On Modal Logics of Hamming Spaces

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Abstract

With a set S of words in an alphabet A we associate the frame (S, H), where sHt iff s and t are words of the same length and $h(s,t) = 1$ for the Hamming distance h. We investigate some unimodal logics of these frames. We show that if the length of words n is fixed and finite, the logics are closely related to many-dimensional products $\mathbf{S5}^n$, so in many cases they are undecidable and not finitely axiomatizable.
The relation H can be extended to infinite sequences. In this case we prove some completeness theorems characterizing the well-known modal logics **DB** and **TB** in terms of the Hamming distance.

Keywords: Decidability, undecidability, finite axiomatizability, Hamming distance, product of modal logics.

1 Introduction

According to a rather old viewpoint on modalities, a proposition is possible in a world if it is true in a similar world. An abstract 'similarity' relation should be reflexive and symmetric, thus the corresponding modal logic is **TB**.

Similarity can be treated more specifically in different ways — e.g. one of them was developed by Z. Pawlak and E. Orlowska and others within the framework of information systems.

In this paper we study another, rather modest, but mathematically explicit approach to similarity of possible worlds — every world is presented by a sequence of properties (coded by symbols of a certain alphabet), and two sequences are similar if they coincide at all positions but one. So we investigate unimodal logics of the following relation on words: sHt iff s and t are words of the same length and $h(s,t) = 1$, where h is the Hamming distance. Recall that the Hamming distance between two words of equal length is the number of positions, where they differ.

Unimodal logics of this kind are closely related to many-dimensional modal logics. If we consider the n-th power (cf. [3]) of an inequality frame $(A, \neq)$

$$(A, \neq)^n = (A^n, \neq_0, \ldots, \neq_{n-1}),$$

we can see that $(A^n, H) = (A^n, \neq_0 \cup \dots \cup \neq_{n-1})$. Thus the $\Box$-operator in the logic of (A^n, H) can be interpreted as the conjunction of all $\Box$-operators in the logic of $(A, \neq)^n$.

If $n = 2$, we obtain a particular case of the $\curlyvee$-product studied in the recent paper [11]. There it was shown that under certain conditions the product modalities can be expressed in terms of the "Hamming-like" $\Box$-operator. In this paper we extend the corresponding construction to higher dimensions.

If the length of words (the dimension of the product) is fixed, the modal operator corresponding to H is rather expressive and as we will show, it can be used to describe various geometric objects in A^n (like spheres or hyperplanes). As usual in many-dimensional logic, there is a price for this richness – logics of these frames for dimensions higher than two are undecidable; the Hamming modal operator allows us to express all modalities of the n-dimensional product of **S5** (up to permutation). Hence we obtain undecidability for logics of the relation H on words of fixed length greater than two.

However, for "long" words (ω-sequences) there are some positive results. We show that the relation H on binary sequences yields the well-known logic **DB** (the minimal symmetric serial logic). This result can also be formulated in terms of a certain relation on sets — 'the symmetric difference of two sets is a singleton'. For $\overline{H}$, the reflexive closure of H, the results are extended to sequences over any infinite I, and A with $|A| > 1$ — the corresponding logic is always **TB**, the minimal symmetric reflexive logic.

2 Preliminaries

2.1 Syntax and semantics

We skip standard definitions concerning propositional modal logics (all logics are supposed normal). $ML(\Diamond_0, \dots, \Diamond_{n-1})$ denotes the set of all modal formulas constructed from the countable set of propositional variables PV using the classical connectives $\wedge$, $\neg$ and the unary connectives $\Diamond_0, \dots, \Diamond_{n-1}$. Other connectives are defined in the usual way, in particular, $\Box_i\varphi = \neg\Diamond_i\neg\varphi$. $\Diamond$ and $\Box$ abbreviate $\Diamond_0$ and $\Box_0$, respectively.

$PV(\varphi)$ denotes the set of all variables occurring in a formula φ. The *modal depth* of a formula φ is denoted by $md(\varphi)$.

K denotes the minimal unimodal logic. For a logic L and a set of formulas Γ, $L + \Gamma$ is the minimal logic containing L and Γ. $L + \varphi$ abbreviates $L + \{\varphi\}$. Our basic logics are

$$\mathbf{KB} := \mathbf{K} + \Diamond\Box p \to p, \quad \mathbf{TB} := \mathbf{KB} + p \to \Diamond p, \quad \mathbf{DB} := \mathbf{KB} + \Diamond\top.$$

The notions of a frame, a (Kripke) model, validity and the truth are standard, cf. [1]. The truth of a formula φ at a world w in a model M is denoted by $\mathsf{M}, w \vDash \varphi$; the validity of φ in a frame F by $\mathsf{F} \vDash \varphi$. For a set of formulas Ψ, $\mathsf{F} \vDash \Psi$ means $\mathsf{F} \vDash \varphi$ for all $\varphi \in \Psi$. F is called an *L-frame* for a logic L if $\mathsf{F} \vDash L$.

$Log(\mathfrak{F})$ denotes the logic of the class $\mathfrak{F}$ (i.e., the set of all valid formulas). For a frame $\mathsf{F} = (W, R)$, $Log(W, R)$ and $Log(\mathsf{F})$ abbreviate $Log(\{\mathsf{F}\})$.

A formula φ is *satisfiable in a frame* F *at a point* w (or briefly, *satisfiable at* F, w) if $(\mathsf{F}, \theta), w \vDash \varphi$ for some valuation θ. For a class of frames $\mathfrak{F}$, φ is *satisfiable in* $\mathfrak{F}$ (or $\mathfrak{F}$*-satisfiable*), if φ is satisfiable in F for some $\mathsf{F} \in \mathfrak{F}$. A formula φ is L*-satisfiable* if φ is satisfiable in some L-frame.

For a relation R on a set W, R^n denotes its n-th iteration. So R^0 is the identity relation on W and $R^{n+1} = R \circ R^n$ for $n \geq 0$, where $\circ$ is the composition of relations. Put $R^{\leq n} := R^0 \cup \dots \cup R^n$.

R^{-1} is the converse to R, and $R^{\pm} := R \cup R^{-1}$ is the symmetric closure of R. For $x \in W$, $R(x) := \{y \mid xRy\}$.

The cardinality of a set W is denoted by $|W|$. The restriction of a function (frame, model) f to a set S is denoted by $f|S$. If (W, R) is a frame and $W \supseteq V \neq \varnothing$, the restriction $(W, R)|V$ $(= (V, R|V) = (V, R \cap (V \times V)))$ is usually denoted by (V, R).

A *point-generated subframe* with a root u of a frame $\mathsf{F} = (W, R)$ is

$$\mathsf{F} \uparrow u := \mathsf{F} | \bigcup_{n \geq 0} R^n(u).$$

If $\mathsf{F} = \mathsf{F} \uparrow u$, u is called a *root* of F. Recall that

$$\mathit{Log}(\mathsf{F}) = \bigcap_{u \in W} \mathit{Log}(\mathsf{F} \uparrow u),$$

by the generation lemma.

A frame (W, R) with a root u is called a *tree* if $R^{-1}(u) = \varnothing$ and $R^{-1}(x)$ is a singleton for any $x \neq u$.

$f : \mathsf{F} \twoheadrightarrow \mathsf{G}$ denotes that f is a p-morphism from F onto G; the existence of a p-morphism is denoted by $\mathsf{F} \twoheadrightarrow \mathsf{G}$.

Recall that $\mathsf{F} \twoheadrightarrow \mathsf{G}$ implies $\mathit{Log}(\mathsf{F}) \subseteq \mathit{Log}(\mathsf{G})$, by the p-morhism lemma.

The next proposition is easily proved by induction on k (cf. e.g. [1]).

Proposition 2.1 *Let* M *be a Kripke model over a frame* (W, R)*. Then for any* x *in* M*, for any* φ *of modal depth* $\leq k$

$$\mathsf{M}, x \vDash \varphi \iff \mathsf{M} | R^{\leq k}(x), x \vDash \varphi.$$

2.2 Words and trees

An *alphabet* is an arbitrary nonempty set. $A^* := \{\lambda\} \cup A \cup A^2 \cup \dots$ is the set of all *words* in an alphabet A, where λ denotes the empty word. st denotes the concatenation of words s and t; $|s|$ is the length of a word s.

For $s, t \in A^*$, put [1]

$$\begin{aligned} s \lhd t &\iff t = as \text{ for some } a \in A, \\ s \unlhd t &\iff s \lhd t \text{ or } s = t, \\ s \sqsubseteq t &\iff t = rs \text{ for some } r \in A^*. \end{aligned}$$

[1] Note that in this paper words are ordered ‘from the end’. This is done for representing numbers written in numeral systems, see the proof of Theorem 5.3.

We regard natural numbers as ordinals. For $n, k > 0$, $T_{n,k}$ is the set of all words of length at most n in the alphabet $k = \{0, \ldots, k-1\}$.

A nonempty set of words W (or a frame $(W, \lhd)$) is a *standard tree* if $(W, \sqsubseteq)$ has the least element u_0 and W is downwards closed:

$$\forall u \forall v\, (u \in W \;\&\; u \neq u_0 \;\&\; v \lhd u \Longrightarrow v \in W).$$

Proposition 2.2

(i) *If φ is **DB**-satisfiable, then there exists $n \geq 2$ such that φ is satisfiable at $(T_{n,n}, \lhd^{\pm}), \lambda$.*

(ii) *If φ is **TB**-satisfiable, then there exists $n \geq 2$ such that φ is satisfiable at $(T_{n,n}, \unlhd^{\pm}), \lambda$.*

Proof. (1) Suppose φ is **DB**-satisfiable. It is well-known that **DB**-frames are serial symmetric and **DB** has the finite model property. So by the generation lemma we obtain a finite **DB**-frame $\mathsf{F} = (W, R)$ with a root u such that φ is satisfiable at F, u.

Now we use unravelling (cf. [3]). Given a finite **DB**-frame $\mathsf{F} = (W, R)$ with a root u such that φ is satisfiable at F, u , we construct a tree $\mathsf{F}^\sharp = (W^\sharp, R^\sharp)$, which consists of all R-paths in F starting at u; $\alpha R^\sharp \beta$ iff β is obtained by adding a world at the end of α. There is p-morphism $\mathsf{F}^\sharp \twoheadrightarrow \mathsf{F}$ sending every path to its end.

Put $n := \max(|W|, md(\varphi))$. Since F is serial, it follows that $1 \leq |R^\sharp(\alpha)| \leq n$ for any $\alpha \in W^\sharp$.

We may further assume that $|R^\sharp(\alpha)| = n$ for all α in $\mathsf{F}^\sharp$ (in fact, $1 \leq |R^\sharp(\alpha)| \leq n$, but $R^\sharp(\alpha)$ can be extended by adding virtual copies of one of its elements; we skip the routine details here). Therefore $\mathsf{F}^\sharp$ is isomorphic to $(n^*, \lhd)$.

Since F is symmetric, it readily follows that $(n^*, \lhd^{\pm}) \twoheadrightarrow \mathsf{F}$, and a p-morphism sends the root to the root. So φ is satisfiable at $(n^*, \lhd^{\pm}), \lambda$. Since $md(\varphi) \leq n$, by Proposition 2.1, it is satisfiable at $(n^*, \lhd^{\pm})|(\lhd^{\pm})^{\leq n}(\lambda), \lambda$. The latter frame is exactly $(T_{n,n}, \lhd^{\pm})$.

(2) The argument for **TB** is almost the same — take the reflexive symmetric closure instead of the symmetric closure. $\square$

2.3 Distances, products, and powersets

For a frame $\mathsf{F} = (W, R)$ we define *the distance function*

$$d(u, v) := \min\{k \geq 0 \mid u(R^{\pm})^k v\}$$

if the latter set is nonempty and ∞ otherwise. If F is connected (i.e., $(W, R^{\pm})$ is rooted), $d(u, v)$ is always finite.

Proposition 2.3 *Let T be a standard tree, d the distance in $(T, \lhd)$, $V \subseteq T$, $|V| > 1$, and $d(v_1, v_2) = 2$ for all distinct $v_1, v_2 \in V$. Then there exists a unique $u \in T$ such that $u \lhd^{\pm} v$ for all $v \in V$.*

Proof. If $v_1, v_2 \in V$, $v_1 \neq v_2$, then there exists a unique point u such that $u \lhd^{\pm} v_1$ and $u \lhd^{\pm} v_2$. Also, for any $v \in V$, if $d(v, v_2) = d(v, v_1) = 2$, then $u \lhd^{\pm} v$. □

For $\mathbf{x}, \mathbf{y} \in A^n$, by $h(\mathbf{x}, \mathbf{y})$ we denote the Hamming distance between $\mathbf{x}$ and $\mathbf{y}$:

$$h(\mathbf{x}, \mathbf{y}) = |\{i \mid x_i \neq y_i\}|.$$

For $\mathbf{x}, \mathbf{y} \in A^*$, put

$$\mathbf{x} H \mathbf{y} \iff |\mathbf{x}| = |\mathbf{y}| \ \& \ h(\mathbf{x}, \mathbf{y}) = 1.$$

So h is the distance in the frame (A^*, H).

$\mathcal{P}(U)$ denotes the power set of a set U; $U \triangle V$ denotes the symmetric difference of sets U, V. The frame $(\{0,1\}^n, H)$ is obviously isomorphic to $(\mathcal{P}(n), \triangle_1)$, where

$$U \triangle_1 V \quad \iff \quad |U \triangle V| = 1.$$

Consider an inequality frame [2] $(A, \neq)$ and its power

$$(A, \neq)^n = (A^n, \neq_0, \ldots, \neq_{n-1});$$

so

$$\mathbf{x} \neq_i \mathbf{y} \iff x_i \neq y_i \text{ and } x_j = y_j \text{ for all } j \neq i.$$

Clearly, $(A^n, H) = (A^n, \neq_0 \cup \cdots \cup \neq_{n-1})$.

Consider the translation $t^{(n)} : ML(\Diamond) \to ML(\Diamond_0, \ldots, \Diamond_{n-1})$ preserving the atoms and the boolean connectives and such that

$$t^{(n)}(\Diamond\varphi) = \Diamond_0 t^{(n)}(\varphi) \vee \cdots \vee \Diamond_{n-1} t^{(n)}(\varphi).$$

The following is trivial:

Proposition 2.4 *For any $A \neq \varnothing$, $n > 0$, $\theta : PV \to A^n$, φ, $\mathbf{x} \in A^n$,*

$$((A, \neq)^n, \theta), \mathbf{x} \vDash t^{(n)}(\varphi) \iff ((A^n, H), \theta), \mathbf{x} \vDash \varphi.$$

Theorem 2.5 (i) *For any infinite A, $n > 0$*

$$Log(A^n, H) = Log(\omega^n, H).$$

(ii) *$Log(\omega^n, H)$ is recursively axiomatizable.*

Proof. We apply Theorem 4.3 from [4].

Let $\mathcal{C}$ be the class $\{(A, \neq) \mid A \text{ is infinite }\}$, which is obviously axiomatizable by a recursive set Σ of classical pure equality formulas. Let

$$\mathcal{C}^{[n]} = \{\mathsf{F}^n \mid \mathsf{F} \in \mathcal{C}\}.$$

[2] A more precise notation is $(A, \neq_A)$.

Recall that for any n-modal formula φ there is a 'cubic translation' $\varphi^n(x_0, \dots, x_{n-1})$; it translates a propositional variable p_j as an atom $P_j(x_0, \dots, x_{n-1})$, and

$$(\Box_i\varphi)^n(x_0, \dots, x_{n-1}) = \forall y(y \neq x_i \to [y/x_i]\varphi^n(x_0, \dots, x_{n-1})).$$

Then we have

$$\mathcal{C}^{[n]} \models \varphi \Leftrightarrow \mathcal{C} \models \overline{\forall}\varphi^n \Leftrightarrow \Sigma \vdash \varphi^n,$$

where $\overline{\forall}\varphi^n$ denotes the universal second-order sentence obtained by quantifying over all parameters and predicate symbols occurring in φ^n. This implies that $Log(\mathcal{C}^{[n]})$ is RE.

On the other hand, it follows that $Log(\mathcal{C}^{[n]}) = Log((A, \neq)^n)$ for any infinite A. In fact, the above equivalence also holds for $\mathcal{C} = \{(A, \neq)\}$, by the Löwenheim – Skolem theorem.

Since the H-modality is recursively encoded in this language, the result follows. $\Box$

3 Non-finitely axiomatizable logics

In this section we recall some known results on logics of Hamming frames: for an infinite alphabet A, the logics $Log(A^n, H)$ are non-finitely axiomatizable for all $n > 0$ (in particular, this holds for the inequality relation on an infinite set). As the paper [8] is published only in Russian, we reproduce its relevant parts here.

Theorems on non-finite axiomatizability for many-dimensional modal logics usually require an intricate technique (see e.g. [5]). In our case we prove a stronger result (non-axiomatizability in finitely many variables) by using a rather simple construction. We base on the approach from [10], as far as we know the first result of this kind in the context of modal logic.

Definition 3.1 *For a modal logic L we define its m-*fragments*:*

$$L\lceil m := \{\varphi \in L \mid PV(\varphi) \subseteq \{p_1, \dots, p_m\}\} \text{ for } m \geq 1,$$
$$L\lceil 0 := \{\varphi \in L \mid PV(\varphi) = \varnothing\}.$$

L is called m-variable axiomatizable *if $\boldsymbol{K} + L\lceil m = L$. A logic is called* finite-variable axiomatizable *if it is m-variable axiomatizable for some m.*

Frames F, G are called (modally) m-equivalent *(in symbols, $\mathsf{F} \sim_m \mathsf{G}$) if $Log(\mathsf{F})\lceil m = Log(\mathsf{G})\lceil m$.*

Lemma 3.2 *[10],[12] Let Λ be a logic, $m \geq 0$, and suppose there exist frames G, G' such that $G \sim_m G'$, $\Lambda \subseteq Log(G)$, $\Lambda \not\subseteq Log(G')$. Then Λ is not m-variable axiomatizable.*

The next fact is a slight modification of the Jankov-Fine lemma (cf. [2], [12]).

Lemma 3.3 *(see [2], [12]) Let $\mathsf{F} = (W, R)$ be a frame such that $R^{\leq k}$ is transitive for some $k > 0$ and let G be a finite frame. Then $Log(\mathsf{F}) \subseteq Log(\mathsf{G})$ iff there exists a point w in F such that $\mathsf{F}\uparrow w$ $(= \mathsf{F}|R^{\leq k}(w)) \twoheadrightarrow \mathsf{G}$.*

Proposition 3.4 *Let f be a monotonic[3] map from a frame (W, R) to a frame (V, S). If there exists $W_0 \subseteq W$ such that $|W_0| > |V|$ and $w_1 R w_2$ for all different w_1, w_2 from W_0, then (V, S) contains a reflexive point.*

Proof. Since $|W_0| > |V|$, there exist $w_1, w_2 \in W_0$ such that $w_1 \neq w_2$ and $f(w_1) = f(w_2)$. Then $w_1 R w_2$ by assumption, so $f(w_1) S f(w_2) = f(w_1)$ by the monotonicity of f. □

For $m > 0$ consider the frames $\mathsf{K}_m := (m, \neq_m)$, $\mathsf{K}'_m := (m, R_m)$, where $\neq_m$ is the inequality relation on m; $R_m := \{(m-1, m-1)\} \cup \neq_m$. Thus K_m is an irreflexive clique with m points, and K'_m is obtained from K_{m+1} by sticking two points into one reflexive point.

Lemma 3.5 *[12] $\mathsf{K}_{2^m+1} \sim_m \mathsf{K}'_{2^m}$ for any $m \geq 0$.*

Theorem 3.6 *[8] If A is infinite, then for any $n > 0$ the logic $Log(A^n, H)$ is not finite-variable axiomatizable.*

Proof. By Theorem 2.5, we may assume that A is the set of integers.

By Lemmas 3.2, 3.5, and 3.3, it is sufficient to show that for any $m > 1$
(i) $(A^n, H) \not\twoheadrightarrow \mathsf{K}_m$,
(ii) $(A^n, H) \twoheadrightarrow \mathsf{K}'_m$.

Note that (i) follows from Proposition 3.4, since A is infinite.

Let $a_0, \ldots, a_{m-2}$ be different elements of A.

If $n = 1$, then we define $f(a_i) = i$ for $m-1$ different elements $a_0, \ldots, a_{m-2}$ of A and $f(w) = m-1$ for all other elements from A. It is easy to check that $f : (A, \neq) \twoheadrightarrow \mathsf{K}'_m$.

Suppose $n > 1$. Let

$$\begin{aligned} &V_i := \{\mathbf{x} \mid x_0 + \cdots + x_{n-1} = a_i\} \text{ for } 0 \leq i < m-1, \\ &V_{m-1} := A^n - \bigcup_{0 \leq i < m-1} V_i. \end{aligned}$$

Clearly, A^n is the disjoint union of these sets. Moreover, for all $i < m-1$, we have

$$\text{if } \mathbf{x} \in V_i, \text{ then } H(\mathbf{x}) \cap V_i = \varnothing; \tag{1}$$

$$\text{if } \mathbf{x} \notin V_i, \text{ then } H(\mathbf{x}) \cap V_i \neq \varnothing; \tag{2}$$

$$\text{for each } \mathbf{x} \in A^n,\ H(\mathbf{x}) \cap V_{m-1} \neq \varnothing. \tag{3}$$

Let us check (1). Let $i < m$, $\mathbf{x} \in V_i$, $\mathbf{x} H \mathbf{y}$. If follows that for some l we have $x_l \neq y_l$, and $x_j = y_j$ for all $j \neq l$. Then $\sum\limits_{0 \leq j < n} x_j \neq \sum\limits_{0 \leq j < n} y_j$, so $\mathbf{y} \notin V_i$.

To obtain (2), take $\mathbf{y}$ such that $x_i = y_i$ for $i > 0$ and $y_0 = a_i - \sum\limits_{1 \leq j < n} x_j$. Then $\mathbf{y} \in V_i$ and $\mathbf{x} H \mathbf{y}$.

Let us check (3). Consider the set

$$U := \{\mathbf{y} \mid x_0 \neq y_0 \ \&\ \forall j > 0\, (y_j = x_j)\}.$$

[3] I.e., f is a homomorphism.

Then $U \subseteq H(\mathbf{x})$. Since U is infinite, and for each $i < m-1$, $U \cap V_i$ is a singleton, we have $U \cap V_{m-1} \neq \varnothing$.

Now we define a map $f : A^n \to m$ by putting $f(\mathbf{x}) := i$ iff $\mathbf{x} \in V_i$. Since every V_i is nonempty, f is surjective. If $\mathbf{x}H\mathbf{y}$ and $\mathbf{x} \notin V_{m-1}$, then using (1) we get $f(\mathbf{x}) \neq f(\mathbf{y})$; hence, f is monotonic. From (2) and (3) it follows that f satisfies the 'lift property', i.e.

$$f(x)R_m i \Rightarrow \exists y \in H(x)\ f(y) = i.$$

Thus f is a p-morphism. $\square$

Note that if $n = 1$ then $(A, H) = (A, \neq)$ and Theorem 3.6 yields

Corollary 3.7 *If A is an infinite set, then $Log(A, \neq)$ is not finite-variable axiomatizable.*

Remark 3.8 This gives us a simple example of a modal logic which is not finitely axiomatizable, but has a finitely axiomatizable conservative extension. Namely, by Corollary 3.7 the logic $Log(\mathbb{R}^2, \neq)$ is not f.a., whereas the topological modal logic with the difference modality of $\mathbb{R}^2$ has a finite axiomatization [7].

Let C_0 and C_1 be a reflexive and an irreflexive singleton, respectively. From Lemma 3.5 it follows that for any $m \geq 0$

$$\mathsf{K}_{2^m+1} \times \mathsf{C}_1 \sim_m \mathsf{K}'_{2^m} \times C_1, \quad \mathsf{K}_{2^m+1} \times \mathsf{C}_0 \sim_m \mathsf{K}'_{2^m} \times \mathsf{C}_0.$$

Let A be an infinite set. If $|B| > 1$, then $(B, \neq) \twoheadrightarrow \mathsf{C}_1$, so

$$\begin{array}{ll} (A, \neq) \times (B, \neq) \twoheadrightarrow \mathsf{K}'_m \times \mathsf{C}_1, & (A, \neq) \times (B, \neq) \not\twoheadrightarrow \mathsf{K}_m \times \mathsf{C}_1, \\ (A, \neq) \times \mathsf{C}_0 \twoheadrightarrow \mathsf{K}'_m \times \mathsf{C}_0, & (A, \neq) \times \mathsf{C}_0 \not\twoheadrightarrow \mathsf{K}_m \times \mathsf{C}_0, \end{array}$$

which leads to

Corollary 3.9 *[8] Suppose B is nonempty, A is infinite. Then $Log((A, \neq) \times (B, \neq))$ is not finite-variable axiomatizable.*

Recently [6] a similar result was obtained for products with the minimal difference logic **DL**: all logics in the interval between $\mathbf{K} \times \mathbf{DL}$ and $\mathbf{S5} \times \mathbf{DL}$ are not finite-variable axiomatizable. However, the logics described by the previous corollary are not in this interval.

4 Undecidability

In this section we prove undecidability results on logics of Hamming spaces.

Fix $n \geq 2$. Our aim is to define unimodal operators (formulas) which will emulate $\mathbf{S5}^n$-modalities in frames (A^n, H). For this purpose we use a formula $sets^{(n)}$ encoding the product structure on (A^n, H) up to permutation of coordinates.

Put $\Box^0\varphi = \varphi$, $\Box^{l+1}\varphi = \Box\Box^l\varphi$, $\Box^{\leq l}\varphi = \bigwedge\limits_{0 \leq i \leq l} \Box^i\varphi$, $\Diamond^l\varphi = \neg\Box^l\neg\varphi$, $\Diamond^{\leq l}\varphi = \neg\Box^{\leq l}\neg\varphi$. Note that the operator $\Box^{\leq n}$ acts like the universal modality on (A^n, H): $((A^n, H), \theta), \mathbf{x} \vDash \Box^{\leq n}\varphi$ for some $\mathbf{x}$ iff $((A^n, H), \theta), \mathbf{x} \vDash \varphi$ for all $\mathbf{x} \in A^n$.

For each set $U \subseteq n$ we fix a variable p_U. Let $sets^{(n)}$ be the conjunction of the following formulas:

$$p_\varnothing \wedge \neg\Diamond p_\varnothing \tag{4}$$

$$\Box^{\leq n}\left(\bigvee_{U\subseteq n} p_U \wedge \bigwedge_{U,V\subseteq n,\ U\neq V} (p_U \to \neg p_V)\right) \tag{5}$$

$$\Box^{\leq n}\left(\bigwedge_{U,V\subseteq n,\ |U\triangle V|=1} (p_U \to \Diamond p_V)\right) \tag{6}$$

$$\Box^{\leq n}\left(\bigwedge_{U,V\subseteq n,\ |U\triangle V|>1} (p_U \to \neg\Diamond p_V)\right) \tag{7}$$

(Note that if we also add the conjuncts $p_U \to \neg\Diamond p_U$ for all nonempty $U \subseteq n$, then we obtain the frame formula for the frame $(\mathcal{P}(n), \triangle_1)$ at the point $\varnothing$.)

For $\mathbf{x}, \mathbf{y} \in A^n$ and a permutation $\sigma : n \to n$, let

$$D_\sigma(\mathbf{x}, \mathbf{y}) = \{i \mid x_{\sigma(i)} \neq y_{\sigma(i)}\}.$$

The meaning of the formula $sets^{(n)}$ is explained by the following key fact.

Lemma 4.1 *Let $|A| > 1$, $((A^n, H), \theta), \mathbf{r} \vDash sets^{(n)}$. Then there exists a unique permutation $\sigma : n \to n$ such that for any $\mathbf{x} \in A^n$ and $V \subseteq n$,*

$$D_\sigma(\mathbf{r}, \mathbf{x}) = V \iff ((A^n, H), \theta), \mathbf{x} \vDash p_V. \tag{8}$$

Proof. For any $\mathbf{x}$ in A^n we define sets $A_0(\mathbf{x}), \ldots, A_{n-1}(\mathbf{x})$ ($\mathbf{x}$-*axis*):

$$A_i(\mathbf{x}) = \{\mathbf{y} \mid \mathbf{y}H\mathbf{x} \ \&\ y_i \neq x_i\}.$$

Clearly, $H(\mathbf{x})$ is the disjoint union of the sets $A_i(\mathbf{x})$, and for $0 \leq i \neq j < n$,

$$\mathbf{y}H\mathbf{z} \text{ for no } \mathbf{y} \in A_i,\ \mathbf{z} \in A_j. \tag{9}$$

Put $\mathsf{M} = ((A^n, H), \theta)$. Let σ be the binary relation on n such that

$$(i, j) \in \sigma \iff \mathsf{M}, \mathbf{y} \vDash p_{\{i\}} \text{ for some } \mathbf{y} \in A_j(\mathbf{r}).$$

First, let us show that σ is a permutation $n \to n$.

By (6), $\mathsf{M}, \mathbf{r} \vDash \Diamond p_{\{0\}} \wedge \ldots \wedge \Diamond p_{\{n-1\}}$. Let $i < n$. Then for some $j < n$, $\mathbf{y} \in A_j(\mathbf{r})$, we have $\mathsf{M}, \mathbf{y} \vDash p_{\{i\}}$. Suppose $k \neq i$, $k < n$; by (7), $\mathsf{M}, \mathbf{y} \vDash \neg\Diamond p_{\{k\}}$; by (5), $\mathsf{M}, \mathbf{y} \vDash \neg p_{\{k\}}$; so we have $\mathsf{M}, \mathbf{z} \nvDash p_{\{k\}}$ for all $\mathbf{z} \in A_j(\mathbf{r})$. It means that σ is a function from n to n. If $\mathbf{z} \in H(\mathbf{r})$ then $\mathsf{M}, \mathbf{z} \vDash \neg p_\varnothing$ by (4), and if $|V| > 1$ then $\mathsf{M}, \mathbf{z} \vDash \neg p_V$ by (7). By (5), $\mathsf{M}, \mathbf{z} \vDash p_V$ for some V, and it follows that V is a singleton. It means that σ is surjective, i.e., it is a permutation.

Since σ is a bijection, then $h(\mathbf{r}, \mathbf{x}) = |D_\sigma(\mathbf{r}, \mathbf{x})|$ for any $\mathbf{x}$ in M.

Let us check (8). By (5), for every $\mathbf{x}$ in M there exists a unique $U \subseteq n$ such that $\mathbf{x} \vDash p_U$. It follows that we only have to prove the left-to-right direction.

We proceed by induction on $h(\mathbf{r}, \mathbf{x})$.

If $h(\mathbf{r}, \mathbf{x}) = 0$, then $\mathbf{x} = \mathbf{r}$, $D_\sigma(\mathbf{r}, \mathbf{x}) = \varnothing$; by (4), $\mathsf{M}, \mathbf{r} \vDash p_\varnothing$.

For the induction step, let $0 \le l < n-1$ and suppose that (8) holds for all $\mathbf{y} \in H^{\le l}(\mathbf{r})$. Let

$$h(\mathbf{x}, \mathbf{r}) = l + 1,\ \mathsf{M}, \mathbf{x} \vDash p_U,\ D_\sigma(\mathbf{r}, \mathbf{x}) = V.$$

We have to show that $U = V$.

If $l = 0$, then $\mathbf{x} \in H(\mathbf{r})$ and, as shown above, $U = \{i\}$ for some i; then, by the definition of σ, $\mathbf{x} \in A_{\sigma(i)}(\mathbf{r})$, which means $V = \{i\}$.

Suppose $l > 0$. Then $|V| = h(\mathbf{r}, \mathbf{x}) = l + 1$. Let $V = \{i_0, \ldots, i_l\}$. Take points $\mathbf{y}^{(0)}, \ldots, \mathbf{y}^{(l)}$ such that for $j \le l$

$$\mathbf{y}^{(j)} \in A_{\sigma(i_j)}(\mathbf{x}) \text{ and } D_\sigma(\mathbf{r}, \mathbf{y}^{(j)}) = V - \{i_j\},$$

that is $y_k^{(j)} = x_k$ for $k \ne \sigma(i_j)$, $y_k^{(j)} = r_k$ for $k = \sigma(i_j)$.

Then $h(\mathbf{r}, \mathbf{y}^{(j)}) = l$, and by the induction hypothesis,

$$\mathsf{M}, \mathbf{y}^{(j)} \vDash p_{V - \{i_j\}} \tag{10}$$

Since $\mathbf{y}^{(j)} \in H(\mathbf{x})$, then by (7) we have

$$|U \bigtriangleup (V - \{i_j\})| \le 1 \text{ for all } j \le l. \tag{11}$$

If $U = V - \{i_j\}$ for some $j \le l$, then $\mathsf{M}, \mathbf{y}^{(k)} \vDash \Diamond p_{V - \{i_j\}}$ for some $k \ne j$ (note that $l > 0$), which contradicts (7). Thus $U \bigtriangleup_1 (V - \{i_j\})$ for all $j \le l$, and $|U| = l - 1$ or $|U| = l + 1$.

In the first case, by (11) we have $U \subseteq V - \{i_j\}$ for all j, so $U \subseteq \bigcap_{0 \le j \le l} (V - \{i_j\}) = \varnothing$. It follows that $l = 1$. Let $\mathbf{y} = \mathbf{y}^{(0)}$, $\{i\} = D_\sigma(\mathbf{r}, \mathbf{y})$. For each $k \ne i$ we choose a point $\mathbf{z}_k$ such that $\mathbf{y} H \mathbf{z}_k$ and $\mathsf{M}, \mathbf{z} \vDash p_{\{i,k\}}$ (such points exist, because (6) and (10) imply $\mathsf{M}, \mathbf{y} \vDash \Diamond p_{\{i,k\}}$). Let Z be the set of these points. Since $|Z| = n - 1$, by the pigeonhole principle there exist two points from the set $Z \cup \{\mathbf{r}, \mathbf{x}\}$ which belong to the same $\mathbf{y}$-axis. If these points are $\mathbf{r}$ and $\mathbf{x}$, we have a contradiction with (4), in all other cases we have a contradiction with (7).

It follows that $|U| = l + 1$, so $U \supseteq V - \{i_j\}$ for all j, so

$$U \supseteq \bigcup_{0 \le j \le l} (V - \{i_j\}) = V.$$

Since $|U| = |V|$, we obtain $U = V$. $\square$

If $((A^n, H), \theta), \mathbf{r} \vDash \mathit{sets}^{(n)}$, then the permutation satisfying (8) is called *the* $(\theta, \mathbf{r})$*-permutation.*

For $0 \le i < n$, put
$$plane_i^{(n)} = \bigvee_{U \subseteq n,\ i \notin U} p_U.$$

From the above lemma we immediately obtain

Proposition 4.2 *Let $|A| > 1$, $((A^n, H), \theta), \mathbf{r} \vDash sets^{(n)}$, σ be the $(\theta, \mathbf{r})$-permutation. Then for any $\mathbf{x} \in A^n$, $0 \le i < n$,*

$$((A^n, H), \theta), \mathbf{x} \vDash plane_i^{(n)} \iff r_{\sigma(i)} = x_{\sigma(i)}. \tag{12}$$

Finally, we define formulas $\Diamond_i^{(n)}$, $0 \le i < n$:

$$\begin{array}{rcl} \Diamond_i^{(n)} & = & s \vee ((plane_i^{(n)} \to \Diamond(\neg plane_i^{(n)} \wedge s)) \wedge \\ & & \wedge(\neg plane_i^{(n)} \to \Diamond(plane_i^{(n)} \wedge (s \vee (\Diamond(\neg plane_i^{(n)} \wedge s)))))). \end{array}$$

For a formula φ, let $\Diamond_i^{(n)}\varphi$ denote the result of substitution of s for φ in $\Diamond_i^{(n)}$.

Let us illustrate the meaning of the above formulas. $\Diamond_i^{(n)}\varphi$ is true at a point $\mathbf{x}$ iff either φ is true at $\mathbf{x}$, or φ is true at a point $\mathbf{y} \in H(\mathbf{x})$ such that the following holds: if $\mathbf{x} \in plane_i^{(n)}$, then $\mathbf{y} \notin plane_i^{(n)}$; if $\mathbf{x} \notin plane_i^{(n)}$, then either $\mathbf{y} \in plane_i^{(n)}$, or $\mathbf{y} \notin plane_i^{(n)}$ and there exists $\mathbf{z} \in plane_i^{(n)}$ such that $\mathbf{x}H\mathbf{z}H\mathbf{y}$. In all cases we "move" along $\sigma(i)$-direction; on the other hand, any point in $A_{\sigma(i)}(\mathbf{x})$ is reachable. It means that the set of all "possible" points is

$$\{\mathbf{y} \mid x_j = y_j \text{ for all } j \neq \sigma(i)\},$$

and we have

Proposition 4.3 *Let $|A| > 1$, $((A^n, H), \theta), \mathbf{r} \vDash sets^{(n)}$, σ be the $(\theta, \mathbf{r})$-permutation. Then for any $\mathbf{x} \in A^n$,*

$$((A^n, H), \theta), \mathbf{x} \vDash \Diamond_i^{(n)} \iff \exists \mathbf{y}\, (D_\sigma(\mathbf{x}, \mathbf{y}) \subseteq \{i\} \;\&\; ((A^n, H), \theta), \mathbf{y} \vDash s)$$

For a formula φ in the n-modal language $ML(\Diamond_0, \dots, \Diamond_{n-1})$, we define the unimodal formula $[\varphi]^{(n)}$:

$$[p]^{(n)} = p \text{ for } p \in PV; \quad [\phi \wedge \psi]^{(n)} = [\phi]^{(n)} \wedge [\psi]^{(n)}; \quad [\neg\phi]^{(n)} = \neg([\phi]^{(n)});$$
$$[\Diamond_i \phi]^{(n)} = \Diamond_i^{(n)}[\phi]^{(n)}.$$

Lemma 4.4 *Let $|A| > 1$, $((A^n, H), \theta), \mathbf{r} \vDash sets^{(n)}$, σ be the $(\theta, \mathbf{r})$-permutation, $(A, A \times A)^n = (A^n, R_0, \dots R_{n-1})$. Then for any n-modal formula φ with $PV(\varphi) \cap PV(sets^{(n)}) = \varnothing$, for any $\mathbf{x} \in A^n$, we have*

$$((A^n, R_{\sigma(0)}, \dots R_{\sigma(n-1)}), \theta), \mathbf{x} \vDash \varphi \iff ((A^n, H), \theta), \mathbf{x} \vDash [\varphi]^{(n)} \tag{13}$$

Proof. Note that $R_{\sigma(i)} = \{(\mathbf{x}, \mathbf{y}) \mid D_\sigma(\mathbf{x}, \mathbf{y}) \subseteq \{i\}\}$. Thus using Proposition 4.3, the proof can be obtained by straightforward induction on the length of φ (see e.g. [11, Lemma 3.5] for details). □

Theorem 4.5 *For $|A| > 1$, $n \geq 2$, for any n-modal formula φ that does not share variables with $sets^{(n)}$, we have:*

$$\varphi \text{ is } (A, A \times A)^n\text{-satisfiable} \iff sets^{(n)} \wedge [\varphi]^{(n)} \text{ is } (A^n, H)\text{-satisfiable.}$$

Proof. Let $\mathsf{F} = (A, A \times A)^n = (A^n, R_0, \ldots, R_{n-1})$.

($\Longleftarrow$). For any permutation $\sigma : n \to n$, the frames $(A^n, R_{\sigma(0)}, \ldots R_{\sigma(n-1)})$ and F are isomorphic, so by Lemma 4.4, φ is F-satisfiable.

($\Longrightarrow$). Suppose $(\mathsf{F}, \theta), \mathbf{r} \vDash \varphi$. Let σ be the identity map on n. Put $\eta(p) = \theta(p)$ for all $p \notin PV(sets^{(n)})$; for $V \subseteq n$, put

$$\eta(p_V) = \{\mathbf{x} \mid D_\sigma(\mathbf{r}, \mathbf{x}) = V\}.$$

Since $PV(\varphi) \cap PV(sets^{(n)}) = \varnothing$, $(\mathsf{F}, \eta), \mathbf{r} \vDash \varphi$. By a straightforward argument, $((A^n, H), \eta), \mathbf{r} \vDash sets^{(n)}$. By Lemma 4.4, $((A^n, H), \eta), \mathbf{r} \vDash [\varphi]^{(n)}$, so $sets^{(n)} \wedge [\varphi]^{(n)}$ is (A^n, H)-satisfiable. $\square$

Since all logics $\mathbf{S5}^n$, $n > 2$, are undecidable [9], and

$$\mathbf{S5}^n = Log(\{(A, A \times A)^n \mid A \neq \varnothing\}) = Log((\omega, \omega \times \omega)^n),$$

we have the following corollaries.

Corollary 4.6 *For any $n > 2$, the logic $Log(\{(A^n, H) \mid A \neq \varnothing\})$ is undecidable.*

Corollary 4.7 *For any $n > 2$, the logic $Log(\omega^n, H)$ is undecidable.*

5 Completeness

In spite of undecidability proved in section 4, there are some positive results on logics of Hamming frames over "long" words. For functions $f, g : I \to A$, put

$$f\, H\, g \iff |\{i \mid i \in I,\ f(i) \neq g(i)\}| = 1.$$

Our first positive result is the completeness of **DB** with respect to the Hamming frame of infinite (0,1)-sequences $(2^\omega, H)$. We formulate it in terms of sets of natural numbers. Clearly, the frame $(2^\omega, H)$ is isomorphic to the frame $(\mathcal{P}(\omega), \triangle_1)$.

Definition 5.1 *Let (W, R), (V, S) be frames, $x \in W$, $f : W \to V$, $n > 0$. f is an* n-reduction at x from (W, R) to (V, S), *if the restriction of f on $R^{\leq n}(x)$ is monotonic, and for any $y \in R^{\leq n-1}(x)$, $z \in V$, if $f(y) S z$ then $y R y'$ and $f(y') = z$ for some y'.*

Lemma 5.2 *Let (W, R), (V, S) be frames, $x \in W$, f be an n-reduction at x from (W, R) to (V, S). Then for any φ with $md(\varphi) \leq n$, if φ is satisfiable at $f(x)$ in (V, S), then φ is satisfiable at x in (W, R).*

Proof. Suppose $((V, S), \theta), f(x) \vDash \varphi$. For $p \in PV$, put $\eta(p) = f^{-1}(\theta(p))$. Then by induction on the modal depth of a formula, it is easy to check that

$$((W, R), \eta), y \vDash \psi \iff ((V, S), \theta), f(y) \vDash \psi$$

for any ψ, l, y such that $0 \leq l \leq n - md(\psi)$ and $y \in R^{\leq l}(x)$. $\square$

For a set U, $\mathcal{P}_{fin}(U)$ denotes the set of all finite subsets of U.

Theorem 5.3 $Log(\mathcal{P}(\omega), \triangle_1)) = Log(\mathcal{P}_{fin}(\omega), \triangle_1)) = \boldsymbol{DB}$.

Proof. First, we introduce some auxiliary definitions. Fix $n \geq 2$. For $x \in \omega$, let $\overline{x}$ be the leaf $a_n \dots a_1 \in T_{n,n}$ such that $x = mn^n + \sum_{1 \leq i \leq n} a_i n^{i-1}$ for some $m \geq 0$.

Let d denote the distance in $(T_{n,n}, \lhd)$.

Let $x \in \omega$, $\overline{x} = a_n \dots a_1$. For $w = b_r \dots b_1 \in T_{n-1,n}$, let $[x\!:\!w]$ be the word $u \in T_{n,n}$ such that $d(\overline{x}, u) = d(\overline{x}, w) - 1$. Clearly, $[x\!:\!w]$ exists and unique: if $w \sqsubseteq \overline{x}$, then $[x\!:\!w] = a_{r+1} \dots a_1$ $(= b_{r+1} \dots b_1)$, otherwise (in this case $r > 0$) $[x\!:\!w] = b_{r-1} \dots b_1$. Note that $w \lhd^{\pm} [x\!:\!w]$. By a straightforward argument, for any $u, v \in T_{n-1,n}$

$$u \lhd^{\pm} v \ \&\ d(\overline{x}, u) < d(\overline{x}, v) \quad \Longleftrightarrow \quad [x\!:\!v] = u, \tag{14}$$

$$u \lhd^{\pm} v \quad \Longrightarrow \quad d(\overline{x}, u) \neq d(\overline{x}, v). \tag{15}$$

For $l \geq -1$, put $\mathcal{P}_l(\omega) = \{V \mid V \subset U,\ |V| \leq l\}$. By induction, we construct a sequence of functions $f_i : \mathcal{P}_i(\omega) \to T_{i,n}$, $0 \leq i \leq n$, such that:

(i) if $S, S' \in \mathcal{P}_i(\omega)$, $S \triangle_1 S'$, then $f_i(S) \lhd^{\pm} f_i(S')$;

(ii) if $S \in \mathcal{P}_{i-1}(\omega)$, $x > \max(S)$, then $f_i(S \cup \{x\}) = [x\!:\!f_i(S)]$;

Put $f_0(\varnothing) = \lambda$. Clearly, f_0 satisfies (i) and (ii).

Let $l < n$ and suppose f_l is already constructed and satisfies (i), (ii). We define f_{l+1} as follows.

Consider $S \subset \omega$. If $|S| \leq l$, put $f_{l+1}(S) = f_l(S)$. Suppose $|S| = l + 1$. Let $S = \{x_0, \dots, x_l\}$, $S_i = S - \{x_i\}$. For any $i, j \leq l$, if $i \neq j$, then $S_i \triangle_1 (S_i \cap S_j) \triangle_1 S_j$, so $d(f_l(S_i), f_l(S_j)) \in \{0, 2\}$ by (i).

If $f_l(S_i) \neq f_l(S_j)$ for some $i, j \leq l$, then by Proposition 2.3 there exists a unique u such that $u \lhd^{\pm} f(S_k)$ for all $k \leq l$. We put

$$f_{l+1}(S) = u.$$

Note that $|u| < |f(S_i)|$ or $|u| < |f(S_j)|$, so $f_{l+1}(S) \in T_{l-1,n}$.

If $f_l(S_i) = f_l(S_j)$ for all $i, j \leq l$, we put

$$f_{l+1}(S) = [\max(S)\!:\!f_l(S_0)].$$

Since $[\max(S)\!:\!f_l(S_0)] \lhd^{\pm} f_l(S_0)$, then $f_{l+1}(S) \in T_{l+1,n}$.

Let us check that f_{l+1} satisfies (i) and (ii).

To show (i), suppose $S \triangle_1 S'$, $S, S' \in \mathcal{P}_{l+1}(\omega)$.

If both S, S' in $\mathcal{P}_l(\omega)$, then (i) holds by the induction hypothesis. Suppose $|S| = l + 1$. If $f(S - \{x\}) \neq f(S - \{y\})$ for some $x, y \in S$, then by the above definition $f(S) \lhd^{\pm} f(S - \{x\})$ for any $x \in S$, and (i) holds, since $S' = S - \{x\}$

for some $x \in S$. Otherwise, $f(S) = [\max(S) : f_l(S')] \lhd^{\pm} f_l(S')$, which proves (i).

The key step in our proof is to check (ii). To this end, suppose $S \in \mathcal{P}_l(\omega)$, $x > \max(S)$. Consider the following two cases.

Case 1. $f_{l+1}(S) = f_{l+1}((S - \{y\}) \cup \{x\})$ for all $y \in S$. Then, by the definition of f_{l+1},

$$f_{l+1}(S \cup \{x\}) = [\max(S \cup \{x\}) : f_l(S)] = [x : f_l(S)].$$

Case 2. $f_{l+1}(S) \neq f_{l+1}(S' \cup \{x\})$, where $S' = S - \{y\}$ for some $y \in S$. Since $x > \max(S')$, by applying the induction hypothesis to $f_l(S' \cup \{x\})$, we obtain

$$f_{l+1}(S' \cup \{x\}) = f_l(S' \cup \{x\}) \stackrel{IH}{=} [x : f_l(S')] = [x : f_{l+1}(S')].$$

It follows that $f_{l+1}(S) \neq [x : f_{l+1}(S')]$. By (i), $f_{l+1}(S) \lhd^{\pm} f_{l+1}(S')$, so (14) implies $d(\overline{x}, f_{l+1}(S)) \geq d(\overline{x}, f_{l+1}(S'))$; by (15), $d(\overline{x}, f_{l+1}(S)) > d(\overline{x}, f_{l+1}(S'))$, and by (14) again, we obtain $[x : f_{l+1}(S)] = f_{l+1}(S')$. Thus

$$f_{l+1}(S) \lhd^{\pm} [x : f_{l+1}(S)] = f_{l+1}(S') \lhd^{\pm} [x : f_{l+1}(S')] = f_{l+1}(S' \cup \{x\}). \quad (16)$$

By the construction of f_{l+1}, $f_{l+1}(S \cup \{x\}) = u$, where

$$u \lhd^{\pm} f_l(S) = f_{l+1}(S), \quad u \lhd^{\pm} f_l(S' \cup \{x\}) = f_{l+1}(S' \cup \{x\}).$$

So by (16) we obtain $f_{l+1}(S \cup \{x\}) = [x : f_l(S)]$, q.e.d.

Let us check that f_n is an n-reduction at $\varnothing$ from $(\mathcal{P}_{fin}(\omega), \triangle_1)$ to $(T_{n,n}, \lhd^{\pm})$. f_n is monotonic by (i). Suppose $S \in \mathcal{P}_{n-1}(\omega)$, $u \in T_{n,n}$, $f_n(S) \lhd^{\pm} u$. If $f_n(S) \lhd u$, take x such that $x > \max(S)$, $u \sqsubseteq \overline{x}$; if $u \lhd f_n(S)$, then $f_n(S) \neq \lambda$ and there exists x such that $x > \max(S)$, $f_n(S) \not\sqsubseteq \overline{x}$. In both cases we obtain $f_n(S \cup \{x\}) = [x : f_n(S)] = u$.

By Propositions 5.2 and 2.2, we obtain that any **DB**-satisfiable formula is satisfiable in $(\mathcal{P}_{fin}(\omega), \triangle_1)$.

It follows that $Log(\mathcal{P}_{fin}(\omega), \triangle_1) \subseteq \mathbf{DB}$. Since $(\mathcal{P}_{fin}(\omega), \triangle_1)$ is a generated subframe of $(\mathcal{P}(\omega), \triangle_1)$, then $Log(\mathcal{P}(\omega), \triangle_1) \subseteq \mathbf{DB}$. The converse inclusions are trivial. $\square$

The above proof gives us the following semantic characterization of the logic **TB**.

For sets S, S', put

$$S \triangle_{\leq 1} S' \iff |S \triangle S'| \leq 1.$$

Since an n-reduction between two frames is also an n-reduction between their reflexive closures, we have

Corollary 5.4 *$Log(\mathcal{P}(\omega), \triangle_{\leq 1}) = Log(\mathcal{P}_{fin}(\omega), \triangle_{\leq 1}) =$ **TB**.*

Put $f \, \overline{H} \, g \iff f \, H \, g$ or $f = g$.

Theorem 5.5 *Let I be an infinite set, $|A| > 1$. Then $Log(A^I, \overline{H}) = \boldsymbol{TB}$.*

Proof. Note that the considered frame is reflexive and symmetric, so $Log(A^I, \overline{H}) \supseteq \mathbf{TB}$.

To prove completeness, assume without any loss of generality that $\{0, 1\} \subseteq A$, $\omega \subseteq I$.

We define $F : A^I \to 2^I$. For $f \in A^I$, to define $F(f) : I \to 2$, we put

$$F(f)(x) = \begin{cases} 0 & f(x) = 0 \\ 1 & f(x) \neq 0 \end{cases}$$

One can easily see that $F : (A^I, \overline{H}) \twoheadrightarrow (2^I, \overline{H})$, so $Log(A^I, \overline{H}) \subseteq Log(2^I, \overline{H})$.

For $f : I \to 2$, let $G(f)$ be the restriction of f to ω. Clearly, $G : (2^I, \overline{H}) \twoheadrightarrow (2^\omega, \overline{H})$, so $Log(2^I, \overline{H}) \subseteq Log(2^\omega, \overline{H})$. By Corollary 5.4, $Log(2^\omega, \overline{H}) = \mathbf{TB}$. It follows that

$$\mathbf{TB} \subseteq Log(A^I, \overline{H}) \subseteq Log(2^I, \overline{H}) \subseteq \mathbf{TB}.$$

□

Theorem 5.6 *For any infinite set I, $Log(\mathcal{P}(I), \triangle_{\leq 1}) = \boldsymbol{TB}$.*

Proof. Follows from Theorem 5.5 for $A = 2$. □

6 Some open questions

For a set I, let $\mathfrak{H}(I) = \{(A^I, H) \mid |A| > 1\}$, $\overline{\mathfrak{H}}(I) = \{(A^I, \overline{H}) \mid |A| > 1\}$.

For any $I \supseteq J$, we have $Log(A^I, \overline{H}) \subseteq Log(A^J, \overline{H})$, since the restriction (projection) operator is a p-morphism. The classes $\mathfrak{H}(n)$, $\overline{\mathfrak{H}}(n)$ are elementary, so their logics have the countable frame property. It follows that $Log(\overline{\mathfrak{H}}(n)) = Log(\omega^n, \overline{H})$ (see the proof of Theorem 5.5). So we have

$$\begin{aligned} \mathbf{S5} = Log(\omega, \omega \times \omega) = Log(\overline{\mathfrak{H}}(1)) \supsetneq Log(\overline{\mathfrak{H}}(2)) = Log(\omega^2, \overline{H}) \supsetneq \\ \supsetneq Log(\overline{\mathfrak{H}}(3)) = Log(\omega^3, \overline{H}) \supsetneq \cdots \supsetneq Log(\overline{\mathfrak{H}}(\omega)) = Log(2^\omega, \overline{H}) = \mathbf{TB}. \end{aligned}$$

(the fact that inclusions are strict is very simple: for example, these logics can be distinguished by formulas of finite width, see e.g. [2]).

The irreflexive case is even less clear. However, for both cases we have

Problem 6.1
$Log(\overline{\mathfrak{H}}(\omega)) \stackrel{?}{=} \bigcap_{n<\omega} Log(\omega^n, \overline{H})$;

$Log(\mathfrak{H}(\omega)) \stackrel{?}{=} \bigcap_{n<\omega} Log(\mathfrak{H}(n))$.

Since $\mathbf{S5}^2$ is decidable (see e.g. [5]), $Log(\omega^2, \overline{H})$ is also decidable.

Problem 6.2 *Is there a decidable logic $Log(\omega^n, \overline{H})$, $n = 3, 4, \ldots$?*

The logic $Log(\mathfrak{H}(2))$ is a fragment of the logic $Log(\{(A, \neq)^2 \mid |A| > 1\})$.

Problem 6.3 *Is the logic $Log(\mathfrak{H}(2))$ decidable?*

Problem 6.4 *Does there exist a finitely axiomatizable logic $Log(\omega^n, \overline{H})$, for $n = 2, 3, \ldots$?*

Finding reflexive analogues of frames K_m, K'_m (if they exist) seems nontrivial.

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