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Research article

The numerical solutions for the nonhomogeneous Burgers' equation with the generalized Hopf-Cole transformation

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Abstract: In this paper, with the help of the generalized Hopf-Cole transformation, we first convert the nonhomogeneous Burgers' equation into an equivalent heat equation with the derivative boundary conditions, in which Neumann boundary conditions and Robin boundary conditions can be viewed as its special cases. For easy derivation and numerical analysis, the reduction order method is used to convert the problem into an equivalent first-order coupled system. Next, we establish a box scheme for this first-order system. By the technical energy analysis method, we obtain the prior estimate of the numerical solution for the box scheme. Furthermore, the solvability and convergence are obtained directly from the prior estimate. The extensive numerical examples are carried out, which verify the developed box scheme can achieve global second-order accuracy for both homogeneous and nonhomogeneous Burgers' equations.

Keywords: Nonhomogeneous Burgers' equation; Reduction order method; Difference scheme; Prior estimate; The generalized Hopf-Cole transformation

1. Introduction

Burgers' equation plays an important role in analyzing fluid turbulence since it has much in common with the Navier-Stokes equation. It was introduced by an English mathematician H. Bateman in 1915 [1] with its corresponding homogeneous boundary conditions as

$$\begin{cases} u_t + uu_x = \mu u_{xx}, \quad 0 < x < L,\ 0 < t \leqslant T, & (1.1a) \\ u(x,0) = \varphi(x), \quad 0 \leqslant x \leqslant L, & (1.1b) \\ u(0,t) = 0,\ u(L,t) = 0, \quad 0 < t \leqslant T, & (1.1c) \end{cases}$$

where $\mu > 0$ is the kinematic viscosity. A Dutch physicist J.M. Burgers explained the mathematical simulation of turbulence with the help of Eqs (1.1a)–(1.1c) in 1948 [2], which made this equation famous. In honor of his work, the equation is named omit the Burgers' equation. In order to efficiently solve Eqs (1.1a)–(1.1c), E. Hopf [3] and J.D. Cole [4] independently introduced a transformation to convert Eqs (1.1a)–(1.1c) into a heat equation with Neumann boundary condition, which made the exact solution explicitly for arbitrary initial conditions. The transformation was well-known as the Hopf-Cole transformation $u(x,t) = -2\mu\frac{w_x}{w}$, where $w(x,t)$ satisfied the following equation

$$\begin{cases} w_t = \mu w_{xx}, \quad 0 < x < L,\ 0 < t \leqslant T, \\ w(x,0) = \exp(-\dfrac{1}{2\mu}\displaystyle\int_0^x \varphi(s)ds), \quad 0 \leqslant x \leqslant L, \\ w_x(0,t) = 0, \quad w_x(L,t) = 0, \quad 0 \leqslant t \leqslant T. \end{cases}$$

During the last few decades, many significant efforts have been carried out oriented towards the robust numerical schemes for Burgers' equation, which forms a benchmark problem in parallel and distributed computation for the partial differential equations solvers [5,6]. Among the various solvers, there are analytical methods involving classical Hopf-Cole transformation [3,4]. There are also many numerical solvers involving finite difference methods [7–11], finite element methods [12,13], spectral methods [14–16] and classification to name a few. A nice and systematic literature for Burgers' equation is referred to in the recent review in [17].

In the paper, based on the Hopf-Cole transformation, we devote to designing an effective numerical method for the Burgers' equation with the nonhomogeneous source function and nonhomogeneous boundary conditions [18] as

$$\begin{cases} u_t + uu_x = \mu u_{xx} + f(x,t), \quad 0 < x < L,\ 0 < t \leqslant T, & (1.3a) \\ u(x,0) = \varphi(x), \quad 0 \leqslant x \leqslant L, & (1.3b) \\ u(0,t) = \alpha(t), \quad u(L,t) = \beta(t), \quad 0 < t \leqslant T, & (1.3c) \end{cases}$$

where $\varphi(x)$, $\alpha(t)$ and $\beta(t)$ are arbitrary given smooth functions. $f(x,t)$ is the trivial source function. In order to satisfy the compatibility condition, we require $\varphi(0) = \alpha(0)$ and $\varphi(L) = \beta(0)$.

The fundamental difficulty lies in analyzing the prior estimate for solving problem Eqs (1.3a)–(1.3c) since the analysis of problems with the derivative boundary conditions are totally different from that with the Robin boundary conditions. In this paper, the emerging difficulties are overcome via the help of the reduction order method and the principle of boundary homogeneity for the transformed problem.

The main novelty of this paper aims at we convert Eqs (1.3a)–(1.3c) into an equivalent heat equation with the derivative boundary conditions for the first time, in which Neumann boundary conditions and Robin boundary conditions can be viewed as special cases. The concrete contributions are listed as follows

- The generalized exponential transformation links the classic Hopf-Cole transformation to the exponential transformation for the constant convection term or variable convection term. In fact, the classic Hopf-Cole transformation can be viewed as the special case of the generalized exponential transformation, see the formula Eq (2.1) in Section 2. When the first u in nonlinear convection term uu_x is considered as a constant coefficient, the application of the generalized exponential transformation is referred to [19, 20]. And when it is viewed as a variable coefficient, its application is referred to [21,22].
- We strictly show that the Burgers' equation with nonhomogeneous boundary conditions and nonhomogeneous right-hand side term is equivalent to a heat equation with the derivative boundary conditions.

- An equivalent box scheme is offered for easy implementation. The solvability and convergence of the established box scheme are analyzed in detail by the technical energy argument.
- Numerical errors and the convergence orders for the homogeneous and nonhomogeneous problems are displayed and verify the effectiveness of the proposed box scheme.
- Compared with other papers on the Burgers equation [23, 24], the main difference or advantage of the method presented in this paper is that we deal with the non-homogeneous boundary problem, while most other methods are only suitable for solving burgers equation with homogeneous boundary.

The rest of the paper is arranged as follows. Section 2 presents the equivalent form for the nonhomogeneous Burgers' equation based on the generalized Hopf-Cole transformation. Section 3 is the main part of the paper, which focuses on the analysis and derivation of the difference scheme. More concretely, it involves some useful notations and lemmas, the reduced order method, a priori estimate of the difference scheme, solvability and convergence. The numerical experiments are carried out in Section 4, followed by some conclusions in Section 5.

To facilitate numerical analysis in what follows, we suppose there exists a constant c_0 such that

$$|\alpha(t)| \leqslant c_0, \quad |\beta(t)| \leqslant c_0, \quad |f(x,t)| \leqslant c_0 \tag{1.4}$$

throughout the whole paper, .

2. An equivalent form

Introducing the generalized exponential transformation

$$w(x,t) = \exp\left(-\frac{1}{2\mu}\int_0^x u(s,t)ds\right), \tag{2.1}$$

and taking the derivative of both sides with respect to x in Eq (2.1), then noting the boundary conditions Eq (1.1c), we have the classical Hopf-Cole transformation [3]

$$u(x,t) = -2\mu\frac{w_x(x,t)}{w(x,t)}. \tag{2.2}$$

Via the help of Eq (2.2), we have

$$\begin{cases} u_t = -2\mu\left(\frac{w_x}{w}\right)_t = -2\mu\frac{w_{xt}w - w_x w_t}{w^2} = -2\mu\left(\frac{w_t}{w}\right)_x, & (2.3a)\\ u_x = -2\mu\left(\frac{w_x}{w}\right)_x, & (2.3b)\\ u_{xx} = -2\mu\left(\frac{w_x}{w}\right)_{xx}. & (2.3c)\end{cases}$$

Substituting Eq (2.2) and Eq (2.3) into Eq (1.3a), we have

$$\left(\frac{w_t}{w}\right)_x - \mu\left[\left(\frac{w_x}{w}\right)^2\right]_x = \mu\left(\frac{w_x}{w}\right)_{xx} - \frac{1}{2\mu}f(x,t),$$

or

$$\left[\frac{w_t}{w}-\mu\left(\frac{w_x}{w}\right)^2-\mu\left(\frac{w_x}{w}\right)_x\right]_x=-\frac{1}{2\mu}f(x,t).$$

Furthermore, we have

$$\left(\frac{w_t-\mu w_{xx}}{w}\right)_x=-\frac{1}{2\mu}f(x,t). \tag{2.4}$$

Integrating for x from 0 to x on both sides of Eq (2.4), we have

$$w_t-q(t)w=\mu w_{xx}+F(x,t)w, \tag{2.5}$$

where

$$q(t)=\frac{w_t(0,t)-\mu w_{xx}(0,t)}{w(0,t)},\qquad F(x,t)=-\frac{1}{2\mu}\int_0^x f(s,t)ds.$$

Multiplying Eq (2.5) by $\exp(-\int_0^t q(s)ds)$ on both sides, we have

$$\left(\exp\Big(-\int_0^t q(s)ds\Big)w\right)_t=\exp\Big(-\int_0^t q(s)ds\Big)\cdot(\mu w_{xx}+F(x,t)w).$$

Let

$$\widetilde{w}(x,t)=w(x,t)\exp\Big(-\int_0^t q(s)ds\Big).$$

Then, we have

$$-2\mu\frac{\widetilde{w}_x}{\widetilde{w}}=-2\mu\frac{w_x}{w}=u(x,t). \tag{2.6}$$

In other words, for the arbitrary $q(t)$, $u(x,t)$ is independent of $q(t)$. Thus, we take $q(t)=0$ for brevity. Meanwhile, Eq (2.5) is simplified as

$$w_t=\mu w_{xx}+F(x,t)w,\quad 0<x<L,\quad 0<t\leqslant T. \tag{2.7}$$

By Eq (2.1), we obtain the initial condition

$$\begin{aligned}w(x,0)&=\exp\left(-\frac{1}{2\mu}\int_0^x u(s,0)ds\right)\\&=\exp\left(-\frac{1}{2\mu}\int_0^x \varphi(s)ds\right)\\&=:\widetilde{\varphi}(x),\quad 0\leqslant x\leqslant L.\end{aligned} \tag{2.8}$$

Noticing Eq (2.6), we have

$$-2\mu\frac{w_x(0,t)}{w(0,t)}=u(0,t)=\alpha(t),\quad 0\leqslant t\leqslant T.$$

Thus, the left boundary condition reads

$$2\mu w_x(0,t) + \alpha(t)w(0,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{2.9}$$

Similarly, we have the right boundary condition

$$2\mu w_x(L,t) + \beta(t)w(L,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{2.10}$$

Above procedures are invertible, thus Eqs (1.3a)–(1.3c) is equivalent to Eqs (2.7)–(2.10).

Remark 1. *We make some explanations about the exponential transformation Eq* (2.1) *and the boundary conditions Eqs* (2.9)–(2.10).

(*a*) *The classical Hopf-Cole transformation Eq* (2.2) *can be viewed as a special case of Eq* (2.1). *The first* $u(x,t)$ *of* uu_x *in Eq* (1.3a) *is supposed to be a "ghost constant coefficient". In this hypothetical situation, the constant coefficient convection term and variable coefficient convection term (e.g.,* [19,22]*) are diminished by the generalized exponential transformation Eq* (2.1).

(*b*) *The boundary conditions Eq* (2.9) *and Eq* (2.10) *are called derivative boundary conditions. To assure that solution of Eqs* (2.7)–(2.10) *is stable and unique, one usually requires* $\alpha(t) \leqslant 0$, $\beta(t) \geqslant 0$. *Under the above constraint, Eq* (2.9) *and Eq* (2.10) *are called Neumann boundary conditions if* $\alpha(t)+\beta(t) \equiv 0$ *and Robin boundary conditions if* $\alpha(t)+\beta(t) \not\equiv 0$*, see e.g.,* [25]. *The arbitrariness of* $\alpha(t)$ *and* $\beta(t)$ *makes the numerical analysis of the problem with derivative boundary conditins difficult compared with that with Robin boundary conditions. In present paper, we will analyze the general cases .*

(*c*) *The right-hand side function* $F(x,t)$ *in Eq* (2.7) *can be computed by Simpson formula numerically when it can not be expressed explicitly by the elementary functions.*

3. The derivation and analysis of the difference scheme

3.1. Notations and lemmas

Before introducing the finite difference scheme, we divide the domain $[0,L] \times [0,T]$. Take positive integers M and N and let $h = L/M$, $\tau = T/N$. Denote $x_i = ih$, $0 \leqslant i \leqslant M$; $t_k = k\tau$, $0 \leqslant k \leqslant N$; $\Omega_h = \{x_i \,|\, 0 \leqslant i \leqslant M\}$, $\Omega_\tau = \{t_k \,|\, 0 \leqslant k \leqslant N\}$, $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. For any grid function $v = \{u_i^k \,|\, 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N\}$ defined on $\Omega_{h\tau}$, we denote

$$u_i^{k-\frac{1}{2}} = \frac{1}{2}(u_i^k + u_i^{k-1}), \qquad u_{i-\frac{1}{2}}^k = \frac{1}{2}(u_i^k + u_{i-1}^k), \qquad \delta_t u_i^{k-\frac{1}{2}} = \frac{1}{\tau}(u_i^k - u_i^{k-1}),$$

$$\delta_x u_{i-\frac{1}{2}}^k = \frac{1}{h}(u_i^k - u_{i-1}^k), \qquad \delta_x^2 u_i^{k-\frac{1}{2}} = \frac{1}{h}(\delta_x u_{i+\frac{1}{2}}^{k-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{k-\frac{1}{2}}),$$

$$\|u\| = \sqrt{h\sum_{i=1}^{M}(u_{i-\frac{1}{2}}^k)^2}, \qquad \|\delta_x u\| = \sqrt{h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}}^k)^2}, \qquad \|u\|_\infty = \max_{0\leqslant i\leqslant M}|u_i^n|.$$

The following two lemmas come from [25].

Lemma 1. *Let* $\{F^k\}_{i=1}^\infty$ *and* $\{g^k\}_{i=1}^\infty$ *be two non-negative sequences and satisfy*

$$F^{k+1} \leqslant (1 + c\tau)F^k + \tau g^k, \quad k = 0, 1, 2, \cdots,$$

then

$$F^k \leqslant \exp(ck\tau)\left(F^0 + \tau\sum_{l=0}^{k-1} g^l\right), \quad k = 0, 1, 2, \cdots .$$

Lemma 2. *Let $u = (u_0, u_1, \cdots, u_M)$ be a discrete function on Ω_h, then for any $\varepsilon > 0$, we have*

$$\|u\|_\infty^2 \leqslant 2\left(1 + \frac{1}{\varepsilon}\right)\|u\|^2 + \left(2\varepsilon + \frac{h}{2}\right)\|\delta_x u\|^2.$$

3.2. The reduced order method for the difference scheme

In the previous section, we have converted Eqs (1.3a)–(1.3c) into an equivalent heat conduction equation with the derivative boundary conditions by the generalized Hopf-Cole transformation. Then, in this section, we will use the reduction order method to derive the numerical scheme of an equivalent problem.

Let

$$v = \mu w_x - \frac{1}{2L}[(x - L)\alpha(t) - x\beta(t)]w,$$

and denote

$$\begin{cases} \gamma(x,t) = \dfrac{1}{2\mu L}[(x - L)\alpha(t) - x\beta(t)], & (3.1a)\\ \theta(x,t) = \dfrac{1}{2L}(\alpha(t) - \beta(t)) + \mu\gamma^2(x,t) + F(x,t). & (3.1b)\end{cases}$$

Then Eqs (2.7)–(2.10) are equivalent to

$$\begin{cases} w_t = v_x + \gamma(x,t)v + \theta(x,t)w, \quad 0 < x < L,\ 0 < t \leqslant T, & (3.2a)\\ \dfrac{v}{\mu} = w_x - \gamma(x,t)w, \quad 0 < x < L,\ 0 < t \leqslant T, & (3.2b)\\ w(x,0) = \widetilde{\varphi}(x), \quad 0 \leqslant x \leqslant L, & (3.2c)\\ v(0,t) = 0, \quad v(L,t) = 0, \quad 0 \leqslant t \leqslant T. & (3.2d)\end{cases}$$

Defining the grid functions on $\Omega_{h\tau}$ as

$$W_i^k = w(x_i, t_k), \quad V_i^k = v(x_i, t_k), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N.$$

Considering Eq (3.2a) at the point $(x_{i-\frac{1}{2}}, t_{k-\frac{1}{2}})$ and Eq (3.2b) at the point $(x_{i-\frac{1}{2}}, t_k)$, and with the help of the Taylor expansion, we have

$$\begin{cases} \delta_t W_{i-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x V_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}} V_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} W_{i-\frac{1}{2}}^{k-\frac{1}{2}} + (r_1)_{i-\frac{1}{2}}^{k-\frac{1}{2}}, & \\ \qquad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, & (3.3a)\\ \dfrac{1}{\mu}V_{i-\frac{1}{2}}^k = \delta_x W_{i-\frac{1}{2}}^k - \gamma_{i-\frac{1}{2}}^k W_{i-\frac{1}{2}}^k + (r_2)_{i-\frac{1}{2}}^k, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, & (3.3b)\\ W_i^0 = \widetilde{\varphi}(x_i), \quad 0 \leqslant i \leqslant M, & (3.3c)\\ V_0^k = 0, \quad V_M^k = 0, \quad 0 \leqslant k \leqslant N, & (3.3d)\end{cases}$$

where $\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}} = \gamma(x_{i-\frac{1}{2}}, t_{k-\frac{1}{2}})$ and $\theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} = \theta(x_{i-\frac{1}{2}}, t_{k-\frac{1}{2}})$, and there exists a constant c_1 such that the local truncation errors satisfy

$$\begin{cases} \left|(r_1)_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right| \leqslant c_1(\tau^2 + h^2), \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, & (3.4a) \\ \left|(r_2)_{i-\frac{1}{2}}^{k}\right| \leqslant c_1 h^2, \quad 1 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N. & (3.4b) \end{cases}$$

Omitting the small terms in Eq (3.3a) and Eq (3.3b), a box scheme for Eqs (3.2a)–(3.2d) reads

$$\begin{cases} \delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x v_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}} v_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} w_{i-\frac{1}{2}}^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, & (3.5a) \\ \dfrac{1}{\mu} v_{i-\frac{1}{2}}^{k} = \delta_x w_{i-\frac{1}{2}}^{k} - \gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k}, \quad 1 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N, & (3.5b) \\ w_i^0 = \widetilde{\varphi}(x_i), \quad 0 \leqslant i \leqslant M, & (3.5c) \\ v_0^k = 0, \quad v_M^k = 0, \quad 1 \leqslant k \leqslant N. & (3.5d) \end{cases}$$

According to Eq (2.2), we have

$$u(x_{i-\frac{1}{2}}, t_k) = -2\mu \frac{\delta_x W_{i-\frac{1}{2}}^{k}}{W_{i-\frac{1}{2}}^{k}} + (r_3)_{i-\frac{1}{2}}^{k}, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N. \tag{3.6}$$

There exists a constant c_3 such that

$$|(r_3)_{i-\frac{1}{2}}^{k}| \leqslant c_3 h^2, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N.$$

Let

$$\widehat{u}_{i-\frac{1}{2}}^{k} = -2\mu \frac{\delta_x w_{i-\frac{1}{2}}^{k}}{w_{i-\frac{1}{2}}^{k}}, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N. \tag{3.7}$$

We can view $\widehat{u}_{i-\frac{1}{2}}^{k}$ as the second-order numerical approximation of $u(x_{i-\frac{1}{2}}, t_k)$ according to Eq (3.6).

Theorem 1. *The difference scheme Eqs* (3.5a)–(3.5d) *is equivalent to*

$$\begin{aligned} \delta_t w_{\frac{1}{2}}^{k-\frac{1}{2}} = {}& \frac{2\mu}{h}\left[\delta_x w_{\frac{1}{2}}^{k-\frac{1}{2}} - \frac{1}{2}\left(\gamma_{\frac{1}{2}}^{k} w_{\frac{1}{2}}^{k} + \gamma_{\frac{1}{2}}^{k-1} w_{\frac{1}{2}}^{k-1}\right)\right] \\ & + \mu\gamma_{\frac{1}{2}}^{k-\frac{1}{2}} \delta_x w_{\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\gamma_{\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{\frac{1}{2}}^{k} w_{\frac{1}{2}}^{k} + \gamma_{\frac{1}{2}}^{k-1} w_{\frac{1}{2}}^{k-1}\right) \\ & + \theta_{\frac{1}{2}}^{k-\frac{1}{2}} w_{\frac{1}{2}}^{k-\frac{1}{2}}, \quad 1 \leqslant k \leqslant N, \\ \frac{1}{2}\left(\delta_t w_{i+\frac{1}{2}}^{k-\frac{1}{2}} + \delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) = {}& \mu\delta_x^2 w_i^{k-\frac{1}{2}} \\ & - \frac{\mu}{2h}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1} - \gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} - \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) \\ & + \frac{\mu}{2}\left(\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}} \delta_x w_{i+\frac{1}{2}}^{k-\frac{1}{2}} + \gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}} \delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&- \frac{\mu}{4}\left[\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1}\right) + \gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right)\right] \\
&+ \frac{1}{2}\left(\theta_{i+\frac{1}{2}}^{k-\frac{1}{2}} w_{i+\frac{1}{2}}^{k-\frac{1}{2}} + \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} w_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right), \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N, \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
\delta_t w_{M-\frac{1}{2}}^{k-\frac{1}{2}} = &\frac{2\mu}{h}\left[-\delta_x w_{M-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{1}{2}\left(\gamma_{M-\frac{1}{2}}^{k} w_{M-\frac{1}{2}}^{k} + \gamma_{M-\frac{1}{2}}^{k-1} w_{\frac{1}{2}}^{k-1}\right)\right] \\
&+ \mu\gamma_{M-\frac{1}{2}}^{k-\frac{1}{2}} \delta_x w_{M-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\gamma_{M-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{M-\frac{1}{2}}^{k} w_{M-\frac{1}{2}}^{k} + \gamma_{M-\frac{1}{2}}^{k-1} w_{M-\frac{1}{2}}^{k-1}\right) \\
&+ \theta_{M-\frac{1}{2}}^{k-\frac{1}{2}} w_{M-\frac{1}{2}}^{k-\frac{1}{2}}, \quad 1 \leqslant k \leqslant N, \qquad (3.10)
\end{aligned}
$$

$$
w_i^0 = \widetilde{\varphi}(x_i), \quad 0 \leqslant i \leqslant M, \qquad (3.11)
$$

and

$$
v_{i-\frac{1}{2}}^{0} = \mu\delta_x w_{i-\frac{1}{2}}^{0} - \mu\gamma_{i-\frac{1}{2}}^{0} w_{i-\frac{1}{2}}^{0}, \quad 1 \leqslant i \leqslant M,
$$

$$
\begin{aligned}
v_i^{k-\frac{1}{2}} = &\mu\delta_x w_{i+\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1}\right) \\
&- \frac{h}{2}\left[\delta_t w_{i+\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}} \delta_x w_{i+\frac{1}{2}}^{k-\frac{1}{2}}\right. \\
&\left.+\frac{\mu}{2}\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1}\right) - \theta_{i+\frac{1}{2}}^{k-\frac{1}{2}} w_{i+\frac{1}{2}}^{k-\frac{1}{2}}\right], \\
&\qquad 0 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N.
\end{aligned}
$$

$$
\begin{aligned}
v_M^{k-\frac{1}{2}} = &\mu\delta_x w_{M-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{M-\frac{1}{2}}^{k} w_{M-\frac{1}{2}}^{k} + \gamma_{M-\frac{1}{2}}^{k-1} w_{M-\frac{1}{2}}^{k-1}\right) \\
&+ \frac{h}{2}\left[\delta_t w_{M-\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{M-\frac{1}{2}}^{k-\frac{1}{2}} \delta_x w_{M-\frac{1}{2}}^{k-\frac{1}{2}}\right. \\
&\left.+\frac{\mu}{2}\gamma_{M-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{M-\frac{1}{2}}^{k} w_{M-\frac{1}{2}}^{k} + \gamma_{M-\frac{1}{2}}^{k-1} w_{M-\frac{1}{2}}^{k-1}\right) - \theta_{M-\frac{1}{2}}^{k-\frac{1}{2}} w_{M-\frac{1}{2}}^{k-\frac{1}{2}}\right], \\
&\qquad 1 \leqslant k \leqslant N.
\end{aligned}
$$

Proof. First, we know that Eq (3.5b) is equivalent to

$$
v_{i-\frac{1}{2}}^{0} = \mu\delta_x w_{i-\frac{1}{2}}^{0} - \mu\gamma_{i-\frac{1}{2}}^{0} w_{i-\frac{1}{2}}^{0}, \quad 1 \leqslant i \leqslant M, \qquad (3.12)
$$

$$
v_{i-\frac{1}{2}}^{k-\frac{1}{2}} = \mu\delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right), \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N. \qquad (3.13)
$$

Substituting Eq (3.13) into Eq (3.5a), we obtain

$$
\begin{aligned}
\delta_x v_{i-\frac{1}{2}}^{k-\frac{1}{2}} &= \delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}} v_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} w_{i-\frac{1}{2}}^{k-\frac{1}{2}} \\
&= \delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}} \delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} \\
&\quad + \frac{\mu}{2}\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) - \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} w_{i-\frac{1}{2}}^{k-\frac{1}{2}},
\end{aligned}
$$

$$1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N. \tag{3.14}$$

Multiplying Eq (3.14) by $\frac{h}{2}$ and adding the result with Eq (3.13), we obtain

$$\begin{aligned} v_i^{k-\frac{1}{2}} &= \mu\delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) \\ &+ \frac{h}{2}\Big[\delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} \\ &+ \frac{\mu}{2}\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) - \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} w_{i-\frac{1}{2}}^{k-\frac{1}{2}}\Big], \\ &1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N. \end{aligned} \tag{3.15}$$

Then, multiplying Eq (3.14) by $\frac{h}{2}$ and subtracting the result with Eq (3.13), we obtain

$$\begin{aligned} v_{i-1}^{k-\frac{1}{2}} &= \mu\delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) \\ &- \frac{h}{2}\Big[\delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} \\ &+ \frac{\mu}{2}\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) - \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} w_{i-\frac{1}{2}}^{k-\frac{1}{2}}\Big], \\ &1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N. \end{aligned} \tag{3.16}$$

Or equivalently,

$$\begin{aligned} v_i^{k-\frac{1}{2}} &= \mu\delta_x w_{i+\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1}\right) \\ &- \frac{h}{2}\Big[\delta_t w_{i+\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}}\delta_x w_{i+\frac{1}{2}}^{k-\frac{1}{2}} \\ &+ \frac{\mu}{2}\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1}\right) - \theta_{i+\frac{1}{2}}^{k-\frac{1}{2}} w_{i+\frac{1}{2}}^{k-\frac{1}{2}}\Big], \\ &0 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N. \end{aligned} \tag{3.17}$$

It follows from Eq (3.15) and Eq (3.17) with $1 \leqslant i \leqslant M-1$, we get

$$\begin{aligned} &\mu\delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) \\ &\quad + \frac{h}{2}\left[\delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\delta_x w_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \frac{\mu}{2}\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k} w_{i-\frac{1}{2}}^{k} + \gamma_{i-\frac{1}{2}}^{k-1} w_{i-\frac{1}{2}}^{k-1}\right) - \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} w_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right] \\ &= \mu\delta_x w_{i+\frac{1}{2}}^{k-\frac{1}{2}} - \frac{\mu}{2}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1}\right) \\ &\quad - \frac{h}{2}\left[\delta_t w_{i+\frac{1}{2}}^{k-\frac{1}{2}} - \mu\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}}\delta_x w_{i+\frac{1}{2}}^{k-\frac{1}{2}} + \frac{\mu}{2}\gamma_{i+\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i+\frac{1}{2}}^{k} w_{i+\frac{1}{2}}^{k} + \gamma_{i+\frac{1}{2}}^{k-1} w_{i+\frac{1}{2}}^{k-1}\right) - \theta_{i+\frac{1}{2}}^{k-\frac{1}{2}} w_{i+\frac{1}{2}}^{k-\frac{1}{2}}\right] \\ &\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N. \end{aligned}$$

That is

$$\frac{1}{2}\left(\delta_t w_{i+\frac{1}{2}}^{k-\frac{1}{2}} + \delta_t w_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) = \mu\delta_x^2 w_i^{k-\frac{1}{2}}$$

$$
\begin{aligned}
&-\frac{\mu}{2h}\left(\gamma_{i+\frac12}^{k}w_{i+\frac12}^{k}+\gamma_{i+\frac12}^{k-1}w_{i+\frac12}^{k-1}-\gamma_{i-\frac12}^{k}w_{i-\frac12}^{k}-\gamma_{i-\frac12}^{k-1}w_{i-\frac12}^{k-1}\right)\\
&+\frac{\mu}{2}\left(\gamma_{i+\frac12}^{k-\frac12}\delta_x w_{i+\frac12}^{k-\frac12}+\gamma_{i-\frac12}^{k-\frac12}\delta_x w_{i-\frac12}^{k-\frac12}\right)\\
&-\frac{\mu}{4}\left[\gamma_{i+\frac12}^{k-\frac12}\left(\gamma_{i+\frac12}^{k}w_{i+\frac12}^{k}+\gamma_{i+\frac12}^{k-1}w_{i+\frac12}^{k-1}\right)+\gamma_{i-\frac12}^{k-\frac12}\left(\gamma_{i-\frac12}^{k}w_{i-\frac12}^{k}+\gamma_{i-\frac12}^{k-1}w_{i-\frac12}^{k-1}\right)\right]\\
&+\frac{1}{2}\left(\theta_{i+\frac12}^{k-\frac12}w_{i+\frac12}^{k-\frac12}+\theta_{i-\frac12}^{k-\frac12}w_{i-\frac12}^{k-\frac12}\right),\quad 1\leqslant i\leqslant M-1,\ 1\leqslant k\leqslant N.
\end{aligned}
$$

In Eq (3.17), for $i=0$, Eq (3.5d) is equivalent to

$$
\begin{aligned}
\delta_t w_{\frac12}^{k-\frac12}=&\frac{2}{h}\left[\mu\delta_x w_{\frac12}^{k-\frac12}-\frac{\mu}{2}\left(\gamma_{\frac12}^{k}w_{\frac12}^{k}+\gamma_{\frac12}^{k-1}w_{\frac12}^{k-1}\right)\right]\\
&+\mu\gamma_{\frac12}^{k-\frac12}\delta_x w_{\frac12}^{k-\frac12}-\frac{\mu}{2}\gamma_{\frac12}^{k-\frac12}\left(\gamma_{\frac12}^{k}w_{\frac12}^{k}+\gamma_{\frac12}^{k-1}w_{\frac12}^{k-1}\right)\\
&+\theta_{\frac12}^{k-\frac12}w_{\frac12}^{k-\frac12},\quad 1\leqslant k\leqslant N.
\end{aligned}
$$

In Eq (3.15), for $i=M$, Eq (3.5d) is equivalent to

$$
\begin{aligned}
\delta_t w_{M-\frac12}^{k-\frac12}=&\frac{2}{h}\left[-\mu\delta_x w_{M-\frac12}^{k-\frac12}+\frac{\mu}{2}\left(\gamma_{M-\frac12}^{k}w_{M-\frac12}^{k}+\gamma_{M-\frac12}^{k-1}w_{M-\frac12}^{k-1}\right)\right]\\
&+\mu\gamma_{M-\frac12}^{k-\frac12}\delta_x w_{M-\frac12}^{k-\frac12}-\frac{\mu}{2}\gamma_{M-\frac12}^{k-\frac12}\left(\gamma_{M-\frac12}^{k}w_{M-\frac12}^{k}+\gamma_{M-\frac12}^{k-1}w_{M-\frac12}^{k-1}\right)\\
&+\theta_{M-\frac12}^{k-\frac12}w_{M-\frac12}^{k-\frac12},\quad 1\leqslant k\leqslant N.
\end{aligned}
$$

This completes the proof. □

3.3. A prior estimate for the difference scheme

In the following theorem, we use the energy method to give a prior estimate for the difference scheme Eqs (3.5a)–(3.5d).

Theorem 2. *Let $\{\tilde{w}_i^k,\tilde{v}_i^k \mid 0\leqslant i\leqslant M, 1\leqslant k\leqslant N\}$ be the solution of*

$$
\begin{cases}
\delta_t\tilde{w}_{i-\frac12}^{k-\frac12}=\delta_x\tilde{v}_{i-\frac12}^{k-\frac12}+\gamma_{i-\frac12}^{k-\frac12}\tilde{v}_{i-\frac12}^{k-\frac12}+\theta_{i-\frac12}^{k-\frac12}\tilde{w}_{i-\frac12}^{k-\frac12}+S_{i-\frac12}^{k-\frac12},\quad 1\leqslant i\leqslant M,\ 1\leqslant k\leqslant N, & (3.18a)\\
\frac{1}{\mu}\tilde{v}_{i-\frac12}^{k}=\delta_x\tilde{w}_{i-\frac12}^{k}-\gamma_{i-\frac12}^{k}\tilde{w}_{i-\frac12}^{k}+T_{i-\frac12}^{k},\quad 1\leqslant i\leqslant M,\ 0\leqslant k\leqslant N, & (3.18b)\\
\tilde{v}_0^k=0,\quad \tilde{v}_M^k=0,\quad 0\leqslant k\leqslant N, & (3.18c)\\
\tilde{w}_i^0=\widehat{\varphi}(x_i),\quad 0\leqslant i\leqslant M. & (3.18d)
\end{cases}
$$

Then we have

$$
G^k\leqslant e^{3c_2T}\left(\|\widehat{\varphi}\|^2+\frac{1}{\mu}\|\tilde{v}^0\|^2+\frac{3}{2}\tau\sum_{l=0}^{k-1}Q^l\right),\quad 1\leqslant k\leqslant N,
$$

where

$$\begin{cases} G^k = \|\tilde{w}^k\|^2 + \dfrac{1}{\mu}\|\tilde{v}^k\|^2, & (3.19a)\\ S^{k-\frac{1}{2}}_{i-\frac{1}{2}} = \dfrac{1}{2}\left(S^k_{i-\frac{1}{2}} + S^{k-1}_{i-\frac{1}{2}}\right), & (3.19b)\\ \delta_t T^{k-\frac{1}{2}}_{i-\frac{1}{2}} = \dfrac{1}{\tau}\left(T^k_{i-\frac{1}{2}} - T^{k-1}_{i-\frac{1}{2}}\right), & (3.19c)\\ Q^k = 3\|S^{k-\frac{1}{2}}\|^2 + 4\mu\|T^{k-\frac{1}{2}}\|^2 + \|\delta_t T^{k-\frac{1}{2}}\|^2. & (3.19d) \end{cases}$$

Proof. **Step 1**: Averaging Eq (3.18b) and Eq (3.18c) with superscripts k and $k-1$, we obtain

$$\frac{1}{\mu}\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}} = \delta_x \tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}} - \left(\gamma^k_{i-\frac{1}{2}}\tilde{w}^k_{i-\frac{1}{2}} + \gamma^{k-1}_{i-\frac{1}{2}}\tilde{w}^{k-1}_{i-\frac{1}{2}}\right) + T^{k-\frac{1}{2}}_{i-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, \tag{3.20}$$

$$\tilde{v}^{k-\frac{1}{2}}_0 = 0, \quad \tilde{v}^{k-\frac{1}{2}}_M = 0, \quad 1 \leqslant k \leqslant N. \tag{3.21}$$

Multiplying Eq (3.18a) by $2\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}$, Eq (3.20) by $2\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}$ and adding the results, we have

$$\begin{aligned} &\frac{1}{\tau}\Big[\left(\tilde{w}^k_{i-\frac{1}{2}}\right)^2 - \left(\tilde{w}^{k-1}_{i-\frac{1}{2}}\right)^2\Big] + \frac{2}{\mu}\left(\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right)^2 \\ =&2\left(\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\delta_x\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}} + \tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\delta_x\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right) + 2\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\left(\gamma^{k-\frac{1}{2}}_{i-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}} + \theta^{k-\frac{1}{2}}_{i-\frac{1}{2}}\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right) \\ &- 2\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\left(\gamma^k_{i-\frac{1}{2}}\tilde{w}^k_{i-\frac{1}{2}} + \gamma^{k-1}_{i-\frac{1}{2}}\tilde{w}^{k-1}_{i-\frac{1}{2}}\right) + 2\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}S^{k-\frac{1}{2}}_{i-\frac{1}{2}} + 2\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}T^{k-\frac{1}{2}}_{i-\frac{1}{2}} \\ \leqslant&\frac{2}{h}\left(\tilde{v}^{k-\frac{1}{2}}_i\tilde{w}^{k-\frac{1}{2}}_i - \tilde{v}^{k-\frac{1}{2}}_{i-1}\tilde{w}^{k-\frac{1}{2}}_{i-1}\right) + 2c_2\left|\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right| + 2c_2\left|\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 \\ &+ 2c_2\left|\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|\left(\left|\tilde{w}^k_{i-\frac{1}{2}}\right| + \left|\tilde{w}^{k-1}_{i-\frac{1}{2}}\right|\right) + 2\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}S^{k-\frac{1}{2}}_{i-\frac{1}{2}} + 2\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}T^{k-\frac{1}{2}}_{i-\frac{1}{2}} \\ \leqslant&\frac{2}{h}\left(\tilde{v}^{k-\frac{1}{2}}_i\tilde{w}^{k-\frac{1}{2}}_i - \tilde{v}^{k-\frac{1}{2}}_{i-1}\tilde{w}^{k-\frac{1}{2}}_{i-1}\right) + \frac{1}{4\mu}\left|\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 + 4\mu c_2^2\left|\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 + 2c_2\left|\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 \\ &+ \frac{1}{2\mu}\left|\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 + 4\mu c_2^2\left(\left|\tilde{w}^k_{i-\frac{1}{2}}\right|^2 + \left|\tilde{w}^{k-1}_{i-\frac{1}{2}}\right|^2\right) + \left|\tilde{w}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 + \left|S^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 \\ &+ \frac{1}{4\mu}\left|\tilde{v}^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2 + 4\mu\left|T^{k-\frac{1}{2}}_{i-\frac{1}{2}}\right|^2, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, \end{aligned} \tag{3.22}$$

where c_2 is a positive constant.
Multiplying Eq (3.22) by h, summing up for i from 1 to M and noticing Eq (3.21), we obtain

$$\begin{aligned} &\frac{1}{\tau}\left(\|\tilde{w}^k\|^2 - \|\tilde{w}^{k-1}\|^2\right) \\ \leqslant&\left(6\mu c_2^2 + c_2 + \frac{1}{2}\right)\left(\|\tilde{w}^k\|^2 + \|\tilde{w}^{k-1}\|^2\right) + \|S^{k-\frac{1}{2}}\|^2 + 4\mu\|T^{k-\frac{1}{2}}\|^2, \\ &\qquad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N. \end{aligned} \tag{3.23}$$

Step 2: Subtracting Eq (3.18b) and Eq (3.18c) with superscripts k and $k-1$, dividing the results by τ on both sides, we have

$$\frac{1}{\mu}\delta_t\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_t\delta_x\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{1}{\tau}\left(\gamma_{i-\frac{1}{2}}^{k}\tilde{w}_{i-\frac{1}{2}}^{k} - \gamma_{i-\frac{1}{2}}^{k-1}\tilde{w}_{i-\frac{1}{2}}^{k-1}\right) + \delta_t T_{i-\frac{1}{2}}^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, \tag{3.24}$$

$$\delta_t\tilde{v}_0^{k-\frac{1}{2}} = 0, \quad \delta_t\tilde{v}_M^{k-\frac{1}{2}} = 0, \quad 1 \leqslant k \leqslant N. \tag{3.25}$$

Multiplying Eq (3.18a) by $2\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}$, Eq (3.24) by $2\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}$ and adding the results, we obtain

$$\begin{aligned}
&2\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + \frac{1}{\mu}\cdot\frac{1}{\tau}\left[\left(\tilde{v}_{i-\frac{1}{2}}^{k}\right)^2 - \left(\tilde{v}_{i-\frac{1}{2}}^{k-1}\right)^2\right] \\
=&\, 2\left[\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)\left(\delta_x\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) + \tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\delta_x\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)\right] \\
&+ 2\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k}\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}}\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) - \frac{2}{\tau}\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k}\tilde{w}_{i-\frac{1}{2}}^{k} - \gamma_{i-\frac{1}{2}}^{k-1}\tilde{w}_{i-\frac{1}{2}}^{k-1}\right) \\
&+ 2\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)S_{i-\frac{1}{2}}^{k-\frac{1}{2}} + 2\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\delta_t T_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) \\
=&\, \frac{2}{h}\left[\left(\tilde{v}_i^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_i^{k-\frac{1}{2}}\right) - \left(\tilde{v}_{i-1}^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_{i-1}^{k-\frac{1}{2}}\right)\right] \\
&+ 2\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k}\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}}\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) - 2\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \delta_t\gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}}\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) \\
&+ 2\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)S_{i-\frac{1}{2}}^{k-\frac{1}{2}} + 2\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\left(\delta_t T_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) \\
\leqslant&\, \frac{2}{h}\left[\left(\tilde{v}_i^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_i^{k-\frac{1}{2}}\right) - \left(\tilde{v}_{i-1}^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_{i-1}^{k-\frac{1}{2}}\right)\right] \\
&+ 2c_2\left|\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right| + 2c_2\left|\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right| + 2c_2\left|\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right|\left(\left|\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right| + \left|\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right|\right) \\
&+ \frac{1}{2}\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + 2\left(S_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + \left(\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + \left(\delta_t T_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 \\
\leqslant&\, \frac{2}{h}\left[\left(\tilde{v}_i^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_i^{k-\frac{1}{2}}\right) - \left(\tilde{v}_{i-1}^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_{i-1}^{k-\frac{1}{2}}\right)\right] \\
&+ \frac{1}{2}\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + 2c_2^2\left(\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + \frac{1}{2}\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right) + 2c_2^2\left(\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 \\
&+ \frac{1}{2}\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + 2c_2^2\left(\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + c_2^2\left(\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + \left(\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 \\
&+ \frac{1}{2}\left(\delta_t\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + 2\left(S_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + \left(\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2 + \left(\delta_t T_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2.
\end{aligned} \tag{3.26}$$

After simplifying Eq (3.26), it becomes

$$\begin{aligned}
&\frac{1}{\mu}\cdot\frac{1}{\tau}\left[\left(\tilde{v}_{i-\frac{1}{2}}^{k}\right)^2 - \left(\tilde{v}_{i-\frac{1}{2}}^{k-1}\right)^2\right] \\
\leqslant&\, \frac{2}{h}\left[\left(\tilde{v}_i^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_i^{k-\frac{1}{2}}\right) - \left(\tilde{v}_{i-1}^{k-\frac{1}{2}}\right)\left(\delta_t\tilde{w}_{i-1}^{k-\frac{1}{2}}\right)\right]
\end{aligned}$$

$$+\left(5c_2^2+1\right)\left(\tilde{v}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2+\left(2c_2^2+1\right)\left(\tilde{w}_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2+2\left(S_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2+\left(\delta_t T_{i-\frac{1}{2}}^{k-\frac{1}{2}}\right)^2. \tag{3.27}$$

Multiplying Eq (3.27) by h, summing up for i from 1 to M and noticing Eq (3.21), we obtain

$$\begin{aligned}
&\frac{1}{\mu}\cdot\frac{1}{\tau}\left(\|\tilde{v}^k\|^2-\|\tilde{v}^{k-1}\|^2\right)\\
\leqslant\ &2\left(\tilde{v}_M^{k-\frac{1}{2}}\delta_t\tilde{w}_M^{k-\frac{1}{2}}-\tilde{v}_0^{k-\frac{1}{2}}\delta_t\tilde{w}_0^{k-\frac{1}{2}}\right)+\frac{1}{2}(5c_2^2+1)\left(\|\tilde{v}^k\|^2+\|\tilde{v}^{k-1}\|^2\right)\\
&+\frac{1}{2}(2c_2^2+1)\left(\|\tilde{w}^k\|^2+\|\tilde{w}^{k-1}\|^2\right)+2\|S^{k-\frac{1}{2}}\|^2+\|\delta_t T^{k-\frac{1}{2}}\|^2,\quad 1\leqslant k\leqslant N.
\end{aligned}\tag{3.28}$$

Step 3: Adding Eq (3.23) and Eq (3.28), we have

$$\begin{aligned}
&\frac{1}{\tau}\left(\|\tilde{w}^k\|^2-\|\tilde{w}^{k-1}\|^2\right)+\frac{1}{\mu}\cdot\frac{1}{\tau}\left(\|\tilde{v}^k\|^2-\|\tilde{v}^{k-1}\|^2\right)\\
\leqslant\ &\frac{1}{2}(5c_2^2+1)(\|\tilde{v}^k\|^2+\|\tilde{v}^{k-1}\|^2)+(6\mu c_2^2+c_2^2+c_2+1)(\|\tilde{w}^k\|^2+\|\tilde{w}^{k-1}\|^2)\\
&+3\|S^{k-\frac{1}{2}}\|^2+\|\delta_t T^{k-\frac{1}{2}}\|^2+4\mu\|T^{k-\frac{1}{2}}\|^2,\quad 1\leqslant k\leqslant N.
\end{aligned}\tag{3.29}$$

Furthermore, we have

$$\begin{aligned}
&\frac{1}{\tau}\left[(\|\tilde{w}^k\|^2+\frac{1}{\mu}\|\tilde{v}^k\|^2)-(\|\tilde{w}^{k-1}\|^2+\frac{1}{\mu}\|\tilde{v}^{k-1}\|^2)\right]\\
\leqslant\ &\left(\frac{5}{2}c_2^2+\frac{1}{2}\right)\left(\|\tilde{v}^k\|^2+\|\tilde{v}^{k-1}\|^2\right)+\left(6\mu c_2^2+c_2^2+c_2+1\right)\left(\|\tilde{w}^k\|^2+\|\tilde{w}^{k-1}\|^2\right)\\
&+3\|S^{k-\frac{1}{2}}\|^2+4\mu\|T^{k-\frac{1}{2}}\|^2+\|\delta_t T^{k-\frac{1}{2}}\|^2,\quad 1\leqslant m\leqslant N.
\end{aligned}\tag{3.30}$$

Noticing that Eq (3.19), Eq (3.30) becomes

$$\frac{1}{\tau}(G^k-G^{k-1})\leqslant c_3(G^k+G^{k-1})+Q^k,\quad 1\leqslant m\leqslant N.$$

When $\tau\leqslant\frac{1}{3c_3}$, using Gronwall inequality in Lemma 1 yields

$$G^k\leqslant e^{3c_3T}\left(G^0+\frac{3}{2}\tau\sum_{l=0}^{k-1}Q^l\right),\quad 1\leqslant k\leqslant N.$$

Equivalently,

$$\begin{aligned}
\|\tilde{w}^k\|^2+\frac{1}{\mu}\|\tilde{v}^k\|^2&\leqslant e^{3c_3T}\left(\|\tilde{w}^0\|^2+\frac{1}{\mu}\|\tilde{v}^0\|^2+\frac{3}{2}\tau\sum_{l=0}^{k-1}Q^l\right)\\
&=e^{3c_2T}\left(\|\widehat{\varphi}\|^2+\frac{1}{\mu}\|\tilde{v}^0\|^2+\frac{3}{2}\tau\sum_{l=0}^{k-1}Q^l\right),\quad 1\leqslant k\leqslant N.
\end{aligned}$$

This completes the proof. □

3.4. Solvability and convergence

Theorem 3 (Solvability)**.** *The difference scheme Eqs* (3.5a)–(3.5d) *is uniquely solvable.*

Define

$$e_i^k = W_i^k - w_i^k, \quad 1 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N,$$

$$f_i^k = V_i^k - v_i^k, \quad 1 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N.$$

Subtracting Eqs (3.3a)–(3.3d) from Eqs (3.5a)–(3.5d), we have the error system

$$\begin{cases} \delta_t e_{i-\frac{1}{2}}^{k-\frac{1}{2}} = \delta_x f_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \gamma_{i-\frac{1}{2}}^{k-\frac{1}{2}} f_{i-\frac{1}{2}}^{k-\frac{1}{2}} + \theta_{i-\frac{1}{2}}^{k-\frac{1}{2}} e_{i-\frac{1}{2}}^{k-\frac{1}{2}} + (r_1)_{i-\frac{1}{2}}^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N, \\ \dfrac{1}{\mu} f_{i-\frac{1}{2}}^k = \delta_x e_{i-\frac{1}{2}}^k - \gamma_{i-\frac{1}{2}}^k e_{i-\frac{1}{2}}^k + (r_2)_{i-\frac{1}{2}}^k, \quad 1 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N, \\ e_i^0 = 0, \quad 0 \leqslant i \leqslant M, \\ f_0^k = 0, \quad f_M^k = 0, \quad 1 \leqslant k \leqslant N. \end{cases}$$

Noticing that Eqs (3.4a)–(3.4b) and Lemma 2, similar to the proof of the prior estimate in Theorem 2, we have the following convergence results for the above error system.

Theorem 4 (Convergence)**.** *Let* $\alpha(t)$*,* $\beta(t) \in C^2[0,T]$*,* $w(x,t) \in C^4([0,L] \times [0,T])$ *and suppose the condition Eq* (1.4) *is satisfied. Then the solution of the difference scheme Eqs* (3.5a)–(3.5d) *is convergent to the solution of Eqs* (3.2a)–(3.2d) *with the order of convergence* $O(\tau^2 + h^2)$*.*

4. Numerical tests

In this section, we will testify to the accuracy and the convergence order for the box scheme Eqs (3.5a)–(3.5d). Based on the discussion in Section 2 and using the equivalence relation in Theorem 1, we give the algorithm flow chart of the box scheme Eqs (3.5a)–(3.5d) for the Burgers' equation Eqs (1.3a)–(1.3c):

Algorithm 1: The box scheme Eqs (3.5a)–(3.5d) for solving Eqs (1.3a)–(1.3c)

Input: parameters $\mu,\ \tilde{\varphi}(x),\ \gamma(x,t),\ \theta(x,t),\ L,\ T,\ M,\ N.$
Output: $\{\widehat{u}_{i-\frac{1}{2}}^k \mid 1 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$.

Compute $u_i^0 = \tilde{\varphi}(x_i), \quad 0 \leqslant i \leqslant M.$
Compute $h = L/M,\ \tau = T/N.$
for $k = 1, \ldots, N$ **do**
| Solving the linear system of Eqs (3.8)–(3.11).
end
Using the discretized transformation Eq (3.7) to recover $\widehat{u}_{i-\frac{1}{2}}^k$.

Denote

$$\mathrm{E}(h,\tau) = \sqrt{h \sum_{i=1}^{M} (u(x_{i-\frac{1}{2}}, t_k) - \widehat{u}_{i-\frac{1}{2}}^k)^2}.$$

Define the spatial convergence order and temporal convergence order respectively by

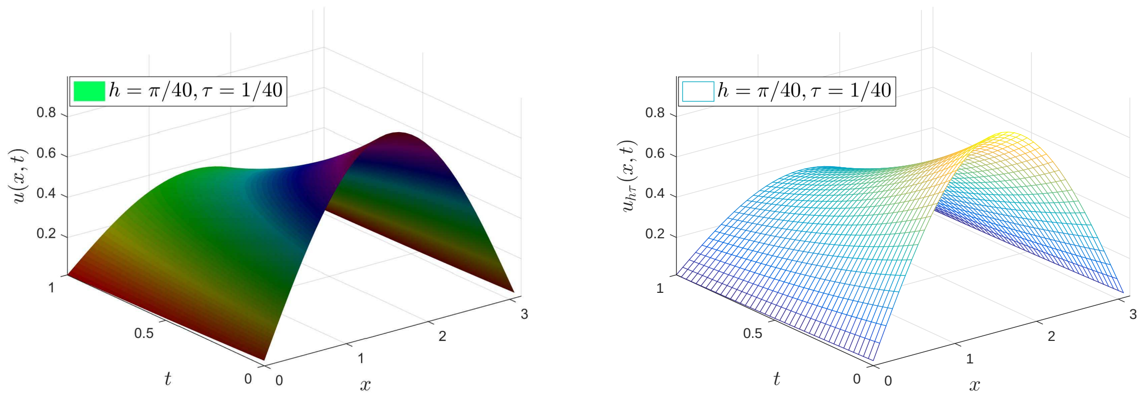
$$\text{Ord}_h = \mathbf{log}_2 \frac{\text{E}(h,\tau)}{\text{E}(h/2,\tau)}, \qquad \text{Ord}_\tau = \mathbf{log}_2 \frac{\text{E}(h,\tau)}{\text{E}(h,\tau/2)}.$$

Example 1. *We first consider the problem with the homogeneous boundary conditions as*

$$\begin{cases} u_t + uu_x = \mu u_{xx} + 0.5\exp(-2t)\sin 2x - (1-\mu)\exp(-t)\sin x, & 0 < x < \pi,\ 0 < t \leqslant 1, \\ u(x,0) = \sin x, \quad 0 \leqslant x \leqslant \pi, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \quad 0 < t \leqslant 1. \end{cases}$$

The exact solution for the above problem is $u(x,t) = \exp(-t)\sin x$. *After simple calculation by Eq* (2.8) *and Eq* (3.1a)*and Eq* (3.1b), *we have* $w(x,0) = 1/(2\mu)\exp(\cos x - 1)$, $\gamma(x,t) = 0$, $\theta(x,t) = 0.125/\mu \cdot \exp(-2t)(\cos 2x - 1) - (1-\mu)/\mu \cdot \exp(-t)(\cos x - 1)$.

The numerical results are listed in Tables 1,2 and exact and numerical surfaces are respectively displayed in Figure 1. Table 1 gives the numerical solutions, exact solutions and their absolute errors, which show that the box scheme is efficient even if the coefficient of viscosity is very small. The convergence orders in time and space are displayed in Table 2. We first fix the spatial step size and test the temporal convergence order in the third column and fourth column, which confirm that the temporal convergence order approaches order two. The numerical results in the last two columns verify that the convergence order in spatial convergence order is order two, which is also consistent with our theoretical results. We further compare the numerical solutions and exact solutions in Figure 2, which again demonstrates that the effectiveness of the box scheme whether the coefficient of viscosity is large or small. Moreover, we also notice that the numerical errors increase in a steep fashion, which means that the stiffness of the system intensifies. In these cases, we should use small step sizes to capture the solution profiles accurately.



(a) Exact solution

(b) Numerical solution

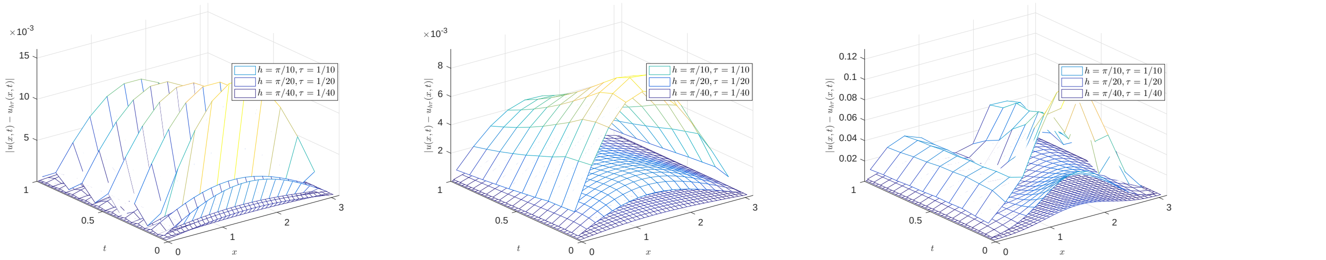
Figure 1. The comparison of the exact solution and the numerical solution with $\mu = 1$.

Table 1. The comparison between numerical solutions and exact solutions for Example 1 with the grid sizes $h = \pi/1000$ and $\tau = 1/1000$.

μ	(x, t)	$\widehat{u}_i^k$	$u(x_i, t_k)$	$\lvert u(x_i, t_k) - \widehat{u}_i^k \rvert$
1000	$(\pi/2, 1/5)$	0.818728888520359	0.818729743009609	0.000000854489250
	$(\pi/2, 2/5)$	0.670318521533638	0.670319219061600	0.000000697527962
	$(\pi/2, 3/5)$	0.548810391409158	0.548810959024948	0.000000567615790
	$(\pi/2, 4/5)$	0.449327947995460	0.449328409779945	0.000000461784485
	$(\pi/2, 1)$	0.367878606492961	0.367878987318467	0.000000380825505
10	$(\pi/2, 1/5)$	0.818728829761817	0.818729743009609	0.000000913247790
	$(\pi/2, 2/5)$	0.670318405694616	0.670319219061600	0.000000813366983
	$(\pi/2, 3/5)$	0.548810292205593	0.548810959024948	0.000000666819355
	$(\pi/2, 4/5)$	0.449327868975789	0.449328409779945	0.000000540804156
	$(\pi/2, 1)$	0.367878548750915	0.367878987318467	0.000000438567552
0.1	$(\pi/2, 1/5)$	0.818719664696976	0.818729743009609	0.000010078312633
	$(\pi/2, 2/5)$	0.670314965926731	0.670319219061600	0.000004253134869
	$(\pi/2, 3/5)$	0.548809375483928	0.548810959024948	0.000001583541020
	$(\pi/2, 4/5)$	0.449327846412115	0.449328409779945	0.000000563367830
	$(\pi/2, 1)$	0.367878673622388	0.367878987318467	0.000000313696078

Table 2. The numerical errors and convergence orders in time and space respectively for Example 1 with different coefficients μ of viscosity.

μ	(h, τ)	$E(h, \tau)$	Ord_τ	(h, τ)	$E(h, \tau)$	Ord_h
1000	$(\pi/1000, 1/10)$	4.6583E − 3	*	$(\pi/10, 1/1000)$	4.9044E − 3	*
	$(\pi/1000, 1/20)$	1.1723E − 3	1.9905	$(\pi/20, 1/1000)$	1.2299E − 4	1.9967
	$(\pi/1000, 1/40)$	2.8666E − 4	2.0319	$(\pi/40, 1/1000)$	3.0795E − 4	1.9972
	$(\pi/1000, 1/80)$	6.7986E − 5	2.0760	$(\pi/80, 1/1000)$	7.7048E − 5	1.9915
	$(\pi/1000, 1/160)$	1.5769E − 5	2.1081	$(\pi/160, 1/1000)$	1.9270E − 6	1.9668
10	$(\pi/1000, 1/10)$	1.7715E − 3	*	$(\pi/10, 1/1000)$	4.7569E − 3	*
	$(\pi/1000, 1/20)$	4.3060E − 4	2.0406	$(\pi/20, 1/1000)$	1.2110E − 4	2.0013
	$(\pi/1000, 1/40)$	1.0790E − 4	1.9966	$(\pi/40, 1/1000)$	3.0556E − 4	1.9978
	$(\pi/1000, 1/80)$	2.7493E − 5	1.9726	$(\pi/80, 1/1000)$	7.6747E − 5	1.9936
	$(\pi/1000, 1/160)$	7.4277E − 6	1.8881	$(\pi/160, 1/1000)$	1.9231E − 6	1.9751
0.1	$(\pi/1000, 1/10)$	3.9668E − 2	*	$(\pi/10, 1/1000)$	7.6766E − 2	*
	$(\pi/1000, 1/20)$	9.4193E − 3	2.0743	$(\pi/20, 1/1000)$	1.9248E − 2	1.9250
	$(\pi/1000, 1/40)$	2.3266E − 3	2.0174	$(\pi/40, 1/1000)$	4.8072E − 3	1.9807
	$(\pi/1000, 1/80)$	5.8019E − 4	2.0036	$(\pi/80, 1/1000)$	1.1916E − 4	1.9952
	$(\pi/1000, 1/160)$	1.4528E − 4	1.9977	$(\pi/160, 1/1000)$	2.8748E − 4	1.9988



(a) $\mu = 1000$ (b) $\mu = 10$ (c) $\mu = 0.1$

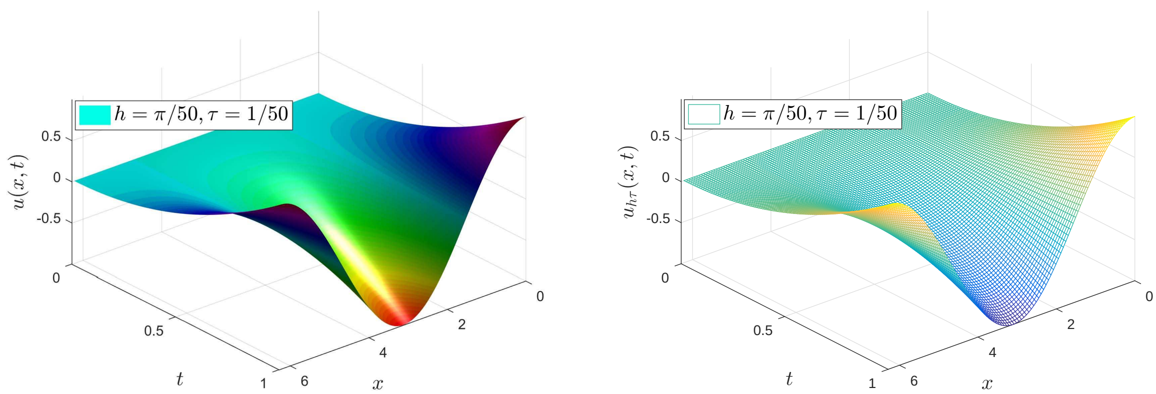
Figure 2. Numerical errors between exact solutions and numerical solutions with different temporal and spatial step sizes and coefficients of viscosity.

Example 2. *Then we consider the problem with the nonhomogeneous boundary conditions as*

$$\begin{cases} u_t + uu_x = \mu u_{xx} + 2t\cos x + \mu t^2\cos x - 0.5t^4\sin 2x, \quad 0 < x < 2\pi,\ 0 < t \leqslant 1, \\ u(x,0) = 0, \quad 0 \leqslant x \leqslant 2\pi, \\ u(0,t) = t^2, \quad u(2\pi,t) = t^2, \quad 0 < t \leqslant 1. \end{cases}$$

The exact solution is $u(x,t) = t^2\cos x$. *Here we easily have* $w(x,0) = 1$, $\gamma(x,t) = -0.5/(2\mu)\cdot t^2$, *and* $\theta(x,t) = 0.25/\mu\cdot t^4 - t/\mu\sin x - 0.5t^2\sin x - 0.125/\mu\cdot t^4\cos 2x + 0.125/\mu\cdot t^4$ *calculated by using Eq* (2.8) *and Eq* (3.1a) *and Eq* (3.1b), *respectively.*

In this example, the boundary conditions and right-hand side terms are both nonhomogeneous. The exact solution surface (left) and numerical solution surface (right) are respectively displayed in Figure 3. The numerical results and error surfaces for different coefficients of viscosity are respectively shown in Tables 3,4 and Figure 4. Similar results to Example 1 can be observed, which further verify the correctness of our theoretical results.



(a) Exact solution (b) Numerical solution

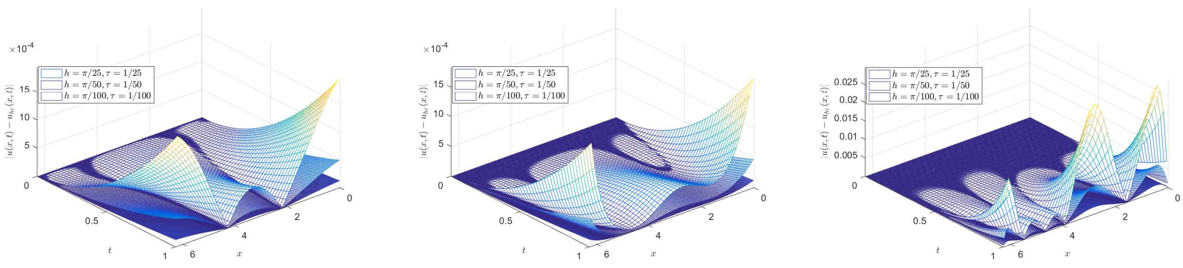
Figure 3. The comparison of the exact solution and the numerical solution with $\mu = 1$.

Table 3. The comparison between numerical solutions and exact solutions for Example 1 with the grid sizes $h = \pi/500$ and $\tau = 1/1000$.

μ	(x, t)	$\widehat{u}_i^k$	$u(x_i, t_k)$	$\lvert u(x_i, t_k) - \widehat{u}_i^k \rvert$
10	$(\pi, 1/5)$	−0.039999587661271	−0.039999802608074	0.000000214946803
	$(\pi, 2/5)$	−0.159998843947508	−0.159999210432297	0.000000366484790
	$(\pi, 3/5)$	−0.359997635176921	−0.359998223472669	0.000000588295748
	$(\pi, 4/5)$	−0.639995934067770	−0.639996841729189	0.000000907661419
	$(\pi, 1)$	−0.999993720851624	−0.999995065201858	0.000001344350234
1	$(\pi, 1/5)$	−0.039999872175453	−0.039999802608074	0.000000069567378
	$(\pi, 2/5)$	−0.159999525361536	−0.159999210432297	0.000000314929239
	$(\pi, 3/5)$	−0.359998865423127	−0.359998223472669	0.000000641950458
	$(\pi, 4/5)$	−0.639997774181156	−0.639996841729189	0.000000932451966
	$(\pi, 1)$	−0.999996062051233	−0.999995065201858	0.000000996849375
0.1	$(\pi, 1/5)$	−0.039999922657609	−0.039999802608074	0.000000120049535
	$(\pi, 2/5)$	−0.159999408166085	−0.159999210432297	0.000000197733788
	$(\pi, 3/5)$	−0.359995956611092	−0.359998223472669	0.000022668615773
	$(\pi, 4/5)$	−0.639980403043749	−0.639996841729189	0.000016438685440
	$(\pi, 1)$	−0.999931362684693	−0.999995065201858	0.000063702517165

Table 4. The numerical errors and convergence orders in time and space respectively for Example 2 with different coefficients μ of viscosity.

μ	(h, τ)	$E(h, \tau)$	Ord_τ	(h, τ)	$E(h, \tau)$	Ord_h
10	$(\pi/1000, 1/10)$	4.8916E − 3	*	$(\pi/50, 1/1000)$	4.9044E − 4	*
	$(\pi/1000, 1/20)$	1.2233E − 3	1.9995	$(\pi/100, 1/1000)$	1.2299E − 4	1.9956
	$(\pi/1000, 1/40)$	3.0648E − 4	1.9970	$(\pi/200, 1/1000)$	3.0795E − 5	1.9978
	$(\pi/1000, 1/80)$	7.7282E − 5	1.9876	$(\pi/400, 1/1000)$	7.7048E − 6	1.9989
	$(\pi/1000, 1/160)$	1.9985E − 5	1.9512	$(\pi/800, 1/1000)$	1.9270E − 6	1.9994
1	$(\pi/1000, 1/10)$	4.2425E − 3	*	$(\pi/50, 1/1000)$	4.7569E − 4	*
	$(\pi/1000, 1/20)$	1.0639E − 3	1.9955	$(\pi/100, 1/1000)$	1.2110E − 4	1.9738
	$(\pi/1000, 1/40)$	2.6562E − 4	2.0019	$(\pi/200, 1/1000)$	3.0556E − 5	1.9867
	$(\pi/1000, 1/80)$	6.5816E − 5	2.0128	$(\pi/400, 1/1000)$	7.6747E − 6	1.9933
	$(\pi/1000, 1/160)$	1.5857E − 5	2.0533	$(\pi/800, 1/1000)$	1.9231E − 6	1.9966
0.1	$(\pi/1000, 1/10)$	5.3579E − 2	*	$(\pi/50, 1/1000)$	7.6766E − 3	*
	$(\pi/1000, 1/20)$	1.2219E − 2	2.1325	$(\pi/100, 1/1000)$	1.9248E − 3	1.9957
	$(\pi/1000, 1/40)$	2.9922E − 3	2.0299	$(\pi/200, 1/1000)$	4.8072E − 4	2.0015
	$(\pi/1000, 1/80)$	7.4401E − 4	2.0078	$(\pi/400, 1/1000)$	1.1916E − 4	2.0123
	$(\pi/1000, 1/160)$	1.8631E − 4	1.9976	$(\pi/800, 1/1000)$	2.8748E − 5	2.0514



(a) $\mu = 10$ (b) $\mu = 1$ (c) $\mu = 0.1$

Figure 4. The numerical errors between exact solutions and numerical solutions with different temporal and spatial step sizes and coefficients of viscosity.

5. Conclusion

In summary, it was demonstrated that the Burgers' equation subject to nonhomogeneous Dirichlet boundary conditions is equivalent to a heat equation with derivative boundary conditions based on the Hopf-Cole transformation. We further convert the heat equation into a first-order system with homogeneous boundary conditions via the help of the linear transformation and the reduced-order method. An efficient box scheme is established for the converted first-order system. We further prove that the box scheme is solvable and convergent.

The relationship between the nonlinear convection term uu_x and variable (or constant) coefficient convection term $c(x)u(x,t)$ ($c(x)$ is the variable coefficient) in other references is linked by a generalized exponential transformation. Moreover, the exponential transformation can be applied to solve other partial differential equations with nonlinear convection terms involving fractional differential equations, delay Sobolev equations and delay functional differential equations with Burger-type nonlinear terms [26, 27]. These will leave to our future research work.

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Conflict of interest

The authors declare there is no conflict of interest.

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