AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4512–4528.
DOI:10.3934/math.2020290
Received: 31 March 2020
Accepted: 13 May 2020
Published: 21 May 2020

Research article

A sharp double inequality involving generalized complete elliptic integral of the first kind

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Abstract: In the article, we establish a sharp double inequality involving the ratio of generalized complete elliptic integrals of the first kind, which is the improvement and generalization of some previously known results.

Keywords: Gaussian hypergeometric function; generalized elliptic integral; gamma function; psi function
Mathematics Subject Classification: 33C05, 33E05

1. Introduction

Let $r \in (0,1)$. Then the Legendre complete elliptic integral $\mathcal{K}(r)$ [1–4] of the first kind is given by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}.$$

It is well-known that the complete elliptic integral $\mathcal{K}(r)$ is the particular case of the Gaussian hypergeometric function [5–10]

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \quad |x| < 1 \tag{1.1}$$

where $(a,0)=1$ for $a \neq 0$, and $(a,n) = a(a+1)(a+2)\cdots(a+n-1)$ for $n \in \mathbb{N}$ is the shifted factorial function. Indeed

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2},\frac{1}{2};1;r^2\right).$$

It is well-known that the Legendre complete elliptic integrals play very important roles in many branches of pure and applied mathematics [11–22]. Recently, the complete elliptic integrals have attracted the attention of many researchers [23–35] due to their extreme importance. In particular, and many remarkable properties, inequalities and applications for the complete elliptic integrals and their related special functions can be found in the literature [36–61].

For $r \in (0,1)$ and $a \in (0,1)$, the generalized elliptic integral $\mathcal{K}_a(r)$ of the first kind [62] is defined by

$$\mathcal{K}_a = \mathcal{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1, r^2). \tag{1.2}$$

Clearly, $\mathcal{K}_a(0) = \pi/2$ and $\mathcal{K}_a(1^-) = \infty$. In what follows, we assume that $a \in (0, 1/2]$ by the symmetry of (1.2).

For $p \in (1,\infty)$ and $r \in (0,1)$, the complete p-elliptic integral $\mathfrak{K}_p(r)$ of the first kind [63] is defined by

$$\mathfrak{K}_p(r) = \int_0^{\pi_p/2} \frac{d\theta}{(1 - r^p \sin_p^p \theta)^{1-1/p}} = \int_0^1 \frac{dt}{\sqrt{(1-t^p)^{1p}(1-r^p t^p)^{1-1/p}}}, \tag{1.3}$$

where $\sin_p \theta$ is the generalized trigonometric function [64] and

$$\pi_p = 2\int_0^1 \frac{dt}{(1-t^p)^{1/p}}$$

is the generalized circumference ratio.

From (1.2) and (1.3) we clearly see that $\mathcal{K}_a(r)$ and $\mathfrak{K}_p(r)$ reduce to the complete elliptic integral $\mathcal{K}(r)$ of the first kind if $a = 1/2$ and $p = 2$. Takeuchi [65] proved that

$$\mathfrak{K}_p(r) = \frac{\pi_p}{2} F\left(\frac{1}{p}, 1 - \frac{1}{p}; 1; r^p\right).$$

Therefore, it follows from (1.2) that

$$\mathcal{K}_{1/p}(r) = \frac{\pi}{\pi_p} \mathfrak{K}_p(r^{2/p}). \tag{1.4}$$

Recently, the generalized elliptic integrals and complete p-elliptic integrals have attracted the attention of many mathematicians. For their recent research progress, we recommend the literature [65–78] to readers.

Anderson et al. [79] proved that the inequality

$$\frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} > \frac{1}{1+r} \tag{1.5}$$

holds for all $r \in (0,1)$.

In [80], Alzer and Richards proved that

$$\frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} > \frac{1}{1+r/4} \tag{1.6}$$

for $r \in (0,1)$, which is an improvement of inequality (1.5).

Motivated by the inequality (1.6), Yin et al. [81] generalized (1.6) to $\mathfrak{K}_p(r)$ and proved that the double inequality

$$\frac{1}{1+\frac{1}{p}(1-\frac{1}{p})r} < \frac{\mathfrak{K}_p(r)}{\mathfrak{K}_p(\sqrt[p]{r})} < 1 \tag{1.7}$$

holds for $r \in (0,1)$ and $p \in (1,2]$.

The main purpose of this paper is to generalized the inequality (1.6) to $\mathcal{K}_a$ and provide an improvement for inequality (1.7). Our main result is the following Theorem 1.1.

We denote by $\sigma = \sigma(a) = a(1-a)$ and $\tau = \tau(a) = [a(1-a)(a^2-a+2)]/4$ for short, which will be often used later. For $a \in (0, 1/2]$, it is easy to know $0 < \sigma(a) \le 1/4$ and keep this in mind.

Theorem 1.1. *Let $a \in (0,1/2]$ and $r \in (0,1)$. Then the double inequality*

$$\hat{\lambda}(a) < \frac{[1+\sigma(a)r]\mathcal{K}_a(r) - [1+\tau(a)r^2]\mathcal{K}_a(\sqrt{r})}{r^3\mathcal{K}_a(\sqrt{r})} < \hat{\mu}(a) \tag{1.8}$$

holds for all $r \in (0,1)$ if and only if $\hat{\lambda}(a) \le \lambda(a)$ and $\hat{\mu}(a) \ge \mu(a)$, where

$$\lambda(a) = -\frac{a(1-a^2)(2-a)(4a^2-4a+3)}{18} \quad \text{and} \quad \mu(a) = \frac{a(1-a^2)(2-a)}{4}.$$

In particular, the double inequality

$$\frac{1+\tau(a)r^2+\lambda(a)r^3}{1+\sigma(a)r} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1+\tau(a)r^2+\mu(a)r^3}{1+\sigma(a)r}. \tag{1.9}$$

holds for all $r \in (0,1)$.

As is known, $\mathcal{K}_a(r)$ reduces to the complete elliptic integral of the first kind $\mathcal{K}(r)$ if $a = 1/2$. The following corollary can be derived from (1.9) of Theorem 1.1.

Corollary 1.2. *The double inequality*

$$\frac{1+[r^2(7-4r)]/64}{1+r/4} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} < \frac{1+[r^2(7+9r)]/64}{1+r/4}$$

holds for $r \in (0,1)$.

It is easy to see that

$$\frac{1+[r^2(7-4r)]/64}{1+r/4} > \frac{1}{1+r/4}$$

for $r \in (0,1)$ and no upper bound for $\mathcal{K}(r)/\mathcal{K}(\sqrt{r})$ was given in (1.6), in other words, the bounds given in Corollary 1.2 are better than that given in (1.6).

From (1.4) and the monotonicity of $\mathfrak{K}_p(r)$, we clearly see that

$$\frac{\mathcal{K}_{1/p}(r)}{\mathcal{K}_{1/p}(\sqrt{r})} = \frac{\mathfrak{K}_p(r^{2/p})}{\mathfrak{K}_p(\sqrt[p]{r})} \le (\ge) \frac{\mathfrak{K}_p(r)}{\mathfrak{K}_p(\sqrt[p]{r})}$$

for all $r \in (0,1)$ and $1 < p \le 2$ ($p \ge 2$), which in conjunction with Theorem 1.1 gives the Corollary 1.3.

Corollary 1.3. *Let $p \in (1,2]$. Then the double inequality*

$$\frac{1+\tau(1/p)r^2+\lambda(1/p)r^3}{1+\sigma(1/p)r} < \frac{\mathfrak{R}_p(r)}{\mathfrak{R}_p(\sqrt[p]{r})} < 1 \tag{1.10}$$

hold for $r \in (0,1)$. If $p \in [2,\infty)$, then the inequality

$$\frac{\mathfrak{R}_p(r)}{\mathfrak{R}_p(\sqrt[p]{r})} < \frac{1+\tau(1/p)r^2+\mu(1/p)r^3}{1+\sigma(1/p)r} \tag{1.11}$$

holds for $r \in (0,1)$.

Note that if $p \in (1,2]$ and $r \in (0,1)$, then it follows from $\lambda(1/p) < 0$ that

$$\tau(1/p)+\lambda(1/p)r > \tau(1/p)+\lambda(1/p) = \frac{(p-1)[8(p-1)^2+p^2(p-1)+6p^4]}{36p^6} > 0,$$

which enables us to know that the lower bound of (1.10) is better than that of (1.7) and it also gives an improvement of [81, Theorem 1.1]. Moreover, it follows easily from $\sigma(1/p) = \tau(1/p)+\mu(1/p)$ that

$$\frac{1+\tau(1/p)r^2+\mu(1/p)r^3}{1+\sigma(1/p)r} < 1$$

for $r \in (0,1)$, which leads to the conclusion that inequality (1.11) has a better upper bound than that of (1.7) for $p \in [2,\infty)$.

2. Preliminaries

In this section, we introduce some more notations and present some technical lemmas, which will be used to prove the main theorem.

For $x \in (0,\infty)$, the classical gamma function $\Gamma(x)$ [82, 83] and psi (digamma) function $\Psi(x)$ [84] are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \Psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively.

The following well-known formulas for $\Gamma(x)$ and $\Psi^{(n)}(x)$ $(n \geq 0)$ are presented in [85]

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq \mathbb{Z}, \tag{2.1}$$

$$\Psi^{(n)}(x) = \begin{cases} -\gamma - \frac{1}{x} + \sum\limits_{k=1}^{\infty} \frac{x}{k(k+x)}, & n = 0 \\ (-1)^{n+1}n!\sum\limits_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}, & n \geq 1, \end{cases} \tag{2.2}$$

where $\gamma = \lim\limits_{n\to\infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.577215\cdots$ is the Euler-Mascheroni constant [86, 87].

For $a \in (0,1/2]$, we clearly see from (1.2) and (2.1) that $\mathcal{K}_a(r)$ can be expressed in terms of power series as

$$\mathcal{K}_a(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(a,n)(1-a,n)}{(n!)^2}r^{2n} = \frac{\sin(\pi a)}{2}\sum_{n=0}^{\infty}\mathcal{W}_n(a)r^{2n}, \tag{2.3}$$

where

$$\mathcal{W}_n = \mathcal{W}_n(a) = \frac{\Gamma(a+n)\Gamma(1-a+n)}{\Gamma(n+1)^2} \tag{2.4}$$

is the generalized Wallis type ratio due to $\sqrt{\mathcal{W}_n(1/2)/\pi}$ is the classical Wallis ratio.

It is easy to verify that $\mathcal{W}_n$ satisfies the recurrence relation

$$\frac{\mathcal{W}_{n+1}}{\mathcal{W}_n} = \frac{(n+a)(n+1-a)}{(n+1)^2} \tag{2.5}$$

and also $\mathcal{W}_n$ is strictly decreasing with respect to $n \geq 0$.

Lemma 2.1. (1) *The function $\mathcal{W}_n(a)$ is strictly decreasing on $(0, 1/2]$ for each $n \in \mathbb{N}$;*
(2) *The function $\mathcal{W}_n(a)/\mathcal{W}_m(a)$ is strictly decreasing on $(0, 1/2]$ for fixed $m > n \geq 1$. In particular,*

$$\frac{\mathcal{W}_n(a)}{\mathcal{W}_m(a)} < \frac{m}{n}. \tag{2.6}$$

Proof. Taking the logarithm, we dente by $f_n(a) = \log \mathcal{W}_n(a)$ and $g_{n,m}(a) = \log[\mathcal{W}_n(a)/\mathcal{W}_m(a)]$.

Differentiation yields

$$f_n'(a) = \Psi(a+n) - \Psi(1-a+n), \tag{2.7}$$

$$g_{n,m}'(a) = \Psi(a+n) - \Psi(1-a+n) - \Psi(a+m) + \Psi(1-a+m). \tag{2.8}$$

From (2.7) and (2.8), we clearly see that

$$f_n'(1/2) = g_{n,m}'(1/2) = 0. \tag{2.9}$$

Moreover, it follows from (2.2), (2.7) and (2.8) that

$$f_n''(a) = \Psi'(a+n) + \Psi'(1-a+n) = \sum_{k=0}^{\infty}\left[\frac{1}{(a+n+k)^2} + \frac{1}{(1-a+n+k)^2}\right] > 0, \tag{2.10}$$

$$\begin{aligned} g_{n,m}''(a) &= \Psi'(a+n) + \Psi'(1-a+n) - \Psi'(a+m) - \Psi'(1-a+m) \\ &= \sum_{k=0}^{\infty}\left[\frac{1}{(a+n+k)^2} - \frac{1}{(a+m+k)^2} + \frac{1}{(1-a+n+k)^2} - \frac{1}{(1-a+m+k)^2}\right] > 0 \end{aligned} \tag{2.11}$$

for $a \in (0, 1/2]$ and $m > n \geq 1$.

Therefore, the monotonicity of $f_n(a)$ and $g_{n,m}(a)$ follows easily from (2.9)–(2.11). □

Lemma 2.2. *For $a \in (0, 1/2]$, define*

$$h_n(a) = [1 + \sigma(a)]\mathcal{W}_{n+2} - 2\tau(a)\mathcal{W}_{2n+2}.$$

Then $h_n(a) > 1/(n+2)$ for $n \geq 2$.

Proof. We first prove

$$\frac{\pi a}{\sin(\pi a)} > 1 + \frac{\pi^2 a^2}{6} + \frac{7\pi^4 a^4}{360} > 1 + \frac{41a^2}{25} + \frac{37a^4}{20} \tag{2.12}$$

for $a \in (0, 1/2]$. Indeed, in terms of power series, one has

$$\pi a - \left(1 + \frac{\pi^2 a^2}{6} + \frac{7\pi^4 a^4}{360}\right)\sin(\pi a) = \frac{(\pi a)^7}{90}\sum_{n=0}^{\infty}(-1)^n \alpha_n (\pi a)^{2n}$$

$$= \frac{(\pi a)^7}{90}\sum_{k=0}^{\infty}[\alpha_{2k} - \alpha_{2k+1}(\pi a)^2](\pi a)^{4k} > \frac{(\pi a)^7}{90}\sum_{k=0}^{\infty}(\alpha_{2k} - 3\alpha_{2k+1})(\pi a)^{4k} \tag{2.13}$$

for $a \in (0, 1/2]$, where

$$\alpha_n = \frac{(n+1)(n+2)(465 + 224n + 28n^2)}{(2n+7)!}.$$

Moreover,

$$\alpha_{2k} - 3\alpha_{2k+1} = \frac{2(k+1)(4k+7)(3861 + 15150k + 15536k^2 + 6272k^3 + 896k^4)}{(4k+9)!} > 0$$

for $k \geq 0$. This in conjunction with (2.13) yields the inequality (2.12) is valid.

Let $\xi(a) = 1800 + 600a - 100a^2 - 952a^3 + 357a^4 + 140a^5 - 42a^6 - 4a^7 + a^8$. Combing this with (2.1) and (2.12), we rewrite $h_2(a)$ as

$$h_2(a) = \frac{(4-a)(9-a^2)(4-a^2)(1-a^2)\xi(a)}{1036800} \cdot \frac{\pi a}{\sin(\pi a)} - \frac{1}{4}$$

$$> \frac{(4-a)(9-a^2)(4-a^2)(1-a^2)\xi(a)}{1036800}\left(1 + \frac{41a^2}{25} + \frac{37a^4}{20}\right) - \frac{1}{4}$$

$$> \frac{a}{103680000}\Big[4007308a^7 + 16451833a^8 + 20105580a^7(1-a^2) + 10501395a^8(1-a^2)$$
$$+6832264a^{11} + 999840a^{12} + 1242460a^{11}(1-a^2) + 32390a^{14} + 67672a^{15}$$
$$+ 7236a^{15}(1-a) + 1480a^{15}(1-a^2) + 185a^{18} + \hat{\xi}(a)\Big] > \frac{a\hat{\xi}(a)}{103680000}, \tag{2.14}$$

where

$$\hat{\xi}(a) = 2160000 + 3628800a - 12746400a^2 + 7736800a^3$$
$$- 2977632a^4 - 48200080a^5 - 3912980a^6.$$

Differentiation of $\hat{\xi}(a)$ yields

$$\hat{\xi}''(a) = -\left[2282400 + 46420800\left(\frac{1}{2} - a\right) + 8a^2(4466448 + 120500200a + 14673675a^2)\right] < 0$$

for $a \in (0, 1/2]$, which implies that $\hat{\xi}(a)$ is strictly concave on $(0, 1/2]$.

From the concavity property of $\hat{\xi}(a)$, we clearly see that

$$\hat{\xi}(a) \geq \min\{\hat{\xi}(0), \hat{\xi}(1/2)\} = \frac{22483}{16} > 0 \tag{2.15}$$

for $a \in (0, 1/2]$.

Therefore, $h_2(a) > 0$ for $a \in (0, 1/2]$ follows from (2.14) and (2.15).

Next, we prove Lemma 2.2 by mathematical induction on n. Assume the induction hypothesis that $h_n(a) > 1/(n+2)$, in other words,

$$[1 + \sigma(a)]\mathcal{W}_{n+2} > 2\tau(a)\mathcal{W}_{2n+2} + \frac{1}{n+2}. \tag{2.16}$$

The recurrence relation (2.5) and (2.16) yield

$$\begin{aligned}
h_{n+1}(a) - \frac{1}{n+3} &= [1 + \sigma(a)]\mathcal{W}_{n+3} - 2\tau(a)\mathcal{W}_{2n+4} - \frac{1}{n+3} \\
&> 2\tau(a)\mathcal{W}_{2n+2}\left(\frac{\mathcal{W}_{n+3}}{\mathcal{W}_{n+2}} - \frac{\mathcal{W}_{2n+4}}{\mathcal{W}_{2n+2}}\right) + \frac{\mathcal{W}_{n+3}}{(n+2)\mathcal{W}_{n+2}} - \frac{1}{n+3} \\
&= \tau(a)\mathcal{W}_{2n+2} \cdot \frac{\zeta_n(a)}{2(2+n)^2(3+n)^2(3+2n)^2} + \frac{a(1-a)}{(n+2)(n+3)^2} > 0
\end{aligned}$$

for $a \in (0, 1/2]$, where

$$\begin{aligned}
\zeta_n(a) = {} & 9(6+\sigma)(4-\sigma) + 6[78 + \sigma(2-\sigma)]n \\
& + [372 + \sigma(58-\sigma)]n^2 + 8(5\sigma + 16)n^3 + 8(\sigma + 2)n^4.
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *For $a \in (0, 1/2]$, we define*

$$\begin{aligned}
A_n &= \mathcal{W}_{n+2} - \lambda(a)\mathcal{W}_{2n+1} - \tau(a)\mathcal{W}_{2n+2} - \mathcal{W}_{2n+4}, \\
B_n &= \sigma(a)\mathcal{W}_{n+2} - \lambda(a)\mathcal{W}_{2n+2} - \tau(a)\mathcal{W}_{2n+3} - \mathcal{W}_{2n+5}.
\end{aligned}$$

Then (i) $A_n > 0$; (ii) $A_n + B_n > 0$ for $n \geq 0$.

Proof. (i) It is easy to know that $(1+x)^n > 1 + nx$ for $n > 0$ and $x > 0$. Combining this with the definition of $\mathcal{W}_n$ and its recurrence relation, we clearly see that

$$\begin{aligned}
\frac{\mathcal{W}_{n+2}}{\mathcal{W}_{2n+1}} &= \frac{\Gamma(a+n+2)\Gamma(1-a+n+2)}{\Gamma(n+3)^2} \cdot \frac{\Gamma(2n+2)^2}{\Gamma(a+2n+1)\Gamma(1-a+2n+1)} \\
&= \frac{(1+2n)^2}{(a+2n)(1-a+2n)} \cdot \frac{(1+2n-1)^2}{(a+2n-1)(1-a+2n-1)} \cdots \frac{(1+n+2)^2}{(a+n+2)(1-a+n+2)} \\
&\geq \left[\frac{(1+2n)^2}{(a+2n)(1-a+2n)}\right]^{n-1} \geq 1 + \frac{(n-1)(1-a+a^2+2n)}{(a+2n)(1-a+2n)}
\end{aligned}$$

and

$$\frac{\mathcal{W}_{2n+2}}{\mathcal{W}_{2n+1}} = \frac{(2n+1+a)(2n+2-a)}{(2n+2)^2},$$

$$\frac{\mathcal{W}_{2n+4}}{\mathcal{W}_{2n+1}} = \frac{(2n+1+a)[(2n+2)^2-a^2][(2n+3)^2-a^2](2n+4-a)}{[(2n+2)(2n+3)(2n+4)]^2}.$$

This yields

$$\begin{aligned}\frac{A_n}{\mathcal{W}_{2n+1}} &= \frac{\mathcal{W}_{n+2}}{\mathcal{W}_{2n+1}} - \lambda(a) - \tau(a)\frac{\mathcal{W}_{2n+2}}{\mathcal{W}_{2n+1}} - \frac{\mathcal{W}_{2n+4}}{\mathcal{W}_{2n+1}} \\ &\geq 1 + \frac{(n-1)(1-a+a^2+2n)}{(a+2n)(1-a+2n)} - \lambda(a) - \frac{\tau(a)(2n+1+a)(2n+2-a)}{(2n+2)^2} \\ &\quad - \frac{(2n+1+a)[(2n+2)^2-a^2][(2n+3)^2-a^2](2n+4-a)}{[(2n+2)(2n+3)(2n+4)]^2} \\ &= \frac{1}{144(a+2n)(1-a+2n)(1+n)^2(2+n)^2(3+2n)^2}\sum_{k=0}^{8}\varepsilon_k n^k, \end{aligned} \tag{2.17}$$

where the coefficients are given by

$\varepsilon_0 = -5184+9072\sigma-540\sigma^2-1620\sigma^3-837\sigma^4$, $\varepsilon_1 = -19872+31104\sigma-5904\sigma^2-6240\sigma^3-4236\sigma^4$,

$\varepsilon_2 = -6624+32400\sigma-20604\sigma^2-17176\sigma^3-8207\sigma^4$, $\varepsilon_3 = 72432-6744\sigma-33692\sigma^2-36712\sigma^3-8004\sigma^4$,

$\varepsilon_4 = 146592-41352\sigma-30604\sigma^2-50016\sigma^3-4220\sigma^4$, $\varepsilon_5 = 128592-36912\sigma-16496\sigma^2-40672\sigma^3-1152\sigma^4$,

$\varepsilon_6 = 58464-15936\sigma-5248\sigma^2-19200\sigma^3-128\sigma^4$, $\varepsilon_7 = 13248-3648\sigma-896\sigma^2-4864\sigma^3$,

$$\varepsilon_8 = 1152-384\sigma-64\sigma^2-512\sigma^3.$$

For $a \in (0, 1/2]$, we clearly see that $0 < \sigma \leq 1/4$. This enables us to know easily that $\varepsilon_j > 0$ for $3 \leq j \leq 8$. Moreover, we can verify

$$\begin{aligned}\varepsilon_0+\varepsilon_3 &= 67248+2328\sigma-34232\sigma^2-38332\sigma^3-8841\sigma^4 > \frac{16505607}{256}, \\ \varepsilon_1+\varepsilon_4 &= 4\left(31680-2562\sigma-9127\sigma^2-14064\sigma^3-2114\sigma^4\right) > \frac{3870855}{32}, \\ \varepsilon_2+\varepsilon_5 &= 121968-4512\sigma-37100\sigma^2-57848\sigma^3-9359\sigma^4 > \frac{30100689}{256},\end{aligned}$$

which yields

$$\begin{aligned}\sum_{k=0}^{8}\varepsilon_k n^k &= (\varepsilon_0+\varepsilon_3 n^3)+(\varepsilon_1 n+\varepsilon_4 n^4)+(\varepsilon_2 n^2+\varepsilon_5 n^5)+\varepsilon_6 n^6+\varepsilon_7 n^7+\varepsilon_8 n^8 \\ &\geq (\varepsilon_0+\varepsilon_3)+(\varepsilon_1+\varepsilon_4)n+(\varepsilon_2+\varepsilon_5)n^2+\varepsilon_6 n^6+\varepsilon_7 n^7+\varepsilon_8 n^8 > 0\end{aligned} \tag{2.18}$$

for $n \geq 1$.

Combining with (2.17) and (2.18), we clearly see that $A_n > 0$ for $n \geq 1$. On the other hand,

$$A_0 = \frac{\pi a(1-a^2)(2-a)}{192\sin(a\pi)}\left[22+2\sigma+\sigma^2+32\left(\frac{1}{16}-\sigma^2\right)\right] > 0$$

for $a \in (0, 1/2]$. This completes the first assertion.

(ii) We first compute $A_0 + B_0$ and $A_1 + B_1$. Simple calculations together with (2.1) and (2.4) lead to

$$A_0 + B_0 = \frac{\pi a(1-a^2)(2-a)}{14400\sin(\pi a)}\left[360 + \frac{40535\sigma}{16} + 2963\sigma\left(\frac{1}{4} - \sigma\right) + 701\sigma\left(\frac{1}{16} - \sigma^2\right)\right] > 0,$$

$$\begin{aligned} A_1 + B_1 = \frac{\pi a(1-a^2)(4-a^2)(3-a)}{705600\sin(\pi a)}\Bigg[&\frac{32939}{32} + 14570\sigma + 4091\left(\frac{1}{16} - \sigma^2\right) \\ &+7293\left(\frac{1}{64} - \sigma^3\right) + 260\left(\frac{1}{256} - \sigma^4\right)\Bigg] > 0 \end{aligned}$$

for $a \in (0, 1/2]$.

For $n \geq 2$, it follows from Lemma 2.1*(i)* and Lemma 2.2 together with $\lambda(a) < 0$ and the monotonicity of $\mathcal{W}_n$ with respect to n that

$$\begin{aligned} A_n + B_n &= [1 + \sigma(a)]\mathcal{W}_{n+2} - \lambda(a)(\mathcal{W}_{2n+1} + \mathcal{W}_{2n+2}) - \tau(a)(\mathcal{W}_{2n+2} + \mathcal{W}_{2n+3}) - (\mathcal{W}_{2n+4} + \mathcal{W}_{2n+5}) \\ &\geq [1 + \sigma(a)]\mathcal{W}_{n+2} - 2\tau(a)\mathcal{W}_{2n+2} - 2\mathcal{W}_{2n+4} \\ &= h_n(a) - 2\mathcal{W}_{2n+4}(a) > \frac{1}{n+2} - 2\cdot\frac{1}{2n+4} = 0 \end{aligned}$$

for $a \in (0, 1/2]$. □

Lemma 2.4. *For $a \in (0, 1/2]$, we define*

$$\begin{aligned} C_n &= \sigma(a)\mathcal{W}_{n+1} - \mu(a)\mathcal{W}_{2n} - \tau(a)\mathcal{W}_{2n+1} - \mathcal{W}_{2n+3}, \\ D_n &= \mathcal{W}_{n+2} - \mu(a)\mathcal{W}_{2n+1} - \tau(a)\mathcal{W}_{2n+2} - \mathcal{W}_{2n+4}. \end{aligned}$$

Then (i) $D_n > 0$; (ii) $C_n + D_n < 0$ for $n \geq 0$.

Proof. (i) From the similar argument as in the proof of Lemma 2.3*(i)*, we clearly see that

$$\begin{aligned} \frac{D_n}{\mathcal{W}_{2n+1}} &= \frac{\mathcal{W}_{n+2}}{\mathcal{W}_{2n+1}} - \mu(a) - \tau(a)\frac{\mathcal{W}_{2n+2}}{\mathcal{W}_{2n+1}} - \frac{\mathcal{W}_{2n+4}}{\mathcal{W}_{2n+1}} \\ &\geq 1 + \frac{(n-1)(1-a+a^2+2n)}{(a+2n)(1-a+2n)} - \mu(a) - \frac{\tau(a)(2n+1+a)(2n+2-a)}{(2n+2)^2} \\ &\quad - \frac{(2n+1+a)[(2n+2)^2-a^2][(2n+3)^2-a^2](2n+4-a)}{[(2n+2)(2n+3)(2n+4)]^2} \\ &= \frac{1}{16(a+2n)(1-a+2n)(1+n)^2(2+n)^2(3+2n)^2}\sum_{k=0}^{8}\epsilon_k n^k, \end{aligned} \tag{2.19}$$

where the coefficients are given by

$$\epsilon_0 = -576 + 1008\sigma - 540\sigma^2 - 164\sigma^3 + 35\sigma^4, \quad \epsilon_1 = -2208 + 2496\sigma - 2704\sigma^2 - 368\sigma^3 + 84\sigma^4,$$

$$\epsilon_2 = -736 - 2480\sigma - 5780\sigma^2 - 164\sigma^3 + 73\sigma^4, \quad \epsilon_3 = 8048 - 16456\sigma - 6660\sigma^2 + 224\sigma^3 + 28\sigma^4,$$

$$\epsilon_4 = 16288 - 26248\sigma - 4452\sigma^2 + 276\sigma^3 + 4\sigma^4, \quad \epsilon_5 = 14288 - 21408\sigma - 1736\sigma^2 + 112\sigma^3,$$
$$\epsilon_6 = 6496 - 9824\sigma - 368\sigma^2 + 16\sigma^3, \quad \epsilon_7 = 1472 - 2432\sigma - 32\sigma^2, \quad \epsilon_8 = 128 - 256\sigma.$$

Since $0 < \sigma \le 1/4$, it is easy to verify that $\epsilon_j > 0$ for $3 \le j \le 8$. Moreover, we have

$$\epsilon_0 + \epsilon_3 = 7472 - 15448\sigma - 7200\sigma^2 + 60\sigma^3 + 63\sigma^4 > 3160,$$
$$\epsilon_1 + \epsilon_4 = 14080 - 23752\sigma - 7156\sigma^2 - 92\sigma^3 + 88\sigma^4 > \frac{123093}{16},$$
$$\epsilon_2 + \epsilon_5 = 13552 - 23888\sigma - 7516\sigma^2 - 52\sigma^3 + 73\sigma^4 > \frac{113751}{16},$$

which yields

$$\begin{aligned}\sum_{k=0}^{8} \epsilon_k n^k &= (\epsilon_0 + \epsilon_3 n^3) + (\epsilon_1 n + \epsilon_4 n^4) + (\epsilon_2 n^2 + \epsilon_5 n^5) + \epsilon_6 n^6 + \epsilon_7 n^7 + \epsilon_8 n^8 \\ &\ge (\epsilon_0 + \epsilon_3) + (\epsilon_1 + \epsilon_4) n + (\epsilon_2 + \epsilon_5) n^2 + \epsilon_6 n^6 + \epsilon_7 n^7 + \epsilon_8 n^8 > 0 \qquad (2.20)\end{aligned}$$

for $n \ge 1$.

From (2.19) and (2.20), we clearly see that $D_n > 0$ for $n \ge 1$. For $n = 0$, we verify directly

$$D_0 = \frac{\pi a(1-a^2)(2-a)}{576\sin(a\pi)}\left[\frac{27}{2} + 234\left(\frac{1}{4} - \sigma\right) + 35\sigma^2\right] > 0$$

for $a \in (0, 1/2]$. This complete the proof of *(i)*.

(ii) For $n \ge 0$, it follows from (2.6) and $\sigma(a) = \tau(a) + \mu(a)$ together with the monotonicity of $\mathcal{W}_n$ with respect to n that

$$\begin{aligned}C_n + D_n &= \sigma(a)\mathcal{W}_{n+1} + \mathcal{W}_{n+2} - \mu(a)(\mathcal{W}_{2n} + \mathcal{W}_{2n+1}) - \tau(a)(\mathcal{W}_{2n+1} + \mathcal{W}_{2n+2}) - (\mathcal{W}_{2n+3} + \mathcal{W}_{2n+4}) \\ &< \sigma(a)\mathcal{W}_{n+1} - 2[\tau(a) + \mu(a)]\mathcal{W}_{2n+2} + \mathcal{W}_{n+2} - 2\mathcal{W}_{2n+4} \\ &= \sigma(a)(\mathcal{W}_{n+1} - 2\mathcal{W}_{2n+2}) + \mathcal{W}_{n+2} - 2\mathcal{W}_{2n+4} < 0\end{aligned}$$

for $a \in (0, 1/2]$. This completes the proof. □

3. Proof of Theorem 1.1

Proof. Define

$$\varphi_a(r) = [1 + \sigma(a)r]\mathcal{K}_a(r) - [1 + \tau(a)r^2 + \lambda(a)r^3]\mathcal{K}_a(\sqrt{r})$$

and

$$\phi_a(r) = [1 + \sigma(a)r]\mathcal{K}_a(r) - [1 + \tau(a)r^2 + \mu(a)r^3]\mathcal{K}_a(\sqrt{r}).$$

In order to prove the inequalities (1.8) is valid, it suffices to show $\varphi_a(r) > 0$ and $\phi_a(r) < 0$ for $r \in (0, 1)$.

From (2.3), we can rewrite $\varphi_a(r)$ and $\phi_a(r)$, in terms of power series, as

$$\begin{aligned}\frac{2}{\sin(\pi a)}\varphi_a(r) &= [1+\sigma(a)r]\sum_{n=0}^{\infty}\mathcal{W}_n r^{2n} - [1+\tau(a)r^2+\lambda(a)r^3]\sum_{n=0}^{\infty}\mathcal{W}_n r^n \\ &= r^4\left[\sum_{n=0}^{\infty}(A_n+B_n r)r^{2n}\right], \end{aligned}\tag{3.1}$$

$$\begin{aligned}\frac{2}{\sin(\pi a)}\phi_a(r) &= [1+\sigma(a)r]\sum_{n=0}^{\infty}\mathcal{W}_n r^{2n} - [1+\tau(a)r^2+\mu(a)r^3]\sum_{n=0}^{\infty}\mathcal{W}_n r^n \\ &= r^3\left[\sum_{n=0}^{\infty}(C_n+D_n r)r^{2n}\right], \end{aligned}\tag{3.2}$$

where A_n, B_n and C_n, D_n are defined as in Lemma 2.3 and Lemma 2.4, respectively.

- If $B_n \geq 0$, then it follows from Lemma 2.3*(i)* that $A_n + B_n r > A_n > 0$ for $r \in (0,1)$. If $B_n < 0$, then Lemma 2.3*(ii)* enables us to know that $A_n + B_n r > A_n + B_n > 0$ for $r \in (0,1)$. This in conjunction with (3.1) yields $\varphi_a(r) > 0$ for $r \in (0,1)$.
- From Lemma 2.4, we clearly see that $C_n + D_n r < C_n + D_n < 0$ for $r \in (0,1)$. This in conjunction with (3.2) implies that $\phi_a(r) < 0$ for $r \in (0,1)$.

We now prove that $\lambda(a)$ and $\mu(a)$ are the best possible constants.

Let

$$\Phi_a(r) = \frac{[1+\sigma(a)r]\mathcal{K}_a(r) - [1+\tau(a)r^2]\mathcal{K}_a(\sqrt{r})}{r^3\mathcal{K}_a(\sqrt{r})}. \tag{3.3}$$

If $\lambda(a) < \delta(a) < \mu(a)$, then it follows from $\Phi_a(0^+) = \lambda(a) < \delta(a)$ and $\Phi_a(1^-) = \mu(a) > \delta W(a)$ that there exist sufficiently small $r_1, r_2 \in (0,1)$ such that $\Phi_a(r) < \delta(a)$ for $r \in (0, r_1)$ and $\Phi_a(r) > \delta(a)$ for $r \in (1-r_2, 1)$.

□

For $a \in (0, 1/2]$, computer experiments enable us to know that $\Phi_a(r)$ is strictly increasing on $(0,1)$ and we leave it to the reader as an open problem.

Open Problem. For $a \in (0, 1/2]$, $\Phi_a(r)$ is defined as in (3.3). Then $\Phi_a(r)$ is strictly increasing from $(0,1)$ onto $(\lambda(a), \mu(a))$.

4. Conclusion

We establish a sharp double inequality involving the ratio of generalized complete elliptic integrals of the first kind, more precisely, the double inequality

$$\frac{1+\tau(a)r^2+\lambda(a)r^3}{1+\sigma(a)r} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1+\tau(a)r^2+\mu(a)r^3}{1+\sigma(a)r}$$

holds for all $r \in (0,1)$, where

$$\sigma(a) = a(1-a), \quad \tau(a) = \frac{a(1-a)(a^2-a+2)}{4},$$

$$\lambda(a) = -\frac{a(1-a^2)(2-a)(4a^2-4a+3)}{18}, \quad \mu(a) = \frac{a(1-a^2)(2-a)}{4},$$

which is the improvement and generalization of some previously known results.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article.

The research was supported by the National Natural Science Foundation of China (Grant Nos. 11971142, 11871202, 61673169, 11701176, 11626101, 11601485) and the Natural Science Foundation of Zhejiang Province (Grant No. LY19A010012).

Conflict of interest

The authors declare that they have no competing interests.

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