Morphisms on Bi-approximation Semantics

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Abstract

In the present paper, we introduce bounded morphisms on bi-approximation semantics, show the so-called p-morphism lemma on bi-approximation semantics, and investigate the dual representation of the morphisms. In addition, we study three properties, namely B-embedding, B-separating and B-reflecting, to preserve validity of sequents on frames. These bounded morphisms do not look like embedding, surjective and isomorphic p-morphisms on Kripke semantics in modal logic. Nevertheless, with help of auxiliary relations or properties held via the dual representation, we notice that the notion of our bounded morphisms on bi-approximation semantics is a natural generalisation of the one on Kripke semantics in modal logic.

Keywords: Bi-approximation semantics, substructural logic, Stone-style representation, Ghilardi and Meloni's canonicity methodology.

1 Introduction

What is the right notion of relational semantics of non-distributive, i.e. not necessarily distributive, lattice-based logics, such as for example substructural logic? The main problem to deal with non-distributive lattice-based logics on relational semantics is how to avoid validating the distributive law, i.e. $\phi \wedge (\psi \vee \chi)$ implies $(\phi \wedge \psi) \vee (\phi \wedge \chi)$. That is, on a Kripke-style semantics, if we interpret conjunction $\wedge$ as "and" and disjunction $\vee$ as "or" as in the case of modal logic, each state naturally satisfies the distributive law. In the literature, we can find some relational-type semantics for non-distributive lattice-based logics by introducing non-standard interpretations of disjunction $\vee$ to reject the distributivity, see e.g. [10,13,12,8], also [15].

Bi-approximation semantics has been introduced in [17] to investigate another potential relational-type semantics for non-distributive lattice-based logics. Unlike what happens in normal relational semantics, we reason about logical consequences (sequents), instead of formulae, on bi-approximation semantics. Note that, recently, a relational-type semantics, named *residuated frames* [6], was introduced as a complete analog of the proof system of substructural logic, which can be useful to the decidability property. On the other hand,

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bi-approximation semantics was introduced as a canonicity-friendly framework to characterise *Ghilardi and Meloni's canonicity methodology* [9], which is applicative to other lattice-based logics. Furthermore, since bi-approximation semantics is canonicity-friendly, we can also discuss a Sahlqvist theorem on bi-approximation semantics [16].

In [17], we have investigated the idea of reasoning about sequents on bi-approximation semantics, and shown the object-level Stone-style representation theorem between p-frames and FL-algebras. However, we have not studied how morphisms are defined on bi-approximation semantics, and if they exist, how they are related to bounded morphisms in Kripke-style semantics. The main purpose of the current paper is to give a possible answer for those questions. In this paper, we shall introduce bounded morphisms to preserve truth values of formulae, and discuss the so-called *p-morphism lemma*, e.g. [11,2,3]. Based on these bounded morphisms, we shall think about invariance of validity of sequents on p-frames via specific morphisms, which are not exactly the same as the validity preserving p-morphisms in Kripke semantics, called *embedding*, *surjective* and *isomorphic* p-morphisms. On the other hand, we shall analyse how our bounded morphisms are related to p-morphisms on Kripke-style semantics. Besides, we shall also prove the dual representation of morphisms between the category of lattice expansions and strict homomorphisms and the category of p-frames and bounded morphisms. In the end, we observe that our bounded morphism is a natural generalisation of p-morphisms on Kripke-style semantics.

We outline the structure of the current paper. We briefly recall the basic terminology of substructural logic in Section 2. In Section 3, bi-approximation semantics are summarised and the fundamental results of bi-approximation semantics are reviewed. Bounded morphisms for bi-approximation semantics are introduced in Section 4. Further, we shall prove the preservation of truth value on bi-approximation models, the so-called p-morphism lemma, for the three different satisfaction relations. Notice that there are three satisfaction relations: one is for assumptions (formulae), the other is for conclusions (formulae), and the last one is for truth value of sequents. In addition, we shall introduce the notion of the validity preserving bounded morphisms on bi-approximation semantics, and show the invariance of validity of sequents. In Section 5, the dual representation of morphisms between lattice expansions and bi-approximation semantics will be argued. As a result, we shall notice that our notion of bounded morphisms also satisfies the same properties as the dual representation of morphisms between Boolean algebras and Kripke semantics via the Stone representation. Finally, in Section 6, we shall give some conclusive remarks. We also note that, because of the strict page restriction, many proofs are in the appendix.

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2 Substructural logic

In this paper, we denote propositional variables by $p, q, r, p_1, \ldots$, the set of all propositional variables by Φ, and $\mathbf{t}$ and $\mathbf{f}$ are logical constants representing *true* and *false*, respectively. As logical connectives, we use *disjunction* $\vee$, *conjunction* $\wedge$, *fusion* (*multiplication*) $\circ$, *implications* (*residuals*) $\rightarrow$ and $\leftarrow$. Formulae of substructural logic are denoted by ϕ, ψ, χ, $\phi_1, \ldots$, and $\psi_1, \ldots$, and the set of all formulae is denoted by Λ. The following BNF generates formulae of substructural logic.

$$\phi ::= p \mid \mathbf{t} \mid \mathbf{f} \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \circ \phi \mid \phi \rightarrow \phi \mid \phi \leftarrow \phi$$

Γ, Δ, Σ, Π are (possibly empty) finite lists of formulae, and θ is a list of at most one formula. Then, we call $\Gamma \Rightarrow \theta$ a *sequent*, see e.g. [14].

For sequents, we consider a sequent calculus for substructural logic, called *FL* as in Fig. 1. In the sequent calculus FL, a formula ϕ is *provable in FL* if

Initial sequents.

$$\phi \Rightarrow \phi \qquad\qquad \Rightarrow \mathbf{t} \qquad\qquad \mathbf{f} \Rightarrow$$

Cut rule.

$$\frac{\Gamma \Rightarrow \phi \qquad \Sigma, \phi, \Pi \Rightarrow \theta}{\Sigma, \Gamma, \Pi \Rightarrow \theta}\ (\text{cut})$$

Rules for logical constants.

$$\frac{\Gamma, \Delta \Rightarrow \theta}{\Gamma, \mathbf{t}, \Delta \Rightarrow \theta}\ (\mathbf{t}\text{w}) \qquad\qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{f}}\ (\mathbf{f}\text{w})$$

Rules for logical connectives.

$$\frac{\Gamma, \phi, \Delta \Rightarrow \theta \qquad \Gamma, \psi, \Delta \Rightarrow \theta}{\Gamma, \phi \vee \psi, \Delta \Rightarrow \theta}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi}\ (\Rightarrow \vee_1) \qquad\qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}\ (\Rightarrow \vee_2)$$

$$\frac{\Gamma, \phi, \Delta \Rightarrow \theta}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \theta}\ (\wedge_1 \Rightarrow) \qquad\qquad \frac{\Gamma, \psi, \Delta \Rightarrow \theta}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \theta}\ (\wedge_2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \phi, \psi, \Delta \Rightarrow \theta}{\Gamma, \phi \circ \psi, \Delta \Rightarrow \theta}\ (\circ \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \phi \qquad \Sigma \Rightarrow \psi}{\Gamma, \Sigma \Rightarrow \phi \circ \psi}\ (\Rightarrow \circ)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Sigma, \psi, \Pi \Rightarrow \theta}{\Sigma, \Gamma, \phi \rightarrow \psi, \Pi \Rightarrow \theta}\ (\rightarrow\Rightarrow) \qquad\qquad \frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \rightarrow \psi}\ (\Rightarrow\rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Sigma, \psi, \Pi \Rightarrow \theta}{\Sigma, \psi \leftarrow \phi, \Gamma, \Pi \Rightarrow \theta}\ (\leftarrow\Rightarrow) \qquad\qquad \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \psi \leftarrow \phi}\ (\Rightarrow\leftarrow)$$

Fig. 1. The sequent calculus FL

the sequent $\Rrightarrow \phi$ is derivable in FL. The substructural logic **FL** is the set of all provable formulae in FL.

Proposition 2.1 *For all formulae ϕ and ψ, we have*

(i) *ϕ is provable if and only if $\mathbf{t} \Rrightarrow \phi$ is derivable,*

(ii) *$\phi \Rrightarrow \psi$ is derivable if and only if $\phi \to \psi$ is provable in FL if and only if $\psi \leftarrow \phi$ is provable in FL,*

(iii) *$\phi_1, \ldots, \phi_n \Rrightarrow \theta$ is derivable in FL if and only if $\phi_1 \circ \cdots \circ \phi_n \Rrightarrow \theta$ is derivable in FL.*

By Proposition 2.1, we sometimes state that the substructural logic **FL** is the set of all sequents derivable in FL. Besides, we also think about a sequent $\phi_1, \ldots, \phi_n \Rrightarrow \theta$ as a pair of two formulae $\phi_1 \circ \cdots \circ \phi_n$ and θ. In particular, if the left-hand side is empty, we let the first formula $\mathbf{t}$, and if the right-hand side is empty, we let the second formula $\mathbf{f}$. Hence, hereinafter, we may state that a sequent $\phi \Rrightarrow \psi$ is a pair of two formulae.

The algebraic counterparts of the substructural logic **FL** are known as FL-algebras [7].

Definition 2.2 (FL-algebra) *An 8-tuple $\mathbb{A} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0\rangle$ is an* FL-algebra*, if $\langle A, \vee, \wedge\rangle$ forms a lattice, $\langle A, *, 1\rangle$ is a monoid, 0 is a constant in A, and for all $a, b, c \in A$, we have*

$$a * b \leq c \iff b \leq a \backslash c \iff a \leq c/b.$$

On FL-algebras, each formula ϕ is interpreted to the corresponding term t as usual, i.e. $\circ$, $\to$, $\leftarrow$, $\mathbf{t}$ and $\mathbf{f}$ are interpreted to $*$, $\backslash$, $/$, 1 and 0. The corresponding term of a formula ϕ is denoted by s_ϕ or by t_ϕ. Furthermore, each sequent $\phi_1, \ldots, \phi_n \Rrightarrow \theta$ is interpreted as an inequality $s_{\phi_1} * \cdots * s_{\phi_n} \leq t_\theta$. We sometimes abuse the notation for a list of formulae, i.e. s_Γ is the term $s_{\phi_1} * \cdots * s_{\phi_n}$, when Γ is the finite list of formulae $\phi_1, \ldots, \phi_n$. Note that if the left hand side of a sequent is empty then the corresponding term is 1, and if the right hand side of a sequent is empty then the corresponding term is 0.

3 Bi-approximation semantics

Bi-approximation semantics is designed as a canonicity-friendly relational semantics for substructural logic [17]. A novelty of this semantics is to evaluate not only formulae but also sequents based on polarities. A *polarity* is a triple $\langle X, Y, B\rangle$ where X and Y are not-necessarily disjoint and non-empty sets and B is a binary relation on $X \times Y$, see e.g. [1,4,8] for polarities. Note that, for each polarity $\langle X, Y, B\rangle$, the binary relation B is naturally extended to a preorder order $\leq$ on $X \cup Y$, see [8,17]. For $x_1, x_2 \in X$, we let $x_1 \leq x_2 \iff \forall y \in Y. [x_2 B y \Longrightarrow x_1 B y]$ and for $y_1, y_2 \in Y$, we let $y_1 \leq y_2 \iff \forall x \in X. [x B y_1 \Longrightarrow x B y_2]$. Hence, we may call the triple $\langle X, Y, \leq\rangle$ with the extended preorder order $\leq$ a *polarity.* Unlike the standard relational semantics like Kripke semantics or Routley-Meyer semantics, bi-approximation semantics reasons about sequents on a polarity $\langle X, Y, \leq\rangle$ as follows: we evaluate *premises* on X, *conclusions* on Y and sequents (logical consequences) as the binary relation $\leq$ between premises and conclusions.

Definition 3.1 *A* polarity frame for substructural logic*,* p-frame *for short, is an 8-tuple* $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y\rangle$*, where* $\langle X, Y, \leq\rangle$ *is a polarity,* R *is a ternary relation on* $X \times X \times Y$*,* O_X *is a non-empty subset of* X*,* N_X *is a subset of* X*,* O_Y *and* N_Y *are subsets of* Y*, and* $\mathbb{F}$ *satisfies*

R-order: *for all* $x, x' \in X$*,*

$$x' \leq x \text{ if and only if } \exists o \in O_X.\big[R^\circ(x, o, x') \text{ or } R^\circ(o, x, x')\big];$$

R-identity: *for each* $x \in X$*,*

$$\exists o_2 \in O_X.\big[R^\circ(x, o_2, x)\big] \text{ and } \exists o_1 \in O_X.\big[R^\circ(o_1, x, x)\big];$$

R-transitivity: *for all* $x_1, x'_1, x_2, x'_2 \in X$ *and* $y, y' \in Y$*,*

$$x'_1 \leq x_1, x'_2 \leq x_2, y \leq y' \text{ and } R(x_1, x_2, y) \Longrightarrow R(x'_1, x'_2, y');$$

R-associativity: *for all* $x_1, x_2, x_3, x \in X$*,*

$$\exists x' \in X.\big[R^\circ(x_1, x', x) \text{ and } R^\circ(x_2, x_3, x')\big]$$
$$\Longleftrightarrow \exists x'' \in X.\big[R^\circ(x_1, x_2, x'') \text{ and } R^\circ(x'', x_3, x)\big];$$

O: $O_X = \{x \in X \mid \forall y \in O_Y. x \leq y\}$ *and* $O_Y = \{y \in Y \mid \forall x \in O_X. x \leq y\}$*;*

N: $N_X = \{x \in X \mid \forall y \in N_Y. x \leq y\}$ *and* $N_Y = \{y \in Y \mid \forall x \in N_X. x \leq y\}$*;*

$\circ$***-tightness:*** *for all* $x_1, x_2 \in X$ *and* $y \in Y$*,*

$$\forall x \in X.\big[R^\circ(x_1, x_2, x) \Longrightarrow x \leq y\big] \Longrightarrow R(x_1, x_2, y);$$

$\rightarrow$***-tightness:*** *for all* $x_1, x_2 \in X$ *and* $y \in Y$*,*

$$\forall y_2 \in Y.\big[R^\rightarrow(x_1, y_2, y) \Longrightarrow x_2 \leq y_2\big] \Longrightarrow R(x_1, x_2, y);$$

$\leftarrow$***-tightness:*** *for all* $x_1, x_2 \in X$ *and* $y \in Y$*,*

$$\forall y_1 \in Y.\big[R^\leftarrow(y_1, x_2, y) \Longrightarrow x_1 \leq y_1\big] \Longrightarrow R(x_1, x_2, y);$$

where $R^\circ(x_1, x_2, x)$*,* $R^\rightarrow(x_1, y_2, y)$ *and* $R^\leftarrow(y_1, x_2, y)$ *are auxiliary relations of* R *defined as follows:*

R°***:*** $R^\circ(x_1, x_2, x) \iff \forall y \in Y.\big[R(x_1, x_2, y) \Longrightarrow x \leq y\big]$*;*

$R^\rightarrow$***:*** $R^\rightarrow(x_1, y_2, y) \iff \forall x_2 \in X.\big[R(x_1, x_2, y) \Longrightarrow x_2 \leq y_2\big]$*;*

$R^\leftarrow$***:*** $R^\leftarrow(y_1, x_2, y) \iff \forall x_1 \in X.\big[R(x_1, x_2, y) \Longrightarrow x_1 \leq y_1\big]$*.*

Remark 3.2 In [17], the tightness conditions are given by more restricted forms. However, we can weaken those tightness conditions as above to show all results in [17], and they are actually even more direct. But the most important reason to introduce the above tightness conditions is that the slight difference affects the current author's subsequent works.

Remark 3.3 The conditions of p-frames are not completely independent (e.g. the R-transitivity follows from the other conditions). However, we keep the above definition to discuss similarities to Kripke-type semantics for distributive substructural logics.

On p-frames, we define doppelgänger valuations to give compatible truth value to atomic sequents, namely we want to make $p \mapsto p$ valid for each propositional variable $p \in \Phi$.

Definition 3.4 (Doppelgänger valuation) *Given a p-frame* $\mathbb{F}$*, a pair* $V = (V^{\downarrow}, V_{\uparrow})$ *of two functions* $V^{\downarrow}\colon \Phi \to \wp(X)$ *and* $V_{\uparrow}\colon \Phi \to \wp(Y)$*, where* $\wp(X)$ *and* $\wp(Y)$ *are the powersets of* X *and* Y*, is a* doppelgänger valuation *on* $\mathbb{F}$*, if* $V^{\downarrow}(p) = \{x \in X \mid \forall y \in V_{\uparrow}(p).\, x \leq y\}$ *and* $V_{\uparrow}(p) = \{y \in Y \mid \forall x \in V^{\downarrow}(p).\, x \leq y\}$ *for each propositional variable* $p \in \Phi$*.*

Given a p-frame $\mathbb{F}$ and a doppelgänger valuation V on $\mathbb{F}$, we call the pair $\mathbb{M} = \langle \mathbb{F}, V \rangle$ a *bi-approximation model.* On a bi-approximation model $\mathbb{M}$, we inductively define a satisfaction relation $\Vdash$ as follows: for all $x \in X$ and $y \in Y$, we let

X-1 $\mathbb{M} \overset{x}{\Vdash} p \iff x \in V^{\downarrow}(p)$ for each propositional variable $p \in \Phi$,

X-2 $\mathbb{M} \overset{x}{\Vdash} \mathbf{t} \iff x \in O_X$,

X-3 $\mathbb{M} \overset{x}{\Vdash} \mathbf{f} \iff x \in N_X$,

X-4 $\mathbb{M} \overset{x}{\Vdash} \phi \vee \psi \iff \forall y \in Y.\big[\mathbb{M} \underset{y}{\Vdash} \phi \vee \psi \Longrightarrow x \leq y\big]$,

X-5 $\mathbb{M} \overset{x}{\Vdash} \phi \wedge \psi \iff \mathbb{M} \overset{x}{\Vdash} \phi$ and $\mathbb{M} \overset{x}{\Vdash} \psi$,

X-6 $\mathbb{M} \overset{x}{\Vdash} \phi \circ \psi \iff \forall y \in Y.\big[\mathbb{M} \underset{y}{\Vdash} \phi \circ \psi \Longrightarrow x \leq y\big]$,

X-7 $\mathbb{M} \overset{x}{\Vdash} \phi \to \psi \iff \forall x' \in X, y \in Y.\big[\mathbb{M} \overset{x'}{\Vdash} \phi$ and $\mathbb{M} \underset{y}{\Vdash} \psi \Longrightarrow R(x', x, y)\big]$,

X-8 $\mathbb{M} \overset{x}{\Vdash} \psi \leftarrow \phi \iff \forall x' \in X, y \in Y.\big[\mathbb{M} \overset{x'}{\Vdash} \phi$ and $\mathbb{M} \underset{y}{\Vdash} \psi \Longrightarrow R(x, x', y)\big]$,

Y-1 $\mathbb{M} \underset{y}{\Vdash} p \iff y \in V_{\uparrow}(p)$ for each propositional variable $p \in \Phi$,

Y-2 $\mathbb{M} \underset{y}{\Vdash} \mathbf{t} \iff y \in O_Y$,

Y-3 $\mathbb{M} \underset{y}{\Vdash} \mathbf{f} \iff y \in N_Y$,

Y-4 $\mathbb{M} \underset{y}{\Vdash} \phi \vee \psi \iff \mathbb{M} \underset{y}{\Vdash} \phi$ and $\mathbb{M} \underset{y}{\Vdash} \psi$,

Y-5 $\mathbb{M} \underset{y}{\Vdash} \phi \wedge \psi \iff \forall x \in X.\big[\mathbb{M} \overset{x}{\Vdash} \phi \wedge \psi \Longrightarrow x \leq y\big]$,

Y-6 $\mathbb{M} \underset{y}{\Vdash} \phi \circ \psi \iff \forall x_1, x_2 \in X.\big[\mathbb{M} \overset{x_1}{\Vdash} \phi$ and $\mathbb{M} \overset{x_2}{\Vdash} \psi \Longrightarrow R(x_1, x_2, y)\big]$,

Y-7 $\mathbb{M} \underset{y}{\Vdash} \phi \to \psi \iff \forall x \in X.\big[\mathbb{M} \overset{x}{\Vdash} \phi \to \psi \Longrightarrow x \leq y\big]$,

Y-8 $\mathbb{M} \underset{y}{\Vdash} \psi \leftarrow \phi \iff \forall x \in X.\big[\mathbb{M} \overset{x}{\Vdash} \psi \leftarrow \phi \Longrightarrow x \leq y\big]$,

S-1 $\mathbb{M} \overset{x}{\underset{y}{\Vdash}} \phi \mapsto \psi \iff$ if $\mathbb{M} \overset{x}{\Vdash} \phi$ and $\mathbb{M} \underset{y}{\Vdash} \psi$ then $x \leq y$,

S-2 $\mathbb{M} \Vdash \phi \mapsto \psi \iff \forall x \in X, y \in Y.\big[\mathbb{M} \overset{x}{\underset{y}{\Vdash}} \phi \mapsto \psi\big]$,

S-3 $\mathbb{F} \models \phi \mapsto \psi \iff \langle \mathbb{F}, V \rangle \models \phi \mapsto \psi$ for every doppelgänger valuation V.

Definition 3.5 (Truth value) *The satisfaction relation* $\models$ *is interpreted as follows:*

(i) $\mathbb{M} \overset{x}{\models} \phi$*: a formula* ϕ *is* assumed *at* x *in* $\mathbb{M}$*,*

(ii) $\mathbb{M} \underset{y}{\models} \psi$*: a formula* ψ *is* concluded *at* y *in* $\mathbb{M}$*,*

(iii) $\mathbb{M} \overset{x}{\underset{y}{\models}} \phi \mapsto \psi$*: a sequent* $\phi \mapsto \psi$ *is* true *between* x *and* y *in* $\mathbb{M}$*,*

(iv) $\mathbb{M} \models \phi \mapsto \psi$*: a sequent* $\phi \mapsto \psi$ *is* universally true *on* $\mathbb{M}$*,*

(v) $\mathbb{F} \models \phi \mapsto \psi$*: a sequent* $\phi \mapsto \psi$ *is* valid *on* $\mathbb{F}$*.*

Preceding results on bi-approximation semantics. Thanks to the tightness conditions in Definition 3.1, we obtain the following lemma for auxiliary relations R°, $R^\to$ and $R^\leftarrow$.

Lemma 3.6 (Redefinition of R) *For each p-frame* $\mathbb{F}$*, we have*

(i) $R(x_1, x_2, y) \iff \forall x \in X. \big[R^\circ(x_1, x_2, x) \Longrightarrow x \le y\big]$,

(ii) $R(x_1, x_2, y) \iff \forall y_2 \in Y. \big[R^\to(x_1, y_2, y) \Longrightarrow x_2 \le y_2\big]$,

(iii) $R(x_1, x_2, y) \iff \forall y_1 \in Y. \big[R^\leftarrow(y_1, x_2, y) \Longrightarrow x_1 \le y_1\big]$.

The so-called *Hereditary property* on bi-approximation semantics is given as follows.

Proposition 3.7 (Hereditary) *Let* $\mathbb{M}$ *be a bi-approximation model. For all* $x, x' \in X$ *and* $y, y' \in Y$*, we have*

(i) *if* $x' \le x$ *and* $\mathbb{M} \overset{x}{\models} \phi$ *then* $\mathbb{M} \overset{x'}{\models} \phi$*,*

(ii) *if* $y \le y'$ *and* $\mathbb{M} \underset{y}{\models} \psi$ *then* $\mathbb{M} \underset{y'}{\models} \psi$*.*

The following proposition states that every doppelgänger valuation is naturally extended from propositional variables Φ to all formulae Λ: see also [17, Corollary 3.11].

Proposition 3.8 *Let* $\mathbb{M}$ *be a bi-approximation model. For each* $x \in X$*, each* $y \in Y$ *and all formulae* $\phi, \psi \in \Lambda$*, we have*

(i) $\mathbb{M} \overset{x}{\models} \phi \iff \forall y \in Y. \big[\mathbb{M} \underset{y}{\models} \phi \Longrightarrow x \le y\big]$,

(ii) $\mathbb{M} \underset{y}{\models} \psi \iff \forall x \in X. \big[\mathbb{M} \overset{x}{\models} \psi \Longrightarrow x \le y\big]$.

Furthermore, we can also prove the soundness and Sahlqvist completeness theorem on bi-approximation semantics.

Theorem 3.9 (Soundness [17]) *Let* $\phi \mapsto \psi$ *be a sequent. If* $\phi \mapsto \psi$ *is derivable in the sequent system FL then it is valid on any p-frame.*

Theorem 3.10 (Sahlqvist completeness [18,16]) *Let* Ω *be a set of sequents which have consistent variable occurrence: see [18] for the definition of consistent variable occurrence. The substructural logic* $\mathbf{FL} \oplus \Omega$*, which is* $\mathbf{FL}$

extended by Ω, is elementary and canonical, hence complete with respect to a class of first-order definable p-frames.

4 Bounded morphisms on bi-approximation semantics

In this section, we introduce *bounded morphisms* by focusing on invariance of the satisfaction relation $\Vdash$: see Lemmata 4.5 and 4.6, and Theorem 4.10.

Definition 4.1 (Bounded morphisms) *Given two p-frames* $\mathbb{F} = \langle X_1, Y_1, \leq_1, R_1, O_{X1}, O_{Y1}, N_{X1}, N_{Y1}\rangle$ *and* $\mathbb{G} = \langle X_2, Y_2, \leq_2, R_2, O_{X2}, O_{Y2}, N_{X2}, N_{Y2}\rangle$, *a pair* $\langle\sigma|\tau\rangle$ *of two functions* $\sigma\colon X_1 \to X_2$ *and* $\tau\colon Y_1 \to Y_2$ *is a* bounded morphism *from* $\mathbb{F}$ *to* $\mathbb{G}$, *denoted by* $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$, *if* $\langle\sigma|\tau\rangle$ *satisfies*

(i) *for all* $x \in X_1$ *and* $y \in Y_1$, $\sigma(x) \leq_2 \tau(y) \Longrightarrow x \leq_1 y$;

(ii) *for all* $x \in X_1$ *and* $y' \in Y_2$,

$$\forall y \in Y_1.\big[y' \leq_2 \tau(y) \Longrightarrow x \leq_1 y\big] \Longrightarrow \sigma(x) \leq_2 y';$$

(iii) *for all* $x' \in X_2$ *and* $y \in Y_1$,

$$\forall x \in X_1.\big[\sigma(x) \leq_2 x' \Longrightarrow x \leq_1 y\big] \Longrightarrow x' \leq_2 \tau(y);$$

(iv) *for all* $x_1, x_2 \in X_1$ *and* $y \in Y_1$, $R_2(\sigma(x_1), \sigma(x_2), \tau(y)) \Longrightarrow R_1(x_1, x_2, y)$;

(v) *for all* $x'_1, x'_2 \in X_2$ *and* $y \in Y_1$,

$$\forall x_1, x_2 \in X_1.\big[\sigma(x_1) \leq_2 x'_1 \text{ and } \sigma(x_2) \leq_2 x'_2 \Longrightarrow R_1(x_1, x_2, y)\big] \Longrightarrow R_2(x'_1, x'_2, \tau(y));$$

(vi) *for all* $x'_1 \in X_2$, $x_2 \in X_1$ *and* $y' \in Y_2$,

$$\forall x_1 \in X_1, y \in Y_1.\big[\sigma(x_1) \leq_2 x'_1 \text{ and } y' \leq_2 \tau(y) \Longrightarrow R_1(x_1, x_2, y)\big] \Longrightarrow R_2(x'_1, \sigma(x_2), y');$$

(vii) *for all* $x_1 \in X_1$, $x'_2 \in X_2$ *and* $y' \in Y_2$,

$$\forall x_2 \in X_1, y \in Y_1.\big[\sigma(x_2) \leq_2 x'_2 \text{ and } y' \leq_2 \tau(y) \Longrightarrow R_1(x_1, x_2, y)\big] \Longrightarrow R_2(\sigma(x_1), x'_2, y');$$

(viii) *for each* $x \in X_1$,

$$\big[x \in O_{X1} \iff \sigma(x) \in O_{X2}\big] \text{ and } \big[x \in N_{X1} \iff \sigma(x) \in N_{X2}\big];$$

(ix) *for each* $y \in Y_1$,

$$\big[y \in O_{Y1} \iff \tau(y) \in O_{Y2}\big] \text{ and } \big[y \in N_{Y1} \iff \tau(y) \in N_{Y2}\big].$$

Moreover, a bounded morphisms $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ *is a bounded morphism on bi-approximation models from* $\langle\mathbb{F}, U\rangle$ *to* $\langle\mathbb{G}, V\rangle$, *if* $\langle\sigma|\tau\rangle$ *additionally satisfies*

(x) for each $x \in X_1$, $x \in U^{\downarrow}(p) \iff \sigma(x) \in V^{\downarrow}(p)$,

(xi) for each $y \in Y_1$, $y \in U_{\uparrow}(p) \iff \tau(y) \in V_{\uparrow}(p)$,

for each propositional variable $p \in \Phi$.

Remark 4.2 Bounded morphisms for polarities are given by the first three conditions.

Remark 4.3 In [8], morphisms on generalized Kripke frames are induced by the complete homomorphisms on the dual algebras. On the setting, we cannot define morphisms on two-sorted frames in our setting unless they are surjective. On the other hand, our setting allows us to have complete homomorphisms on the dual morphisms which cannot be represented by our bounded morphisms, and this is not the case of modal logic.

At first, one may feel that the conditions of bounded morphisms for bi-approximation semantics are far from those for Kripke semantics. For example, a p-morphism $\rho\colon \langle W, R\rangle \to \langle W', R'\rangle$ for Kripke frames satisfies the following condition: for all $w_1, w_2 \in W_1$, if $R(w_1, w_2)$ then $R'(\rho(w_1), \rho(w_2))$. This condition looks completely opposite of (i) in Definition 4.1. However, with help of the definition of $\leq$ and the auxiliary relations R°, $R^\rightarrow$ and $R^\leftarrow$ (see Definition 3.1), we can find some similarities.

Proposition 4.4 *Let* $\mathbb{F}$ *and* $\mathbb{G}$ *be p-frames, and* $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ *a bounded morphism. Then, we have*

(i) *for all* $x_1, x_2 \in X_1$, $x_1 \leq_1 x_2 \Longrightarrow \sigma(x_1) \leq_2 \sigma(x_2)$, [2]

(ii) *for all* $y_1, y_2 \in Y_1$, $y_1 \leq_1 y_2 \Longrightarrow \tau(y_1) \leq_2 \tau(y_1)$,

(iii) *for all* $x_1, x_2, x \in X_1$, $R_1^\circ(x_1, x_2, x) \Longrightarrow R_2^\circ(\sigma(x_1), \sigma(x_2), \sigma(x))$,

(iv) *for all* $x_1 \in X_1$, $y_2, y \in Y_1$, $R_1^\rightarrow(x_1, y_2, y) \Longrightarrow R_2^\rightarrow(\sigma(x_1), \tau(y_2), \tau(y))$,

(v) *for all* $x_2 \in X_2$, $y_1, y \in Y_1$, $R_1^\leftarrow(y_1, x_2, y) \Longrightarrow R_2^\leftarrow(\tau(y_1), \sigma(x_2), \tau(y))$.

Now we shall show the so-called p-morphism lemma for bi-approximation models. However, unlike the p-morphism lemma for Kripke models, in our case, there are three satisfaction relations which we should respect, i.e. two types of $\Vdash$ for formulae and one $\Vdash$ for sequents.

Lemma 4.5 (for formulae) *Let* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *be bi-approximation models, and* $\langle\sigma|\tau\rangle\colon \mathbb{M}_1 \to \mathbb{M}_2$ *a bounded morphism. For all formulae* $\phi, \psi \in \Lambda$, *each* $x \in X_1$ *and each* $y \in Y_1$, *we have*

(i) $\mathbb{M}_1 \overset{x}{\Vdash} \phi \iff \mathbb{M}_2 \overset{\sigma(x)}{\Vdash} \phi$,

(ii) $\mathbb{M}_1 \underset{y}{\Vdash} \psi \iff \mathbb{M}_2 \underset{\tau(y)}{\Vdash} \psi$.

Next, we show the p-morphism lemma for sequents.

Lemma 4.6 (for sequents) *Let* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *be bi-approximation models, and* $\langle\sigma|\tau\rangle\colon \mathbb{M}_1 \to \mathbb{M}_2$ *a bounded morphism. For every sequent* $\phi \mapsto \psi$, *we have*

$$\mathbb{M}_1 \Vdash \phi \mapsto \psi \iff \forall x \in X_1, y \in Y_1. \big[\mathbb{M}_2 \overset{\sigma(x)}{\underset{\tau(y)}{\Vdash}} \phi \mapsto \psi\big].$$

[2] This follows from (iii) and R-order on the current setting. However, this holds on the setting of bounded morphisms for polarities as well.

Proof. ($\Rightarrow$). For arbitrary $x \in X_1$ and $y \in Y_1$, suppose that $\mathbb{M}_2 \Vdash^{\sigma(x)} \phi$ and $\mathbb{M}_2 \Vdash_{\tau(y)} \psi$. To show that $\sigma(x) \leq_2 \tau(y)$, we shall use (ii) of Definition 4.1. That is, it suffices to show that, for each $y' \in Y_1$, if $\tau(y) \leq_2 \tau(y')$ then $x \leq_1 y'$. By the Hereditary condition, if $\tau(y) \leq_2 \tau(y')$, we have $\mathbb{M}_2 \Vdash_{\tau(y')} \psi$. Thanks to Lemma 4.5, we obtain that $\mathbb{M}_1 \Vdash^{x} \phi$ and $\mathbb{M}_1 \Vdash_{y'} \psi$. Since $\mathbb{M}_1 \models \phi \mapsto \psi$, we get $x \leq_1 y'$. Therefore, $\sigma(x) \leq_2 \tau(y)$.

($\Leftarrow$). For arbitrary $x \in X_1$ and $y \in Y_1$, if $\mathbb{M}_1 \Vdash^{x} \phi$ and $\mathbb{M}_1 \Vdash_{y} \psi$, by Lemma 4.5, we have that $\mathbb{M}_2 \Vdash^{\sigma(x)} \phi$ and $\mathbb{M}_2 \Vdash_{\tau(y)} \psi$. By our assumption $\mathbb{M}_2 \Vdash^{\sigma(x)}_{\tau(y)} \phi \mapsto \psi$, we obtain that $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, we conclude $x \leq_1 y$. □

Remark 4.7 Unlike what happens in the setting of modal logic, our bounded morphisms are not strong enough to show the local correspondence property, i.e. $\mathbb{M}_1 \Vdash^{x}_{y} \phi \mapsto \psi \iff \mathbb{M}_2 \Vdash^{\sigma(x)}_{\tau(y)} \phi \mapsto \psi$, in general. To prove it, we need an extra condition: see Lemma 4.9.

Next, our discussion is heading toward invariance of validity of sequents, namely p-morphism lemmata for p-frames. To do so, we first introduce the following special bounded morphisms. Let $\mathbb{F}$ and $\mathbb{G}$ be p-frames, and $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ a bounded morphism.

B-embedding: for all $x \in X_1$ and $y \in Y_1$, $x \leq_1 y \Longrightarrow \sigma(x) \leq_2 \tau(y)$;

B-separating: for all $x' \in X_2$ and $y' \in Y_2$,

$$\forall x \in X_1, y \in Y_1.\big[\sigma(x) \leq_2 x' \text{ and } y' \leq_2 \tau(y) \Longrightarrow x \leq_1 y\big] \Longrightarrow x' \leq_2 y';$$

B-reflecting: both B-embedding and B-separating.

Intuitively, B-embedding and B-separating are sort of the order-embedding and the surjectivity for bi-approximation semantics, respectively: see also Theorem 4.10. Therefore, B-reflecting is sort of the isomorphism. However, on bi-approximation semantics, since truth value of sequents is *approximated*, we may have states which do not affect to evaluate sequents. In other words, the surjectivity is not vital to argue invariance of validity of sequents. B-reflecting is designed to capture the essence of the approximation. So, B-reflecting is not always isomorphic, but it is perfectly describing the approximation of sequents on $\mathbb{F}$ as the approximation of sequents on $\mathbb{G}$, and vise versa. For B-embedding bounded morphisms, we can obtain the following.

Proposition 4.8 *Let $\mathbb{F}$ and $\mathbb{G}$ be p-frames, and $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ a B-embedding bounded morphism. Then we have*

(i) *for all $x_1, x_2 \in X_1$, $\sigma(x_1) \leq_2 \sigma(x_2) \Longrightarrow x_1 \leq_1 x_2$,*

(ii) *for all $y_1, y_2 \in Y_1$, $\tau(y_1) \leq_2 \tau(y_2) \Longrightarrow y_1 \leq_1 y_2$.*

Proof. Here, we show (i) only, but (ii) can be analogously proved. For arbitrary $x_1, x_2 \in X_1$, assume $\sigma(x_1) \leq_2 \sigma(x_2)$. For any $y \in Y_1$, if $x_2 \leq_1 y$ then, as

$\langle\sigma|\tau\rangle$ is B-embedding, $\sigma(x_2) \leq_2 \tau(y)$. By transitivity, we have $\sigma(x_1) \leq \tau(y)$. By (i) of Definition 4.1, we obtain $x_1 \leq_1 y$, which concludes $x_1 \leq_1 x_2$. □

Lemma 4.9 (Local p-morphism lemma for sequents) *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be bi-approximation models, and $\langle\sigma|\tau\rangle\colon \mathbb{M}_1 \to \mathbb{M}_2$ a B-embedding bounded morphism. For each sequent $\phi \Mapsto \psi$, each $x \in X_1$ and each $y \in Y_1$, we have*

$$\mathbb{M}_1 \Vdash^{x}_{y} \phi \Mapsto \psi \iff \mathbb{M}_2 \Vdash^{\sigma(x)}_{\tau(y)} \phi \Mapsto \psi.$$

Proof. ($\Rightarrow$). Suppose that $\mathbb{M}_2 \Vdash^{\sigma(x)} \phi$ and $\mathbb{M}_2 \Vdash_{\tau(y)} \psi$. Thanks to Lemma 4.5, we have that $\mathbb{M}_1 \Vdash^{x} \phi$ and $\mathbb{M}_1 \Vdash_{y} \psi$. By our assumption, we obtain $x \leq_1 y$. Here, as $\langle\sigma|\tau\rangle$ is B-embedding, $\sigma(x) \leq_2 \tau(y)$ holds.

($\Leftarrow$). Assume that $\mathbb{M}_1 \Vdash^{x} \phi$ and $\mathbb{M}_1 \Vdash_{y} \psi$. Thanks to Lemma 4.5, we have that $\mathbb{M}_2 \Vdash^{\sigma(x)} \phi$ and $\mathbb{M}_2 \Vdash_{\tau(y)} \psi$. As $\mathbb{M}_2 \Vdash^{\sigma(x)}_{\tau(y)} \phi \Mapsto \psi$, we get $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, $x \leq_1 y$, which concludes $\mathbb{M}_1 \Vdash^{x}_{y} \phi \Mapsto \psi$. □

Now, we show the following invariance of validity of sequents on bi-approximation semantics.

Theorem 4.10 *Let $\mathbb{F}$ and $\mathbb{G}$ be p-frames, and $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ a bounded morphism. For each sequent $\phi \Mapsto \psi$, we have*

(i) *if $\langle\sigma|\tau\rangle$ is B-embedding then $\mathbb{G} \models \phi \Mapsto \psi \Longrightarrow \mathbb{F} \models \phi \Mapsto \psi$,*

(ii) *if $\langle\sigma|\tau\rangle$ is B-separating then $\mathbb{F} \models \phi \Mapsto \psi \Longrightarrow \mathbb{G} \models \phi \Mapsto \psi$,*

(iii) *if $\langle\sigma|\tau\rangle$ is B-reflecting then $\mathbb{F} \models \phi \Mapsto \psi \iff \mathbb{G} \models \phi \Mapsto \psi$.*

Proof. Here, we prove only (i), by contraposition. Suppose $\mathbb{F} \not\models \phi \Mapsto \psi$. Then, there exists a doppelgänger valuation U on $\mathbb{F}$, $x \in X_1$ and $y \in Y_1$ such that $\mathbb{F}, U \not\Vdash^{x}_{y} \phi \Mapsto \psi$. Firstly, we claim that there exists a doppelgänger valuation V on $\mathbb{G}$ which makes $\langle\sigma|\tau\rangle$ a bounded morphism from $\langle\mathbb{F}, U\rangle$ to $\langle\mathbb{G}, V\rangle$.

For the doppelgänger valuation U on $\mathbb{F}$, we let

(i) $V^{\downarrow}(p) := \{x' \in X_2 \mid \forall y \in U_{\uparrow}(p).\, x' \leq_2 \tau(y)\}$,

(ii) $V_{\uparrow}(p) := \{y' \in Y_2 \mid \forall x' \in V^{\downarrow}(p).\, x' \leq_2 y'\}$,

for each propositional variable $p \in \Phi$. Now, we show that V is a doppelgänger valuation on $\mathbb{G}$. It suffices to show

$$V^{\downarrow}(p) = \{x' \in X_2 \mid \forall y' \in V_{\uparrow}(p).\, x' \leq_2 y'\}.$$

($\subseteq$). For arbitrary $x' \in V^{\downarrow}(p)$ and $y' \in V_{\uparrow}(p)$, by the definition of $V_{\uparrow}(p)$, we have $x' \leq_2 y'$. ($\supseteq$). We prove this by contraposition. Suppose $x' \notin V^{\downarrow}(p)$. Then, there exists $y \in U_{\uparrow}(p)$ such that $x' \not\leq_2 \tau(y)$. By definition, we have $\tau[U_{\uparrow}(p)] \subseteq V_{\uparrow}(p)$. Hence, $\tau(y) \in V_{\uparrow}(p)$, which shows the statement. Therefore, V is a doppelgänger valuation on $\mathbb{G}$.

Next, we show $\langle\sigma|\tau\rangle$ is a bounded morphism from $\langle\mathbb{F}, U\rangle$ to $\langle\mathbb{G}, V\rangle$. That is, $x \in U^{\downarrow}(p) \iff \sigma(x) \in V^{\downarrow}(p)$ and $y \in U_{\uparrow}(p) \iff \tau(y) \in V_{\uparrow}(p)$.

$x \in U^{\downarrow}(p) \iff \sigma(x) \in V^{\downarrow}(p)$. ($\Rightarrow$). For arbitrary $x \in U^{\downarrow}(p)$ and $y \in U_{\uparrow}(p)$, we have $x \leq_1 y$. Since $\langle \sigma | \tau \rangle$ is B-embedding, we obtain $\sigma(x) \leq_2 \tau(y)$, so $\sigma(x) \in V^{\downarrow}(p)$. ($\Leftarrow$). For arbitrary $\sigma(x) \in V^{\downarrow}(p)$ and $y \in U_{\uparrow}(p)$, by the definition of $V^{\downarrow}(p)$, we obtain $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, $x \leq_1 y$, hence $x \in U^{\downarrow}(p)$.

$y \in U_{\uparrow}(p) \iff \tau(y) \in V_{\uparrow}(p)$. ($\Rightarrow$). For arbitrary $y \in U_{\uparrow}(p)$ and $x' \in V^{\downarrow}(p)$, by definition, we have $x' \leq_2 \tau(y)$, hence $\tau(y) \in V_{\uparrow}(p)$. ($\Leftarrow$). For arbitrary $\tau(y) \in V_{\uparrow}(p)$ and $x \in U^{\downarrow}(p)$, because $x \in U^{\downarrow}(p) \iff \sigma(x) \in V^{\downarrow}(p)$ (as we saw above), we have $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, we obtain $x \leq_1 y$, hence $y \in U_{\uparrow}(p)$. Therefore, $\langle \sigma | \tau \rangle$ is a bounded morphism from $\langle \mathbb{F}, U \rangle$ to $\langle \mathbb{G}, V \rangle$.

Now, by Lemma 4.9 and our assumption, i.e. $\mathbb{F}, U \mathrel{\substack{x \\ \not\models \\ y}} \phi \mapsto \psi$, we obtain that $\mathbb{G}, V \mathrel{\substack{\sigma(x) \\ \not\models \\ \tau(y)}} \phi \mapsto \psi$, which derives $\mathbb{G} \not\models \phi \mapsto \psi$. □

Remark 4.11 To prove (i) of Theorem 4.10, we construct a doppelgänger valuation V on $\mathbb{G}$ from a doppelgänger valuation U on $\mathbb{F}$. There are two natural way to do this. One is in the proof of Theorem 4.10. The other is the following: for each proposition variable $p \in \Phi$,

(i) $V^{\downarrow}(p) := \{x' \in X_2 \mid \forall y' \in V_{\uparrow}(p).\, x' \leq_2 y'\}$,

(ii) $V_{\uparrow}(p) := \{y' \in Y_2 \mid \forall x \in U^{\downarrow}(p).\, \sigma(x) \leq_2 y'\}$.

In general, these two doppelgänger valuations do not coincide. However, when $\langle \sigma | \tau \rangle$ satisfies B-separating as well, i.e. B-reflecting, they are always identical.

5 The dual representation

Here, we state that a homomorphism $h \colon \mathbb{A} \to \mathbb{B}$ is *strict*, if for each element $b \in B$ there exist $\underline{a}, \overline{a} \in A$ such that $h(\underline{a}) \leq_B b$ and $b \leq_B h(\overline{a})$. Note that, on bounded lattices (bounded lattice expansions), every homomorphism is strict, because it preserves the constants top $\top$ and bottom $\bot$.

To discuss the dual representation of bi-approximation semantics, we introduce the following categories.

- $\mathcal{FL}$: the category of FL-algebras and strict homomorphisms.
- $\mathcal{BFL}$: the category of bounded FL-algebras and homomorphisms.
- $\mathcal{POL}$: the category of p-frames and bounded morphisms.

We mention that $\mathcal{POL}$ is named after *polarity frames*.

Proposition 5.1 *$\mathcal{BFL}$ is a full and faithful subcategory of $\mathcal{FL}$.*

Below we briefly recall the *object-level* representation of bi-approximation semantics in [17].

Dual algebras of p-frames. For each p-frame $\mathbb{F}$, we construct two isomorphic FL-algebras *in parallel* based on a Galois connection between $\wp(X)$ and $\wp(Y)^{\partial}$, where $\wp(X)$ is the poset of the powerset of X and the set-inclusion $\subseteq$ and $\wp(Y)^{\partial}$ is the poset of the powerset of Y and the set-reverse-inclusion $\supseteq$. Note that the superscript $\_^{\partial}$ indicates that the order is the reverse of the standard inclusion $\subseteq$. We introduce the following two functions $\lambda\colon \wp(X) \to \wp(Y)^\partial$ and $\upsilon\colon \wp(Y)^\partial \to \wp(X)$: for each $\mathfrak{X} \in \wp(X)$ and each $\mathfrak{Y} \in \wp(Y)^\partial$, we let

(i) $\lambda(\mathfrak{X}) := \{y \in Y \mid \forall x \in \mathfrak{X}.\, x \leq y\}$,

(ii) $\upsilon(\mathfrak{Y}) := \{x \in X \mid \forall y \in \mathfrak{Y}.\, x \leq y\}$.

Since λ and υ form a Galois connection, the images are isomorphic as posets, i.e. $\upsilon[\wp(Y)^\partial] \cong \lambda[\wp(X)]$. Hereafter, we denote the images $\upsilon[\wp(Y)^\partial]$ by $\mathbb{D}$ and $\lambda[\wp(X)]$ by $\mathbb{U}$. Note that each element in $\mathbb{D}$ is a downward closed subset of X and every element in $\mathbb{U}$ is an upward closed subset of Y. To extend $\mathbb{D}$ and $\mathbb{U}$ as two isomorphic FL-algebras, we define the operations $\vee$, $\wedge$, $*$, $\backslash$ and $/$, on top of $\mathbb{D}$ and $\mathbb{U}$ as follows: for all $\alpha^\downarrow, \beta^\downarrow \in \mathbb{D}$ and $\alpha_\uparrow, \beta_\uparrow \in \mathbb{U}$ (note that $\alpha^\downarrow$ and $\alpha_\uparrow$, and $\beta^\downarrow$ and $\beta_\uparrow$ are corresponding elements in $\mathbb{D}$ and $\mathbb{U}$)

(i) $\alpha^\downarrow \vee \beta^\downarrow := \upsilon(\alpha_\uparrow \vee \beta_\uparrow)$; $\alpha_\uparrow \vee \beta_\uparrow := \alpha_\uparrow \cap \beta_\uparrow$;

(ii) $\alpha^\downarrow \wedge \beta^\downarrow := \alpha^\downarrow \cap \beta^\downarrow$; $\alpha_\uparrow \wedge \beta_\uparrow := \lambda(\alpha^\downarrow \wedge \beta^\downarrow)$;

(iii) $\alpha^\downarrow * \beta^\downarrow := \upsilon(\alpha_\uparrow * \beta_\uparrow)$; $\alpha_\uparrow * \beta_\uparrow := \{y \in Y \mid \forall x_1 \in \alpha^\downarrow, x_2 \in \beta^\downarrow.\, R(x_1, x_2, y)\}$;

(iv) $\alpha^\downarrow \backslash \beta^\downarrow := \{x_2 \in X \mid \forall x_1 \in \alpha^\downarrow, y \in \beta_\uparrow.\, R(x_1, x_2, y)\}$; $\alpha_\uparrow \backslash \beta_\uparrow := \lambda(\alpha^\downarrow \backslash \beta^\downarrow)$;

(v) $\beta_\uparrow / \alpha^\downarrow := \{x_1 \in X \mid \forall x_2 \in \alpha^\downarrow, y \in \beta_\uparrow.\, R(x_1, x_2, y)\}$; $\beta_\uparrow / \alpha_\uparrow := \lambda(\beta^\downarrow / \alpha^\downarrow)$.

Theorem 5.2 ([17]) $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, O_X, N_X \rangle$ *and* $\langle \mathbb{U}, \vee, \wedge, *, \backslash, /, O_Y, N_Y \rangle$ *are FL-algebras. Furthermore, they are isomorphic.*

Definition 5.3 (Dual algebra) *Let $\mathbb{F}$ be a p-frame.* The dual algebra of $\mathbb{F}$ *is an abstract FL-algebra* $\mathbb{F}^+ = \langle A, \vee, \wedge, *, \backslash, /, 1, 0 \rangle$ *which is isomorphic to* $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, O_X, N_X \rangle$ *and* $\langle \mathbb{U}, \vee, \wedge, *, \backslash, /, O_Y, N_Y \rangle$*.*[3]

Theorem 5.4 ([17]) *Let $\mathbb{F}$ be a p-frame. For each sequent $\phi \mapsto \psi$, we have*

$$\mathbb{F} \Vdash \phi \mapsto \psi \iff \mathbb{F}^+ \models s_\phi \leq t_\psi.$$

Recall that s_ϕ and t_ψ are the algebraic terms for ϕ and ψ: see Section 2.

Dual frames of FL-algebras. Here we show the construction of the dual frames of FL-algebras. We mention that the dual frames correspond to *the intermediate level* in [9] but see also [5,18].

Let $\mathbb{A} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0 \rangle$ be an FL-algebra. On $\mathbb{A}$, we introduce the following polarity $\langle \mathcal{F}, \mathcal{I}, \sqsubseteq \rangle$, where $\mathcal{F}$ is the set of all filters of $\mathbb{A}$, $\mathcal{I}$ is the set of all ideals of $\mathbb{A}$, and $F \sqsubseteq I \iff F \cap I \neq \emptyset$ for $F \in \mathcal{F}$ and $I \in \mathcal{I}$. Note that we have $F_1 \sqsubseteq F_2 \iff F_2 \subseteq F_1$ and $I_1 \sqsubseteq I_2 \iff I_1 \subseteq I_2$ by definition. Now, on top of this polarity, we put extra structures: R, $O_\mathcal{F}$, $O_\mathcal{I}$, $N_\mathcal{F}$ and $N_\mathcal{I}$ as follows: for all $F_1, F_2 \in \mathcal{F}$ and $I \in \mathcal{I}$, we let $R(F_1, F_2, I) \iff F_1 * F_2 \sqsubseteq I$, where $F_1 * F_2 := \{a \in A \mid \exists f_1 \in F_1, f_2 \in F_2.\, f_1 * f_2 \leq a\}$. And, we let $O_\mathcal{F}$, $O_\mathcal{I}$, $N_\mathcal{F}$ and $N_\mathcal{I}$ be the set of all filters containing 1, the set of all ideals containing 1, the set of all filters containing 0 and the set of all ideals containing 0. Then, we call the 8-tuple $\mathbb{A}_+ = \langle \mathcal{F}, \mathcal{I}, \sqsubseteq, R, O_\mathcal{F}, O_\mathcal{I}, N_\mathcal{F}, N_\mathcal{I} \rangle$ *the dual frame of* $\mathbb{A}$.

[3] Yet another concrete construction of the dual algebra is suggested by a reviewer. But, we keep this definition at least in the current paper.

Theorem 5.5 ([17]) *For any FL-algebra $\mathbb{A}$, the dual frame $\mathbb{A}_+$ is a p-frame.*

Also, we mention that, on the dual frame $\mathbb{A}_+$, we have that

(i) $F_1 * F_2 := \{a \in A \mid \exists f_1 \in F_1, f_2 \in F_2.\, f_1 * f_2 \leq a\}$ is a filter,

(ii) $F \backslash I := \{a \in A \mid \exists f \in F, i \in I.\, a \leq f \backslash i\}$ is an ideal,

(iii) $I/F := \{a \in A \mid \exists f \in F, i \in I.\, a \leq i/f\}$ is an ideal,

(iv) $R^{\circ}(F_1, F_2, F) \iff F \sqsubseteq F_1 * F_2$,

(v) $R^{\rightarrow}(F_1, I_2, I) \iff F_1 \backslash I \sqsubseteq I_2$,

(vi) $R^{\leftarrow}(I_1, F_2, I) \iff I/F_2 \sqsubseteq I_1$.

Theorem 5.6 ([17]) *Let $\mathbb{A}$ be an FL-algebra. For each sequent $\phi \mapsto \psi$,*

$$\mathbb{A} \models s_\phi \leq t_\psi \iff \mathbb{A}_+ \Vdash \phi \mapsto \psi,$$

where s_ϕ and t_ψ are the algebraic terms corresponding to ϕ and ψ.

The dual representation of morphisms. Now, we consider the dual representation of morphisms based on the object-level dual representation.

Let $\mathbb{A}$ and $\mathbb{B}$ be FL-algebras, $\mathcal{F}_A$ the set of all filters of $\mathbb{A}$, $\mathcal{F}_B$ the set of all filters of $\mathbb{B}$, $\mathcal{I}_A$ the set of all ideals of $\mathbb{A}$ and $\mathcal{I}_B$ the set of all ideals of $\mathbb{B}$. For every strict homomorphism $h\colon \mathbb{A} \to \mathbb{B}$, we define a pair of maps $h_+\colon \mathcal{F}_B \to \mathcal{F}_A$ and $h_-\colon \mathcal{I}_B \to \mathcal{I}_A$ as follows:

(i) for each $F \in \mathcal{F}_B$, we let $h_+(F) := \{a \in A \mid h(a) \in F\}$,

(ii) for each $I \in \mathcal{I}_B$, we let $h_-(I) := \{a \in A \mid h(a) \in I\}$.

It is straightforward to show that h_+ and h_- are well-defined. But, note that the strictness is mandatory to prove the non-emptiness of $h_+(F)$ and $h_-(I)$.

Theorem 5.7 *Let $\mathbb{A}$ and $\mathbb{B}$ be FL-algebras, and $h\colon \mathbb{A} \to \mathbb{B}$ a strict homomorphism. The pair of maps h_+ and h_- forms a bounded morphism from $\mathbb{B}_+$ to $\mathbb{A}_+$, i.e. $\langle h_+ | h_- \rangle\colon \mathbb{B}_+ \to \mathbb{A}_+$.*

Conversely, for p-frames $\mathbb{F}$ and $\mathbb{G}$ and a bounded morphism $\langle \sigma | \tau \rangle\colon \mathbb{F} \to \mathbb{G}$, we introduce two maps $\sigma^+\colon \mathbb{D}_2 \to \mathbb{U}_1$ and $\tau^-\colon \mathbb{U}_2 \to \mathbb{D}_1$, where $\mathbb{F}^+ \cong \mathbb{D}_1 \cong \mathbb{U}_1$ and $\mathbb{G}^+ \cong \mathbb{D}_2 \cong \mathbb{U}_2$: for each $\alpha \in \mathbb{G}^+$ which is $\alpha^{\downarrow} \in \mathbb{D}$ and $\alpha_{\uparrow} \in \mathbb{U}$, we let

(i) $\sigma^+(\alpha^{\downarrow}) := \{y \in Y_1 \mid \forall x \in \sigma^{-1}[\alpha^{\downarrow}].\, x \leq_1 y\}$,

(ii) $\tau^-(\alpha_{\uparrow}) := \{x \in X_1 \mid \forall y \in \tau^{-1}[\alpha_{\uparrow}].\, x \leq_1 y\}$,

where σ^{-1} and τ^{-1} are the inverse images of σ and τ. For these maps, we can prove the following important facts.

Proposition 5.8 (Coherence) *Let $\mathbb{F}$ and $\mathbb{G}$ be p-frames, and $\langle \sigma | \tau \rangle\colon \mathbb{F} \to \mathbb{G}$ a bounded morphism. For all $\alpha^{\downarrow} \in \mathbb{D}_2$ and $\alpha_{\uparrow} \in \mathbb{U}_2$, if they are corresponding points, i.e. $\lambda(\alpha^{\downarrow}) = \alpha_{\uparrow}$ and $\upsilon(\alpha_{\uparrow}) = \alpha^{\downarrow}$, we have*

(i) $\sigma^+(\alpha^{\downarrow}) = \tau^{-1}[\alpha_{\uparrow}]$,

(ii) $\tau^-(\alpha_{\uparrow}) = \sigma^{-1}[\alpha^{\downarrow}]$.

Proof. Here, we show (ii) only, but (i) can be analogously proved. ($\subseteq$). For every $x \in \tau^-(\alpha_\uparrow)$, it suffices to show that $\sigma(x) \leq_2 y'$ for each $y' \in \alpha_\uparrow$. Let y' be any element in $\alpha_\uparrow$. We use (ii) of Definition 4.1. For every $y \in Y_1$, if $y' \leq_2 \tau(y)$, since $\alpha_\uparrow$ is upward closed, $\tau(y) \in \alpha_\uparrow$, hence $y \in \tau^{-1}[\alpha_\uparrow]$. Now, by the definition of τ^-, we obtain $x \leq_1 y$. Therefore, $\sigma(x) \leq_2 y'$. ($\supseteq$). Let $x \in \sigma^{-1}[\alpha^\downarrow]$, i.e. $\sigma(x) \in \alpha^\downarrow$. For each $y \in \tau^{-1}[\alpha_\uparrow]$, since $\tau(y) \in \alpha_\uparrow$, we have $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, $x \leq_1 y$, hence $x \in \tau^-(\alpha_\uparrow)$. □

Proposition 5.9 (Interdefinability) *Let $\mathbb{F}$ and $\mathbb{G}$ be p-frames, and $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ a bounded morphism. σ^+ and τ^- coincide on the dual algebras. That is, for each $\alpha \in \mathbb{G}^+$, i.e. $\alpha^\downarrow \in \mathbb{D}_2$ and $\alpha_\uparrow \in \mathbb{U}_2$, we have $\sigma^+(\alpha^\downarrow) := \{y \in Y_1 \mid \forall x \in \tau^-(\alpha_\uparrow).\, x \leq_1 y\}$ and $\tau^-(\alpha_\uparrow) := \{x \in X_1 \mid \forall y \in \sigma^+(\alpha^\downarrow).\, x \leq_1 y\}$.*

With respect to the coherence of σ^+ and τ^- in Proposition 5.8 and the interdefinability of σ^+ and τ^- in Proposition 5.9, hereafter, we treat the two maps σ^+ and τ^- as a map, denoted by $\langle\sigma^+|\tau^-\rangle$. We sum up the maps in Fig. 2.

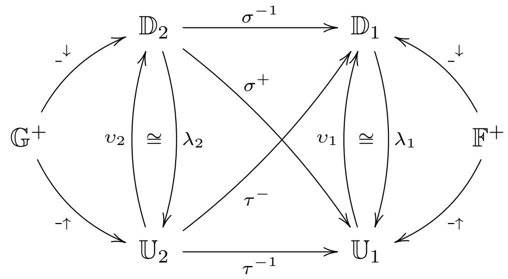

Fig. 2. The relationships of σ^+, τ^-, σ^{-1} and τ^{-1}

Theorem 5.10 *Let $\mathbb{F}, \mathbb{G}$ be p-frames and $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ a bounded morphism. The map $\langle\sigma^+|\tau^-\rangle\colon \mathbb{G}^+ \to \mathbb{F}^+$ is a strict homomorphism from $\mathbb{G}^+$ to $\mathbb{F}^+$.*

Theorem 5.11 *Let $\mathbb{A}$ and $\mathbb{B}$ be FL-algebras, $h\colon \mathbb{A} \to \mathbb{B}$ a strict homomorphism, $\mathbb{F}$ and $\mathbb{G}$ p-frames, and $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ a bounded morphism.*

(i) *If $h\colon \mathbb{A} \to \mathbb{B}$ is injective, $\langle h_+|h_-\rangle\colon \mathbb{B}_+ \to \mathbb{A}_+$ is B-separating.*

(ii) *If $h\colon \mathbb{A} \to \mathbb{B}$ is surjective, $\langle h_+|h_-\rangle\colon \mathbb{B}_+ \to \mathbb{A}_+$ is B-embedding.*

(iii) *If $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ is B-separating, $\langle\sigma^+|\tau^-\rangle\colon \mathbb{G}^+ \to \mathbb{F}^+$ is injective.*

(iv) *If $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ is B-embedding, $\langle\sigma^+|\tau^-\rangle\colon \mathbb{G}^+ \to \mathbb{F}^+$ is surjective.*

6 Conclusion

In the current paper, we have introduced the notion of bounded morphisms on bi-approximation semantics by focusing on invariance of the satisfaction relation on bi-approximation models. Based on the notion of bounded morphisms, we have investigated the so-called p-morphism lemma on bi-approximation semantics. Apart from Kripke models, on bi-approximation semantics, we evaluate not only formulae (assumptions and conclusions) but also sequents (logical consequences). Nevertheless, we have shown that the bounded morphism can preserve all three satisfaction relations on bi-approximation models. Also, we have shown the similarity to p-morphisms in Kripke semantics. In addition, we have discussed invariance of validity of sequents on p-frames via B-embedding, B-separating and B-reflecting bounded morphisms as well. As we have seen in Section 4, the concepts of those bounded morphisms are not exactly the same as those on Kripke semantics. However, the dual representation of morphisms between lattice expansions and bi-approximation semantics satisfies the same properties as the dual representation of morphisms between modal algebras and Kripke semantics in modal logic, e.g. the dual morphisms coincide with the inverse maps. Therefore, the bounded morphisms can be seen as a natural generalisation of p-morphisms on Kripke semantics. Further results on bi-approximation semantics have already discussed by means of bounded morphisms, which will appear in the current author's future work.

Appendix

Proof. [Proposition 4.4] (i). For arbitrary $x_1, x_2 \in X_1$, let $x_1 \leq_1 x_2$. By the definition of $\sigma(x_1) \leq_2 \sigma(x_2)$, we want to show that, for each $y' \in Y_2$, if $\sigma(x_2) \leq_2 y'$ then $\sigma(x_1) \leq_2 y'$. To show that, we use (ii) of Definition 4.1. Assume that $\sigma(x_2) \leq_2 y'$. For each $y \in Y_1$, if $y' \leq_2 \tau(y)$, by transitivity, $\sigma(x_2) \leq_2 \tau(y)$. By (i) of Definition 4.1, we have $x_2 \leq_1 y$. Again, by transitivity, we obtain that $x_1 \leq_1 y$. So, we have $\sigma(x_1) \leq_2 y'$, which concludes $\sigma(x_1) \leq_2 \sigma(x_2)$.

(ii). For arbitrary $y_1, y_2 \in Y_1$, suppose that $y_1 \leq_1 y_2$. Assume that $x' \leq_2 \tau(y_1)$. For any $x \in X_1$, if $\sigma(x) \leq_2 x'$, by transitivity, $\sigma(x) \leq_2 \tau(y_1)$. By (i) of Definition 4.1, we have $x \leq_1 y_1$, hence $x \leq_1 y_2$ by transitivity. Therefore, we obtain $x' \leq_2 \tau(y_2)$, which means $\tau(y_1) \leq_2 \tau(y_2)$.

(iii). For arbitrary $x_1, x_2, x \in X_1$, suppose that $R_1^{\circ}(x_1, x_2, x)$. We want to use (ii) of Definition 4.1 to show $R_2^{\circ}(\sigma(x_1), \sigma(x_2), \sigma(x))$. For any $y \in Y_1$ and $y' \in Y_2$, if $y' \leq_2 \tau(y)$ and $R_2(\sigma(x_1), \sigma(x_2), y')$ then, by R_2-transitivity, $R_2(\sigma(x_1), \sigma(x_2), \tau(y))$. By (iv) of Definition 4.1, we obtain $R_1(x_1, x_2, y)$. By the definition of R_1°, we get $x \leq_1 y$, hence $\sigma(x) \leq_2 y'$ by (ii) of Definition 4.1, which concludes $R_2^{\circ}(\sigma(x_1), \sigma(x_2), \sigma(x))$.

(v). For arbitrary $x_2 \in X_1$, $y_1, y \in Y_1$, assume $R_1^{\leftarrow}(y_1, x_2, y)$. For any $x_1' \in X_2$ satisfying $R_2(x_1', \sigma(x_2), \tau(y))$, and each $x_1 \in X_1$, if $\sigma(x_1) \leq_2 x_1'$, by R_2-transitivity, $R_2^{\leftarrow}(\sigma(x_1), \sigma(x_2), \tau(y))$, so $R_1(x_1, x_2, y)$. By the definition of $R_1^{\leftarrow}$, we obtain $x_1 \leq_1 y_1$. By (iii) of Definition 4.1, we get $x_1' \leq_2 \tau(y_1)$, which means $R_2^{\leftarrow}(\tau(y_1), \sigma(x_2), \tau(y))$. □

Proof. [Lemma 4.5] Parallel induction. Base cases hold by definition.

Inductive steps: for each $x \in X_1$ and each $y \in Y_1$,

$\vee$: (ii). $\mathbb{M}_1 \models_{y} \phi \vee \psi$ is, by definition, $\mathbb{M}_1 \models_{y} \phi$ and $\mathbb{M}_1 \models_{y} \psi$. By induction hypothesis, they are equivalent to $\mathbb{M}_2 \models_{\tau(y)} \phi$ and $\mathbb{M}_2 \models_{\tau(y)} \psi$, which means $\mathbb{M}_2 \models_{\tau(y)} \phi \vee \psi$.

(i). (⇒). Assume $\mathbb{M}_1 \overset{x}{\VDash} \phi \vee \psi$. For any $y' \in Y_2$, suppose $\mathbb{M}_2 \underset{y'}{\VDash} \phi \vee \psi$. We use (ii) of Definition 4.1. For each $y \in Y_1$, if $y' \leq_2 \tau(y)$, by the Hereditary, we have $\mathbb{M}_2 \underset{\tau(y)}{\VDash} \phi \vee \psi$. By (ii), we have $\mathbb{M}_1 \underset{y}{\VDash} \phi \vee \psi$, hence $x \leq_1 y$ by our assumption. So, we get $\sigma(x) \leq_2 y'$, which concludes $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \vee \psi$. (⇐). Assume $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \vee \psi$. For any $y \in Y_1$, if $\mathbb{M}_1 \underset{y}{\VDash} \phi \vee \psi$, by (ii), we have $\mathbb{M}_2 \underset{\tau(y)}{\VDash} \phi \vee \psi$, hence $\sigma(x) \leq_2 \tau(y)$ by our assumption. By (i) of Definition 4.1, we obtain $x \leq_1 y$, which derives $\mathbb{M}_1 \overset{x}{\VDash} \phi \vee \psi$.

∧: (i). $\mathbb{M}_1 \overset{x}{\VDash} \phi \wedge \psi$ is, by definition, $\mathbb{M}_1 \overset{x}{\VDash} \phi$ and $\mathbb{M}_1 \overset{x}{\VDash} \psi$. By induction hypothesis, they are equivalent to $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi$ and $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \psi$, which concludes $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \wedge \psi$.

(ii). (⇒). Assume $\mathbb{M}_1 \underset{y}{\VDash} \phi \wedge \psi$. For any $x' \in X_2$, suppose $\mathbb{M}_2 \overset{x'}{\VDash} \phi \wedge \psi$. We use (iii) of Definition 4.1. For every $x \in X_1$, if $\sigma(x) \leq_2 x'$, by the Hereditary, we have $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \wedge \psi$. By our assumption, we obtain $\sigma(x) \leq_2 \tau(y)$, hence $x \leq_1 y$. So, $x' \leq_2 \tau(y)$, which concludes $\mathbb{M}_2 \underset{\tau(y)}{\VDash} \phi \wedge \psi$. (⇐). Assume $\mathbb{M}_2 \underset{\tau(y)}{\VDash} \phi \wedge \psi$. For any $x \in X_1$, if $\mathbb{M}_1 \overset{x}{\VDash} \phi \wedge \psi$, by (i), we have $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \wedge \psi$, hence $\sigma(x) \leq_2 \tau(y)$, by our assumption. So, we obtain $x \leq_1 y$, which derives $\mathbb{M}_1 \underset{y}{\VDash} \phi \wedge \psi$.

∘: (ii). (⇒). Assume $\mathbb{M}_1 \underset{y}{\VDash} \phi \circ \psi$. For arbitrary $x'_1, x'_2 \in X_2$, suppose that $\mathbb{M}_2 \overset{x'_1}{\VDash} \phi$ and $\mathbb{M}_2 \overset{x'_2}{\VDash} \psi$. We use (v) of Definition 4.1. For all $x_1, x_2 \in X_1$, if $\sigma(x_1) \leq_2 x'_1$ and $\sigma(x_2) \leq_2 x'_2$, by the Hereditary, we have that $\mathbb{M}_2 \overset{\sigma(x_1)}{\VDash} \phi$ and $\mathbb{M}_2 \overset{\sigma(x_2)}{\VDash} \psi$. By induction hypothesis, we have that $\mathbb{M}_1 \overset{x_1}{\VDash} \phi$ and $\mathbb{M}_1 \overset{x_2}{\VDash} \psi$. By our assumption, we obtain $R_1(x_1, x_2, y)$. Therefore, we get $R_2(x'_1, x'_2, \tau(y))$, which concludes $\mathbb{M}_2 \underset{\tau(y)}{\VDash} \phi \circ \psi$. (⇐). Assume that $\mathbb{M}_2 \underset{\tau(y)}{\VDash} \phi \circ \psi$. For arbitrary $x_1, x_2 \in X_1$, if $\mathbb{M}_1 \overset{x_1}{\VDash} \phi$ and $\mathbb{M}_1 \overset{x_2}{\VDash} \psi$, by induction hypothesis, we have that $\mathbb{M}_2 \overset{\sigma(x_1)}{\VDash} \phi$ and $\mathbb{M}_2 \overset{\sigma(x_2)}{\VDash} \psi$, hence $R_1(x_1, x_2, y)$, which derives $\mathbb{M}_1 \underset{y}{\VDash} \phi \circ \psi$.

(ii). This is the same as the case of $\mathbb{M}_1 \overset{x}{\VDash} \phi \vee \psi \iff \mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \vee \psi$.

→: (i). (⇒). Assume $\mathbb{M}_1 \overset{x}{\VDash} \phi \to \psi$. For arbitrary $x'_1 \in X_2$ and $y' \in Y_2$, suppose $\mathbb{M}_2 \overset{x'_1}{\VDash} \phi$ and $\mathbb{M}_2 \underset{y'}{\VDash} \psi$. We use (vi) of Definition 4.1. For all $x_1 \in X_1$ and $y \in Y_1$, if $\sigma(x_1) \leq_2 x'_1$ and $y' \leq_2 \tau(y)$, by the Hereditary, we have that $\mathbb{M}_2 \overset{\sigma(x_1)}{\VDash} \phi$ and $\mathbb{M}_2 \underset{\tau(y)}{\VDash} \psi$. By induction hypothesis, we obtain that $\mathbb{M}_1 \overset{x_1}{\VDash} \phi$ and $\mathbb{M}_1 \underset{y}{\VDash} \psi$. Since $\mathbb{M}_1 \overset{x}{\VDash} \phi \to \psi$, we get $R_1(x_1, x, y)$. Therefore, $R_2(x'_1, \sigma(x), y')$, which means $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \to \psi$. (⇐). Assume $\mathbb{M}_2 \overset{\sigma(x)}{\VDash} \phi \to \psi$. For arbitrary $x_1 \in X_1$ and $y \in Y_1$, if $\mathbb{M}_1 \overset{x_1}{\VDash} \phi$ and

$\mathbb{M}_1 \models_y \psi$, by induction hypothesis, we have that $\mathbb{M}_2 \overset{\sigma(x_1)}{\models} \phi$ and $\mathbb{M}_2 \underset{\tau(y)}{\models} \psi$. Because of $\mathbb{M}_2 \overset{\sigma(x)}{\models} \phi \to \psi$, we obtain $R_2(\sigma(x_1), \sigma(x), \tau(y))$. Hence, $R_1(x_1, x, y)$, which concludes $\mathbb{M}_1 \overset{x}{\models} \phi \to \psi$.

(ii). This is analogous to (ii) of $\leftarrow$.

$\leftarrow$: (i). This is analogous to (i) of $\to$.

(ii). ($\Rightarrow$). Assume $\mathbb{M}_1 \models_y \psi \leftarrow \phi$. For any $x' \in X_2$, suppose $\mathbb{M}_2 \overset{x'}{\models} \psi \leftarrow \phi$. We use (iii) of Definition 4.1. For each $x \in X_1$, if $\sigma(x) \leq_2 x'$, by the Hereditary, we have $\mathbb{M}_2 \overset{\sigma(x)}{\models} \psi \leftarrow \phi$. By (i), we obtain $\mathbb{M}_1 \overset{x}{\models} \psi \leftarrow \psi$. By our assumption, we get $x \leq_1 y$. So, $x' \leq_2 \tau(y)$, which means $\mathbb{M}_2 \underset{\tau(y)}{\models} \psi \leftarrow \phi$.

($\Leftarrow$). Suppose $\mathbb{M}_2 \underset{\tau(y)}{\models} \psi \leftarrow \phi$. For any $x \in X_1$, if $\mathbb{M}_1 \overset{x}{\models} \psi \leftarrow \phi$, by (i), $\mathbb{M}_2 \overset{\sigma(x)}{\models} \psi \leftarrow \phi$. Since $\mathbb{M}_2 \underset{\tau(y)}{\models} \psi \leftarrow \phi$, we obtain $\sigma(x) \leq_2 \tau(y)$, hence $x \leq_1 y$, which concludes $\mathbb{M}_1 \models_y \psi \leftarrow \phi$.

□

Proof. [(ii) and (iii) of Theorem 4.10] (ii). We prove by contraposition. Suppose $\mathbb{G} \not\models \phi \mapsto \psi$. Then, there exists a doppelgänger valuation V on $\mathbb{G}$ such that $\mathbb{G}, V \not\models \phi \mapsto \psi$. For the doppelgänger valuation V, we induce a doppelgänger valuation U on $\mathbb{F}$, which makes $\langle\sigma|\tau\rangle$ is a bounded morphism from $\langle\mathbb{F}, U\rangle$ to $\langle\mathbb{G}, V\rangle$. For any propositional variable $p \in \Phi$, we let

(1) $U^{\downarrow}(p) := \{x \in X_1 \mid \forall y' \in V_{\uparrow}(p).\, \sigma(x) \leq_2 y'\}$,

(2) $U_{\uparrow}(p) := \{y \in Y_1 \mid \forall x' \in V^{\downarrow}(p).\, x' \leq_2 \tau(y)\}$.

Now, we show that $\langle\sigma|\tau\rangle$ is a bounded morphism from $\langle\mathbb{F}, U\rangle$ to $\langle\mathbb{G}, V\rangle$, i.e. $x \in U^{\downarrow}(p) \iff \sigma(x) \in V^{\downarrow}(p)$ and $y \in U_{\uparrow}(p) \iff \tau(y) \in V_{\uparrow}(p)$.

($x \in U^{\downarrow}(p) \iff \sigma(x) \in V^{\downarrow}(p)$). ($\Rightarrow$). For arbitrary $x \in U^{\downarrow}(p)$ and $y' \in V_{\uparrow}(p)$, we use (ii) of Definition 4.1. For any $y \in Y_1$, if $y' \leq_2 \tau(y)$, as $V_{\uparrow}(p)$ is upward closed, we have $\tau(y) \in V_{\uparrow}(p)$. By the definition of $U^{\downarrow}(p)$, we get $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, $x \leq_1 y$. So, $\sigma(x) \leq_2 y'$, hence $\sigma(x) \in V^{\downarrow}(p)$ ($\Leftarrow$). This is trivial by definition.

($y \in U_{\uparrow}(p) \iff \tau(y) \in V_{\uparrow}(p)$). ($\Rightarrow$). For arbitrary $y \in U_{\uparrow}(p)$ and $x' \in V^{\downarrow}(p)$, we use (iii) of Definition 4.1. For any $x \in X_1$, if $\sigma(x) \leq_2 x'$, as $V^{\downarrow}(p)$ is downward closed, we have $\sigma(x) \in V^{\downarrow}(p)$. By the definition of $U_{\uparrow}(p)$, we obtain that $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, $x \leq_1 y$. So, $x' \leq_2 \tau(y)$, hence $\tau(y) \in V_{\uparrow}(p)$. ($\Leftarrow$). This is trivial by definition.

Next, we prove that U is a doppelgänger valuation on $\mathbb{F}$. That is,

(1) $U^{\downarrow}(p) = \{x \in X_1 \mid \forall y \in U_{\uparrow}(p).\, x \leq_1 y\}$

($\subseteq$). For arbitrary $x \in U^{\downarrow}(p)$ and $y \in U_{\uparrow}(p)$, since $\sigma(x) \in V^{\downarrow}(p)$ and $\tau(y) \in V_{\uparrow}(p)$ (as we saw above) and V is a doppelgänger valuation on $\mathbb{G}$, we obtain $\sigma(x) \leq_2 \tau(y)$, hence $x \leq_1 y$ by (i) of Definition 4.1. ($\supseteq$). Contraposition. Suppose $x \notin U^{\downarrow}(p)$. There exists $y' \in V_{\uparrow}(p)$ such that $\sigma(x) \not\leq_2 y'$. By (ii) of Definition 4.1, there exists $y \in Y_1$ such that $y' \leq_2 \tau(y)$ but $x \not\leq_1 y$. As $V_{\uparrow}(p)$ is upward closed, $\tau(y) \in V_{\uparrow}(p)$, hence $y \in U_{\uparrow}(p)$.

Therefore, there exists $y \in U_{\uparrow}(p)$ such that $x \not\leq_1 y$.

(2) $U_{\uparrow}(p) = \{y \in Y_1 \mid \forall x \in U^{\downarrow}(p).\, x \leq_1 y\}$

($\subseteq$). For arbitrary $y \in U_{\uparrow}(p)$ and $x \in U^{\downarrow}(p)$, since $\tau(y) \in V_{\uparrow}(p)$ and $\sigma(x) \in V^{\downarrow}(p)$ (as we saw above) and V is a doppelgänger valuation on $\mathbb{G}$, we obtain $\sigma(x) \leq_2 \tau(y)$, hence $x \leq_1 y$. ($\supseteq$). Contraposition. Suppose $y \notin U_{\uparrow}(p)$. There exists $x' \in V^{\downarrow}(p)$ such that $x' \not\leq_2 \tau(y)$. By (iii) of Definition 4.1, there exists $x \in X_1$ such that $\sigma(x) \leq_2 x'$ but $x \not\leq_1 y$. Since $V^{\downarrow}(p)$ is downward closed, $\sigma(x) \in V^{\downarrow}(p)$, hence $x \in U^{\downarrow}(p)$. So, there exists $x \in U^{\downarrow}(p)$ such that $x \not\leq_1 y$.

Therefore, U is a doppelgänger valuation on $\mathbb{F}$.

By our assumption, i.e. $\mathbb{G}, V \not\Vdash \phi \Mapsto \psi$, there exists $x' \in X_2$ and $y' \in Y_2$ such that $\mathbb{G}, V \not\Vdash^{x'}_{y'} \phi \Mapsto \psi$, which means $\mathbb{G}, V \Vdash^{x'} \phi$, $\mathbb{G}, V \Vdash_{y'} \psi$ but $x' \not\leq_2 y'$. Now, as $\langle \sigma | \tau \rangle$ is B-separating, there exist $x \in X_1$ and $y \in Y_1$ such that $\sigma(x) \leq_2 x'$, $y' \leq_2 \tau(y)$ and $x \leq_1 y$. By the Hereditary, we obtain that $\mathbb{G}, V \Vdash^{\sigma(x)} \phi$ and $\mathbb{G}, V \Vdash_{\tau(y)} \psi$. So, by Lemma 4.5, we also have that $\mathbb{F}, U \Vdash^{x} \phi$ and $\mathbb{F}, U \Vdash_{y} \psi$. However, since $x \not\leq_1 y$, we conclude $\mathbb{F}, U \not\Vdash^{x}_{y} \phi \Mapsto \psi$, hence $\mathbb{F} \not\Vdash \phi \Mapsto \psi$.

(iii). This follows directly from (i) and (ii). □

Proof. [Theorem 5.7] By definition, h_+ is a function from $\mathcal{F}_B$ to $\mathcal{F}_A$ and h_- is a function from $\mathcal{I}_B$ to $\mathcal{I}_A$. All we need to show here is to check the conditions in Definition 4.1.

(i). For arbitrary $F \in \mathcal{F}_B$ and $I \in \mathcal{I}_B$, if $h_+(F) \sqsubseteq_A h_-(I)$, there exists $a \in A$ such that $a \in h_+(F) \cap h_-(I)$. By definition, $h(a) \in F$ and $h(a) \in I$, hence $h(a) \in F \cap I$, which concludes $F \sqsubseteq_B I$.

(iii). Contraposition. For arbitrary $G \in \mathcal{F}_A$ and $I \in \mathcal{I}_B$, suppose $G \not\sqsubseteq_A h_-(I)$. Let F be the generated filter by the image $h[G]$, i.e. $F := {\uparrow} h[G]$. By definition, we have $h[G] \subseteq_B F$, hence $h_+(F) \sqsubseteq_A G$. Plus, by our assumption $G \not\sqsubseteq_A h_-(I)$, we obtain $F \not\sqsubseteq_B I$ (otherwise, it contradicts to $G \not\sqsubseteq_A h_-(I)$). Therefore, there exists $F \in \mathcal{F}_B$ such that $h_+(F) \sqsubseteq_B G$ but $F \not\sqsubseteq_B I$.

(iv). For all $F_1, F_2 \in \mathcal{F}_B$ and $I \in \mathcal{I}_B$, assume $R_A(h_+(F_1), h_+(F_2), h_-(I))$, namely $h_+(F_1) *_A h_+(F_2) \sqsubseteq_A h_-(I)$. Then, there exist $a_1 \in h_+(F_1)$ and $a_2 \in h_+(F_2)$ such that $a_1 *_A a_2 \in h_-(I)$, which means $h(a_1 *_A a_2) \in I$. Since h is homomorphic, we have $h(a_1) *_B h(a_2) \in I$. Moreover, as $h(a_1) \in F_1$ and $h(a_2) \in F_2$, we conclude $F_1 *_B F_2 \sqsubseteq_B I$, i.e. $R_B(F_1, F_2, I)$.

(v). Contraposition. For arbitrary $G_1, G_2 \in \mathcal{F}_A$ and $I \in \mathcal{I}_B$, suppose that $R_A(G_1, G_2, h_-(I))$ does not hold, i.e. $G_1 *_A G_2 \not\sqsubseteq_A h_-(I)$. Let F_1 and F_2 be the generated filters by the images $h[G_1]$ and $h[G_2]$, that is, $F_1 := {\uparrow} h[G_1]$ and $F_2 := {\uparrow} h[G_2]$. By definition, we obtain that $h[G_1] \subseteq_B F_1$ and $h[G_2] \subseteq_B F_2$, hence $h_+(F_1) \sqsubseteq_A G_1$ and $h_+(F_2) \sqsubseteq_A G_2$. In addition, for any $b \in F_1 *_B F_2$, there exist $a_1 \in G_1$ and $a_2 \in G_2$ such that $h(a_1) *_B h(a_2) \leq_B b$. As h is homomorphic, we have $h(a_1 *_A a_2) \leq_B b$. Now, if $F_1 *_B F_2 \sqsubseteq_B I$ then $h(a_1 *_A a_2) \in I$, which contradicts to $G_1 *_A G_2 \not\sqsubseteq_A h_-(I)$. Therefore, $F_1 *_B F_2 \not\sqsubseteq_B I$, i.e. $R_B(F_1, F_2, I)$ does not hold.

(vi). Contraposition. For arbitrary $G_1 \in \mathcal{F}_A$, $F_2 \in \mathcal{F}_B$ and $J \in \mathcal{I}_A$, suppose that $R_A(G_1, h_+(F_2), J)$ does not hold, i.e. $G_1 *_A h_+(F_2) \not\sqsubseteq_A J$. Let F_1 be the generated filter by the image $h[G_1]$, i.e. $F_1 := \uparrow h[G_1]$, and I the generated ideal by the image $h[J]$, i.e. $I := \downarrow h[J]$. By definition, we have that $h[G_1] \subseteq_B F_1$ and $h[J] \subseteq_B I$, hence $h_+(F_1) \sqsubseteq_A G_1$ and $J \sqsubseteq_A h_-(I)$. Furthermore, if $F_1 *_B F_2 \sqsubseteq_B I$, there exist $a_a \in G_1$ and $a \in I$ such that $h(a_1)\backslash_B h(a) = h(a_1 \backslash_A a) \in F_2$, which contradicts to $G_1 *_A h_+(F_2) \not\sqsubseteq_A J$. Therefore, $F_1 *_B F_2 \not\sqsubseteq_B I$, i.e. $R_B(F_1, F_2, I)$ does not hold.

(viii). For any $F \in O_{\mathcal{F}_B}$, by definition $1_B \in F$. Because h is homomorphic, $h(1_A) = 1_B \in F$, which derives $1_1 \in h_+(F)$. So, $h_+(F) \in O_{\mathcal{F}_A}$. Conversely, if $h_+(F) \in O_{\mathcal{F}_A}$ then $1_A \in h_+(F)$, so $h(1_A) = 1_B \in F$. Therefore, $F \in O_{\mathbb{F}_B}$. The other case is analogous. □

Proof. [Proposition 5.9] By the definition of σ^+ and τ^-, and Proposition 5.8.□

Proof. [Theorem 5.10] It suffices to show that $\langle \sigma^+ | \tau^- \rangle \colon \mathbb{G}^+ \to \mathbb{F}^+$ is homomorphic. Note that the strictness follows from the preservability of top $\top$ and bottom $\bot$, because $\mathbb{F}^+$ and $\mathbb{G}^+$ are bounded.

($\vee$). To prove $\langle \sigma^+ | \tau^- \rangle(\alpha \vee \beta) = \langle \sigma^+ | \tau^- \rangle(\alpha) \vee \langle \sigma^+ | \tau^- \rangle(\beta)$, it suffices to show that $\sigma^+(\alpha^\downarrow \vee \beta^\downarrow) = \sigma^+(\alpha^\downarrow) \vee \sigma^+(\beta^\downarrow)$, i.e. $\tau^{-1}[\alpha_\uparrow \cap \beta_\uparrow] = \tau^{-1}[\alpha_\uparrow] \cap \tau^{-1}[\beta_\uparrow]$ by Proposition 5.8. But, this is straightforward.

($\wedge$). To prove $\langle \sigma^+ | \tau^- \rangle(\alpha \wedge \beta) = \langle \sigma^+ | \tau^- \rangle(\alpha) \wedge \langle \sigma^+ | \tau^- \rangle(\beta)$, it suffices to show that $\tau^-(\alpha_\uparrow \wedge \beta_\uparrow) = \tau^-(\alpha_\uparrow) \wedge \tau^-(\beta_\uparrow)$, i.e. $\sigma^{-1}[\alpha^\downarrow \cap \beta^\downarrow] = \sigma^{-1}[\alpha^\downarrow] \cap \sigma^{-1}[\beta^\downarrow]$ by Proposition 5.8. But, this is straightforward.

($*$). To prove $\langle \sigma^+ | \tau^- \rangle(\alpha * \beta) = \langle \sigma^+ | \tau^- \rangle(\alpha) * \langle \sigma^+ | \tau^- \rangle(\beta)$, it suffices to show $\sigma^+(\alpha^\downarrow * \beta^\downarrow) = \sigma^+(\alpha^\downarrow) * \sigma^+(\beta^\downarrow)$, i.e. $\tau^{-1}[\alpha_\uparrow * \beta_\uparrow] = \tau^{-1}[\alpha_\uparrow] * \tau^{-1}[\beta_\uparrow]$ by Proposition 5.8. ($\subseteq$). For each $y \in \tau^{-1}[\alpha_\uparrow * \beta_\uparrow]$, to prove $y \in \tau^{-1}[\alpha_\uparrow] * \tau^{-1}[\beta_\uparrow]$, we need to show that $R_1(x_1, x_2, y)$ for arbitrary $x_1 \in \upsilon_1(\tau^{-1}[\alpha_\uparrow])$ and $x_2 \in \upsilon_1(\tau^{-1}[\beta_\uparrow])$. Let y, x_1 and x_2 be arbitrary elements in $\tau^{-1}[\alpha_\uparrow * \beta_\uparrow]$, $\upsilon_1(\tau^{-1}[\alpha_\uparrow])$ and $\upsilon_1(\tau^{-1}[\beta_\uparrow])$. By Proposition 5.9 and Proposition 5.8, $x_1 \in \sigma^{-1}[\alpha^\downarrow]$ and $x_2 \in \sigma^{-1}[\beta^\downarrow]$. By definition, we have that $\tau(y) \in \alpha_\uparrow * \beta_\uparrow$, $\sigma(x_1) \in \alpha^\downarrow$ and $\sigma(x_2) \in \beta^\downarrow$. Further, by the definition of $*$ on $\mathbb{U}_2$, we get $R_2(\sigma(x_1), \sigma(x_2), \tau(y))$. By (iv) of Definition 4.1, we obtain $R_1(x_1, x_2, y)$. ($\supseteq$). For each $y \in \tau^{-1}[\alpha_\uparrow] * \tau^{-1}[\beta_\uparrow]$, we want to show that $R_2(x'_1, x'_2, \tau(y))$ for arbitrary $x'_1 \in \alpha^\downarrow$ and $x'_2 \in \beta^\downarrow$. Let y, x'_1 and x'_2 be arbitrary elements in $\tau^{-1}[\alpha_\uparrow * \beta_\uparrow]$, $\alpha^\downarrow$ and $\beta^\downarrow$. We use (v) of Definition 4.1. For all $x_1, x_2 \in X_1$, if $\sigma(x_1) \leq_2 x'_1$ and $\sigma(x_2) \leq_2 x'_2$, since $\alpha^\downarrow$ and $\beta^\downarrow$ are downward closed, $\sigma(x_1) \in \alpha^\downarrow$ and $\sigma(x_2) \in \beta^\downarrow$, hence $x_1 \in \sigma^{-1}[\alpha^\downarrow]$ and $x_2 \in \sigma^{-1}[\beta^\downarrow]$. By Proposition 5.8 and Proposition 5.9, we have that $x_1 \in \upsilon_1(\tau^{-1}[\alpha_\uparrow]])$ and $x_2 \in \upsilon_1(\tau^{-1}[\beta_\uparrow])$. As $y \in \tau^{-1}[\alpha_\uparrow] * \tau^{-1}[\beta_\uparrow]$, we obtain $R_1(x_1, x_2, y)$, which concludes $R_2(x'_1, x'_2, \tau(y))$.

($\backslash$). To prove $\langle \sigma^+ | \tau^- \rangle(\alpha \backslash \beta) = \langle \sigma^+ | \tau^- \rangle(\alpha) \backslash \langle \sigma^+ | \tau^- \rangle(\beta)$, it suffices to show $\tau^-(\alpha \backslash \beta) = \tau^-(\alpha) \backslash \tau^-(\beta)$, that is, $\sigma^{-1}(\alpha^\downarrow \backslash \beta^\downarrow) = \sigma^{-1}[\alpha^\downarrow] \backslash \sigma^{-1}[\beta^\downarrow]$ by Proposition 5.8. ($\subseteq$). Let $x_2 \in \sigma^{-1}[\alpha^\downarrow \backslash \beta^\downarrow]$. To show $x_2 \in \sigma^{-1}[\alpha^\downarrow] \backslash \sigma^{-1}[\beta^\downarrow]$, it suffices to show that $R_1(x_1, x_2, y)$ holds for arbitrary $x_1 \in \sigma^{-1}[\alpha^\downarrow]$ and $y \in \lambda_1(\sigma^{-1}[\beta^\downarrow])$. For all $x_1 \in \sigma^{-1}[\alpha^\downarrow]$ and $y \in \lambda_1(\sigma^{-1}[\beta^\downarrow])$, by Proposition 5.8 and Proposition 5.9, we have $y \in \tau^{-1}[\beta_\uparrow]$, hence $\sigma(x_1) \in \alpha^\downarrow$ and $\tau(y) \in \beta_\uparrow$. By our assumption

$x_2 \in \sigma^{-1}[\alpha^{\downarrow}\backslash\beta^{\downarrow}]$, we obtain $R_2(\sigma(x_1),\sigma(x_2),\tau(y))$. By (iv) of Definition 4.1, we conclude $R_1(x_1,x_2,y)$. ($\supseteq$). Let x_2 be any element in $\sigma^{-1}[\alpha^{\downarrow}]\backslash\sigma^{-1}[\beta^{\downarrow}]$. To show $x_2 \in \sigma^{-1}[\alpha^{\downarrow}\backslash\beta^{\downarrow}]$, we need to prove that $\sigma(x_2) \in \alpha^{\downarrow}\backslash\beta^{\downarrow}$. For arbitrary $x_1' \in \alpha^{\downarrow}$ and $y' \in \beta_{\uparrow}$, we use (vi) of Definition 4.1. For all $x_1 \in X_1$ and $y \in Y_1$, if $\sigma(x_1) \leq_2 x_1'$ and $y' \leq_2 \tau(y)$, as $\alpha^{\downarrow}$ is downward closed and $\beta_{\uparrow}$ is upward closed, $\sigma(x) \in \alpha^{\downarrow}$ and $\tau(y) \in \beta_{\uparrow}$. Since $x_2 \in \sigma^{-1}[\alpha^{\downarrow}]\backslash\sigma^{-1}[\beta^{\downarrow}]$, we obtain $R_1(x_1,x_2,y)$, hence $x_2 \in \sigma^{-1}[\alpha^{\downarrow}\backslash\beta^{\downarrow}]$.

($/$). This is analogous to the case of ($\backslash$).

(1). To prove $\langle\sigma^+|\tau^-\rangle(1) = 1$, it suffices to show that $\tau^-(1) = 1$, i.e. $\sigma^{-1}[O_{X2}] = O_{X1}$ by Proposition 5.8. But, this is straightforward.

(0), ($\top$) and ($\bot$) are analogous to the case of (1). □

Proof. [Theorem 5.11] (i). We prove by contraposition. For arbitrary $G \in \mathcal{F}_A$ and $J \in \mathcal{I}_A$, suppose $G \not\sqsubseteq_A J$, i.e. $G \cap J = \emptyset$. Let F be the generated filter by the image $h[G]$, i.e. $F := \uparrow h[G]$, and I the generated ideal by the image $h[J]$, i.e. $I := \downarrow h[J]$. By definition, we have that $h[G] \subseteq_B F$ and $h[J] \subseteq_B I$. So, we have that $h_+(F) \sqsubseteq_A G$ and $J \sqsubseteq_A h_-(I)$. Now, if $F \sqsubseteq_B I$, i.e. $F \cap I \neq \emptyset$, there exist $a_1 \in G$ and $a_2 \in J$ such that $h(a_1) \leq_B h(a_2)$, that is $h(a_1) \vee_B h(a_2) = h(a_1 \vee_A a_2) = h(a_2)$. Since h is injective, we obtain $a_1 \leq_A a_2$. Hence, $G \sqsubseteq_A J$, which contradicts to $G \not\sqsubseteq_A J$. Therefore, $F \not\sqsubseteq_B I$.

(ii). For arbitrary $F \in \mathcal{F}_B$ and $I \in \mathcal{I}_B$, if $F \sqsubseteq_B I$, there exists $b \in F \cap I$. As h is surjective, there exists $a \in A$ such that $h(a) = b \in F \cap I$. Then, $a \in h_+(F)$ and $a \in h_-(I)$, hence $h_+(F) \sqsubseteq_A h_-(I)$.

(iii). Since every lattice is anti-symmetric, it suffices to show that $\langle\sigma^+|\tau^-\rangle$ is order-embedding. That is, we show that, for all $\alpha, \beta \in \mathbb{G}^+$, if $\langle\sigma^+|\tau^-\rangle(\alpha) \leq_1 \langle\sigma^+|\tau^-\rangle(\beta)$ then $\alpha \leq_2 \beta$. We prove it by contraposition. Suppose $\alpha \not\leq_2 \beta$. Then, there exists $x' \in \alpha^{\downarrow}$ and $y' \in \beta_{\uparrow}$ such that $x' \not\leq_2 y'$. Since $\langle\sigma|\tau\rangle$ is B-separating, there exist $x \in X_1$ and $y \in Y_1$ such that $\sigma(x) \leq_2 x'$, $y' \leq_2 \tau(y)$ and $x \not\leq_1 y$. By (i) of Definition 4.1, $\sigma(x) \not\leq_2 \tau(y)$, which derives $\sigma(x) \notin \beta^{\downarrow}$, i.e. $x \notin \sigma^{-1}[\beta^{\downarrow}]$. Moreover, as $\alpha^{\downarrow}$ is downward closed and $\beta_{\uparrow}$ is upward closed, we have that $\sigma(x) \in \alpha^{\downarrow}$ and $\tau(y) \in \beta_{\uparrow}$, hence $x \in \sigma^{-1}[\alpha^{\downarrow}]$. That is, there exists $x \in X_1$ such that $x \in \sigma^{-1}[\alpha^{\downarrow}]$ but $x \notin \sigma^{-1}[\beta^{\downarrow}]$. By Proposition 5.8, we conclude $\tau^-(\alpha_{\uparrow}) \not\subseteq_1 \tau^-(\beta_{\uparrow})$, i.e. $\langle\sigma^+|\tau^-\rangle(\alpha) \not\leq_1 \langle\sigma^+|\tau^-\rangle(\beta)$.

(iv). For each element $\beta \in \mathbb{F}^+$, we let an element $\alpha \in \mathbb{G}^+$ as follows:

(i) $\alpha^{\downarrow} := \{x' \in X_2 \mid \forall y \in \beta_{\uparrow}.\, x' \leq_2 \tau(y)\}$,

(ii) $\alpha_{\uparrow} := \{y' \in Y_2 \mid \forall x' \in \alpha^{\downarrow}.\, x' \leq_2 y'\}$.

Note that there is another natural way to introduce α as we saw in Remark 4.11. Now, we check that they coincide, namely $\alpha^{\downarrow} = \upsilon_2(\alpha_{\uparrow})$ and $\alpha_{\uparrow} = \lambda_2(\alpha^{\downarrow})$. But, the latter is trivial by definition.

($\alpha^{\downarrow} = \upsilon_2(\alpha_{\uparrow})$). ($\subseteq$). For arbitrary $x' \in \alpha^{\downarrow}$ and $y' \in \alpha_{\uparrow}$, by the definition of $\alpha_{\uparrow}$, we trivially have $x' \leq_2 y'$. ($\supseteq$). Contraposition. Suppose $x' \notin \alpha^{\downarrow}$. There exists $y \in \beta_{\uparrow}$ such that $x' \not\leq_2 \tau(y)$. By definition, we have $\tau[\beta_{\uparrow}] \subseteq \alpha_{\uparrow}$, so $\tau(y) \in \alpha_{\uparrow}$. But, $x' \not\leq_2 \tau(y)$, which concludes $x' \notin \upsilon_2(\alpha_{\uparrow})$.

Next we show $\tau^-(\alpha_{\uparrow}) = \beta^{\downarrow}$ and $\sigma^+(\alpha^{\downarrow}) = \beta_{\uparrow}$.

($\tau^-(\alpha_{\uparrow}) = \beta^{\downarrow}$). By Proposition 5.8, $\tau^-(\alpha_{\uparrow}) = \sigma^{-1}[\alpha^{\downarrow}]$. ($\subseteq$). For arbitrary

$x \in \sigma^{-1}[\alpha^{\downarrow}]$ and $y \in \beta_{\uparrow}$, we have $\sigma(x) \in \alpha^{\downarrow}$. By the definition of $\alpha^{\downarrow}$, we obtain $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, $x \leq_1 y$, hence $x \in \beta^{\downarrow}$. ($\supseteq$). For arbitrary $x \in \beta^{\downarrow}$ and $y \in \beta_{\uparrow}$, we have $x \leq_1 y$. As $\langle \sigma | \tau \rangle$ is B-embedding, we obtain $\sigma(x) \leq_2 \tau(y)$, hence $\sigma(x) \in \alpha^{\downarrow}$, which means $x \in \sigma^{-1}[\alpha^{\downarrow}]$.

($\sigma^+(\alpha^{\downarrow}) = \beta_{\uparrow}$). By Proposition 5.8, $\sigma^+(\alpha^{\downarrow}) = \tau^{-1}[\alpha_{\uparrow}]$. ($\subseteq$). For arbitrary $y \in \tau^{-1}[\alpha_{\uparrow}]$ and $x \in \beta^{\downarrow}$, we have $\tau(y) \in \alpha_{\uparrow}$. Further, as we saw above, $\sigma^{-1}[\alpha^{\downarrow}] = \beta^{\downarrow}$, hence $\sigma(x) \in \alpha^{\downarrow}$. By the definition of $\alpha_{\uparrow}$, we obtain $\sigma(x) \leq_2 \tau(y)$. By (i) of Definition 4.1, $x \leq_1 y$. Therefore, $y \in \beta_{\uparrow}$. ($\supseteq$). For arbitrary $y \in \beta_{\uparrow}$ and $x' \in \alpha^{\downarrow}$, by the definition of $\alpha^{\downarrow}$, we have $x' \leq_2 \tau(y)$. Therefore, $\tau(y) \in \alpha_{\uparrow}$, i.e. $y \in \tau^{-1}[\alpha_{\uparrow}]$.

As a conclusion, for each $\beta \in \mathbb{F}^+$, there exists $\alpha \in \mathbb{G}^+$ such that $\langle \sigma^+ | \tau^- \rangle(\alpha) = \beta$, hence $\langle \sigma^+ | \tau^- \rangle$ is surjective. □

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