilon\frac{\partial\chi_1}{\partial t}\cdot\nabla v(x^\varepsilon,t) + \varepsilon^2\frac{\partial\chi_2}{\partial t}\cdot\nabla\nabla v(x^\varepsilon,t) \\ &+ \varepsilon\chi_1\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\right)\cdot\nabla\frac{\partial v}{\partial t}(x^\varepsilon,t) \\ &+ \varepsilon^2\chi_2\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\right)\otimes\left(-\frac{\varpi}{\varepsilon}\right)\cdot\nabla\nabla\nabla v(x^\varepsilon,t) \\ &+ \varepsilon^2\chi_2\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\right)\cdot\nabla\nabla\frac{\partial v}{\partial t}(x^\varepsilon,t), \end{aligned} \tag{6.1}$$

$$\begin{aligned}\phi_\varepsilon^{(space)}(x,t) &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^r})v\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z\right)\Big\{-\frac{\varepsilon^2}{2}\nabla\nabla v(x^\varepsilon,t)\cdot z\otimes z \\ &+ \varepsilon^2\int_0\nabla\nabla v(x^\varepsilon-\varepsilon zq,t)\cdot z\otimes z(1-q)dq \\ &+ \varepsilon^3\chi_1\left(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}\right)\int_0^1\nabla\nabla\nabla v(x^\varepsilon-\varepsilon zq,t)\,z\otimes z(1-q)dq \\ &- \varepsilon^3\chi_2\left(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}\right)\int_0^1\nabla\nabla\nabla v(x^\varepsilon-\varepsilon zq,t)\,zdq\Big\}dz; \end{aligned} \tag{6.2}$$

we have

$$\left\|\phi_\varepsilon^{(space)}\right\|_2 \to 0 \text{ and } \left\|\phi_\varepsilon^{(time)}\right\|_2 \to 0 \text{ as } \varepsilon \to 0, \tag{6.3}$$

where $\|\cdot\|_2$ is the norm in $L^2\left((0,T), L^2\left(\mathbb{R}^d\right)\right)$ and $x^\varepsilon = x - \frac{\varpi}{\varepsilon}t$.

Proof. The convergence for $\phi_\varepsilon^{(time)}$ immediately follows from the representation (6.1) for this function. For the function $\phi_\varepsilon^{(space)}$, the proof is completely analogous to the proof of [13, Proposition 5]. The proof of Proposition 2 is done.

Together with Proposition 2, we get Eq (4.28), that is, $\lim_{\varepsilon\to 0^+} \|\phi_\varepsilon\|_{L^2(I,L^2(\mathbb{R}^d))} = 0$.

Let $u^0(x,t)$ be a solution of Eq (3.4) with $u^0(x,0) = \varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$. For any $T > 0$, then $v(x,t) \in C^\infty\left((0,T), \mathcal{S}\left(\mathbb{R}^d\right)\right)$ and we bring it into the equation satisfied by u^ε by constructing the approximate auxiliary functions (4.4)–(4.6). It follows from Lemma 4.2 that w^ε satisfies the following equation

$$\begin{aligned} \frac{\partial w^\varepsilon(x,t)^{\frac{1}{p}}}{\partial t} - L^\varepsilon w^\varepsilon &= \frac{1}{p}|w^\varepsilon(x,t)|^{\frac{1-p}{p}} \frac{\partial u^0}{\partial t}(x^\varepsilon,t) - \Theta \cdot \nabla\nabla u^0(x^\varepsilon,t) + \phi_\varepsilon(x,t) \\ &= \Phi^\varepsilon(x,t), \\ w^\varepsilon(x,0) &= \varphi(x) + \psi^\varepsilon(x), \end{aligned}$$

where $x^\varepsilon = x - \frac{\varpi}{\varepsilon}t$ and

$$\psi^\varepsilon(x) = \varepsilon\chi_1\left(\frac{x}{\varepsilon},0\right)\cdot\nabla\varphi(x) + \varepsilon^2\chi_2\left(\frac{x}{\varepsilon},0\right)\cdot\nabla\nabla\varphi(x) \in L^2\left(\mathbb{R}^d\right).$$

Consequently, the difference $\hat{v}^\varepsilon(x,t) = w^\varepsilon(x,t) - v^\varepsilon(x,t)$, where v^ε is the solution of Eq (3.26), which satisfies the following problem:

$$\frac{\partial((w^\varepsilon)^{\frac{1}{p}} - (v^\varepsilon)^{\frac{1}{p}})}{\partial t} - L^\varepsilon\hat{v}^\varepsilon(x,t) = \Phi^\varepsilon(x,t),\ v^\varepsilon(x,0) = \psi^\varepsilon(x). \tag{6.4}$$

Notice that, with Proposition 2, we have that $\|\psi^\varepsilon\|_{L^2(\mathbb{R}^d)} = O(\varepsilon)$ and $\|\Phi^\varepsilon\|_2 = o(1)$ when $u(x,t) \neq 0$.

We will show that $(v^\varepsilon)^{1/p} - (w^\varepsilon)^{1/p}$ tends to zero in $L^\infty((0,T); L^2_{loc}(\mathbb{R}^d)))$ as $\varepsilon \to 0$. Denote

$$Z = L^\infty_{loc}((0,T), BV(\mathbb{R}^d))) \cap C((0,T), L^1_{loc}(\mathbb{R}^d)),\ \mathcal{Z} = L^1((0,T), L^1_{loc}(\mathbb{R}^d)).$$

Proposition 3. *Let $v^\varepsilon \in Z$ be the solution of Eq (6.4) with a small ψ^ε and ϕ^ε:*

$$\|\phi_\varepsilon\|_{L^2((0,T),L^2(\mathbb{R}^d))} = o(1),\ \|\psi^\varepsilon\|_{L^2(\mathbb{R}^d)} = O(\varepsilon) \quad as\ \varepsilon \to 0.$$

Then, we have

$$\|(w^\varepsilon)^{1/p} - (v^\varepsilon)^{1/p}\|_{L^\infty((0,T),L^2_{loc}(\mathbb{R}^d))} \to 0 \ as\ \varepsilon \to 0.$$

Proof. For $p = 1$, please refer to [14]; we mainly discuss the case that $p \neq 1$. Let $(w^\varepsilon)^{1/p} = v_1$, and let $(v^\varepsilon)^{1/p} = u^\varepsilon = v_2$ satisfy the following equations

$$\begin{cases} \frac{\partial v_1^{1/p}}{\partial t} - L^\varepsilon v_1 = \Phi^\varepsilon,\ v_1(x,0) = \varphi(x) + \psi^\varepsilon(x), \\ \frac{\partial v_2^{1/p}}{\partial t} - L^\varepsilon v_2 = 0,\ v_2(x,0) = \varphi(x). \end{cases} \tag{6.5}$$

By subtraction, we have

$$\partial_t\left((\delta I - L^t)\hbar\right) - L^t(v_1 - v_2) = \Phi^\varepsilon, t \in I, \tag{6.6}$$

$\hbar = (\delta I - L^t)^{-1}(v_1^{1/p} - v_2^{1/p})$ and $X = L^2(U)$. By multiplying both sides of Eq (6.6) with the test function $\hbar$ and integrate it, we obtain

$$\begin{aligned}
&(\Phi^\varepsilon, \hbar\rangle_X \\
&= \langle \partial_t ((\delta I - L^t)\hbar), \hbar\rangle_X + \langle (\delta I - L^t)(v_1 - v_2), \hbar\rangle_X - \delta \langle (v_1 - v_2), \hbar\rangle_X \\
&= \langle \partial_t ((\delta I - L^t)\hbar), \hbar\rangle_X + \langle v_1^{1/p} - v_2^{1/p}, v_1 - v_2\rangle_X - \delta \langle (v_1 - v_2), \hbar\rangle_X .
\end{aligned}$$

Applying the Cauchy-Schwartz inequality and using $(\Phi^\varepsilon, \hbar\rangle_X \doteq o(\varepsilon)$, the monotonicity of $v \mapsto v^{1/p}$ yields

$$\begin{aligned}
o(\varepsilon) &\geq \langle \partial_t ((\delta I - L^t)\hbar), \hbar\rangle_X - \delta \langle (v_1 - v_2), \hbar\rangle_X \\
&= \frac{d}{dt} \langle (\delta I - L^t)\hbar, \hbar\rangle_X - \langle (\delta I - L^t)\hbar, \partial_t \hbar\rangle_X - \delta \langle (v_1 - v_2), \hbar\rangle_X .
\end{aligned} \tag{6.7}$$

We denote $\hat{J}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^r}) = J\left(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r}\right) v\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) m\left(\frac{x}{\varepsilon}\right)$ for any $U \subset\subset \mathbb{R}^d$; the second term on the right-hand side of the inequality (6.7) is rewritten as

$$\begin{aligned}
&\langle (\delta I - L^t)\hbar, \partial_t \hbar\rangle_X \\
= \;& \frac{\delta}{2}\frac{d}{dt}\|\hbar\|_{L^2} - \int_U \int_{\mathbb{R}^d} \hat{J}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^r})(\hbar(y,t) - \hbar(x,t))(\partial_t \hbar(x,t)) dydx \\
= \;& \frac{\delta}{2}\frac{d}{dt}\|\hbar\|_{L^2} + \frac{1}{2}\int_U \int_{\mathbb{R}^d} \hat{J}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^r})(\hbar(y,t) - \hbar(x,t))(\partial_t \hbar(y,t) - \partial_t \hbar(x,t)) dydx \\
= \;& \frac{\delta}{2}\frac{d}{dt}\|\hbar\|_{L^2} + \frac{1}{2}\int_U \int_{\mathbb{R}^d} \hat{J}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^r})(\hbar(y,t) - \hbar(x,t))(\partial_t \hbar(y,t) - \partial_t \hbar(x,t)) dydx \\
= \;& \frac{\delta}{2}\frac{d}{dt}\|\hbar\|_{L^2} + \frac{1}{4}\int_U \int_{\mathbb{R}^d} \partial_t \Big[\hat{J}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^r})(\hbar(y,t) - \hbar(x,t))^2\Big] dydx \\
- \;& \frac{1}{4}\int_U \int_{\mathbb{R}^d} \partial_t \Big[J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})\Big] v(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) m\left(\frac{x}{\varepsilon}\right)(\hbar(y,t) - \hbar(x,t))^2 dydx \\
= \;& \frac{1}{2}\frac{d}{dt} \langle (\delta I - L^t)\hbar, \hbar\rangle_X \\
- \;& \frac{1}{4\varepsilon^r}\int_U \int_{\mathbb{R}^d} \partial_s \Big[J\left(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r}\right)\Big] v\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) m\left(\frac{x}{\varepsilon}\right)(\hbar(y,t) - \hbar(x,t))^2 dydx.
\end{aligned} \tag{6.8}$$

Combining the inequality (6.7) with Eq (6.8) and using the symmetry of $v\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) m\left(\frac{x}{\varepsilon}\right)$, we can derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \langle ((\delta I - L^t)\hbar), \hbar\rangle_X \\
&\leq \frac{C}{2\varrho\varepsilon^r} \|\partial_s J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})\|_{L^\infty} \langle -L^t \hbar, \hbar\rangle_X + \delta \langle (v_1 - v_2), \hbar\rangle_X + o(\varepsilon) \\
&\leq \frac{C_1}{2} \|\partial_s J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})\|_{L^\infty} \langle (\delta I - L^t)\hbar, \hbar\rangle_X + o(\delta) + o(\varepsilon),
\end{aligned} \tag{6.9}$$

where $C_1 = 1/(\varrho\varepsilon^r)$. Applying Gronwall's inequality to (6.9), for all $t \in \bar{I}$, we have

$$(\beta_1 + \delta) \int_{U \Subset \mathbb{R}^d} \left| \left(v_1^{\frac{1}{p}} - v_2^{\frac{1}{p}}\right)(x,t) \right|^2 dx \overset{(2.6)}{\leq} \langle (\delta I - L^t)\hbar, \hbar\rangle_X$$

$$\leq \Big(\Big\langle(\delta I - L^0)\hbar(0), \hbar(0)\Big\rangle_X + o(\delta,\varepsilon)\Big)\exp\Big(C\int_0^T \Big\|\partial_s J\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big\|_{L^\infty(\mathbb{R}^d)}dt\Big)$$
$$\leq C_1\Big(\int_{U\Subset\mathbb{R}^d} |v_{0,1}^{\frac{1}{p}} - v_{0,2}^{\frac{1}{p}}|^2 dx + o(\delta,\varepsilon)\Big)\exp\Big(C\int_0^T \|\partial_s J\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\|_{L^\infty(\mathbb{R}^d)}dt\Big). \tag{6.10}$$

When $\varepsilon,\delta \to 0$, then $v_{0,1} - v_{0,2} \to 0$; it follows that $v_1^{1/p}(t) - v_2^{1/p}(t) \to 0$ in $L^2_{loc}(\mathbb{R}^d)$ for all $t \in I$. The proof is done.

7. Proof of Theorem 3.2

We now give the proof of Theorem 3.2. We have

$$u^\varepsilon = (v^\varepsilon)^{1/p},\ u^0 = (v^0)^{1/p},\ \|w^\varepsilon(x,t) - u^0(x - \frac{\varpi}{\varepsilon}t, t)\|_Z \to 0 \text{ as } \varepsilon \to 0.$$

Then, the inequality (6.10) immediately yields by Proposition 3:

$$\|u^\varepsilon(x,t) - u^0(x - \frac{\varpi}{\varepsilon}t, t)\|_Z \to 0 \text{ or } \|u^\varepsilon(x + \frac{\varpi}{\varepsilon}t, t) - u^0(x,t)\|_Z \to 0 \ \text{ as } \varepsilon \to 0.$$

Thus, we only prove Theorem 3.2 for a dense set in $L^1\left(\mathbb{R}^d\right)$ of initial data, when $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$. For any $\varphi \in L^1\left(\mathbb{R}^d\right)$ and $\ell > 0$ there exists $\varphi_\ell \in \mathcal{S}\left(\mathbb{R}^d\right)$ such that $\|\varphi - \varphi_\ell\|_{L^1(\mathbb{R}^d)} < \delta$. We denote by u_ℓ^ε and u_ℓ^0 the solution of Eqs (3.2) and (3.4) with initial data φ_ℓ. Because Eq (3.4) is the standard Cauchy problem for a parabolic operator with constant coefficients, the classical upper bound of its solution is given in [31, Theorem E] for any $T > 0$:

$$\left\|u^0(x,t) - u_\ell^0(x,t)\right\|_Z \leq \|\varphi - \varphi_\ell\|_{L^1(\mathbb{R}^d)} < \ell, t \in [0,T]. \tag{7.1}$$

By the estimate in Proposition 3 we obtain

$$\left\|u_\ell^\varepsilon(x,t) - u^\varepsilon(x,t)\right\|_Z \leq C_1\ell. \tag{7.2}$$

For an arbitrarily small $\delta > 0$, the upper bounds of the inequalities (7.1) and (7.2) are valid, and these imply that

$$\begin{aligned}\|u^\varepsilon(x + \frac{\varpi}{\varepsilon}t, t) - u^0(x,t)\|_Z &\leq \|u^\varepsilon(x + \frac{\varpi}{\varepsilon}t, t) - u_\ell^\varepsilon(x + \frac{\varpi}{\varepsilon}t, t)\|_Z \\ &+ \|u_\ell^\varepsilon(x + \frac{\varpi}{\varepsilon}t, t) - u_\ell^0(x,t)\|_Z \\ &+ \|u_\ell^0(x,t) - u^0(x,t)\|_Z \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

This completes the proof of Theorem 3.2.

8. Nonlocal porous medium equation for nonnegative initial values

We first give a framework for nonlocal nonlinear diffusion problems.

Lemma 8.1. *[16, Corollary 1.6] For a given homogeneous jump kernel ρ, the operator $\mathscr{L}_v$ is defined by*

$$(\mathscr{L}_v u)(x) = \int_{\mathbb{R}^d} [u(x) - u(y)]\rho\Big(v(x), v(y); x, y\Big)dy. \tag{8.1}$$

For every initial condition $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the nonlinear nonlocal initial value problem

$$\begin{cases} \partial_t u(x,t) + \mathscr{L}_u u(x,t) = 0, & (x,t) \in \mathbb{R}^d \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{8.2}$$

has a very weak solution $u(x,t)$ such that

$$u \in L^\infty\Big([0,\infty), L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\Big) \cap C\Big([0,\infty), L^1(\mathbb{R}^d)\Big),$$

and the solution has the following properties:
(1) Mass is conserved: $\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u_0(x)dx$ for all $t \geq 0$;
(2) L^p-norms are nonincreasing: $\|u(t)\|_p \leq \|u_0\|_p$ for all $p \in [1,\infty]$ and $t \geq 0$;
(3) If $u_0(x) \geq 0$ for a.e. $x \in \mathbb{R}^d$, then $u(t,x) \geq 0$ for a.e. $x \in \mathbb{R}^d$ and $t \geq 0$.

Homogenization of the local porous medium equation for negative initial values can be seen in [6]. Here we consider the following nonlocal scaling operator with a time-dependent kernel:

$$\mathscr{L}_t^\varepsilon u(x,t) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} J(\frac{x-y}{\varepsilon}, \frac{t}{\varepsilon^r})v(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})\Big(u^p(y,t) - u^p(x,t)\Big)dy, \tag{8.3}$$

where $p > 1$ and the Cauchy problem and its corresponding effective Cauchy problem

$$\begin{cases} \partial_t u^\varepsilon(x,t) - \mathscr{L}_t^\varepsilon u^\varepsilon(x,t) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ u^\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{8.4}$$

$$\begin{cases} \partial_t u^0(x,t) - \mathscr{L}_t^0 u^0(x,t) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ u^0(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{8.5}$$

respectively, where $\varphi(x) \in L^{[1,\infty]}(\mathbb{R}^d) := L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$,

$$\mathscr{L}_t^0 u(x,t) = \Theta(x,t) \cdot \nabla\nabla u^p, \tag{8.6}$$

and the matrix $\Theta(x,t)$ will be given below.

We now describe the main result and give a simple proof.

Theorem 8.1. *Assume that the functions $J(z,s)$ and $v(x,y)$ satisfy the conditions (2.1)–(2.3). Let $u^\varepsilon(x,t)$ be a solution of the nonlocal evolutional Cauchy problem (8.4) and $u^0(x,t)$ be a solution of the local Cauchy problem (8.5). Then there exists a positive definite matrix-valued function $\Theta(x,t)$ such that for any $T > 0$,*

$$\|u^\varepsilon(x,t) - u^0(x,t)\|_{L^1\left((0,T), L^2_{loc}(\mathbb{R}^d)\right)} \to 0 \text{ as } \varepsilon \to 0. \tag{8.7}$$

Proof. Set

$$w^{\varepsilon}(x,t) = u(x,t) + \varepsilon\chi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla u(x,t) + \varepsilon^2\chi_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\nabla\nabla u(x,t), \tag{8.8}$$

$$\begin{aligned} \frac{\partial w^{\varepsilon}(x,t)}{\partial t} &= pu^{p-1}\frac{\partial u}{\partial t}(x,t) + \Big(\frac{1}{\varepsilon}\cdot\varepsilon^{2-r}\frac{\partial\chi_1}{\partial s}\Big)\cdot\nabla u(x,t) \\ &+ \varepsilon^{2-r}\Big(\frac{\partial\chi_2}{\partial s}\Big)\cdot\nabla\nabla u(x,t) + \phi_{\varepsilon}^{(time)}(x,t) \end{aligned} \tag{8.9}$$

with

$$\begin{aligned} &\phi_{\varepsilon}^{(time)}(x,t) = \varepsilon\frac{\partial\chi_1}{\partial t}\cdot\nabla u(x,t) + \varepsilon^2\frac{\partial\chi_2}{\partial t}\cdot\nabla\nabla u(x,t) \\ &+\varepsilon\chi_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\cdot\nabla\frac{\partial u}{\partial t}(x,t) + \varepsilon^2\chi_2\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\cdot\nabla\nabla\frac{\partial u}{\partial t}(x,t). \end{aligned} \tag{8.10}$$

Denote

$$\begin{aligned} &H^{\varepsilon}w^{\varepsilon}(x,t) = \frac{\partial w^{\varepsilon}(x,t)}{\partial t} \\ &-\frac{1}{\varepsilon^{d+2}}\int_{\mathbb{R}^d} J\left(\frac{x-y}{\varepsilon},\frac{t}{\varepsilon^r}\right)\nu\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)((w^{\varepsilon}(y,t))^p - (w^{\varepsilon}(x,t))^p)\,dy; \end{aligned}$$

we have

$$\begin{aligned} &H^{\varepsilon}w^{\varepsilon}(x,t) \\ &= \frac{\partial u}{\partial t}(x,t) + \left(\frac{1}{\varepsilon}\cdot\varepsilon^{2-r}\frac{\partial\chi_1}{\partial s}\right)\cdot\nabla u(x,t) + \varepsilon^{2-r}\left(\frac{\partial\chi_2}{\partial s}\right)\cdot\nabla\nabla u(x,t) \\ &+\varepsilon\frac{\partial\chi_1}{\partial t}\cdot\nabla u(x,t) + \varepsilon^2\frac{\partial\chi_2}{\partial t}\cdot\nabla\nabla u(x,t) + \varepsilon\chi_1\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\right)\cdot\nabla\frac{\partial u}{\partial t}(x,t) \\ &+\varepsilon^2\chi_2\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\right)\cdot\nabla\nabla\frac{\partial u}{\partial t}(x,t) \\ &-\frac{1}{\varepsilon^{d+2}}\int_{\mathbb{R}^d} J(\frac{x-y}{\varepsilon},\frac{t}{\varepsilon^r})\nu(\frac{x}{\varepsilon},\frac{y}{\varepsilon})\Big\{u(y,t) + \varepsilon\chi_1\left(y,t,\frac{y}{\varepsilon},\frac{t}{\varepsilon^r}\right)\cdot\nabla u(y,t) \\ &+\varepsilon^2\chi_2\Big(y,t,\frac{y}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\nabla\nabla u(y,t) - u(x,t) - \varepsilon\chi_1\nabla u\Big(x,t\Big) - \varepsilon^2\chi_2\nabla\nabla u(x,t)\Big\} \\ &\cdot\underbrace{\Big(w^{\varepsilon}(y,t)^{p-1} + w^{\varepsilon}(y,t)^{p-2}w^{\varepsilon}(x,t) + \cdots + w^{\varepsilon}(x,t)^{p-1}\Big)}_{(a^p-b^p)=(a-b)(a^{p-1}+a^{p-2}b\cdots+ab^{p-2}+b^{p-1})}dy \\ &:= \frac{\partial u}{\partial t}(x,t) + \frac{1}{\varepsilon}\mathcal{M}_0(x,t) + \mathcal{M}_{\varepsilon}(x,t) + \phi_{\varepsilon}(x,t). \end{aligned} \tag{8.11}$$

Using the Taylor formula and symmetry of the integral, we directly give

$$\begin{aligned} &\mathcal{M}_0^{\varepsilon}(x,t) = \varepsilon^{2-r}\frac{\partial\chi_1}{\partial s}\nabla u(x,t) - \nabla u(x,t)\Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^r})\nu(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\Big(-z \\ &+\chi_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r}) - \chi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\Big)dz\Big]\Big(pu^{p-1}(x,t) + o(\varepsilon)\Big), \end{aligned} \tag{8.12}$$

$$
\begin{aligned}
&\mathcal{M}_\varepsilon(x,t) = \varepsilon^{2-r}\frac{\partial\chi_2}{\partial s}\nabla\nabla u(x,t)\\
&-\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^r})\nu(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\Big[\Big(-z\chi_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})\\
&+\chi_2(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})-\chi_2(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})+\frac{1}{2}z^2\Big)pu^{p-1}(x,t))\nabla\nabla u(x,t)\\
&+\Big(-z\chi_1(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^r})+\frac{1}{2}z^2\Big)p(p-1)u^{p-2}\nabla u\otimes\nabla u\Big]dz. \qquad (8.13)
\end{aligned}
$$

Similar to the proofs in the above chapters, we can get the corresponding conclusion. That is, due to Eq (8.11), there are two functions χ_1 and χ_2 such that $\mathcal{M}_0 = 0$.

For $0 < r \leq 2$, we have that $\mathcal{M}_\varepsilon \to \Theta \cdot \nabla\nabla u^p(x,t)$ as $\varepsilon \to 0$, where

$$
\begin{aligned}
\Theta^{ij} &= \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{2}(\xi-q)^i(\xi-q)^j J(\xi-q,s)\nu(\xi,q)m(\xi)dqd\xi ds\\
&- \int_0^1\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} J(\xi-q,s)\nu(\xi,q)m(\xi)(\xi-q)^i\chi_1^j(x,t,q,s)dqd\xi ds. \qquad (8.14)
\end{aligned}
$$

For $r > 2$, we can get a result that is similar to Eq (8.14), so we do not repeat the proof here.

9. Stochastic homogenization

9.1. Introduction and statement

The literature on the stochastic homogenization of parabolic equations of local equations can be found in [9, 10]. In this section, we introduce how to deal with the nonlocal parabolic equation model with random statistically homogeneous coefficients, where the ideas and methods mainly come from [41], for which we need some additional measure ergodic theory.

T is a $d-$dimensional dynamical system, (Ω, F, P) has a standard probability space and we assume that $\mu(x,w) = \mu(T_x w), T_x : \Omega \to \Omega$ satisfy the following properties.

(1) $T_{y_1}T_{y_2} = T_{y_1+y_2}$ for all y_1, y_2 in $R^d, T_0 = Id$.
(2) $P(T_y A) = P(A)$ for all $A \in \mathcal{F}$ and all $y \in R^d$.
(3) T_x is a measurable map from $R^d \times \Omega$ to Ω, where R^d is equipped with the Borel $\sigma-$algebra.

We consider the operator

$$
L^\varepsilon v(x,t) = \frac{1}{\varepsilon^{d+2}}\int_{\mathbb{R}^d} J(\frac{x-y}{\varepsilon},\frac{t}{\varepsilon^2},\omega_t)\nu(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\omega_s)\Big(v(y,t)-v(x,t)\Big)dy, \qquad (9.1)
$$

where for $a.e.\ \omega_t$, we have

$$
\begin{aligned}
&J(z,t) \in L^\infty((0,T),L^1(\mathbb{R}^d)), J(\cdot,t) \subset\subset \mathbb{R}^d,\\
&J(z,t) = J(-z,t), J_1(z) \geq J(z,t) \geq 0;\\
&\nu(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\omega_s) = \nu(\frac{x}{\varepsilon},\omega_s)\nu(\frac{y}{\varepsilon},\omega_s), \qquad (9.2)
\end{aligned}
$$

where ω_t and ω_s are random fields to $\mathbb{R}\times\mathbb{R}^d$ that are stationary with respect to time t and space x, respectively. We fix an ergodic environment probability, that is, assume that

$$
\begin{cases}
(\Omega,\mathcal{F},\mathbb{P})\ \textit{is a probability space endowed with an ergodic semigroup},\\
\tau : \mathbb{Z}^d\times\mathbb{R}\times\Omega \to \Omega\ \textit{of measure preserving maps},
\end{cases} \qquad (9.3)
$$

and we denote by $\mathbf{L}^2$ the set of stationary maps $u = u(x, t, \omega)$, meaning that

$$u(x + k, t + s, \omega) = u(x, t, T_{(k,s)}\omega), \forall (k, s, \omega) \in \mathbb{R}^d \times \mathbb{R} \times \Omega \tag{9.4}$$

Notice that spatial variables can be Z^d−stationary in local equations, but they are not applied here and we will see them later. Define norm-$\mathbf{L}^2$

$$||u||_{\mathbf{L}^2} = \mathbb{E}[\int_{\tilde{Q}_1} u^2] < +\infty. \tag{9.5}$$

Note that, if $u \in \mathbf{L}^2$ and U is a bounded measurable subset of $\mathbb{R}^d$, the stationarity in time implies that the limit

$$\mathbb{E}\left[\int_U u(x, t)dx\right] = \lim_{h\to 0^+} \mathbb{E}\left[\frac{1}{2h}\int_U \int_{t-h}^{t} u(x, s)dxds\right]$$

exists for any $t \in \mathbb{R}$ and is independent of t. Let C be the subset of $\mathbf{L}^2$ of maps with smooth and square integrable space and time derivatives of all order belonging to $\mathbf{L^2}$. C is dense in $\mathbf{L}^2$ with respect to the norm in the inequality (9.5).

We denote by $\mathbf{H}^1, \mathbf{H_x^1}$ the closure of C with respect to the norm

$$||u||_{\mathbf{H}^1} = \left(||u||^2_{\mathbf{L}^2} + ||\partial_t u||^2_{\mathbf{L}^2} + ||Du||^2_{\mathbf{L}^2}\right)^{1/2}, \ ||u||_{\mathbf{H}^1_\mathbf{x}} = \left(||u||^2_{\mathbf{L}^2} + ||Du||^2_{\mathbf{L}^2}\right)^{1/2}$$

and $\mathbf{H_x^{-1}}$ is the dual space of $\mathbf{H}^1_\mathbf{x}$. Moreover, $\mathbf{L}^2_{pot}$ is the closure with respect to the $\mathbf{L}^2$-norm of $\{Du : u \in C\}$ in $\left(\mathbf{L}^2(\Omega)\right)^d$.

Set $v(x, t) = |u(x, t)|^{p-1}u(x, t)$, denote $L^\varepsilon v(x, t) = L^\varepsilon(|u|^{p-1}u) := \mathscr{L}_t^\varepsilon u(x, t)$. We now consider the Cauchy problem and its corresponding effective Cauchy problem

$$\begin{cases} \partial_t u^\varepsilon(x, t) - \mathscr{L}_t^\varepsilon u^\varepsilon(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, T), \\ u^\varepsilon(x, 0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{9.6}$$

$$\begin{cases} \partial_t u^0(x, t) - \mathscr{L}_t^0 u^0(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, T), \\ u^0(x, 0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{9.7}$$

where $\varphi(x) \in L^{[1,\infty]}(\mathbb{R}^d) := L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$,

$$\mathscr{L}_t^0 u(x, t) = \Theta(x, t)\nabla\nabla|u|^{p-1}u, \tag{9.8}$$

and the matrix $\Theta(x, t)$ will be given below.

We also transform the problems (9.6) and (9.7) into the following Cauchy problems

$$\begin{cases} \partial_t v^\varepsilon(x, t)^{1/p} - L^\varepsilon v^\varepsilon(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, T), \\ v^\varepsilon(x, 0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{9.9}$$

$$\begin{cases} \partial_t v(x, t)^{1/p} - \Theta \cdot \nabla\nabla v(x, t) = 0, & (x, t) \in \mathbb{R}^d \times (0, T), \\ v(x, 0) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{9.10}$$

9.2. Main results

Theorem 9.1. *Assume that the functions $J(z, s)$ and $\nu(x, y)$ satisfy the condition (9.2). Let $u^{\varepsilon}(x, t)$ be the solution of the evolution Cauchy problem (9.9) and $u^0(x, t)$ be the solution of the effective Cauchy problem (9.10). Then, there exists a positive definite constant matrix Θ such that for any $T > 0$, we have*

$$\left\| u^{\varepsilon}(x,t) - u^0(x,t) \right\|_{L^1((0,T),L^{p+1}_{loc}(\mathbb{R}^d))} \to 0 \ as\ \varepsilon \to 0^+ a.s. \tag{9.11}$$

The homogenized flux $\Theta(x, t)$ can be characterized as follows.
Case I. For $r = 2$ and $p = 1$, the homogenized constant matrix Θ is characterized by

$$\Theta = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\mathbb{R}^d}\int_{\Omega}\Big(z\otimes z - z\otimes\zeta_{-z}(0,s,\omega)\Big)J(z,s)\mu(0,\omega)\mu(-z,\omega)dzdsd\mathbf{P}(\omega), \tag{9.12}$$

where $\chi = \chi(y, s)$ and $(\xi, s) \in \mathbb{T}^d \times \mathbb{T}$ solves the cell problem

$$\begin{cases} \int_{\mathbb{R}^d} J(\xi-q,s)\nu(\xi,q)\Big(q-\xi+\chi(q,s)-\chi(\xi,s)\Big)dq = \partial_s\chi(\xi,s), \\ \chi(y,0) = \chi(y,1),\ \ y\in\mathbb{T}^d. \end{cases} \tag{9.13}$$

Case II. For $r = 2$ and $p \in (0, 1]$, the homogenized matrix $\Theta(x, t)$ is characterized by

$$\begin{aligned} \Theta(x,t) \ &= \ \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\mathbb{R}^d}\int_{\Omega}\Big(z\otimes z - z\otimes\zeta_{-z}(x,t,0,s,\omega)\Big) \\ &\quad \times J(z,s)\mu(0,\omega)\mu(-z,\omega)dzdsd\mathbf{P}(\omega), \end{aligned}$$

where $\chi^k = \chi^k(y, s)$ solves the cell problem

$$\begin{cases} \int_{\mathbb{R}^d} J(\xi-q,s)\nu(\xi,q)\Big(q-\xi+\chi(x,t,q,s)-\chi(x,t,\xi,s)\Big)dq \\ \qquad = \frac{1}{p}|u^0|^{1-p}(\partial_s\chi(x,t,\xi,s)),\ \ (\xi,s)\in\mathbb{T}^d\times\mathbb{T}, \\ \chi(x,t,y,0) = \chi(x,t,y,1),\ \ y\in\mathbb{T}^d. \end{cases} \tag{9.14}$$

Case III. For $r = 2$ and $p \in (1, 2]$, the homogenized matrix Θ is characterized by

$$\chi^k(x,t,y,s) = \begin{cases} p|u^0|^{p-1}\natural^k(x,t,y,s) & \text{if } u^0(x,t)\neq 0, \\ 0 & \text{if } u^0(x,t) = 0, \end{cases}$$

where $\natural^k = \natural^k(x, t, y, s)$ solves the cell-problem for each $(x, t) \in [u_0 \neq 0]$,

$$\begin{cases} \partial_s\natural(x,t,\xi,s) = \int_{\mathbb{R}^d} J(\xi-q,s)\nu(\xi,q)\Big(q-\xi \\ \qquad\qquad +p|u^0|^{p-1}\natural(x,t,q,s) - p|u^0|^{p-1}\natural(x,t,\xi,s)\Big)dq, \\ \natural(x,t,y,0) = \natural_1(x,t,y,1),\ \ y\in\mathbb{T}^d, \end{cases} \tag{9.15}$$

and the measurable set $\Big[u^0 \neq 0\Big] := \Big\{(x,t)\in\mathbb{R}^d\times(0,T) : u^0(x,t)\neq 0\Big\}$.

We need to construct special axillary functions with the following structures.
(i) For $p = 1$,

$$w_0^\varepsilon(x,t) = v(x,t) + \underbrace{\varepsilon\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)}_{First\ corrector} \nabla v(x,t) + \underbrace{v_2^\varepsilon(x,t) + v_3^\varepsilon(x,t).}_{Additional\ terms\ for\ compensation} \tag{9.16}$$

(ii) For $0 < p < 1$,

$$w_0^\varepsilon(x,t) = v(x,t) + \varepsilon\chi(x,t,\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)\nabla\hat{v}(x,t) + \hat{v}_2^\varepsilon(x,t) + \hat{v}_3^\varepsilon(x,t). \tag{9.17}$$

(iii) For $1 < p \le 2$,

$$w_1^\varepsilon(x,t) = v(x,t) + \varepsilon p|v(x,t)|^{\frac{p-1}{p}} \natural(x,t,\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)\nabla v(x,t) + \check{v}_2^\varepsilon(x,t) + \check{v}_3^\varepsilon(x,t). \tag{9.18}$$

The functions v_i^ε and $\check{v}_i^\varepsilon$ are used to eliminate some extra parts that will be mentioned later when we consider some convergence.

Next, we need to bring in the auxiliary function $w_i^\varepsilon(x,t)$ to decompose according to the order of ε.
Case 1. For $p = 1$, similar to the derivations in [13, 14], we only need a new corrector $\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})$. Substituting the expression on the right-hand side of H^ε for w_0^ε in Eq (4.3), we get

$$H^\varepsilon w_0^\varepsilon(x,t) = \frac{\partial v}{\partial t}(x,t) + \frac{1}{\varepsilon}\underbrace{\mathcal{M}_0(x,t)}_{=0} + \underbrace{\mathcal{M}_\varepsilon(x,t)}_{Zero-order\ expansion} + \underbrace{\phi_\varepsilon(x,t)}_{Remainder} \tag{9.19}$$

as $\varepsilon \to 0^+$, where

$$\begin{aligned}
\mathcal{M}_0(x,t) &= \frac{\partial\chi}{\partial s}\nabla v(x,t) - \nabla v(x,t)\Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z) \\
&\quad \cdot\Big(-z + \chi(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2}) - \chi(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\Big)dz\Big], \qquad (9.20)\\
\mathcal{M}_\varepsilon(x,t) &= -\nabla\nabla v(x,t)\Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z) \\
&\quad \cdot\Big(-z\otimes\chi(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2}) + \frac{1}{2}z\otimes z\Big)dz\Big], \qquad (9.21)
\end{aligned}$$

$$\phi_\varepsilon(x,t) = \phi_\varepsilon^{(time)} - \phi_\varepsilon^{(space)},\ \phi_\varepsilon^{(time)}(x,t) = \varepsilon\chi(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\cdot\nabla\frac{\partial v}{\partial t}(x,t), \tag{9.22}$$

$$\begin{aligned}
&\phi_\varepsilon^{(space)} = \int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z\Big) \\
&\cdot\Big(\int_0^1 \nabla\nabla v(x-\varepsilon z\eta,t)\cdot z\otimes z(1-\eta)d\eta - \frac{1}{2}\nabla\nabla v(x,t)\cdot z\otimes z\Big)dz \\
&+\frac{1}{\varepsilon}\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z\right)\chi\left(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2}\right)\Big(\nabla v(x-\varepsilon z,t) - \nabla v(x,t)\Big)dz
\end{aligned}$$

$$+\nabla\nabla v(x,t)\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z\right)z\otimes\chi\left(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2}\right)dz. \tag{9.23}$$

Case 2. For $p \in (0,1)$,

$$\frac{\partial w_1^\varepsilon(x,t)}{\partial t} = \frac{1}{p}|w_1^\varepsilon|^{\frac{1-p}{p}}\Big[\frac{\partial v}{\partial t}(x,t) + \frac{1}{\varepsilon}\frac{\partial\chi}{\partial s}\cdot\nabla v(x,t) + \phi_\varepsilon^{(time)}(x,t)\Big].$$

Using the Taylor expansions we have

$$H^\varepsilon w_1^\varepsilon(x,t) = \frac{1}{p}|w_1^\varepsilon|^{\frac{1-p}{p}}\frac{\partial v}{\partial t}(x,t) + \frac{1}{\varepsilon}\mathcal{M}_0(x,t) + \mathcal{M}_\varepsilon(x,t) + \phi_\varepsilon(x,t) \text{ as } \varepsilon\to 0^+,$$

where

$$\begin{aligned}\mathcal{M}_0(x,t) &= \frac{1}{p}|w^\varepsilon|^{\frac{1-p}{p}}\frac{\partial\chi}{\partial s}\nabla v(x,t) - \nabla v(x,t)\Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ &\quad\cdot\Big(-z+\chi(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2})-\chi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\Big)dz\Big], \qquad (9.24)\\ \mathcal{M}_\varepsilon(x,t) &= -\nabla\nabla v(x,t)\Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)\\ &\quad\cdot\Big(-z\chi(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2})+\frac{1}{2}z^2\Big)dz\Big], \qquad (9.25)\end{aligned}$$

$$\begin{aligned}\phi_\varepsilon(x,t) &= \frac{1}{p}|w_1^\varepsilon|^{\frac{1-p}{p}}\phi_\varepsilon^{(time)} + \phi_\varepsilon^{(space)},\\ \phi_\varepsilon^{(time)}(x,t) &= \varepsilon\frac{\partial\chi}{\partial t}\cdot\nabla v(x,t) + \varepsilon\chi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\cdot\nabla\frac{\partial v}{\partial t}(x,t). \qquad (9.26)\end{aligned}$$

Case 3. For $p \in (1,2]$, we have

$$\frac{\partial w_2^\varepsilon(x,t)}{\partial t} = \frac{1}{p}|w_2^\varepsilon|^{\frac{1-p}{p}}\Big[\frac{\partial v}{\partial t}(x,t) + (\frac{1}{\varepsilon}p|v|^{\frac{p-1}{p}}\frac{\partial\natural_1}{\partial s})\cdot\nabla v(x,t) + \phi_\varepsilon^{(time)}(x,t)\Big], \tag{9.27}$$

$$\begin{aligned}\phi_\varepsilon^{(time)}(x,t) &= \varepsilon(p-1)|v(x,t)|^{-(1+\frac{1}{p})}v(x,t)\frac{\partial v}{\partial t}\natural(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla v(x,t)\\ &+ \varepsilon p|v(x,t)|^{\frac{p-1}{p}}\frac{\partial\natural}{\partial t}\cdot\nabla v(x,t) + \varepsilon p|v(x,t)|^{\frac{p-1}{p}}\natural(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\cdot\nabla\frac{\partial v}{\partial t}(x,t).\end{aligned}$$

and

$$H^\varepsilon w_2^\varepsilon(x,t) = \frac{1}{p}|w_2^\varepsilon|^{\frac{1-p}{p}}\frac{\partial v}{\partial t} + \frac{1}{\varepsilon}\mathcal{M}_0(x,t) + \mathcal{M}_\varepsilon(x,t) + \phi_\varepsilon^{(time)} + \phi_\varepsilon^{(space)} \text{ as } \varepsilon\to 0^+, \tag{9.28}$$

where

$$\mathcal{M}_0(x,t) = \varepsilon^{2-r}\Big|\frac{v}{w_2^\varepsilon}\Big|^{\frac{p-1}{p}}\frac{\partial\natural_1}{\partial s}\nabla v(x,t) - \nabla v\Big[\int_{\mathbb{R}^d} J(z,\frac{t}{\varepsilon^2})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)$$

$$\cdot\Big(-z+p|v(y,t)|^{\frac{p-1}{p}}\natural(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2})-p|v(x,t)|^{\frac{p-1}{p}}\natural(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\Big)dz\Big], \tag{9.29}$$

$$\mathcal{M}_\varepsilon(x,t)=-\nabla\nabla v(x,t)\Big[\int_{\mathbb{R}^d}J(z,\frac{t}{\varepsilon^2})v(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z)$$
$$\cdot\Big(-zp|v(y,t)|^{\frac{p-1}{p}}\natural(y,t,\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2})+\frac{1}{2}z^2\Big)dz\Big]. \tag{9.30}$$

Due to the order of ε, we put the terms with $O(\varepsilon)$ and the higher-order terms with $o(\varepsilon)$ into the remainder as the fourth part.

Next, we will prove the main conclusion. For the convenience of the proof, we first prove the linear case, and then point out some results of the nonlinearity that are different from linearity.

The proof of the linear equation is divided into three parts, where the first part is about the first-order random corrector, the second part is about the zero-order term and remainder, and the last part is the proof of Theorem 9.1.

For any $\delta>0$, let us consider the equation

$$\delta\chi^\delta-\delta\partial_{tt}\chi^\delta+\partial_t\chi^\delta-\delta\Delta\chi^\delta-\mathfrak{A}_\omega\chi^\delta=0, \tag{9.31}$$

where

$$\mathfrak{A}_\omega\chi=\int_{\mathbb{R}^d}J(z,t)v(x,x-z)(-z+\chi(x-z,t)-\chi(x,t))dz. \tag{9.32}$$

We set

$$\zeta_z(\xi,t,\omega)=\chi(\xi-z,t,\omega)-\chi(\xi,t,\omega). \tag{9.33}$$

Throughout the proof, to justify repeated integration by parts and to deal with the unbounded domain, we use the exponential weight $\widehat{\Gamma}_\theta$, which, for $\theta>0$, is given by

$$\widehat{\Gamma}_\theta(x,t)=\exp\left\{-\theta\left(1+|x|^2+t^2\right)^{1/2}\right\}.$$

The first lemma is about the existence of and some a priori bounds for the approximate corrector in a bounded domain.

Theorem 9.2. *Assume that Eqs (9.2)–(9.4) are satisfied; there exists a unique map* χ: $\mathbb{R}^{d+1}\times\Omega\to\mathbb{R}$ *such that*

$$\partial_t\chi^k-\mathfrak{A}_\omega\chi^k=0, k=1,2,\cdots,d, \ in\ \mathbf{L^2} \tag{9.34}$$

and for all $z\in\mathbb{R}^d$, $\zeta_z(\xi,t,\omega)\in\mathbf{L^2}$ *is a stationary field that satisfies*

$$\int_{\widetilde{Q}_1}\zeta_z(x,t,\omega)dxdt=0\ \mathbb{P}-a.s.$$

The positive definite constant matrix Θ *is defined by*

$$\Theta(x,t)\ =\ \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\mathbb{R}^d}\int_\Omega(z\otimes z-z\otimes\zeta_{-z}(0,s,\omega))$$

$$\times J(z,s,\omega)\mu(0,\omega)\mu(-z,\omega)dzdsd\mathbf{P}(\omega). \tag{9.35}$$

Moreover, $\mathbb{P}$*-a.s.* $\chi^{\varepsilon}(x,t,\omega)=\varepsilon\chi\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega\right)$ *is a function satisfying sub-linear growth, that is*

$$\chi^{\varepsilon}(x,t,\omega)\overset{\varepsilon\to 0}{\longrightarrow}0 \textit{ in } L^2_{loc}(\mathbb{R}^{d+1}).$$

The proof is long and technical, so we will follow the diagram step by step.

Step	Task
1	The existence for the approximate corrector in a bounded domain
2	The existence for the approximate corrector in an unbounded domain
3	The convergence of an approximate sequence
4	The existence of a corrector
5	The stationarity of a corrector
6	The uniqueness of a corrector
7	The sublinearity of a corrector

9.3. Uniqueness, existence and sublinear growth of corrector

Denote

$$\begin{aligned}\mathfrak{A}_{\omega}u &= \int_{\mathbb{R}^d} J(z,t)\nu(x,x-z)\Big(u(x-z,t)-u(x,t)\Big)dz\\ &- \int_{\mathbb{R}^d} J(z,t)\nu(x,x-z)zdz := A_{\omega}u-f.\end{aligned} \tag{9.36}$$

For a large enough L and the bounded support sets of u and J, we get

$$(A_{\omega}u,u)_{L^2(\widetilde{Q}_L)}\le 0.$$

Step 1. Approximation sequence for constructing the solution.

Lemma 9.1. *Assume Eqs (9.2)–(9.4) for any* $\omega\in\Omega,\delta>0$ *and sufficiently large* $L>0$*, and let* $\mathfrak{u}_L\in H^1_0\left(\widetilde{Q}_L\right)$ *be the solution of*

$$\delta\mathfrak{u}^{\delta}_L-\delta\partial_{tt}\mathfrak{u}^{\delta}_L+\partial_t\mathfrak{u}^{\delta}_L-\delta\Delta\mathfrak{u}^{\delta}_L-\mathfrak{A}_{\omega}\mathfrak{u}^{\delta}_L=0 \textit{ in } \widetilde{Q}_L,\quad \mathfrak{u}^{\delta}_L=0 \textit{ in } \partial\widetilde{Q}_L. \tag{9.37}$$

Then, we have that $\theta_m>0$*, which depends on* δ *but not on L or* ω*, such that for any* $\theta\in(0,\theta_m]$ *and* $\mathbb{P}$*-a.s.,*

$$\int_{\tilde{Q}_L}\left(\delta(\mathfrak{u}^{\delta}_L)^2+\delta\left(\partial_t\mathfrak{u}^{\delta}_L\right)^2+\left(D\mathfrak{u}^{\delta}_L\right)^2\right)\widehat{\Gamma}_{\theta}\lesssim C_{\Gamma,\delta,\theta}. \tag{9.38}$$

Proof. Using $\widehat{\Gamma}_{\theta}u_L$ as a test function in Eq (9.37), $\widehat{\Gamma}_{\theta}$ satisfies $\left|D\widehat{\Gamma}_{\theta}\right|+\left|\partial_t\widehat{\Gamma}_{\theta}\right|\lesssim\theta\widehat{\Gamma}_{\theta}$; we find that

$$\int_{\widetilde{Q}_L}\left(\delta(\mathfrak{u}^{\delta}_L)^2(x,t,\omega)+\delta\left(\partial_t\mathfrak{u}^{\delta}_L\right)^2(x,t,\omega)-\delta\left(D\mathfrak{u}^{\delta}_L\right)^2(x,t,\omega)\right)\widehat{\Gamma}_{\theta}$$

$$\leq -\int_{\widetilde{Q}_L}\left(\delta\partial_t \mathfrak{u}_L^\delta \mathfrak{u}_L^\delta \frac{\partial_t\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} - \frac{\left(\mathfrak{u}_L^\delta\right)^2 \partial_t\widehat{\Gamma}_\theta}{2\widehat{\Gamma}_\theta} + \mathfrak{u}_L^\delta D\mathfrak{u}_L^\delta \cdot \frac{D\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} + f\mathfrak{u}_L^\delta\right)\widehat{\Gamma}_\theta$$

$$\lesssim \int_{\widetilde{Q}_L}\left(\delta\theta\left|\partial_t \mathfrak{u}_L^\delta\right|\left|\mathfrak{u}_L^\delta\right| + \theta\mathfrak{u}_L^\delta D\mathfrak{u}_L + \theta(\mathfrak{u}_L^\delta)^2 + |f||\mathfrak{u}_L^\delta|\right)\widehat{\Gamma}_\theta; \tag{9.39}$$

by the Cauchy-Schwartz inequality, we can finish the proof.
Step 2. Next we prove the existence of χ^δ.

Lemma 9.2. *Assume that Eqs (9.2)–(9.4) are satisfied. For any $\delta > 0$ and $\theta \in (0, \theta_m)$, in the sense of distributions, there exists a unique stationary solution $\chi^\delta \in \mathbf{H}^1_{\hat{\Gamma}_\theta} \subset \mathbf{H}^1$ of*

$$\delta\chi^\delta - \delta\partial_{tt}\chi^\delta + \partial_t\chi^\delta - \delta\Delta\chi^\delta - \mathfrak{A}_\omega\chi^\delta = 0 \text{ in } \mathbb{R}^{d+1}, \mathbb{P} - a.s. \tag{9.40}$$

It is independent of $\theta \in (0, \theta_m)$. We also have the following estimate

$$\mathbb{E}\int_{\tilde{Q}_1}\left[\delta\left(\chi^\delta\right)^2 + \delta\left(\partial_t\chi^\delta\right)^2 + \left(D\chi^\delta\right)^2\right] \leq C. \tag{9.41}$$

Proof. This equation contains both local and nonlocal terms; our aim is to get an approximate equation in an unbounded domain, so we first show that $\mathfrak{u}_L \to \mathfrak{u}$. Then ω is arbitrarily fixed, and $\mathfrak{u}_L \in \mathbf{H}^1_{\widehat{\Gamma}_\theta}$ for any $\theta \in (0, \theta_m]\,, L \in (0, \infty)$ from Lemma 9.1. According to the rule of diagonals, this produces subsequences, which we still remember as the original notation, then, for some $\mathfrak{u} \in \bigcap_{\theta_1 \in (0,\theta_m]} \mathbf{H}^1_{\widehat{\Gamma}_{\theta_1}}$ and any $\theta \in (0, \theta_m]$ we have $\mathfrak{u}_L \overset{L\to\infty}{\longrightarrow} \mathfrak{u}$ in $\mathbf{H}^1_{\widehat{\Gamma}_\theta}$. Here,

$$\mathfrak{A}_\omega\mathfrak{u}_L - \mathfrak{A}_\omega\mathfrak{u} = \int_{\mathbb{R}^d} J(z,t)\nu(x, x-z)\Big((\mathfrak{u}_L - \mathfrak{u})(x-z,t) - (\mathfrak{u}_L - \mathfrak{u})(x,t)\Big)dz;$$

thus, for any $L > 0$ and $\theta \in (0, \theta_m)$, $\mathfrak{u}_L \to \mathfrak{u}$ in $L^2(\widetilde{Q}_L)$, taking norm in $L^2_{\widehat{\Gamma}_\theta}$ and dividing the integral area into two parts, we have

$$\begin{aligned}
&\|\mathfrak{A}_\omega\mathfrak{u}_L - \mathfrak{A}_\omega\mathfrak{u}\|_{L^2_{\Gamma_\theta}(\mathbb{R}^{d+1})} \\
&\leq \|\mathfrak{A}_\omega\mathfrak{u}_L - \mathfrak{A}_\omega\mathfrak{u}\|_{L^2_{\Gamma_\theta}(\tilde{Q}_L)} + (\sup_{\mathbb{R}^{d+1}\setminus\widetilde{Q}_L}\frac{\Gamma_\theta}{\Gamma_{\theta_m}})\|\mathfrak{A}_\omega\mathfrak{u}_L - \mathfrak{A}_\omega\mathfrak{u}\|_{L^2_{\Gamma_{\theta_m}}(\mathbb{R}^{d+1}\setminus\tilde{Q}_L)} \\
&\leq \|\mathfrak{A}_\omega\mathfrak{u}_L - \mathfrak{A}_\omega\mathfrak{u}\|_{L^2(\tilde{Q}_L)} + (\sup_{\mathbb{R}^{d+1}\setminus\widetilde{Q}_L}\frac{\Gamma_\theta}{\Gamma_{\theta_m}})(\|\mathfrak{A}_\omega\mathfrak{u}_L\|_{L^2_{\Gamma_{\theta_m}}(\mathbb{R}^{d+1})} + \|\mathfrak{A}_\omega\mathfrak{u}\|_{L^2_{\Gamma_{\theta_m}}(\mathbb{R}^{d+1})}) \\
&:= V_1 + V_2.
\end{aligned} \tag{9.42}$$

We know that $\mathfrak{A}_\omega$ is a bounded operator in $L^2(\tilde{Q}_L)$ from [13, Proposition 6]; however,

$$\|\mathfrak{A}_\omega\mathfrak{u}_L - \mathfrak{A}_\omega\mathfrak{u}\|_{L^2_{\Gamma_\theta}(\mathbb{R}^{d+1})} \to 0$$

can be obtained directly through a similar argument in the inequality (9.42).

Note that $V_1, V_2 \to 0$ uniformly as $L \to +\infty$ by using the definition of Γ_θ. Therefore we can assume that, $\Delta\mathfrak{u}_L(x,t,\omega) \to \Delta\mathfrak{u} = \zeta, \mathfrak{A}_\omega\mathfrak{u}_L \to \mathfrak{A}_\omega\mathfrak{u} = \zeta_1$, where $\zeta, \zeta_1 \in \bigcap_{\theta' \in (0,\theta_m]} L^2_{\widehat{\Gamma}_{\theta'}}$ as $L \to \infty$.

It can be seen that, in the sense of distribution, we have

$$\delta\mathfrak{u} - \delta\partial_{tt}\mathfrak{u} + \partial_t\mathfrak{u} - \delta\Delta\mathfrak{u} - \mathfrak{A}_\omega\mathfrak{u} = 0 \text{ in } \mathbb{R}^{d+1}, \tag{9.43}$$

and for all $\theta \in (0, \theta_m]$,

$$\int_{\tilde{Q}} \delta\Big(\mathfrak{u}^2 + (\partial_t\mathfrak{u})^2 + |D\mathfrak{u}|^2\Big)\widehat{\Gamma}_\theta \lesssim C_{\Gamma,\delta,\theta}, \tag{9.44}$$

where $\partial_t\mathfrak{u} \in \mathbf{H}_{\mathbf{x}}^{-1}, D\mathfrak{u} \in \Big(\mathbf{L}^2(\Omega)\Big)^d$.

Next, we check that $\mathfrak{u} \in \bigcap_{\theta' \in [0,\theta_m]} H^1_{\widehat{\Gamma}_{\theta'}}$ is a solution of Eq (9.43) in the sense of distribution. Let $\phi \in C_c^\infty\Big(\mathbb{R}^{d+1}\Big)$. For a large enough L, we have

$$\int_{\tilde{Q}_L} [\delta((\mathfrak{u}_L - \phi)^2 + (\partial_t\mathfrak{u}_L - \partial_t\phi)^2 + |\nabla\mathfrak{u}_L - \nabla\phi|^2) - \mathfrak{A}_\omega(\mathfrak{u}_L^\delta - \phi)(\mathfrak{u}_L^\delta - \phi)]\widehat{\Gamma}_\theta = 0;$$

using $u_L\widehat{\Gamma}_\theta$ as a test function for the equation of u_L, we find that

$$\int_{\tilde{Q}_L} (\delta\mathfrak{u}_L^2 + \delta(\partial_t\mathfrak{u}_L)^2 + \delta\partial_t\mathfrak{u}_L\mathfrak{u}_L\frac{\partial_t\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} + \partial_t\mathfrak{u}_L\mathfrak{u}_L + \delta\mathfrak{u}_L\nabla\mathfrak{u}_L \cdot \frac{D\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} + \delta|\nabla\mathfrak{u}_L|^2)\widehat{\Gamma}_\theta$$
$$- \int_{\tilde{Q}_L}\int_{\mathbb{R}^d} J\mu(x-z,\omega)\mu(x,\omega)(-z + \mathfrak{u}_L(x-z,t,\omega) - \mathfrak{u}_L(x,t,\omega))\mathfrak{u}_L(x,t,\omega)dz\widehat{\Gamma}_\theta = 0;$$

thus, the above two identities subtract yield that

$$\int_{\tilde{Q}_L} \Big(-2\delta\mathfrak{u}_L\phi + \delta\phi^2 - 2\delta\partial_t\mathfrak{u}_L\partial_t\phi + \delta(\partial_t\phi)^2 + \delta(\nabla\phi)^2 - 2\delta\nabla\mathfrak{u}_L \cdot \nabla\phi$$
$$+\mathfrak{A}_\omega\mathfrak{u}_L \cdot \phi + \mathfrak{A}_\omega\phi \cdot (\mathfrak{u}_L - \phi) - \partial_t\mathfrak{u}_L\mathfrak{u}_L - \delta\partial_t\mathfrak{u}_L\mathfrak{u}_L\frac{\partial_t\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} - \delta\mathfrak{u}_L\nabla\mathfrak{u}_L \cdot \frac{D\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta}\Big)\widehat{\Gamma}_\theta = 0.$$

As $L \to \infty, \mathfrak{u}_L \to \mathfrak{u}, \nabla\mathfrak{u}_L \to \nabla\mathfrak{u}, \partial_t\mathfrak{u}_L \to \partial_t\mathfrak{u}$ and $(\zeta_z^\delta)_L \to \zeta_z^\delta$ in $L^2_{\widehat{\Gamma}_\theta}$ and in the sense of L^2_{loc}, we get

$$\int_{\mathbb{R}^{d+1}} \Big(-2\delta\mathfrak{u}\phi + \delta\phi^2 - 2\delta\partial_t\mathfrak{u}\partial_t\phi + \delta(\partial_t\phi)^2 + \delta(\nabla\phi)^2 - 2\delta\nabla\mathfrak{u} \cdot \nabla\phi$$
$$+\mathfrak{A}_\omega\mathfrak{u} \cdot \phi + \mathfrak{A}_\omega\phi \cdot (\mathfrak{u} - \phi) - \partial_t\mathfrak{u}\mathfrak{u} - \delta\partial_t\mathfrak{u}\mathfrak{u}\frac{\partial_t\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} - \delta\mathfrak{u}\nabla\mathfrak{u} \cdot \frac{D\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta}\Big)\widehat{\Gamma}_\theta = 0.$$

By integrating Eq (9.43) against $\phi\widehat{\Gamma}_\theta$, we obtain

$$\int_{\mathbb{R}^{d+1}} \Big(\delta\mathfrak{u}\phi + \delta\partial_t\mathfrak{u}\partial_t\phi + \delta\partial_t\mathfrak{u}\phi\frac{\partial_t\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} + \partial_t\mathfrak{u}\phi + \delta\nabla\mathfrak{u}\nabla\phi + \delta\nabla\mathfrak{u}\phi\frac{D\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} - \mathfrak{A}_\omega\mathfrak{u} \cdot \phi\Big)\widehat{\Gamma}_\theta = 0.$$

Adding the above two equations gives

$$\int_{\mathbb{R}^{d+1}} \Big(-\delta\phi(\mathfrak{u} - \phi) - \delta\partial_t\phi(\partial_t\mathfrak{u} - \partial_t\phi) - \delta\nabla(\mathfrak{u} - \phi)\nabla\phi + \mathfrak{A}_\omega\phi \cdot (\mathfrak{u} - \phi)$$

$$-\partial_t \mathfrak{u}(\mathfrak{u}-\phi) - \delta\partial_t \mathfrak{u}(\mathfrak{u}-\phi)\frac{\partial_t \widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} - \delta(\mathfrak{u}-\phi)\nabla \mathfrak{u}\frac{D\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta}\Big)\widehat{\Gamma}_\theta = 0. \tag{9.45}$$

For every $\psi \in C_c^\infty(\mathbb{R}^{d+1})$, set $\phi = \mathfrak{u} + h\psi$, as $h \to 0^+$ we have

$$\int_{\mathbb{R}^{d+1}} \Big(\delta\mathfrak{u}\psi + \delta\partial_t\mathfrak{u}\partial_t\psi + \delta D\mathfrak{u}D\psi - \mathfrak{A}_\omega\mathfrak{u}\cdot\psi + \partial_t\mathfrak{u}\psi + \delta\partial_t\mathfrak{u}\psi\frac{\partial_t\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta} + \delta\psi\nabla\mathfrak{u}\frac{D\widehat{\Gamma}_\theta}{\widehat{\Gamma}_\theta}\Big)\widehat{\Gamma}_\theta = 0, \tag{9.46}$$

where $\mathfrak{u}$ and $D\mathfrak{u}$ are locally integrable, ψ has a compact support, $D\widehat{\Gamma}_\theta, \partial_t\widehat{\Gamma}_\theta \overset{\theta\to 0}{\longrightarrow} 0$ locally uniformly; thus, we have

$$\int_{\mathbb{R}^{d+1}} \delta\mathfrak{u}\psi + \delta\partial_t\mathfrak{u}\partial_t\psi + \delta D\mathfrak{u}D\psi - \mathfrak{A}_\omega\mathfrak{u}\cdot\psi + \partial_t\mathfrak{u}\psi = 0 \text{ as } \theta \to 0. \tag{9.47}$$

For any $\psi \in C_c^\infty(\mathbb{R}^{d+1})$, Eq (9.47) implies that $\mathfrak{u}$ is a solution of Eq (9.43) in the sense of distributions. Finally $\mathfrak{u}$ is stationary from the uniqueness of Eq (9.43).

Step 3. Approximate the convergence of sequences χ^δ in $\mathbf{L^2}$.

We have known that from the inequality (9.41)

$$\delta\int_{\mathbb{R}^{d+1}}(\partial_t\chi^\delta)^2 dxdt \le C,\ \int_{\mathbb{R}^{d+1}}(\chi^\delta)^2 dxdt \le C,\ \delta\int_{\mathbb{R}^{d+1}}\left|\nabla\chi^\delta\right|^2 dxdt \le C,$$

where the constant C is independent of δ. Letting $\delta \to 0$, for an arbitrarily fixed ω, we have

$$\begin{aligned} \delta\partial_t\chi^\delta &\rightharpoonup \partial_t\chi && \text{weakly in } L^2(\mathbb{R}^{d+1}),\\ \chi^\delta &\rightharpoonup \chi && \text{weakly in } L^2(\mathbb{R}^{d+1}),\\ \delta\nabla\chi^\delta &\rightharpoonup 0 && \text{weakly in } L^2(\mathbb{R}^{d+1}). \end{aligned}$$

Moreover, we have

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\nu(x,y)\left|\chi^\delta(y,t)-\chi^\delta(x,t)\right|^2 dydxdt \le C.$$

Hence, for any measurable subset $E \subset Q\times Q\times Q_1 \subset \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$, using the cauchy-Schwartz inequality we find that

$$\begin{aligned} &\left|\int_E J(x,y)\nu(x,y)\Big(\chi^\delta(y,t)-\chi^\delta(x,t)\Big)dydxdt\right|^2\\ &\le \int_E J(x,y)\nu(x,y)dydxdt\int_E J(x,y)\nu(x,y)\left|\chi^\delta(y,t)-\chi^\delta(x,t)\right|^2 dydxdt\\ &\lesssim_{J,\nu} C. \end{aligned} \tag{9.48}$$

Now, applying the Dunford-Pettis theorem, there exists $\vartheta(x,z,t)$ such that

$$J(z,t)\nu(x,x-z)\left(-z+\chi^\delta(x-z,t)-\chi^\delta(x,t)\right) \rightharpoonup \vartheta(x,z,t)$$

weakly in $L^1(Q\times Q\times Q_1)$. Taking $\mathfrak{u} = \chi^\delta$ and $\psi = \chi$ in Eq (9.47), as $\delta \to 0$ we have

$$\int_{Q_1}\int_Q \partial_t\chi\chi dxdt = \int_{Q_1}\int_Q\int_Q \vartheta\chi dxdzdt.$$

Therefore, to finish the proof we have to show that over $Q* = Q_1 \times Q \times Q$

$$\int_{Q*} J(z,t)\nu(x,x-z)(-z+\chi(x-z,t)-\chi(x,t))\chi dxdzdt = \int_{Q*} \vartheta\chi dxdzdt. \tag{9.49}$$

In fact, taking $u = \chi^\delta$ and $\psi = \chi^\delta$ in Eq (9.47), we have

$$\begin{aligned}&\int_{Q_1}\int_Q \left(\delta(\chi^\delta)^2 + \delta\left(\partial_t\chi^\delta\right)^2 + \partial_t\chi^\delta\chi^\delta + \delta(\nabla\chi^\delta)^2\right) - \int_{Q_1}\int_Q \partial_t\chi\chi \\ = &\int_{Q*} J(z,t)\nu\,(x,x-z)\left(-z+\chi^\delta\,(x-z,t)-\chi^\delta(x,t)\right)\chi^\delta dxdzdt - \int_{Q*}\vartheta\chi dxdzdt;\end{aligned} \tag{9.50}$$

thus,

$$\lim_{\delta\to 0}[\int_{Q*} J(z,t)\nu(x,x-z)\left(-z+\chi^\delta(x-z,t)-\chi^\delta(x,t)\right)\chi^\delta(x,t) - \int_{Q*}\vartheta\chi] = 0. \tag{9.51}$$

Using the monotonicity of the nonlocal operator, for all $\Psi \in L^\infty(Q\times Q_1)$,

$$\begin{aligned}&\int_{Q*} J(z,t)\left(-z+\chi^\delta(x-z,t)-\chi^\delta(x,t)\right)\left(\chi^\delta(x,t)-\Psi(x,t)\right)dzdxdt \\ &\le \int_{Q*}(J(z,t)(-z+\Psi(x-z,t)-\Psi(x,t)))\left(\chi^\delta(x,t)-\Psi(x,t)\right)dzdxdt.\end{aligned} \tag{9.52}$$

Passing to the limit as $\delta \to 0$, by using Eq (9.51), we have

$$\int_{Q*}\vartheta(\chi-\Psi)dzdxdt \le \int_{Q*} J(z,t)(-z+\Psi(x-z,t)-\Psi(x,t))(\chi(x,t)-\Psi(x,t))dzdxdt.$$

Choosing $\Psi = \chi \pm \gamma\chi, \gamma > 0$, and letting $\gamma \to 0$, we get Eq (9.49), and the proof is finished.

From Eq (9.43) and the Cauchy-Schwartz inequality, for any $\psi \in L^2(Q\times Q_1)$ we get

$$\int_{Q\times Q_1} \delta\chi^\delta\psi + \delta\partial_t\chi^\delta\partial_t\psi + \delta D\chi^\delta D\psi \to 0 \; as \; \delta \to 0. \tag{9.53}$$

Passing to the limit as $\delta \to 0$ in Eq (9.43), for a.e. ω the function $\chi\,(z,t,\omega)$ satisfies the equation

$$\begin{aligned}\partial_t\chi \;&-\; \int_{\mathbb{R}^d} J(z,t)\mu\,(x-z,t,\omega)\,\mu\,(x,\omega)\,(\chi(x-z,t,\omega)-\chi(x,t,\omega))dz \\ &+\; \int_{\mathbb{R}^d} zJ(z,t)\mu\,(x,\omega)\,\mu\,(x-z,\omega)\,dz = 0;\end{aligned}$$

thus, we prove that $\chi(x,t,\omega)$ is a solution of Eq (9.34).

Denote $\zeta_z(\xi,t,\omega) = \chi(z+\xi,t,\omega) - \chi(\xi,t,\omega)$; then for $z \in \mathbb{R}^d$, we have

$$\zeta_z(\xi,t,\omega) = \zeta_z\left(0,t,T_\xi\omega\right).$$

Step 4. Stationarity of ζ_z.

For all $z \in \mathbb{R}^d$ and $t \in (0, T)$, the field $\zeta_z(x, t, \omega)$ is statistically homogeneous in x and t, and

$$\zeta_z(0, t, \omega) = \chi(z, t, \omega).$$

Thus, the random function $\chi(x, t, \omega)$ is not stationary, but its increments

$$\zeta_z(\xi, t, \omega) = \chi(\xi + z, t, \omega) - \chi(\xi, t, \omega)$$

form a stationary field for any given z.

We first prove the uniqueness of χ^δ. Let u_1 and u_2 be two solutions and set $\tilde{u} = u_1 - u_2$. Using $\tilde{u}\widehat{\rho_\theta}$ as a test function in Eq (9.43) for $\tilde{u}$, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} (\delta \tilde{u}^2 + \delta(\partial_t \tilde{u})^2 + (D\widetilde{u})^2 - (A_\omega \tilde{u}, \tilde{u}))\widehat{\Gamma}_\theta \\
&= - \int_{\mathbb{R}^{d+1}} (\delta \tilde{u} \partial_t \tilde{u} \partial_t \widehat{\Gamma}_\theta + \tilde{u} D\tilde{u} + f|\tilde{u}|) \cdot D\widehat{\Gamma}_\theta \\
&\leq \theta \int_{\mathbb{R}^{d+1}} (\delta |\tilde{u}||\partial_t \tilde{u}| + C_0 |D\tilde{u}||\tilde{u}|)\widehat{\Gamma}_\theta.
\end{aligned} \tag{9.54}$$

Then a standard argument based on the Cauchy-Schwartz inequality implies that, for θ small enough, $\tilde{u} \equiv 0$.

Next, we will prove that $\zeta_z(x, t, \omega)$ is stationary in x, t.

Proposition 4. *The function $\chi(z, t, \omega)$ can be extended to $\mathbb{R}^d \times \mathbb{R} \times \Omega$ in such a way that $\chi(z, t, \omega)$ satisfies the relation inequality (9.34), i.e., $\chi(x, t, \omega)$ has stationary increments:*

$$\begin{cases} \chi(z + \xi, t, \omega) - \chi(\xi, t, \omega) = \chi\left(z, t, T_\xi \omega\right) = \chi\left(z, t, T_\xi \omega\right) - \chi\left(0, t, T_\xi \omega\right), \\ \chi(z, t + s, \omega) - \chi(z, s, \omega) = \chi\left(z, t, T_s \omega\right) - \chi\left(z, 0, T_s \omega\right). \end{cases} \tag{9.55}$$

Proof. The strong convergence $\left\{\chi^\delta\right\}$ implies that there exists a subsequence of $\left\{\chi^{\delta_{n_k}}\right\}$ that converges a.s. to the same limit $\chi(z, t, \omega)$:

$$\lim_{k \to \infty} \chi^{\delta_{n_k}}(z, t, \omega) = \chi(z, t, \omega) \text{ for } a.e.(z, t, \omega).$$

Since J and ν are stationary, according to the uniqueness of the stationary solution χ^δ, we get

$$\chi^{\delta_{n_k}}(z + \xi, t, \omega) - \chi^{\delta_{n_k}}(\xi, t, \omega) = \chi^{\delta_{n_k}}\left(z, t, T_\xi \omega\right) - \chi^{\delta_{n_k}}\left(0, t, T_\xi \omega\right) = \chi^{\delta_{n_k}}\left(z, t, T_\xi \omega\right).$$

Thus $\left\{\chi^{\delta_{n_k}}\right\}$ in Eq (9.43) and passing to the limit as $k \to \infty$ we obtain Eq (9.34) first only for z_1 and z_2 that z_1, z_2 and $z_1 + z_2$ belong to supp $J(\cdot, t)$. Then, we extend the function $\chi(z, t, \omega)$ to a.e. $z \in \mathbb{R}^d$ due to Eq (9.55): $\chi(z_1 + z_2, t, \omega) = \chi(z_2, t, \omega) + \chi(z_1, t, T_{z_2}\omega)$. The proof of the second formula is similar, so we omit the details. Therefore we get the stationarity of ζ_z.

Step 5. Uniqueness of χ.

We first establish an important lemma.

Let $\varphi = \varphi(x, R(t)) = \varphi(\frac{|x|}{R(t)}) \in C^1\left(\mathbb{R}^d \times [0, +\infty)\right)$ be such that

$$\varphi(x, R) = 0 \text{ in } \mathbb{R}^d \backslash Q_{2R},\ \varphi(x, R) = 1 \text{ in } Q_R,\ \varphi(x, R) = 2 - \frac{|x|}{R(t)} \text{ in } Q_{2R} \backslash Q_R.$$

Denote $\bar\sigma_z = \bar\sigma(x+z,t,\omega) - \bar\sigma(x,t,\omega)$ and

$$\mathbb{A}_1 = \int_{\mathbb{R}^d}\int_{|\xi|>3R} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\bar\sigma_z\bar\sigma(\xi+z,\omega)\left(\varphi\left(\frac{|\xi+z|}{R}\right)-\varphi\left(\frac{|\xi|}{R}\right)\right)dzd\xi,$$

$$\begin{aligned}\mathbb{A}_2 &= \int_{\mathbb{R}^d}\int_{|\xi|\le 3R} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\Big(\bar\sigma(\xi+z,\omega)-\bar\sigma(\xi,\omega)\Big)^2\\ &\quad\times\left(\varphi\left(\frac{|\xi+z|}{R}\right)-\varphi\left(\frac{|\xi|}{R}\right)\right)d\xi dz\\ &= \int_{|z|\le\sqrt{R}}\int_{|\xi|\le 3R} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\Big(\bar\sigma(\xi+z,\omega)-\bar\sigma(\xi,\omega)\Big)^2\\ &\quad\times\left(\varphi\left(\frac{|\xi+z|}{R}\right)-\varphi\left(\frac{|\xi|}{R}\right)\right)d\xi dz\\ &+ \int_{|z|\ge\sqrt{R}}\int_{|\xi|\le 3R} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\Big(\bar\sigma(\xi+z,\omega)-\bar\sigma(\xi,\omega)\Big)^2\\ &\quad\times\left(\varphi\left(\frac{|\xi+z|}{R}\right)-\varphi\left(\frac{|\xi|}{R}\right)\right)d\xi dz = \mathbb{A}_{2<}+\mathbb{A}_{2>},\end{aligned}$$

$$\begin{aligned}\mathbb{A}_3 &= \int_{\mathbb{R}^d}\int_{|\xi|\le 3R} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\bar\sigma_z\bar\sigma(\xi,\omega)\\ &\quad\cdot\left(\varphi\left(\frac{|\xi+z|}{R}\right)-\varphi\left(\frac{|\xi|}{R}\right)\right)d\xi dz\\ &\le \int_{\mathbb{R}^d}\int_{\substack{2R\le|\xi|\le 3R,\\ |\xi|\le R}} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\bar\sigma_z\bar\sigma(\xi,\omega)\\ &\quad\cdot\left(\varphi\left(\frac{|\xi+z|}{R}\right)-\varphi\left(\frac{|\xi|}{R}\right)\right)d\xi dz\\ &+ \int_{\mathbb{R}^d}\int_{R\le|\xi|\le 2R} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\bar\sigma_z\bar\sigma(\xi,\omega)\\ &\quad\cdot\left(\varphi\left(\frac{|\xi+z|}{R}\right)-\varphi\left(\frac{|\xi|}{R}\right)\right)d\xi dz.\end{aligned}$$

Lemma 9.3. *We have the following estimates*

$$\begin{aligned}\mathbb{A}_1 &\le \phi_1(R) + \frac{C}{\alpha|R'(t)}R^{d-1} + \alpha|R'(t)|\int_{R\le|\eta|\le 2R}\frac{|\bar\sigma(\eta,\omega)|^2}{R}\varphi\left(\frac{|\eta|}{R}\right)d\eta,\\ \mathbb{A}_{2<} &\le \frac{C}{\sqrt{R(t)}}R(t)^d, \quad \mathbb{A}_{2>}\le c_2R(t)^d,\\ \mathbb{A}_3 &\le c_sR^d + \frac{C}{\alpha|R'(t)}R^{d-1} + \alpha|R'(t)|\int_{R\le|\eta|\le 2R}|\bar\sigma(\eta,\omega)|^2\frac{|\eta|}{R^2}d\eta,\end{aligned}$$

and

$$\phi\left(T_\eta\omega,t\right) = \int_{\mathbb{R}^d}|z|J(z,t)|\bar\sigma_z(T_{\eta-z}\omega,t)|, \phi_1(R) = \alpha_2^2\int_{|\eta|\le R}\phi\left(T_\eta\omega\right)\frac{|\bar\sigma(\eta,\omega)|}{R}d\eta \lesssim c_dR^d,$$

where c_2, c_d and α are small enough.

The proof is given in Appendix C.

Lemma 9.4. *For a.e. ω and all $\xi \in \mathbb{R}^d$, we have*

$$\mathbf{E}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\mathbb{R}^d} J(z,t)\mu(z,\omega)\mu(0,\omega)\bar{\sigma}_z^2 dzdt = 0. \tag{9.56}$$

Proof. Because the map $t \to \mathbf{E}\int_{\mathbb{R}^d} J(z,t)\mu(z,\omega)\mu(0,\omega)\bar{\sigma}_z^2 dz$ is well-defined, and for all $t \in \mathbb{R}$, in order to prove Eq (9.56) by using the contradiction, we have

$$\mathbf{E}\int_{\mathbb{R}^d} J(z,t)\mu(z,\omega)\mu(0,\omega)\bar{\sigma}_z^2 dz \geq 0. \tag{9.57}$$

Fix $R > 0$; in view of the stationarity of $\bar{\sigma}_z$, there exist $\varepsilon_0 > 0$ and $0 < \kappa < \widehat{\kappa}$ such that, for all $t \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0)$ and $R > 0$,

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(z,t)\mu(x+z,\omega)\mu(x,\omega)(\bar{\sigma}(x+z,t,\omega) - \bar{\sigma}(x,t,\omega))^2\varphi\left(\frac{|x|}{R}\right)dzdx \geq \kappa R^d. \tag{9.58}$$

T is large enough; let $R(t) = (T - \gamma_1 t)^{1/4}$ satisfy $R'(t) \leq 0$ for some $\gamma_1 > 0$; then

$$\begin{aligned}
& \frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\varphi(x,R(t))dx \\
= \ & R'(t)\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\partial_R\varphi(x,R(t))dx + \int_{\mathbb{R}^d}\bar{\sigma}(x,t)\partial_t\bar{\sigma}(x,t)\varphi(x,R(t))dx \\
= \ & R'(t)\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\partial_R\varphi(x,R(t))dx \\
+ \ & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(z,t)\mu(x+z,\omega)\mu(x,\omega)(\bar{\sigma}(x+z,\omega) - \bar{\sigma}(x,\omega))\bar{\sigma}(x,\omega)\varphi\left(\frac{|x|}{R}\right)dzdx \\
= \ & R'(t)\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\partial_R\varphi(x,R(t))dx \\
- \ & \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(z,t)\mu(x+z,\omega)\mu(x,\omega)\Big(\bar{\sigma}(x+z,\omega) - \bar{\sigma}(x,\omega)\Big) \\
& \cdot\left(\bar{\sigma}(x+z,\omega)\varphi\left(\frac{|x+z|}{R}\right) - \bar{\sigma}(x,\omega)\varphi\left(\frac{|x|}{R}\right)\right)dzdx,
\end{aligned}$$

thus,

$$\begin{aligned}
& \frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\varphi(x,R(t))dx = R'(t)\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\partial_R\varphi(x,R(t))dx \\
& -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(z,t)\mu(x+z,\omega)\mu(x,\omega)\Big(\bar{\sigma}(x+z,\omega) - \bar{\sigma}(x,\omega)\Big)^2\varphi\left(\frac{|x|}{R}\right)dzdx \\
& -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(z,t)\mu(x+z,\omega)\mu(x,\omega)\Big(\bar{\sigma}(x+z,\omega) - \bar{\sigma}(x,\omega)\Big)\bar{\sigma}(x+z,\omega) \\
& \cdot\left(\varphi\left(\frac{|x+z|}{R}\right) - \varphi\left(\frac{|x|}{R}\right)\right)dzdx.
\end{aligned} \tag{9.59}$$

According to Lemma 9.3, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\varphi(x,R(t))dx$$
$$\le R'(t)\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\partial_R\varphi(x,R(t))dx-\kappa R(t)^d+|\mathbb{A}_1|+|\mathbb{A}_2|+|\mathbb{A}_3|, \tag{9.60}$$

the fact that the inequality $\partial_R\varphi(x,R(t))=\frac{|x|}{R(t)^2},\varphi(\frac{|x|}{R})\le\frac{|x|}{R}$ when $R\le|x|\le 2R$, if α is small enough, we obtain the following from estimates in Lemma 9.3:

$$\frac{d}{dt}\mathbb{E}\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\varphi(x,R(t))dx+(\kappa-c_2-c_s-c_d)R(t)^d$$
$$\le\frac{C}{\sqrt{R(t)}}R(t)^d+\frac{C}{|R'(t)|}R(t)^{d-1}; \tag{9.61}$$

due to the facts that $\left|Q_{2R(t)}\backslash Q_{R(t)}\right|\lesssim C(R(t))^d$ and $R'(t)=\gamma_1(R(t))^{-1}$, for some $C>0$, we get

$$\frac{d}{dt}\mathbb{E}\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\varphi(x,R(t))dx$$
$$\le -R(t)^d\Big(\kappa-c_2-c_s-c_d-C\gamma_1^{-1}-\frac{C}{\sqrt{R(t)}}\Big). \tag{9.62}$$

Choosing $\gamma_1>1$ large enough and c_2,c_s and c_d small enough such that $\kappa-c_2-c_s-c_d-C\gamma_1^{-1}\ge\hat{\tau}/2$ and $t\le t_T=(T-(4C\hat{\tau}^{-1})^2)\gamma_1^{-1}$, in order to have $\frac{C}{\sqrt{R(t)}}\le\hat{\tau}/4$ on $[0,t_T]$, we have

$$\frac{d}{dt}\mathbb{E}\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t)\varphi(x,R(t))dx\le -R(t)^d\frac{\hat{\tau}}{4}.\quad \textit{for any } t\in[0,t_T].$$

Along with integration in time over $t\in[t_1,\gamma_1^{-1}T]$ for $t_1\in[0,\sqrt{T}]$, suppose that γ_1^{-1} and T are sufficiently large and satisfy $\gamma_1^{-1}T<t_T$; also, given the fact that $\varphi\ge 0$, we have

$$\mathbb{E}\int_{\mathbb{R}^d}\frac{1}{2}\bar{\sigma}^2(x,t_1)\varphi(x,R(t_1))dx\ge\frac{\hat{\tau}}{4}\int_{t_1}^{\gamma_1^{-1}T}R(t)^d dt.$$

Integrating in time over $t_1\in[0,\sqrt{T}]$, since $R(t_1)\le T^{1/4}$ and $\varphi(x,R(t_1))\le 1_{Q_{2T^{1/4}}}$, we get

$$\mathbb{E}\int_0^{T^{1/4}}\int_{Q_{2T^{1/4}}}\frac{1}{2}\bar{\sigma}^2(x,t_1)dxdt_1\ge C^{-1}\hat{\tau}T^{(d+3)/4}.$$

Hence, we can apply Lemma B.1 in Appendix B which implies that, for any $\delta>0$, there exists R_δ such that, for all $R\ge R_\delta$,

$$\mathbb{E}\int_0^R\int_{Q_R}\bar{\sigma}^2(x,t_1)dxdt_1\le\delta R^{d+3}.$$

Here choosing $R=2T^{1/4}$ and T large enough, we obtain

$$C^{-1}\hat{\tau}T^{\frac{d+3}{4}}\le\mathbb{E}\int_0^{T^{1/4}}\int_{Q_{2T^{1/4}}}\frac{1}{2}\bar{\sigma}^2(x,t_1)dxdt_1\le\delta 2^{d+2}T^{\frac{d+3}{4}},$$

as δ is small enough, which yields a contradiction. Hence we get

$$\mathbf{E}\int_{-1/2}^{1/2}\int_{\mathbb{R}^d} J(z,t)\mu(z,\omega)\mu(0,\omega)\bar{\sigma}_z^2 dzdt \le 0. \tag{9.63}$$

By combining the inequality (9.63) with the inequality (9.57), we get Eq (9.56); the proof is done.

Suppose that u and u_1 are the solutions of the equation and set $\mathfrak{T}^1 = \partial_t u^1$ and $\xi^1 = A_\omega u^1$; then $\mathfrak{T} - \mathfrak{T}^1 - (\xi - \xi^1) = 0$; by applying Lemma 9.4 to the pair $(\mathfrak{T} - \mathfrak{T}^1, u - u^1)$, we find that

$$\mathbb{E}\int_{-1/2}^{1/2}(\xi - \xi^1)(u - u^1)dt = 0,$$

which implies the uniqueness of χ.
Step 6. Sublinear growth.

Lemma 9.5. *The family of functions* $\{\chi^\varepsilon(x,t,\omega) = \varepsilon\chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega\right)\}_{\varepsilon>0}$ *is bounded and compact in* $L^2_{loc}(\mathbb{R}^{d+1})$.

Proof. Assume that $\chi^\varepsilon(x,t,\omega) = \varepsilon\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)$ satisfies the equation

$$\partial_t\chi^\varepsilon - \mathfrak{A}_\omega\chi^\varepsilon = 0 \text{ in } \mathbb{R}^d \times \mathbb{R}. \tag{9.64}$$

Denote $\chi_z^\varepsilon(x,t) = z + \chi^\varepsilon(x+z,t,\omega) - \chi^\varepsilon(x,t,\omega)$, $\diamond = J(z,s)\mu(x+z,\omega)\mu(x,\omega)$ and $j_\ell = 4\alpha_2^2 j_0$. Without loss of generality, we assume that $0 < j_\ell \le 1$; we will show that there exists a universal constant C_0 such that, $\mathbb{P}$-a.s. and for any $R, T > 0$,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_0^T\int_{Q_R}(\chi^\varepsilon(x,t))^2\,dxdt \\
&\le C_0 T^3 R^{d-2}\mathbb{E}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\mathbb{R}^d}|z|J(z,s)|\chi_z^\varepsilon(T_{-z}\omega,s)|^2 dzds.
\end{aligned}$$

Fix $\xi \in C^\infty(\mathbb{R};[0,1])$ such that

$$\xi \equiv 0 \text{ in } (-\infty, \frac{3j_\ell}{2} - 2), \quad \xi \equiv 1 \text{ in } [j_\ell - 1, +\infty), \quad |\xi'| \le 2. \tag{9.65}$$

Denote

$$\varphi(x,s,t) = \xi\Big((\frac{3j_\ell}{2} - \frac{sj_\ell}{2t}) - \|x\|_\infty R^{-1}\Big),$$

where $\|x\|_\infty = \max\{|x_i| : i = 1,\cdots,d\}$. Since $1 \le \frac{3}{2} - \frac{s}{2t} \le \frac{3}{2}$ for $s \in [0,t]$, $\varphi(x,s,t) = 1$ in Q_R, while $\varphi(x,s,t) = 0$ in $\mathbb{R}^d \backslash Q_{2R}$.

From Eq (9.64) and Young's inequality, fix $t > 0$ and for any $s \in (0,t)$, we have

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathbb{R}^d}(\chi^\varepsilon)^2\varphi(x,s,t)dx = \int_{\mathbb{R}^d}(\chi^\varepsilon)^2\partial_s\varphi dx - L_\omega\chi^\varepsilon\varphi \\
&= \int_{\mathbb{R}^d}(\chi^\varepsilon)^2\partial_s\varphi - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\diamond\chi_z^\varepsilon\chi^\varepsilon(x+z,s,\omega)(\varphi(x+z,s,t) - \varphi(x,s,t))dzdx
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \diamond\chi_z^{\varepsilon}(\chi^{\varepsilon}(x+z,s,\omega)-\chi^{\varepsilon}(x,s,\omega))\varphi(x,s,t)dzdx \\
&= \int_{\mathbb{R}^d}(\chi^{\varepsilon})^2\partial_s\varphi - \mathbb{B}_1 - \mathbb{B}_2, \qquad (9.66)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{B}_1 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \diamond\chi_z^{\varepsilon}\chi^{\varepsilon}(x+z,s,\omega)\Big(\varphi(x+z,s,t)-\varphi(x,s,t)\Big)dzdx, \\
\mathbb{B}_2 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \diamond\chi_z^{\varepsilon}\Big(\chi^{\varepsilon}(x+z,\omega)-\chi^{\varepsilon}(x,\omega)\Big)\varphi(x,s,t)dzdx = \mathbb{B}_{21}+\mathbb{B}_{22}, \\
\mathbb{B}_{21} &= R^{-1}t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \diamond(\chi_z^{\varepsilon}(T_x\omega))^2\Big(\varphi(x+z,s)-\varphi(x,s)\Big)dxdz \\
&= R^{-1}t\int_{\mathbb{R}^d}\int_{|x|\le 3R} \diamond(\chi_z^{\varepsilon}(T_x\omega))^2\Big(\varphi(x+z,s)-\varphi(x,s)\Big)dxdz \\
&+ R^{-1}t\int_{\mathbb{R}^d}\int_{x|\ge 3R} \diamond(\chi_z^{\varepsilon}(T_x\omega))^2\Big(\varphi(x+z,s)-\varphi(x,s)\Big)dxdz, \\
\mathbb{B}_{22} &= Rt^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \diamond(\chi^{\varepsilon}(x+z,s,\omega))^2\Big(\varphi(x+z,s)-\varphi(x,s)\Big)dxdz.
\end{aligned}
$$

Denote $\tilde{\phi}(\eta,s,\omega) = \int_{\mathbb{R}^d}|z|J(z,s)|\chi_z^{\varepsilon}(T_{\eta-z}\omega,s)|^2dz$; then we have

$$
\begin{aligned}
\mathbb{B}_{21} &\le R^{\frac{d}{2}-2}t\alpha_2^2\Big(\int_{|\eta|\le 2R}\tilde{\phi}^2(T_\eta\omega)d\eta\Big)^{1/2} \\
&+ R^{\frac{d}{2}-2}t\alpha_2^2\Big(\int_{|\xi|\le 3R}\tilde{\phi}^2(T_\xi\omega)d\xi\Big)^{1/2} \\
&\le R^{\frac{d}{2}-2}t\alpha_2^2¥^{\varepsilon}(s). \qquad (9.67)
\end{aligned}
$$

Finally, the penultimate inequality uses Appendix C. So we have

$$
¥^{\varepsilon}(s) = \Big(\int_{|\eta|\le 2R}\tilde{\phi}^2(T_\eta\omega,s)d\eta\Big)^{1/2} + \Big(\int_{|\xi|\le 3R}\tilde{\phi}^2(T_\xi\omega,s)d\xi\Big)^{1/2}, \qquad (9.68)
$$

thus,

$$
\begin{aligned}
\mathbb{B}_{22} &\le \xi' t^{-1}\alpha_2^2\int_{\mathbb{R}^d}J_1(z)|z|dz\int_{\mathbb{R}^d}(\chi^{\varepsilon}(x,s,\omega))^2dx \\
&= \xi' t^{-1}\alpha_2^2 j_0\int_{\mathbb{R}^d}(\chi^{\varepsilon})^2dx. \qquad (9.69)
\end{aligned}
$$

Since $\partial_s\varphi = -2\alpha_2^2 j_0 t^{-1}\xi'$ while $|\varphi(x)-\varphi(y)| \le R^{-1}\xi'|x-y|$, we can absorb the last term on the right-hand side of Eq (9.66) into the first one to obtain

$$
\frac{d}{ds}\int_{\mathbb{R}^d}(\chi^{\varepsilon}(s))^2\varphi(x,s,t)dx = \int_{\mathbb{R}^d}(\chi^{\varepsilon})^2\partial_s\varphi dx - \mathbb{B}_1 - \mathbb{B}_2 \le -\mathbb{B}_1 - \mathbb{B}_{21}. \qquad (9.70)
$$

Integrating in time over $s \in [0, t]$ and using the definition of ϕ we get

$$\int_{Q_R} (\chi^\varepsilon(t))^2 dx \leq -\int_0^t \mathbb{B}_1(s)ds + R^{\frac{d}{2}-2} t \int_0^t ¥^\varepsilon(s)ds + \int_{Q_{2R}} (\chi^\varepsilon(0))^2 dx. \tag{9.71}$$

Integrating in time over $t \in [0, T]$,

$$\begin{aligned}\int_0^T \int_{Q_R} (\chi^\varepsilon(t))^2 dxdt \quad &\leq \quad -\int_0^T \int_0^t \mathbb{B}_1(s)dsdt + R^{\frac{d}{2}-2} \int_0^T t \int_0^t ¥^\varepsilon(s)dsdt \\ &+ \quad T \int_{Q_{2R}} (\chi^\varepsilon(0))^2 \, dx. \end{aligned} \tag{9.72}$$

Let $\varepsilon \to 0$; from Appendix B and the ergodic theorem, we have $\mathbb{P}$-a.s.,

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int_0^T \int_{Q_R} (\chi^\varepsilon(x,t))^2 dxdt \\ &\leq -\int_0^T \int_0^t \int_{\mathbb{R}^d} \mathbb{E}\Big[\int_{\tilde{Q}_1} \mathfrak{A}_\omega \chi \cdot \chi\Big]\varphi(x,s,t)dxdsdt + R^{\frac{d}{2}-2} T \int_0^T \int_0^t ¥(s)dsdt, \end{aligned} \tag{9.73}$$

where

$$\begin{aligned} ¥ \quad &= \quad \int_{-1/2}^{1/2} \Big(\int_{|\eta|\leq 2R} \tilde{\phi}^2(0,s,\omega)d\eta\Big)^{1/2} ds + \int_{-1/2}^{1/2} \Big(\int_{|\xi|\leq 3R} \tilde{\phi}^2(0,s,\omega)d\xi\Big)^{1/2} ds \\ &\leq \quad CR^{\frac{d}{2}} \int_{-1/2}^{1/2} \tilde{\phi}(0,s,\omega)ds. \end{aligned} \tag{9.74}$$

Lemma 9.4 gives that the first term on the right-hand side of the inequality (9.73) vanishes. Thus, using the inequality (9.74),

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int_0^T \int_{Q_R} (\chi^\varepsilon(x,t))^2 dxdt \\ &\lesssim R^{d-2} T^3 \mathbb{E} \int_{-1/2}^{1/2} \int_{\mathbb{R}^d} |z| J(z,s) |\chi_z^\varepsilon(T_{-z}\omega, s)|^2 dzds.\end{aligned}$$

Due to the symmetric property we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int_{-T}^T \int_{Q_R} (\chi^\varepsilon(x,t))^2 dxdt \\ &\leq C_0 T^3 R^{d-2} \mathbb{E} \int_{-1/2}^{1/2} \int_{\mathbb{R}^d} |z| J(z,s) |\chi_z^\varepsilon(T_{-z}\omega, s)|^2 dzds.\end{aligned} \tag{9.75}$$

Next we show that $\chi^\varepsilon(x,t)$ converges to 0 as $\varepsilon \to 0^+$.

Let $\omega \in \Omega$ be such that the inequality (9.75) holds for any $T, R > 0$. The inequality (9.75) implies that the family $(\chi^\varepsilon(x,t))_{\varepsilon>0}$ is bounded in $L^2_{loc}(\mathbb{R}^d \times \mathbb{R})$.

Note that $(\partial_t \chi^\varepsilon(x,t))_{\varepsilon>0}$ is bounded in $L^2_{loc}(\mathbb{R}, L^2)$, and $(\chi^\varepsilon(\cdot,t))_{\varepsilon>0}$ is compact in $L^2_{loc}(\mathbb{R}^d)$ according to [41, Lemma 4.1]. Hence, with the help of the classical Lions-Aubin lemma, the family $(\chi^\varepsilon(x,t))_{\varepsilon>0}$ is relatively compact in $L^2_{loc}(\mathbb{R}^{d+1})$.

Let $(\chi^{\varepsilon_n}(x,t))$ be any convergent subsequence with limit χ in $L^2_{loc}(\mathbb{R}^d\times\mathbb{R})$. Since ζ_z is stationary in an ergodic environment, it converges weakly to a constant. Thus, owing to Eq (9.64), χ solves $\partial_t\chi = 0$ in $\mathbb{R}^d\times\mathbb{R}$. Letting $T\to 0$ in the inequality (9.75) yields that $\chi(\cdot,0)=0$. Therefore, $\chi\equiv 0$, so $\chi^{\varepsilon_n}\to 0$ in $L^2_{loc}(\mathbb{R}^d\times\mathbb{R})$. Lemma 9.5 has been proved.

Proof of Theorem 9.2. Following the above Steps 1–6, we can finish the proof of the main Theorem 9.2 in this section.

10. Additional terms of the asymptotic expansion

Remembering Eq (9.19), as $\varepsilon\to 0^+$, we have an asymptotic expansion for $w^\varepsilon(x,t) = v^0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t)$:

$$H^\varepsilon w_0^\varepsilon(x,t) = \frac{\partial}{\partial t}v^0(x,t) + \frac{1}{\varepsilon}\underbrace{\mathcal{M}_0(x,t)}_{=0} + \underbrace{\mathcal{M}_\varepsilon(x,t)}_{Zero-order\ expansion} + \underbrace{\phi_\varepsilon(x,t)}_{Remainder}.$$

We now give a decomposition of the zero-order ε^0 term in the asymptotic expansion of $H^\varepsilon w_0^\varepsilon(x,t)$.

10.1. Some lemmas in asymptotic analysis

Lemma 10.1. *For the zero-order expansion term $\mathcal{M}_\varepsilon(x,t)$ we have*

$$\mathcal{M}_\varepsilon(x,t) = (D_1 - D_2)\cdot\nabla\nabla v^0 + \Upsilon^\varepsilon + F(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)\cdot\nabla\nabla v^0, \tag{10.1}$$

where $\Upsilon^\varepsilon = (\beta_\varepsilon, v_2^\varepsilon)$, $F(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega) = \aleph_1(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega) - \aleph_2(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)$, and the matrices D_1 and D_2 are

$$\begin{aligned} D_1 &= \int_{-1/2}^{1/2}\int_{\mathbb{R}^d}\frac{1}{2}z\otimes z\mathbf{E}\Big\{J(z,s)\mu(0,\omega)\mu(-z,\omega)\Big\}dzds,\\ D_2 &= \int_{-1/2}^{1/2}\int_{\mathbb{R}^d}\frac{1}{2}z\otimes\mathbf{E}\Big\{J(z,t)\zeta_{-z}(0,t,\omega)\mu(0,\omega)\mu(-z,\omega)\Big\}dzdt;\end{aligned}$$

$F(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)$, $\aleph_1(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)$ and $\aleph_2(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)$ are stationary fields with a zero mean which are given by

$$\begin{aligned}\aleph_1(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega) &= \frac{1}{2}\int_{\mathbb{R}^d} z^2\Big[J(z,\frac{t}{\varepsilon^2})\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\\ &\qquad -\mathbf{E}\Big\{J(z,s,\omega)\mu(0,\omega)\mu(-z,\omega)\Big\}\Big]dz,\end{aligned} \tag{10.2}$$

$$\begin{aligned}\aleph_2(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega) &= \frac{1}{2}\int_{\mathbb{R}^d} z\Big[J(z,\frac{t}{\varepsilon^2})\zeta_{-z}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\\ &\qquad -\mathbf{E}\Big\{J(z,s,\omega)\zeta_{-z}(0,\frac{t}{\varepsilon^2},\omega)\mu(0,\omega)\mu(-z,\omega)\Big\}\Big]dz.\end{aligned} \tag{10.3}$$

For the problems

$$\begin{cases}(\partial_t - L^\varepsilon)v_2^\varepsilon(x,t,\omega) = -\Upsilon^\varepsilon,\\ v_2^\varepsilon(x,0)=0,\end{cases} \tag{10.4}$$

$$\begin{cases}(\partial_t - L^\varepsilon)v_3^\varepsilon(x,t,\omega) = -F^\varepsilon\cdot\nabla\nabla v^0,\\ v_3^\varepsilon(x,0)=0,\end{cases} \tag{10.5}$$

we have that $\|v_i^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^d)}\to 0$ $(i=2,3)$ $\mathbb{P}-a.s.$ as $\varepsilon\to 0^+$.

We need to prove the boundedness of the sequence v_i^ε and then prove its compactness by using the Lions-Aubin lemma; we finally explain that $\|v_i^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^d)}$ converges to 0. The proof requires some technical estimates in [41, Section 5]; we provide the proof in the Appendix C.

Proposition 5. *The matrix* $\Theta = \mathbb{D}_1 - \mathbb{D}_2$ *is positive definite:*

$$\Theta = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\mathbb{R}^d}\int_{\Omega}(z\otimes z - z\otimes\zeta_{-z}(0,s,\omega))J(z,s)\mu(0,\omega)\mu(-z,\omega)dzdsd\mathbf{P}(\omega) > 0.$$

Proof. The procedure of deriving Θ's positive definiteness is basically similar to [41, Proposition 5.1].

10.2. Remainder ϕ_ε

We now consider the estimate of the remainder

$$\phi_\varepsilon(x,t) = \phi_\varepsilon^{(time)}(x,t) + \phi_\varepsilon^{(space)}(x,t),$$

in the asymptotic expansion Eq (9.19), where $\phi_\varepsilon^{(time)}$ and $\phi_\varepsilon^{(space)}$ are defined in Eqs (9.22) and (9.23) respectively.

Proposition 6. *Let* $v \in C^\infty\left((0,T),\mathcal{S}\left(\mathbb{R}^d\right)\right)$*; then, for the functions* $\phi_\varepsilon^{(time)}$ *and* $\phi_\varepsilon^{(space)}$ *we have*

$$\left\|\phi_\varepsilon^{(space)}\right\|_2 \to 0 \text{ and } \left\|\phi_\varepsilon^{(time)}\right\|_2 \to 0 \text{ as } \varepsilon \to 0, \tag{10.6}$$

where $\|\cdot\|_2$ *is the norm in* $L^2((0,T),L^2(\mathbb{R}^d))$.

Proof. The convergence for $\phi_\varepsilon^{(time)}$ immediately follows from the representation Eq (9.22). For the function $\phi_\varepsilon^{(space)}$ given in Eq (9.23), the proof is completely analogous to that of [13, Proposition 5]. The proof of Proposition 6 is done.

10.3. Asymptotic representation of zero-order term

We now give an asymptotic representation of the second term $\mathcal{M}_0(x,t)$ in Eq (9.19), that is,

$$\mathcal{M}_0(x,t) = \hat{L}v^0 + F\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega\right)\nabla\nabla v^0 + \Upsilon^\varepsilon, \tag{10.7}$$

where $F(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)$ is a stationary matrix-field with a zero average and Υ^ε is a non-stationary term; they are defined in Lemma 10.1. Additionally, u_2^ε and u_3^ε satisfy

$$(\partial_t - L^\varepsilon)u_2^\varepsilon = -F(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega)\nabla\nabla v^0,\ (\partial_t - L^\varepsilon)u_3^\varepsilon = -\Upsilon^\varepsilon, \tag{10.8}$$

respectively, and

$$\|u_2^\varepsilon\|_\infty \to 0, \quad \|u_3^\varepsilon\|_\infty \to 0 \text{ as } \varepsilon \to 0.$$

For the corrector χ, from the sublinearity of χ^ε, we have

$$\left\|\varepsilon\chi\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\nabla v^0(x,t)\right\|_{L^2\left(\mathbb{R}^{d+1}\right)} \to 0 \text{ as } \varepsilon \to 0. \tag{10.9}$$

This yields

$$\|w^\varepsilon - v^0\| \to 0 \text{ as } \varepsilon \to 0.$$

With this choice of χ, u_2^ε and u_3^ε, the expression $(\partial_t - L^\varepsilon)w^\varepsilon$ can be rearranged as follows:

$$\begin{aligned}(\partial_t - L^\varepsilon)w^\varepsilon &= (\partial_t - L^\varepsilon)(v^0 + \varepsilon\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega)\nabla v^0) + (\partial_t - L^\varepsilon)(u_2^\varepsilon + u_3^\varepsilon)\\ &= (\partial_t - L^\varepsilon)\, v^\varepsilon + \phi_\varepsilon.\end{aligned} \tag{10.10}$$

From Proposition 6, $\|\phi_\varepsilon\|_2$ vanishes as $\varepsilon \to 0$. This implies that

$$\|w^\varepsilon - v^\varepsilon\|_{L^2((0,T),L^2(\mathbb{R}^d))} \to 0 \text{ as } \varepsilon \to 0,\ \mathbb{P} - a.s. \tag{10.11}$$

10.4. Proof of Theorem 9.1

Proof of Theorem 9.1. The proof is almost the same as that of Theorem 3.2 on the homogenization of nonlinear nonlocal parabolic equations with time dependent coefficients under a periodic and stationary structure that we have discussed in Sections 3–7; the difference is that we replace the periodic structure with the stationary structure in the nonlocal operator. Combining Eq (10.11) and using the triangle inequality and the fact that Schwartz space is dense in L^2, Theorem 9.1 is proved.

11. Nonlinear results for $p \neq 1$

In this section we just give some results of the corresponding equations with a nonlinear nonlocal operator for $p \neq 1$. The proof is rather long and tedious so we omit the proof in details.

Case I. $0 < p < 1$**.**

We need to show that the nonlinear term will make the corrector χ depend on $v(x,t)$ and ε, and that $\chi^\varepsilon \to \chi$ as $\varepsilon \to 0$.

Theorem 11.1. *Assume that the linear condition is satisfied; there exists a unique map* χ*:* $\mathbb{R}^{d+1}\times\Omega \to \mathbb{R}$ *such that*

$$\frac{1}{p}\,|v|^{(1-p)/p}\,\partial_s\chi - \mathfrak{A}_\omega\chi = 0 \text{ in } \mathbf{L}^2, \tag{11.1}$$

and $\forall z \in \mathbb{R}^d$*,* $\zeta_z(x,t,y,s,\omega) = \chi(x,t,y+z,s,\omega) - \chi(x,t,y,s,\omega) \in L^2_{loc}(\mathbb{R}^{d+1}, \mathbf{L}^2)$*, satisfying*

$$\int_{\widetilde{Q}_1} \zeta_z(x,t,y,s,\omega)dyds = 0,\ \mathbb{P} - a.s.$$

We have the convergence

$$\chi^\varepsilon(x,t,\omega) = \varepsilon\chi\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega\Big) \xrightarrow{\varepsilon\to 0} 0 \text{ in } L^2_{loc}(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}),\ \mathbb{P} - a.s.$$

In addition, the positive definite matrix Θ *is defined by*

$$\Theta = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\mathbb{R}^d}\int_\Omega \Big(z\otimes z - z\otimes\zeta_{-z}(x,t,0,s,\omega)\Big)J(z,s)\mu(0,\omega)\mu(-z,\omega)dzdsd\mathbf{P}(\omega).$$

Case II. $1 < p \leq 2$**.**

Theorem 11.2. *Assume that the linear condition is satisfied, there exists a unique map* χ*:* $\mathbb{R}^{d+1}\times\Omega \to \mathbb{R}$ *such that*

$$\partial_s \natural - \mathfrak{A}^1_\omega \natural = 0 \textit{ in } \mathbf{L^2}, \tag{11.2}$$

where

$$\mathfrak{A}^1_\omega \natural = \int_{\mathbb{T}^d} J(\xi - q, t)\nu(\xi, q)\Big(q - \xi + p|v|^{(p-1)/p}\natural(x,t,\xi,s) - p|v|^{(p-1)/p}\natural(x,t,q,s)\Big)dq,$$

$$\chi^k(x,t,y,s) = \begin{cases} p|v^0|^{(p-1)/p}\natural^k(x,t,y,s) & \textit{if} \quad v^0(x,t) \neq 0, \\ 0 & \textit{if} \quad v^0(x,t) = 0. \end{cases}$$

We have the convergence

$$\natural^\varepsilon(x,t,\omega) = \varepsilon\natural(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega) \overset{\varepsilon\to 0}{\longrightarrow} 0 \textit{ in } L^2_{loc}(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}),\ \mathbb{P} - a.s.$$

In addition, the positive definite matrix Θ *is the same as* $0 < p < 1$.

Results of non-self-similar scales are similar to those obtained above.

Remark 4. There are two parts of the proof here that are different from the equation with the linear operator. The first is that the random corrector depends on macroscopic and microscopic variables and the solution u of the equation, which requires more approximations as in the periodic case to obtain the existence of the corrector. The second part is that the heterogeneous solution u^ε converges to a homogeneous solution $u(x,t)$. Usually we can find a corrector χ depending on $u(x,t)$, but this will create the problem of not having enough information about the regularity of the map $u \mapsto \chi(\cdot,\cdot,u,\omega)$, and it requires us to develop some useful tools to overcome the difficulty. In 2022, Cardaliaguet, Dirr and Souganidis [42] dealt with the homogenization of a class of nonlinear parabolic equations and the corresponding random corrector

$$\chi^\varepsilon(x,t,\omega) = \chi(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\nabla u(x,t),\omega).$$

In order to circumvent this difficulty, they introduced a localization argument for the gradient of the corrector, which is based on a piecewise constant approximation of ∇u. Whether a more general equation (like doubly nonlinear equations or fractional diffusion equations) can be used is the direction we will think about in the future.

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of interest

The authors have no conflicts to disclose.

Authors' contributions

Junlong Chen carried out the homogenization theory of partial differential equations, and Yanbin Tang evaluated carried out the reaction diffusion equations and the perturbation theory of partial differential equations. All authors carried out the proofs and conceived the study. All authors read and approved the final manuscript.

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Appendix A

Proof of Lemma 3.1. Let $u, v \in L^{[1,\infty]}(\mathbb{R}^d)$ be such that

$$\max\{\|u\|_1, \|v\|_1, \|u\|_\infty, \|v\|_\infty\} \leq \Lambda$$

and let $\Lambda > 0$ be a given constant. Using the integrability condition and the local Lipschitz continuity with Lipschitz constant L_Λ of $\Gamma^\delta(u(x), u(y), x, y, t)$. If we take $-\Lambda < a, b, c < \Lambda$ such that $|c-b|, |a-b| \geq$

δ, then for a given function f we have

$$\left|\frac{f(a)-f(b)}{a-b}-\frac{f(c)-f(b)}{c-b}\right| \leq \frac{2}{\delta^2}\left(\max_{|\xi|<\Lambda}|f(\xi)|+\Lambda\max_{|\xi|<\Lambda}|f'(\xi)|\right)|a-c|, \tag{A.1}$$

so that we can easily get the local Lipschitz-continuity of Γ^δ and we have

$$\begin{aligned}
&\|\mathcal{L}_u^{t,\delta}u-\mathcal{L}_v^{t,\delta}v\|_\infty \leq \|\mathcal{L}_u^{t,\delta}(u-v)\|_\infty+\|(\mathcal{L}_u^{t,\delta}-\mathcal{L}_v^{t,\delta})v\|_\infty\\
\leq\ & \sup_{x\in\mathbb{R}^d}|\int_{\mathbb{R}^d}(u(x)-u(y)-v(x)+v(y))\Gamma^\delta(u(x),u(y),x,y,t)dy|\\
+\ & \sup_{x\in\mathbb{R}^d}|\int_{\mathbb{R}^d}(v(x)-v(y))(\Gamma^\delta\big(u(x),u(y),x,y,t)-\Gamma^\delta(v(x),v(y),x,y,t))dy|\\
\leq\ & 2\alpha_2\|u-v\|_\infty\sup_{x\in\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}J(x-y,t)\frac{|u(y)|^{p-1}u(y)-|u(x)|^{p-1}u(x)}{u(y)-u(x)}dy\Big|\\
+\ & 2\alpha_2\|v\|_\infty L_\Lambda\int_{\mathbb{R}^d}J(x-y,t)\Big(|u(y)-v(y)|+|u(x)-v(x)|\Big)dy\\
\leq\ & (2\alpha_2\|u-v\|_\infty C_p(\|u\|_\infty+\|v\|_\infty)^{p-1}+2\alpha_2\|v\|_\infty L_\Lambda\|u-v\|_\infty)\int_{\mathbb{R}^d}J(x,t)dx\\
\leq\ & M(p,\alpha_2,\Lambda)\|u-v\|_\infty,
\end{aligned}\tag{A.2}$$

where C_p and $M(p,\alpha_2,\Lambda)$ are constants and $L_\Lambda=\frac{2(p+1)}{\delta^2}\Lambda^p$.

Similarly, we have

$$\begin{aligned}
&\|\mathcal{L}_u^{t,\delta}u-\mathcal{L}_v^{t,\delta}v\|_1 \leq \|\mathcal{L}_u^{t,\delta}(u-v)\|_1+\|(\mathcal{L}_u^{t,\delta}-\mathcal{L}_v^{t,\delta})v\|_1\\
\leq\ & \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\Big(u(x)-u(y)-v(x)+v(y)\Big)\Gamma^\delta\Big(u(x),u(y),x,y,t\Big)dy\Big|dx\\
+\ & \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\Big(v(x)-v(y)\Big)\Big(\Gamma^\delta(u(x),u(y),x,y,t)-\Gamma^\delta(v(x),v(y),x,y,t)\Big)dy\Big|dx\\
\leq\ & 2\alpha_2C_p(\|u\|_\infty+\|v\|_\infty)^{p-1}\int_{\mathbb{R}^d}|u(x)-v(x)|dx\sup_{x\in\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}J(x-y,t)dy\Big|\\
+\ & 2\alpha_2\int_{\mathbb{R}^d}|v(x)|\int_{\mathbb{R}^d}L_\Lambda J(x-y,t)\Big(|u(y)-v(y)|+|u(x)-v(x)|\Big)dydx;
\end{aligned}$$

thus,

$$\begin{aligned}
\|\mathcal{L}_u^{t,\delta}u-\mathcal{L}_v^{t,\delta}v\|_1 \leq\ & 2\alpha_2\|u-v\|_1C_p(\|u\|_\infty+\|v\|_\infty)^{p-1}\int_{\mathbb{R}^d}J(x,t)dx\\
+\ & 2\alpha_2L_\Lambda\Big(\|v\|_\infty\int_{\mathbb{R}^d}J(x,t)dx+\|v\|_1\Big)\|u-v\|_1\\
\leq\ & M(p,\alpha_2,\Lambda)\|u-v\|_1.
\end{aligned}\tag{A.3}$$

This completes the proof of Lemma 3.1.

Appendix B

Lemma B.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\mathfrak{U} : \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}$ *have the time derivative* $\partial_t \mathfrak{U}$ *and increments* $\zeta_z(\xi, t, \omega) = \mathfrak{U}(z + \xi, t, w) - \mathfrak{U}(\xi, t, w)$. *Then,* $\mathbb{P}$*-a.s.,*

$$\lim_{R \to +\infty} R^{-(d+2)} \int_{Q_R} |\mathfrak{U}(x, 0)|^2 dx = 0 \text{ and } \lim_{R \to +\infty} R^{-(d+3)} \int_{\widetilde{Q}_R} |\mathfrak{U}(x, t)|^2 dx dt = 0.$$

That is, given $\mathfrak{U}^\varepsilon(x, t, \omega) = \varepsilon \mathfrak{U}(x/\varepsilon, t/\varepsilon, \omega)$, *for any fixed* $R > 0$, *we have*

$$\lim_{\varepsilon \to 0} \int_{Q_R} |\mathfrak{U}^\varepsilon(x, 0)|^2 dx = 0 \text{ and } \lim_{\varepsilon \to 0} \int_{\widetilde{Q}_R} |\mathfrak{U}^\varepsilon(x, t)|^2 dx dt = 0, \ \mathbb{P} - a.s.$$

Proof. The result is not surprising, as this reflects the sublinear growth property of the corrector. This is the property of the oscillatory function and it can be seen in [41, Lemma 4.1]. We can apply [42, Lemma A.2] and [43, Theorem 5.3] and use the classical nonlocal Poincaré's inequality for any $\omega \in \Omega_0 \subset \Omega$; the family $(\mathfrak{U}^\varepsilon)_{\varepsilon > 0}$ is relatively compact in $L^2_{loc}\left(\mathbb{R}^{d+1}\right)$, thus $\mathfrak{U}^{\varepsilon_k} \to 0$ in $L^2_{loc}\left(\mathbb{R}^{d+1}\right)$ as $k \to \infty$ and $\mathbb{P}$-a.s. Taking in expectation we have $\mathfrak{U}^\varepsilon \to 0$ as $\varepsilon \to 0$. $\mathfrak{U}^\varepsilon(\cdot, 0)$ also satisfies the assertion of [41, Lemma 4.1], so we omit it.

Appendix C

Proof of Lemma 9.3. The idea of the proof comes from [41, Proposition 4.5]; here, we mainly describe our ideas.

If $|\xi| > 3R$, then $\varphi\left(\frac{|\xi|}{R}\right) = 0$. Also $\varphi\left(\frac{|\xi+z|}{R}\right) = 0$ if $|\xi| > 3R$ and $|z| > R$. We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{|\xi| > 3R} J(z, t) \mu(\xi + z, \omega) \mu(\xi, \omega) \left| \bar{\sigma}_z \left(T_\xi \omega \right) \right| |\bar{\sigma}(\xi + z, \omega)| \varphi\left(\frac{|\xi + z|}{R} \right) d\xi dz \\
&\leq \alpha_2^2 \int_{|\eta| \leq 2R} \left(\int_{|z| > R} |z| J(z, t) \left| \bar{\sigma}_z \left(T_{\eta - z} \omega \right) \right| dz \right) \frac{1}{R} |\bar{\sigma}(\eta, \omega)| \varphi\left(\frac{|\eta|}{R} \right) d\eta \\
&\leq \alpha_2^2 \int_{|\eta| \leq 2R} \phi\left(T_\eta \omega \right) \frac{1}{R} |\bar{\sigma}(\eta, \omega)| \varphi\left(\frac{|\eta|}{R} \right) d\eta, \qquad \text{(C.1)}
\end{aligned}$$

where $\eta = \xi + z$ and

$$\phi\left(T_\eta \omega, t\right) = \int_{\mathbb{R}^d} |z| J(z, t) \left| \bar{\sigma}_z \left(T_{\eta - z} \omega \right) \right| dz. \qquad \text{(C.2)}$$

Due to the integrability of $\bar{\sigma}_z(\omega)$ with the weighted kernel function in probability, $\phi(\omega) \in L^2(\Omega)$, we have

$$\begin{aligned}
\mathbb{A}_1 &\leq \alpha_2^2 \int_{|\eta| \leq R} \phi\left(T_\eta \omega \right) \frac{|\bar{\sigma}(\eta, \omega)|}{R} d\eta + \alpha_2^2 \int_{R \leq |\eta| \leq 2R} \phi\left(T_\eta \omega \right) \frac{|\bar{\sigma}(\eta, \omega)|}{R} \varphi\left(\frac{|\eta|}{R} \right) d\eta \\
&\leq \phi_1(R) + \frac{C}{\alpha R |R'(t)|} \int_{|\eta| \leq 2R} \phi^2\left(T_\eta \omega \right) d\eta + \alpha |R'(t)| \int_{R \leq |\eta| \leq 2R} \frac{|\bar{\sigma}(\eta, \omega)|^2}{R} \varphi\left(\frac{|\eta|}{R} \right) d\eta
\end{aligned}$$

$$\leq \phi_1(R) + \frac{CR^{d-1}}{\alpha|R'(t)|} + \alpha|R'(t)| \int_{R\leq|\eta|\leq 2R} \frac{|\bar{\sigma}(\eta,\omega)|^2}{R} \varphi\left(\frac{|\eta|}{R}\right) d\eta. \tag{C.3}$$

Applying the Cauchy-Schwarz inequality, the boundedness of ϕ and the sublinear property of $\bar{\sigma}$ (Lemma 9.5 is proved) gives

$$\phi_1(R) \lesssim R^d \Big(\frac{1}{R^d} \int_{|\eta|\leq R} \phi^2 \left(T_\eta \omega\right) d\eta\Big)^{\frac{1}{2}} \Big(\frac{1}{R^d} \int_{|\eta|\leq R} (\frac{|\bar{\sigma}(\eta,\omega)|}{R})^2 d\eta\Big)^{\frac{1}{2}} \lesssim_R c_d R^d,$$

where c_d is sufficiently small.

The first term on the right-hand side of $\mathbb{A}_3$ satisfies

$$\int_{\mathbb{R}^d} \int_{\substack{2R\leq|\xi|\leq 3R, \\ |\xi|\leq R}} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\bar{\sigma}_z\bar{\sigma}(\xi,\omega)(\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R}))d\xi dz \lesssim_R c_s R^d,$$

where we use the estimate of I_2 in [41, Proposition 4.5] and the sublinear property of $\bar{\sigma}$; and, c_s is sufficiently small. Applying the inequality $\left|\varphi\big(\frac{|x|}{R}\big) - \varphi\big(\frac{|y|}{R}\big)\right| \leq \frac{|x-y|}{R}$, for $|\xi| \geq R$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{R\leq|\xi|\leq 2R} J(z,t)\mu(\xi+z,\omega)\mu(\xi,\omega)\bar{\sigma}_z\bar{\sigma}(\xi,\omega)\Big(\varphi\Big(\frac{|\xi+z|}{R}\Big) - \varphi\Big(\frac{|\xi|}{R}\Big)\Big)d\xi dz \\
&\leq \alpha_2^2 \int_{\mathbb{R}^d} \int_{R\leq|\xi|\leq 2R} J(z,t)\Big|\bar{\sigma}_z\Big(\xi,\omega\Big)\Big|\Big|\bar{\sigma}(\xi,\omega)\Big|\frac{|z|}{R} d\xi dz \\
&\leq \frac{C}{\alpha R|R'(t)|} \int_{|\eta|\leq 2R} \phi^2(T_\eta\omega)d\eta + \alpha|R'(t)| \int_{R\leq|\eta|\leq 2R} \frac{|\bar{\sigma}(\eta,\omega)|^2}{R} d\eta \\
&\leq \frac{CR^{d-1}}{\alpha|R'(t)|} + \alpha|R'(t)| \int_{R\leq|\eta|\leq 2R} \frac{|\bar{\sigma}(\eta,\omega)|^2|\eta|}{R} d\eta.
\end{aligned} \tag{C.4}$$

$\mathbb{A}_{2<} \leq \frac{C}{\sqrt{R(t)}} R(t)^d$ and $\mathbb{A}_{2>} \leq c_2 R(t)^d$ can be directly obtained in [41, Proposition 4.5]. This ends the proof of Lemma 9.3.

Proof of Lemma 10.1. For any $\varphi \in L^2((0,T)\times\mathbb{R}^d)$, we can find that

$$\begin{aligned}
J_2^\varepsilon(\varphi) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & J(z,\frac{t}{\varepsilon^2}) z\mu\Big(\frac{x}{\varepsilon},\omega\Big)\mu\Big(\frac{x}{\varepsilon}-z,\frac{t}{\varepsilon^2},\omega\Big)\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega\Big) \\
&\cdot\Big(\nabla\nabla v^0(x,t)\varphi(x,t) - \nabla\nabla v^0(x-\varepsilon z,t)\varphi(x-\varepsilon z,t)\Big)dxdz
\end{aligned} \tag{C.5}$$

is a bounded linear functional on $L^2(\mathbb{R}\times\mathbb{R}^d)$. Then, by the Riesz theorem for a.e. ω, there exists a function $\beta_\varepsilon \in L^2(\mathbb{R}\times\mathbb{R}^d)$ such that $\Upsilon^\varepsilon = (\beta_\varepsilon, \varphi)$:

$$(\partial_t - L^\varepsilon)v_3^\varepsilon(x,\omega) = -F^\varepsilon(x,t,\omega) = -F\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega\Big)\cdot\nabla\nabla v^0(x,t). \tag{C.6}$$

Since supp$v^0 \subset B$ is a bounded subset of $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} J_1(z)|z|\,|\zeta_{-z}(\omega)|\,dz \in L^2(\Omega)$, by the Birkhoff theorem $v_3^\varepsilon \in L^2((0,T)\times\mathbb{R}^d)$.

Our goal is to prove that $\|v_3^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. We first show that the family $\{v_3^\varepsilon\}$ is bounded in $L^2((0,T)\times\mathbb{R}^d)$. Denote

$$\mathfrak{G}_1^2 = \frac{1}{2\varepsilon^2} \int_0^T \int_{\mathbb{R}^{2d}} J(z,\frac{t}{\varepsilon^2})\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(v_2^\varepsilon(x-\varepsilon z,t) - v_2^\varepsilon(x,t))^2 dzdxdt,$$

$$\mathfrak{G}_2^2 = \int_0^T \int_{\mathbb{R}^d} \left(v_2^\varepsilon(x,t)\right)^2 dxdt.$$

We first give an important lemma.

Lemma C.1. *[41, Lemma 5.1] For v_2^ε in Eq (10.4) and $J_2^\varepsilon(\varphi)$ in Eq (C.5), we have*

$$J_2^\varepsilon(v_2^\varepsilon) \leq (\mathfrak{G}_1 + \mathfrak{G}_2) \cdot o(1) \text{ as } \varepsilon \to 0.$$

Lemma C.2. *For v_2^ε in Eq (10.4) and a.e. ω, we have that $\|v_2^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$.*

Proof. Multiply v_2^ε on both sides in Eq (10.4); then, we have that $((\partial_t - L^\varepsilon)v_2^\varepsilon, v_2^\varepsilon) = (-\Upsilon^\varepsilon, u_2^\varepsilon)$. Consider the second term on the left-hand side; we have

$$\begin{aligned}
&-\int_0^s \iint_{\mathbb{R}^{2d}} J(z, \frac{t}{\varepsilon^2})\mu(\frac{x}{\varepsilon}, \omega)\mu(\frac{x}{\varepsilon} - z, \omega)(v_2^\varepsilon(x - \varepsilon z, t) - v_2^\varepsilon(x,t))dz v_2^\varepsilon(x,t)dxdt \\
&= \frac{1}{2}\int_0^s \iint_{\mathbb{R}^{2d}} J(z, \frac{t}{\varepsilon^2})\mu(\frac{x}{\varepsilon}, \omega)\mu(\frac{x}{\varepsilon} - z, \omega)(v_2^\varepsilon(x - \varepsilon z, t) - v_2^\varepsilon(x,t))^2 dzdxdt.
\end{aligned}$$

For any $s \in (0, T)$, and by integrating over 0 to s,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d} v_2^2(x,s)dx - \int_0^s \int_{\mathbb{R}^d} \Upsilon^\varepsilon(x,t)v_2^\varepsilon(x,t)dxdt = \int_0^s (L^\varepsilon v_2^\varepsilon, v_2^\varepsilon)dt \leqslant 0, \\
&\mathfrak{G}_1^2 + \mathfrak{G}_2^2 \lesssim \|u_2^\varepsilon\|_\infty^2 + \mathfrak{G}_1^2 \lesssim (\mathfrak{G}_1 + \mathfrak{G}_2) \cdot o(1);
\end{aligned}$$

using the Gronwall inequality, we get that $\|v_2^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$.

We focus on $\{v_3^\varepsilon\}$ defined in Eq (10.5). Our goal is to prove that $\|v_3^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^d)} \to 0$. We first prove its compactness in L^2.

Lemma C.3. *$\{v_3^\varepsilon\}$ is compact and $v_3^\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^2((0,T)\times\mathbb{R}^d)$.*

Proof. See details in [41, Lemmas 5.3, 5.4].