Non-finite Axiomatizability of Dynamic Topological Logic

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Abstract

Dynamic topological logic ($\mathcal{DTL}$) is a polymodal logic designed for reasoning about *dynamic topological systems*. These are pairs $\langle X, f \rangle$, where X is a topological space and $f : X \to X$ is continuous. $\mathcal{DTL}$ uses a language L which combines the topological S4 modality $\square$ with temporal modalities from linear temporal logic.

Recently, I gave a sound and complete axiomatization DTL^* for an extension of the logic to the language L^*, where $\diamondsuit$ is allowed to act on finite sets of formulas and is interpreted as a tangled closure operator. No complete axiomatization is known over L, although one proof system, which we shall call KM, was conjectured to be complete by Kremer and Mints.

In this paper we show that, given any language L' such that $\mathsf{L} \subseteq \mathsf{L}' \subseteq \mathsf{L}^*$, the set of valid formulas of L' is not finitely axiomatizable. It follows, in particular, that KM is incomplete.

Keywords: dynamic topological logic, topological semantics, temporal logic, spatial logic

1   Introduction

Finding a transparent axiomatization for Dynamic Topological Logic ($\mathcal{DTL}$) has been an elusive open problem since 2005, when one (which we shall call KM) was proposed by Kremer and Mints in [12] without establishing its completeness. In [7] I offered a complete axiomatization, not over the language L used in [12], but rather in an extended language L^* which allowed the modal $\diamondsuit$ to be applied to finite sets of formulas. It was then interpreted as a 'tangled closure' operator (see Section 3). The resulting logic is called DTL^*.

However, the fact that DTL^* used the unfamiliar 'tangled closure' operation and was substantially less intuitive than KM left the completeness of the latter as a relevant open problem. Actually, the only motivation given in [7] for passing to an extended language was, *There is a completeness proof which*

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works in the extended language but not in the original one; a valid, but not terribly compelling, reason.

The results in this paper will show that indeed the use of the tangled closure is an essential part of this axiomatization, and cannot be removed without extending KM (although it is not clear what such an extension should look like). In fact, we prove more. We show that, given $k < \omega$, there is a formula $\texttt{Trouble}^k \in \mathsf{L}$ such that $\texttt{Trouble}^k$ is derivable in DTL^* only by using formulas of the form $\Diamond \Gamma$ where Γ has at least k elements. This shows that $\mathcal{DTL}$ can be written as a strictly increasing sequence of theories and hence is not finitely axiomatizable; it follows, in particular, that KM is incomplete.

1.1 Previous work on $\mathcal{DTL}$

Dynamic topological logic ($\mathcal{DTL}$) combines the topological S4 with Linear Temporal Logic. The 'topological interior' interpretation of modal logic was already studied by Tarski, McKinsey and others around the 1940s [16] and more recently in works like [2,11,14]. Temporal logic also has a long history, having been studied by Prior before 1960 [15] and received substantial attention since; see [13] for a nice overview.

The purpose of $\mathcal{DTL}$ is to reason about *dynamic topological systems* (dts's); these are pairs $\langle X, f \rangle$, where X is a topological space and $f : X \to X$ is a continuous function, and shall be discussed in greater detail in Section 4. Dynamic Topological Logic was originally introduced as a bimodal logic in [1], where it was called S4C. In our notation, it uses the 'interior' modality $\Box$, interpreted topologically, and 'next-time' modality f, interpreted as a preimage operator; see Section 4 for details. The logic S4C is a rather well-behaved modal logic; it is decidable, axiomatizable and has the finite model property, all of which was established in [1]. Later, [12] showed that a variant, called S4H, was complete for the class of dynamical systems where f is a homeomorphism.

Also in [12], it was noted that by adding the infinitary temporal modality 'henceforth' (here denoted $[f]$), one could reason about long-term behavior of dts's, capturing phenomena such as topological recurrence. Thus they introduced an extension of S4C, which we denote $\mathcal{DTL}$. $\mathcal{DTL}$ turned out to behave much worse than S4C; it was proven to be undecidable in [9], and in [10] it was also shown that, if we restrict to the case where f is a homeomorphism, then the logic becomes non-axiomatizable. Fortunately, with arbitrary continuous functions the logic turned out to be recursively enumerable [4], but the only currently known axiomatization is over L^* [7].

This axiomatization uses the fact, first observed in [3], that L^* is more expressive than L. There, it is shown that, over the class of finite S4 models, $\mathsf{L}^*_\Diamond$ (i.e., the fragment of L^* without temporal modalities) is equally expressive to the bisimulation-invariant fragment of both first-order logic and monadic second-order logic, while $\mathsf{L}_\Diamond$ is weaker. This added expressive power is used in an important way in [7], although it is not proven that such an extension is *necessary*.

1.2   Layout of the paper

Sections 2-5 give a general review of dynamic topological logic and the known results relevant for this paper. In Section 2, we introduce topological spaces and show how one can see preorders as a special case. Section 3 gives the main properties of the tangled closure operator, an important part of $\mathcal{DTL}^*$, introduced in Section 4. Section 5 then reviews the axiomatization from [7] and defines some important sublogics.

Section 6 introduces tangled bisimulations, which are based on those presented in [3]. These are used to show that $\bigdiamond_{i=1}^{k+1} \gamma_i$ cannot in general be defined using exclusively k-adic occurrences of $\diamond$.

Section 7 defines the formulas $\mathtt{Trouble}^k$ which are derivable in DTL^k (see also Appendix A), as well as other formulas which are useful for our purposes. Finally, Section 8 shows that, indeed, DTL^k is consistent with $\neg\mathtt{Trouble}^{k+1}$, thus establishing our main result.

2   Topologies and preorders

In this section we shall very briefly review some basic notions from topology. As is well-known, topological spaces provide an interpretation of the modal logic $\mathsf{S4}$, generalizing its well-known Kripke semantics.

Let us recall the definition of a *topological space*:

Definition 2.1 *A* topological space *is a pair* $\mathfrak{X} = \langle |\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}} \rangle$, *where* $|\mathfrak{X}|$ *is a set and* $\mathcal{T}_{\mathfrak{X}}$ *a family of subsets of* $|\mathfrak{X}|$ *satisfying*

(i) $\varnothing, |\mathfrak{X}| \in \mathcal{T}_{\mathfrak{X}}$;

(ii) *if* $U, V \in \mathcal{T}_{\mathfrak{X}}$ *then* $U \cap V \in \mathcal{T}_{\mathfrak{X}}$ *and*

(iii) *if* $\mathcal{O} \subseteq \mathcal{T}_{\mathfrak{X}}$ *then* $\bigcup \mathcal{O} \in \mathcal{T}_{\mathfrak{X}}$.

The elements of $\mathcal{T}_{\mathfrak{X}}$ *are called* open sets. *Complements of open sets are* closed sets.

Given a set $A \subseteq |\mathfrak{X}|$, its *interior*, denoted A°, is defined by

$$A^\circ = \bigcup \left\{ U \in \mathcal{T}_{\mathfrak{X}} : U \subseteq A \right\}.$$

Dually, we define the closure $\overline{A}$ as $|\mathfrak{X}| \setminus (|\mathfrak{X}| \setminus A)^\circ$; this is the smallest closed set containing A.

Topological spaces generalize transitive, reflexive Kripke frames. Recall that these are pairs $\mathfrak{W} = \langle |\mathfrak{W}|, \preccurlyeq_{\mathfrak{W}} \rangle$, where $\preccurlyeq_{\mathfrak{W}}$ is a preorder on the set $|\mathfrak{W}|$. We will write $\preccurlyeq$ instead of $\preccurlyeq_{\mathfrak{W}}$ whenever this does not lead to confusion.

To see a preorder as a special case of a topological space, define

$$\downarrow w = \left\{ v : v \preccurlyeq w \right\}.$$

Then consider the topology $\mathcal{T}_{\preccurlyeq}$ on $|\mathfrak{W}|$ given by setting $U \subseteq |\mathfrak{W}|$ to be open if and only if, whenever $w \in U$, we have $\downarrow w \subseteq U$ (so that all sets of the form $\downarrow w$

provide a basis for $\mathcal{T}_{\preccurlyeq}$). A topology of this form is a *preorder topology*[2].

Throughout this text we will often identify preorders with their corresponding topologies, and many times do so tacitly.

We will also use the notation

- $w \prec v$ for $w \preccurlyeq v$ but $v \not\preccurlyeq w$ and

- $w \approx v$ for $w \preccurlyeq v$ and $v \preccurlyeq w$.

The relation $\approx$ is an equivalence relation; the equivalence class of a point $x \in |\mathfrak{W}|$ is usually called a *cluster*, and we will denote it by $[x]$.

3 The tangled closure

The *tangled closure* is an important component of DTL^*. It was introduced in [3] for Kripke frames and has also appeared in [5,6,7,8].

Definition 3.1 *Let $\mathfrak{X}$ be a topological space and $\mathcal{S} \subseteq 2^{|\mathfrak{X}|}$.*

Given $E \subseteq |\mathfrak{X}|$, we say $\mathcal{S}$ is tangled in E *if, for all $A \in \mathcal{S}$, $A \cap E$ is dense in E.*

We define the tangled closure of $\mathcal{S}$, *denoted $\mathcal{S}^*$, to be the union of all sets E such that $\mathcal{S}$ is tangled in E.*

It is important for us to note that the tangled closure is defined over any topological space; however, we will often be concerned with locally finite preorders in this paper. Here, the tangled closure is relatively simple.

Lemma 3.1 *Let $\langle S, \preccurlyeq \rangle$ be a finite preorder, $x \in S$ and $\mathcal{O} \subseteq 2^S$. Then, $x \in \mathcal{O}^*$ if and only if there exist $\langle y_A \rangle_{A \in \mathcal{O}}$ such that $y_A \in A$, $y_A \preccurlyeq x$ for all $A \in \mathcal{O}$ and $y_A \approx y_B$ for all $A, B \in \mathcal{O}$.*

Proof. A proof can be found in any of [5,6,7,8]. $\square$

4 Dynamic Topological Logic

The language L^* is built from propositional variables in a countably infinite set PV using the Boolean connectives $\wedge$ and $\neg$ (all other connectives are to be defined in terms of these), the unary modal operators f ('next') and $[f]$ ('henceforth'), along with a polyadic modality $\Diamond$ which acts on finite sets, so that if Γ is a finite set of formulas then $\Diamond \Gamma$ is also a formula. Note that this is a modification of the usual language of $\mathcal{DTL}$, where $\Diamond$ acts on single formulas only. We write $\square$ as a shorthand for $\neg \Diamond \neg$; similarly, $\langle f \rangle$ denotes the dual of $[f]$. We also write $\Diamond \gamma$ instead of $\Diamond \{\gamma\}$; its meaning is identical to that of the usual $\mathsf{S4}$ modality [8]. We will often write $\bigDiamond_{i \leq I} \gamma_i$ instead of $\Diamond \{\gamma_i\}_{i \leq I}$.

Given a formula ϕ, the *depth* of ϕ, denoted $\mathrm{dpt}(\phi)$, is the modal nesting depth of ϕ, while its *width*, $\mathrm{wdt}(\phi)$, denotes the maximal k such that ϕ has a subformula of the form $\bigDiamond_{i=1}^{k} \gamma_i$. For $k < \omega$, L^k denotes the sublanguage of L^* where all formulas have width at most k. Thus, in particular, $\mathsf{L} = \mathsf{L}^1$.

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[2] Or, more specifically, a *downset topology*. Note that I stray from convention, since most authors use the upset topology here, but I personally find this presentation more natural. This will later be reflected in the semantics for $\square$.

Formulas of L^* are interpreted on dynamical systems over topological spaces, or *dynamic topological systems*.

Definition 4.1 *A* weak dynamic topological system (dts) *is a triple*

$$\mathfrak{X} = \langle |\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}}, f_{\mathfrak{X}} \rangle ,$$

where $\langle |\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}} \rangle$ *is a topological space and*

$$f_{\mathfrak{X}} : |\mathfrak{X}| \to |\mathfrak{X}|.$$

If further $f_{\mathfrak{X}}$ *is continuous*[3] *, we say* $\mathfrak{X}$ *is a* dynamical system*.*

Definition 4.2 *Given a (weak) dynamic topological system* $\mathfrak{X}$*, a* valuation *on* $\mathfrak{X}$ *is a function*

$$[\![\cdot]\!] : \mathsf{L}^* \to 2^{|\mathfrak{X}|}$$

satisfying

$$[\![\alpha \wedge \beta]\!]_{\mathfrak{X}} = [\![\alpha]\!]_{\mathfrak{X}} \cap [\![\beta]\!]_{\mathfrak{X}}$$
$$[\![\neg \alpha]\!]_{\mathfrak{X}} = |\mathfrak{X}| \setminus [\![\alpha]\!]_{\mathfrak{X}}$$
$$[\![f\alpha]\!]_{\mathfrak{X}} = f^{-1} [\![\alpha]\!]_{\mathfrak{X}}$$
$$[\![[f]\alpha]\!]_{\mathfrak{X}} = \bigcap_{n \geq 0} f^{-n} [\![\alpha]\!]_{\mathfrak{X}}$$
$$[\![\diamondsuit \{\alpha_1, ..., \alpha_n\}]\!]_{\mathfrak{X}} = \{[\![\alpha_1]\!]_{\mathfrak{X}}, ..., [\![\alpha_n]\!]_{\mathfrak{X}}\}^* .$$

A *(weak) dynamic topological model* (wdtm/dtm) is a (weak) dynamic topological system $\mathfrak{X}$ equipped with a valuation $[\![\cdot]\!]_{\mathfrak{X}}$. We say a formula ϕ is *valid* on $\mathfrak{X}$ if $[\![\phi]\!]_{\mathfrak{X}} = |\mathfrak{X}|$, and write $\mathfrak{X} \models \phi$. If a formula ϕ is valid on every dynamic topological model, then we write $\models \phi$. $\mathcal{DTL}$ is the set of valid formulas of L under this interpretation, while $\mathcal{DTL}^*$ denotes the set of valid formulas of L^*.

We will often write $\langle \mathfrak{X}, x \rangle \models \phi$ instead of $x \in [\![\phi]\!]_{\mathfrak{X}}$.

5 The axiomatization

We shall distinguish $\mathcal{DTL}^*$, which is defined semantically, from DTL^*, which is a proof system. The two have the same set of theorems, but we will be interested in natural subsystems of DTL^* which are defined syntactically.

Below, note that the modality f is unary, and $f\Gamma$ is merely a shorthand for $\{f\gamma : \gamma \in \Gamma\}$; p denotes a propositional variable and P a finite set of propositional variables. Then, the axiomatization DTL^* consists of the following:

Taut All propositional tautologies.

Topological axioms
  K $\Box(p \to q) \to (\Box p \to \Box q)$
  T $\Box p \to p$

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[3] That is, whenever $U \subseteq |\mathfrak{X}|$ is open, then so is $f^{-1}(U)$

4 $\Box p \to \Box\Box p$

Fix$_\Diamond$ $\Diamond P \to \bigwedge_{q \in P} \Diamond(q \wedge \Diamond P)$

Ind$_\Diamond$ $\bigwedge_{i \in I}(p \to \Diamond(q_i \wedge p)) \to (p \to \bigdiamond_{i \in I} q_i)$

Temporal axioms

Neg$_f$ $\neg fp \leftrightarrow f\neg p$

And$_f$ $f(p \wedge q) \leftrightarrow fp \wedge fq$

Fix$_{[f]}$ $[f]p \to p \wedge f[f]p$

Ind$_{[f]}$ $[f](p \to fp) \to (p \to [f]p)$

Cont* $\Diamond fP \to f\Diamond P$.

Rules

MP Modus ponens

Subs $\dfrac{\phi}{\phi[\boldsymbol{p}/\boldsymbol{\psi}]}$

N$_\Box$ $\dfrac{\phi}{\Box\phi}$ $\qquad\qquad$ N$_f$ $\dfrac{\phi}{f\phi}$ $\qquad\qquad$ N$_{[f]}$ $\dfrac{\phi}{[f]\phi}$

This axiomatization is sound and complete, as proven in [7]:

Theorem 5.1 DTL* *is sound and complete for the class of dynamic topological models.*

There are many subtleties in our proof system, so before continuing we should make a few remarks.

First, let us say a few words about the substitution rule. It is to be understood as 'simulataneous substitution', where $\boldsymbol{p}$ represents a finite sequence of variables, $\boldsymbol{\psi}$ a finite sequence of formulas and each variable is replaced by the respective formula. By standard arguments, this rule preserves validity, as there is nothing in our semantics distinguishing atomic facts from complex propositions; this much is not problematic.

Further, since we are concerned with finite axiomatizability of a logic it is important to include it; otherwise, each substitution instance of any of the axioms would have to be regarded as a new axiom and the finite axiomatizability would fail for obvious reasons. Of course this is not the only possible presentation, as one can also consider axiomatizations by finitely many *schemas*, but here we shall consider different formulas to be different also as axioms.

With this in mind, we should note that the above axiomatization is not finite, nor can it be modified into a finite version in an obvious way. Evidently the set of all propositional tautologies can be replaced by finitely many axioms, but this is not what concerns us. Much more importantly, we need infinitely many axioms for $\Diamond$, and it is only in the metalanguage that we can give them a uniform presentation. In fact, the symbol P representing a finite set of propositional variables is not a symbol of L*, where we would have to write out explicitly $\{p_1, \ldots, p_k\}$ for each given value of k.

Of particular interest is the schema Cont*. This was originally named TCont; we adopt the new notation to stress that the standard 'continuity'

axiom,

$$\mathsf{Cont}^1 : \Diamond f p \to f \Diamond p,$$

is indeed a special case.

Cont^* is really an infinite collection of axioms. To be precise, for $k < \omega$ let

$$\mathsf{Cont}^k = \bigotimes_{i \in [1,k]} f p_i \to f \bigotimes_{i \in [1,k]} p_i.$$

note that Cont^{k+1} extends Cont^k since we can always substitute p_{k+1} by p_k.

We then let DTL^k be the variant of DTL^* where Cont^* is replaced by Cont^k. We denote derivability in DTL^k by $\vdash^k$. DTL^0 denotes the system with no continuity axiom.

Our goal will be to show that $\langle \mathsf{DTL}^k \rangle_{k<\omega}$ gives a sequence of theories of strictly increasing strength. Since DTL^* is the union of these theories, it will follow as a straightforward consequence that DTL^* is not finitely axiomatizable. However, to do this we will need a second refinement, this time of each DTL^k.

For $n, k < \omega$, we let DTL^k_n be the subtheory of DTL^k which restricts the substitution rule in the following ways:

 (i) Subs may only be applied immediately to axioms and

 (ii) if Subs is applied to Cont^k, then each p_i must be replaced by a formula with modal depth at most n.

A very easy induction on derivations shows that any proof in DTL^k may be transformed into one satisfying the above two conditions for some value of n and hence $\mathsf{DTL}^k = \bigcup_{n<\omega} \mathsf{DTL}^k_n$. We denote derivability in DTL^k_n by $\vdash^k_n$.

The reason for passing to DTL^k_n is that the substitution rule, while preserving validity, does not preserve *model* validity; if $\mathfrak{M} \models \phi$, it does not always follow that $\mathfrak{M} \models \phi[p/\psi]$. Later we wish to build specific models of fragments of DTL^*, and to check soundness for these models, DTL^k_n has the advantage that we only need to focus on substitution instances of axioms. This will become relevant in Section 8.

DTL^* is an extension of KM, which can be defined as follows:

Definition 5.2 *The calculus* KM *is the restriction of* DTL^* *to* L^1.

In KM, all appearances of $\Diamond$ must be applied to a single formula; in particular, the axioms $\mathsf{Fix}_\Diamond$ and $\mathsf{Ind}_\Diamond$ are not present, and Cont^* becomes Cont^1. We should note that DTL^1 is very similar, but not identical, to KM. DTL^1 allows formulas of the form $\Diamond \Gamma$ within derivations for Γ arbitrarily large, but Cont^* is also replaced by Cont^1. We do have, however, that $\mathsf{KM} \subseteq \mathsf{DTL}^1$.

Later we shall show that the sequence $\langle \mathsf{DTL}^k \rangle_{k<\omega}$ is strictly increasing in strength, even over L; i.e., there are formulas $\mathtt{Trouble}^k \in \mathsf{L}$ such that $\vdash^{k+1} \mathtt{Trouble}^{k+1}$ but $\nvdash^k \mathtt{Trouble}^{k+1}$. These are defined in Section 7; but first, we need to define *partial tangled bisimulations*, the fundamental tool we shall use to prove our main results.

6   Tangled bisimulations

Many of our proofs are based on partial bisimulation techniques. As we will be working in a polyadic system, we shall need a notion of partial bisimulation which preserves the polyadic $\Diamond$, originally introduced in [3].

We shall only define these partial bisimulations for finite Kripke models, for two reasons. First, this will be sufficient for our purposes, as all the constructions we shall use involve only finite models. Second, our definition exploits the fact that the tangled closure operation is much, much simpler in the finite setting, and would not readily generalize to infinite frames, much less arbitrary topological spaces.

Definition 6.1 *Given models* $\mathfrak{X}, \mathfrak{Y}$, $n < \omega$ *and* $k \leq \omega$, *we define a binary relation* $\leftrightarrow_*^n \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ *by inducion on* n *as follows.*
If $n = 0$, *then* $x \leftrightarrow_*^n y$ *if and only if* x *and* y *satisfy the same set of atoms.*
Otherwise, $x \leftrightarrow_*^{n+1} y$ *if they satisfy the same atoms and*

$\mathsf{Forth}_{\preccurlyeq}$ *whenever* $m < \omega$ *and* $x_1 \approx x_2 \approx \ldots \approx x_m \preccurlyeq x$, *there are* $y_1 \approx y_2 \approx \ldots y_m \preccurlyeq y$ *such that* $x_i \leftrightarrow_*^n y_i$ *for all* $i \leq m$,

$\mathsf{Back}_{\preccurlyeq}$ *whenever* $m < \omega$ *and* $y_1 \approx y_2 \approx \ldots \approx y_m \preccurlyeq y$, *there are* $x_1 \approx x_2 \approx \ldots \approx x_m \preccurlyeq x$ *such that* $x_i \leftrightarrow_*^n y_i$ *for all* $i \leq m$,

Forth_f $f_{\mathfrak{X}}(x) \leftrightarrow_*^n f_{\mathfrak{Y}}(y)$

$\mathsf{Forth}_{[f]}$ *for every* $m < \omega$ *there is* $m' < \omega$ *such that* $f_{\mathfrak{X}}^m(x) \leftrightarrow_*^n f_{\mathfrak{Y}}^{m'}(y)$ *and*

$\mathsf{Back}_{[f]}$ *for every* $m < \omega$ *there is* $m' < \omega$ *such that* $f_{\mathfrak{Y}}^m(y) \leftrightarrow_*^n f_{\mathfrak{X}}^{m'}(x)$

Note there is no 'back' clause for f as it would be identical to Forth_f. When the respective structures are clear from context, we may write $x \leftrightarrow_*^n y$ instead of $\langle \mathfrak{X}, x \rangle \leftrightarrow_*^n \langle \mathfrak{Y}, y \rangle$.

Lemma 6.1 *If* ϕ *is a formula such that* $\mathrm{dpt}(\phi) \leq n$, *then given* $x \leftrightarrow_*^n y$, *we have that* $x \in \llbracket \phi \rrbracket_{\mathfrak{X}}$ *if and only if* $y \in \llbracket \phi \rrbracket_{\mathfrak{Y}}$.

Proof. The proof proceeds by a standard induction on $\mathrm{dpt}(\phi)$ and we omit it. $\square$

7   Trouble formulas

In this section we shall introduce a sequence of formulas $\langle \mathtt{Trouble}^k \rangle_{k < \omega}$ with the property that $\vdash^k \mathtt{Trouble}^k$. As we shall see later, $\nvdash^k \mathtt{Trouble}^{k+1}$, thus establishing that DTL^{k+1} is stronger than DTL^k. The formulas $\mathtt{Trouble}^k$ will all be in L^1.

Before continuing, let us establish some notational conventions. Given natural numbers n, k, we will denote by $|n|_k$ the unique element m of $\{1, \ldots, k\}$ such that $n \equiv m \pmod{k}$. Note that this strays from the standard remainder in that $|k|_k = k$, but it shall simplify several expressions later on. Intervals shall be assumed to be intervals of natural numbers, i.e.

$$[a, b] = \{n \in \mathbb{N} : a \leq n \leq b\}.$$

Further, we shall assume that the set of propositional variables is enumerated by $\{p_k\}_{k<\omega}$.

Definition 7.1 *The following abbreviations shall be used throughout the text.*

$$\texttt{Cycle}^k = \Diamond p_k \to \bigwedge_{i=1}^{k} (p_i \to f p_{|i+1|_k})$$

$$\texttt{Start}_i^k = p_i \wedge [f]\texttt{Cycle}^k$$

$$\texttt{Bundle}^k = \Box \bigwedge_{i=1}^{k} \Diamond \texttt{Start}_i^k$$

$$\texttt{Tangle}^k = \bigdiamond_{i\in[1,k]} \texttt{Start}_i^k$$

$$\texttt{Trouble}^k = \texttt{Bundle}^k \to [f]\Diamond p_k$$

Before continuing, let us give some intuition for these formulas.

The formula $\texttt{Cycle}^k$ states that f 'cycles' the values of $p(x)$; if $p(x) = p_i$, $p(f(x)) = p_{i+1}$, unless $i = k$ in which case $p(f(x)) = p_1$. The formula $\Diamond p_k$ is used as a sort of trigger for this cycling behavior; when $\Diamond p_k$ fails, $p(f(x))$ is unspecified.

$\texttt{Start}_i^k$ is used to begin the cycling behavior described by $\texttt{Cycle}^k$ at p_i; it says that, initially, p_i holds, and from then on, f cycles the values of $p(f^n(x))$, provided that $\Diamond p_k$ holds at each step.

$\texttt{Bundle}^k$ and $\texttt{Tangle}^k$ are similar, but $\texttt{Bundle}^k$ is stronger. As we will mainly be interpreting these formulas over finite Kripke models, let us restrict the discussion to this setting. Here, the meaning of $\texttt{Tangle}^k$ should be familiar; it says there is a cluster where there is a point x_i satisfying each $\texttt{Start}_i^k$.

Proposition 7.2 *Given $k < \omega$, $\vdash^k \texttt{Trouble}^k$.*

Proof. See Appendix A. □

8 Incompleteness of finite fragments

The formula $\texttt{Trouble}^k$ is derivable in DTL^k; let us now see that $\texttt{Trouble}^{k+1}$ is not. To prove this, we shall introduce models $\mathfrak{D}(N, K)$. They will be composed of two submodels; $\mathfrak{C}(K)$, defined later, and $\mathfrak{B}(N, K)$, defined below.

The general idea is that the models $\mathfrak{D}(n+1, k+1)$ will satisfy

$$\mathsf{DTL}_n^k \cup \{\neg\texttt{Trouble}^{k+1}\},$$

thus showing that $\not\vdash_n^k \texttt{Trouble}^{k+1}$ for all n. From this we may conclude that $\not\vdash^k \texttt{Trouble}^{k+1}$.

Below, $\coprod$ denotes a disjoint union.

Definition 8.1 *Let $k < \omega$.*

A preordered model $\mathfrak{S}$ is k-simple if

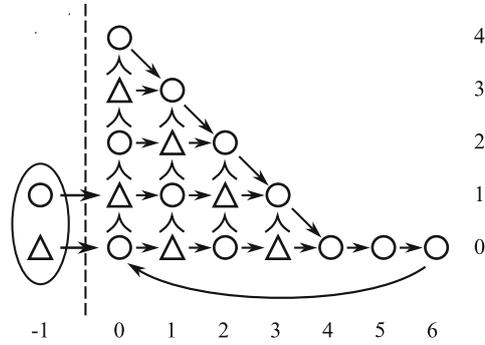$$|\mathfrak{S}| = \coprod_{i=1}^{k} [\![p_i]\!]_{\mathfrak{S}} \,.$$

Fig. 1. The model $\mathfrak{D}(2,2)$, described in Definition 8.5. $\mathfrak{B}(2,2)$, as in Definition 8.2, is the submodel on the right-hand side of the dotted line and $\mathfrak{C}(2)$ is the submodel on its left. Arrows indicate $f_{\mathfrak{D}}$, while $\preccurlyeq$ is the transitive, reflexive closure of the relation represented by $\prec$ together with the ellipse on the left, which represents $\approx$. Points represented by a circle satisfy p_1, by a triangle, p_2.

If $\mathfrak{S}$ is simple and $x \in |\mathfrak{S}|$, we write $p_{\mathfrak{S}}(x)$ for the unique $p \in \{p_1, \dots, p_k\}$ such that $x \in [\![p]\!]_{\mathfrak{S}}$.

As always, we will drop subindices when it does not lead to confusion, writing $p(x)$ instead of $p_{\mathfrak{S}}(x)$. All of the models we shall work with in the rest of the text shall be k-simple for some k.

Before defining our structures formally, let us give a general idea. Consider the model $\mathfrak{D} = \mathfrak{D}(3,2)$ depicted in Figure 1. We will name a point x using triples $(h(x), t(x), k(x))$, where $h(x)$ is the 'spatial' (vertical) coordinate, $t(x)$ the 'temporal' (horizontal) coordinate and $k(x)$ is the index of $p(x)$, which in this case is 1 for points represented by a circle and 2 for triangles. The points on the left of the dotted line will be written $(0, -1, k(x))$.

First, let us observe that $\mathfrak{D} \models \mathtt{Cycle}^2$, since $f_{\mathfrak{D}}$ alternates between circles (which satisfy p_1) and triangles (which satisfy p_2). The exception for this are the points on the main diagonal $h + t = 6$ and on the 'tail' $t \geq 4$, but these points do not satisfy $\diamond p_2$ and thus they also satisfy $\mathtt{Cycle}^2$. From this, one can easily check that $(0, -1, 1)$ satisfies $\neg \mathtt{Trouble}^2$.

Meanwhile, the key aspect of the model is that $f_{\mathfrak{D}}$ is discontinuous, since $(0, -1, 2) \preccurlyeq (0, -1, 1)$ yet

$$f_{\mathfrak{D}}(0, -1, 2) = (0, 0, 1) \not\preccurlyeq (1, 0, 2) = f_{\mathfrak{D}}(0, -1, 1).$$

This discontinuity is easily seen to make the following instance of $\mathtt{Cont}^2$ fail on $(0, -1, 1)$:

$$\diamond f\{p_1, p_2\} \rightarrow f \diamond \{p_1, p_2\}.$$

However, instances of $\mathtt{Cont}^1$ of small modal depth do hold.

Consider, for example,

$$\diamond f p_1 \rightarrow f \diamond p_1.$$

Here we see that $f_{\mathfrak{D}}(0,-1,2)$ satisfies fp_1, so that $(0,-1,1)$ satisfies $\Diamond fp_1$. If $f_{\mathfrak{D}}$ were continuous, we would be able to use $f_{\mathfrak{D}}(0,-1,2)$ as a witness that $(0,-1,1)$ satisfies $f\Diamond p_1$, but in this case we cannot. However, we do have a different witness, namely $(2,0,1)$. More generally, as we shall see in Lemma 8.4, $(2,0,1) \leftrightarrows^1_* (0,0,1)$ so the two satisfy the same formulas of modal depth one.

Thus $\mathfrak{D}$ satisfies DTL^1_1 as well as $\neg\mathtt{Trouble}^2$, from which we conclude that $\nvdash^1_1 \mathtt{Trouble}^2$. To see that $\nvdash^1_n \mathtt{Trouble}^2$, we need to consider a larger model, $\mathfrak{D}(n,2)$, which is built much like $\mathfrak{D}(3,2)$ but is deeper. By varying n, we conclude that $\nvdash^1 \mathtt{Trouble}^2$.

Now, let us give the formal definition of $\mathfrak{B}(N,K)$, which is the submodel of $\mathfrak{D}(N,K)$ on the right of the dotted lines in Figure 1.

Definition 8.2 *Given $N,K < \omega$, we define a K-simple dynamic model $\mathfrak{B} = \mathfrak{B}(N,K)$ by letting*

(i) *$|\mathfrak{B}|$ be the set of all triples of natural numbers (h,t,k) such that either*
  (a) *$h + t \leq NK$, $k \in [1,K]$ and $k \not\equiv h + t \pmod{K}$ or*
  (b) *$h = 0$, $t \in [KN+1, (K+1)N]$ and $k \neq K$.*

(ii) *$(h_1,t_1,k_1) \preccurlyeq_{\mathfrak{B}} (h_2,t_2,k_2)$ if and only if $t_1 = t_2$ and $h_1 \geq h_2$;*

(iii) $f_{\mathfrak{B}}(h,t,k) = \begin{cases} (h, t+1, |k+1|_K) & \text{if } h + t < NK \\\\ (h-1, t+1, k) & \text{if } h+t = NK \text{ and } h > 0 \\\\ (h, t+1, k) & \text{if } h = 0 \text{ and } t < N(K+1) \\\\ (0, 0, |k+1|_K) & \text{if } h = 0 \text{ and } t = N(K+1) \end{cases}$

(iv) *$p(h,t,k) = p_k$.*

We will write $x = (h(x), k(x), t(x))$. We will also write $s(x) = h(x) + t(x)$.

It will be convenient to describe the $\leftrightarrows^m_*$-equivalence classes over $\mathfrak{B}(N,K)$. We shall do this using the relations $\sim^m$, defined below.

Definition 8.3 *For $m < N$, say $x \sim^m y$ if $p(x) = p(y)$ and one (or more) of the following occurs:*

(i) *$s(x) = s(y)$,*

(ii) *$s(x), s(y) \leq K(N-m)$ or*

(iii) *$s(x), s(y) \in [NK, N(K+1) - m]$.*

The models $\mathfrak{B}(k,n)$ are designed to be very homogeneous, so that different points are hard to distinguished using L^*. The relations $\sim^m$ are representative of this.

Lemma 8.1 *For every $x \in |\mathfrak{B}(N,K)|$ and $m < N$,*

(i) *there is $y \sim^m x$ with $h(y) = 0$ and*

(ii) *if $h(y) = 0$ there is $n < \omega$ such that $f_{\mathfrak{B}}^n(x) = y$.*

Proof. The first claim is obvious if we notice that

$$(h, t, k) \sim^m (0, h + t, k).$$

For the second, first we note that $h(f^{N(K+1)+1}(x)) = 0$ independently of x; then note that $f_{\mathfrak{B}}$ is clearly transitive on those elements z with $h(z) = 0$, given that

$$f^{N(K+1)+1}(0, 0, k) = (0, 0, |k + 1|_K),$$

thus 'rotating' $k(z)$. □

Now, let us see that $\sim^m$ does, indeed, guarantee partial bisimulation.

Proposition 8.4 *If $x \sim^m y$ then $x \leftrightarroweq_*^m y$.*

Proof. We work by induction on m, considering each clause of a tangled bisimulation.

Note that $\sim^m$ preserves atoms, in particular covering the case $m = 0$.

Otherwise, suppose $x \sim^{m+1} y$. Clearly we only need to prove the 'forth' clauses, since the 'back' clauses are symmetric.

Forth$_\preccurlyeq$ We shall only consider the case where $s(x), s(y) < NK$; the other case is similar and easier.

Suppose $x_0 \approx x_1 \approx \ldots \approx x_{I-1} \preccurlyeq x$; note that we can assume $I \leq K$, since $\mathfrak{B}$ has cluster width K. Note also that each x_i has $h(x_i) \geq h(x)$ and $t(x_i) = t(x)$.

Consider $h' = h(x_i) + t(y) - t(x)$. If $h' \geq h(y)$, set $h = h'$; otherwise, let h be the least value such that $h \geq h(y)$ and $h + t(y) \equiv h(x_i) + t(x) \pmod{K}$. Then, set $y_i = (h, t(y), k(x_i))$.

First, note that $s(y_i) \equiv s(x_i) \pmod{K}$, so that all y_i are elements of $|\mathfrak{B}|$. Now, we further have that $s(y_i) = s(x_i)$ except in the case that $h' < h(y)$, in which it easily follows that $s(x) \neq s(y)$, so $s(x), s(y) < K(N - (m + 1))$ and thus $s(x_i), s(y_i) < K(N - m)$.

In either case we use our induction hypothesis to see that $y_i \leftrightarroweq^m x_i$, as claimed.

Forth$_f$ This follows from observing that the required (in)equalities are preserved by $f_{\mathfrak{B}}$ and we skip it.

Forth$_{[f]}$ Let $n < \omega$ and consider $z = f_{\mathfrak{B}}^n(x)$. Then, by Lemma 8.1(i), there is $z' \sim^m z$ with $h(z') = 0$, while by Lemma 8.1(ii), there is n' such that $f_{\mathfrak{B}}^{n'}(y) = z'$, as required.

□

Now that we have studied the models $\mathfrak{B}(N, K)$, let us add the 'head' $\mathfrak{C}(K)$, which is where the trouble really lurks. The resulting model will be called $\mathfrak{D}(N, K)$, where points in $\mathfrak{C}(K)$ will map discontinuously onto $\mathfrak{B}(N, K)$. However, these discontinuities will require large formulas to capture in L^{K-1}, given that $\mathfrak{C}(K)$ will consist of a cluster with K points.

Definition 8.5 *We define a model $\mathfrak{C} = \mathfrak{C}(K)$ where*

- $|\mathfrak{C}| = \{0\} \times \{-1\} \times [1, K]$
- $\preccurlyeq_{\mathfrak{C}}$ *is total (i.e., $\mathfrak{C}$ consists of a single cluster)*
- $p(0, -1, k) = p_k$.

We define a model $\mathfrak{D} = \mathfrak{D}(N, K)$ based on $|\mathfrak{C}(K)| \cup |\mathfrak{B}(N, K)|$ with

$$\preccurlyeq_{\mathfrak{D}} = \preccurlyeq_{\mathfrak{C}(K)} \cup \preccurlyeq_{\mathfrak{B}(N,K)}$$

and

$$f_{\mathfrak{D}}(0, -1, k) = \begin{cases} (0, 0, |k + 1|_K) & \text{if } k \neq K - 1 \\ \\ (1, 0, K) & \text{if } k = K - 1. \end{cases}$$

Our strategy now is to show that $\mathfrak{D}(N + 1, K + 1)$ is a model of $\mathsf{DTL}_N^K \cup \{\neg\mathtt{Trouble}^{K+1}\}$; from this we may conclude that $\nvdash^K \mathtt{Trouble}^{K+1}$, given that $\mathsf{DTL}^K = \bigcup_{n < \omega} \mathsf{DTL}_n^K$.

Lemma 8.2 $\mathfrak{D}(N + 1, K + 1) \models \mathsf{DTL}_N^K$.

Proof. All the rules of DTL_N^K preserve model validity, so it suffices to check that $\mathfrak{D}(N + 1, K + 1)$ satisfies all axioms of DTL_N^K; that is, all permitted substitution instances of axioms of DTL^K.

Since $\mathfrak{D}(N + 1, K + 1)$ is a weak dynamical system, it satisfies every axiom of DTL_N^K except possibly for instances of Cont^K.

So, let

$$\sigma = \bigdiamond_{i \leq K} f\delta_i \to f \bigdiamond_{i \leq K} \delta_i$$

be a substitution instance of Cont^K where each δ_i has modal depth at most N.

Let $x \in |\mathfrak{D}|$ and assume that

$$x \in \left[\!\!\left[\bigdiamond_{k \leq K} f\delta_i \right]\!\!\right]_{\mathfrak{D}};$$

since $f_{\mathfrak{D}} \upharpoonright |\mathfrak{B}|$ is continuous, we can suppose that $x \in |\mathfrak{C}|$, for otherwise $x \in [\![\sigma]\!]_{\mathfrak{D}}$.

Then, given $i < K$ there is $x_i = (0, -1, k_i) \approx x$ such that $x_i \in [\![f\delta_i]\!]_{\mathfrak{D}}$. For at least one value of $k_* \in [1, K + 1]$ we have that $k_* \neq |k_i + 1|_K$ for all i; we then have that $y_i = (k_*, 0, |k_i + 1|_K)$ is an element of $|\mathfrak{B}|$ and by Lemma 8.4

$$y_i \leftrightarrow_*^N f_{\mathfrak{D}}(x_i).$$

Meanwhile, $y_i \approx y_j \preccurlyeq f_{\mathfrak{D}}(x)$ for all i, j, so that

$$x \in \left[\!\!\left[f \bigdiamond_{i \leq K} \delta_i \right]\!\!\right]_{\mathfrak{D}},$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Lemma 8.3 *Given $K, N < \omega$ and $k \in [1, k]$,*

$$\langle \mathfrak{D}(N, K), (0, -1, k) \rangle \models \neg \mathtt{Trouble}^K.$$

Proof. Let $\mathfrak{D} = \mathfrak{D}(N, K)$.

First, let us show that every $x \in |\mathfrak{D}|$ satisfies

$$\mathtt{Cycle}^K = \Diamond p_K \to \bigwedge_{k \leq K} (p_k \to f p_{|k+1|_K}).$$

If $s(x) \geq NK$, then $x \notin \llbracket \Diamond p_K \rrbracket_{\mathfrak{D}}$ and thus $x \notin \llbracket \Diamond p_K \rrbracket_{\mathfrak{D}}$. This shows that $x \in \llbracket \mathtt{Cycle}^K \rrbracket_{\mathfrak{D}}$, as required.

Otherwise, letting $y = f_{\mathfrak{D}}(x)$, we note by case-by-case inspection that $k(y) = |k(x)+1|_K$, so that x satisfies $p_{k(x)} \to f p_{|k(x)+1|_K}$, whereas for $k \neq k(x)$, x satisfies $p_k \to f p_{|k+1|_K}$ trivially. Thus $\mathtt{Cycle}^K$ holds everywhere, as claimed.

It follows from this, in particular, that $(0, -1, k)$ satisfies $p_k \wedge [f]\mathtt{Cycle}^K$, i.e. $\mathtt{Start}_k^K$; this shows that $(0, -1, k)$ satisfies $\bigwedge_{1 \leq i \leq K} \Diamond \mathtt{Start}_i^K$ and, given that $k \in [1, K]$ was arbitrary,

$$\langle \mathfrak{D}, (0, -1, k) \rangle \models \Box \bigwedge_{i=1}^{k} \Diamond \mathtt{Start}_i^K = \mathtt{Bundle}^K.$$

It remains to show that $(0, -1, k)$ satisfies $\langle f \rangle \Box \neg p_K$; but this follows from the observation that

$$f_{\mathfrak{D}}^{NK+1}(0, -1, k) = (0, NK, k') \notin \llbracket \Diamond p_K \rrbracket_{\mathfrak{D}}.$$

We conclude that

$$\langle \mathfrak{D}, (0, -1, k) \rangle \models \mathtt{Bundle}^K \wedge \neg[f]\Diamond p_K \equiv \neg \mathtt{Trouble}^K,$$

as claimed. $\square$

The following lemma summarizes our results so far:

Lemma 8.6 *For all $k < \omega$, the formula $\mathtt{Trouble}^{k+1} \in \mathsf{L}^1$ is derivable in DTL^{k+1}, but not in DTL^k.*

Proof. By Proposition 7.2, $\vdash^{k+1} \mathtt{Trouble}^{k+1}$; meanwhile, if $\vdash^k \mathtt{Trouble}^{k+1}$, we would have that $\vdash_n^k \mathtt{Trouble}^{k+1}$ for some n.

But this cannot be, since we have seen that

$$\mathfrak{D}(n+1, k+1) \models \mathsf{DTL}_n^k \cup \{\neg \mathtt{Trouble}^{k+1}\},$$

and thus $\nvdash_n^k \mathtt{Trouble}^{k+1}$. $\square$

With this, we may easily prove our main result.

Theorem 8.7 *Let λ be any language such that $\mathsf{L} \subseteq \lambda \subseteq \mathsf{L}^*$, and let $\mathcal{DTL}[\lambda] = \mathcal{DTL}^* \cap \lambda$.*

Similarly, for $k < \omega$, define $\mathsf{DTL}^k[\lambda] = \mathsf{DTL}^k \cap \lambda$.

Then, given any natural number k, $\mathcal{DTL}[\lambda]$ is not finitely axiomatizable[4] *over $\mathsf{DTL}^k[\lambda]$.*

Proof. Let T be any sound, finite extension of $\mathsf{DTL}^k[\lambda]$, so that without loss of generality we may assume $\mathsf{T} = \mathsf{DTL}^k[\lambda] + \phi$ for some valid formula ϕ.

Since DTL^* is complete, we would have that that $\mathsf{DTL}^* \vdash \phi$, and hence, for some value of K, $\mathsf{DTL}^K \vdash \phi$; obviously, we may take $K \geq k$.

But then, we have by Lemma 8.6 that $\mathsf{DTL}^K \nvdash \texttt{Trouble}^{K+1}$, and hence

$$\mathsf{DTL}^k[\lambda] + \phi \nvdash \texttt{Trouble}^{K+1} \in \mathcal{DTL}[\lambda].$$

Meanwhile, T was arbitrary, so we conclude that $\mathcal{DTL}[\lambda]$ is not finitely axiomatizable over $\mathsf{DTL}^k[\lambda]$.   □

This result is quite general, so it may be convenient to explicitly mention some special cases. The following corollary states some immediate consequences of Theorem 8.7; below, recall that $\mathsf{KM} \subseteq \mathsf{DTL}^1[\mathsf{L}]$.

Corollary 8.8 $\mathcal{DTL}$ *and* $\mathcal{DTL}^*$ *are not finitely axiomatizable. In particular,* KM *is incomplete for the class of dynamic topological models.*

9   Concluding remarks

The axiomatization DTL^* introduced the tangled modality to Dynamic Topological Logic as a sort of scaffolding, to be removed once the appropriate techniques were available. However, I believe the present work to be a convincing argument that indeed it is a central element of the logic; tangled sets affect the behavior of dynamical systems and to be unable to reason about them directly gives a logical formalism an unnecessary handicap.

Of course, none of the results presented here show that a reasonable axiomatization within L^1 is impossible to find. I am not sure how relevant such an axiomatization would be at this point, but it remains an interesting problem.

Meanwhile, I believe a more fruitful direction is to analyze other logics which are hard to axiomatize because of a similar lack in expressive power. In particular, there are many products of modal logics which have very similar models to those of Dynamic Topological Logic; perhaps they too would benefit from a polyadic variant?

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[4] Observe that DTL^{k+1} is finitely axiomatizable over DTL^k, but it does not necessarily follow from this that $\mathsf{DTL}^{k+1}[\lambda]$ is finitely axiomatizable over $\mathsf{DTL}^k[\lambda]$ for all λ.

followed from the results presented in the submitted version. The original version only observed that DTL^k was incomplete for all k.

Appendix

A   Derivability of Trouble formulas

In this Appendix, we shall give a syntactic proof of Proposition 7.2. Recall that this proposition states that, for all $k < \omega$, $\vdash^k \mathtt{Trouble}^k$.

Proof. Reasoning within $\mathsf{S4}$ one readily sees that, for any $i < k$, $\vdash^0 \mathtt{Bundle}^k \to \Diamond(\mathtt{Start}_i^k \wedge \mathtt{Bundle}^k)$; thus we may apply the rule $\mathsf{Ind}_\Diamond$ to derive $\mathtt{Bundle}^k \to \bigotimes_{i=1}^k \mathtt{Start}_i^k$ and obtain

$$\vdash^0 \mathtt{Bundle}^k \to \mathtt{Tangle}^k. \tag{A.1}$$

Further, we note that

$$\vdash^0 \mathtt{Tangle}^k \to \Diamond p_k, \tag{A.2}$$

since this is a consequence of the axiom $\mathsf{Fix}_\Diamond$.

For any $i \in [1, k]$ we may use $\mathsf{Fix}_\Diamond$ to see that

$$\vdash^0 \mathtt{Tangle}^k \to \Diamond(\mathtt{Start}_i^k \wedge \mathtt{Tangle}^k).$$

Using (A.2), this imples

$$\vdash^0 \mathtt{Tangle}^k \to \Diamond(\mathtt{Start}_i^k \wedge \Diamond p_k \wedge \mathtt{Tangle}^k),$$

i.e.

$$\vdash^0 \mathtt{Tangle}^k \to \Diamond(p_i \wedge [f]\mathtt{Cycle}^k \wedge \Diamond p_k \wedge \mathtt{Tangle}^k).$$

Now, by $\mathsf{Fix}_{[f]}$, $\vdash^0 [f]\mathtt{Cycle}^k \to (\mathtt{Cycle}_i^k \wedge f[f]\mathtt{Cycle}^k)$, whereas

$$\vdash^0 \Diamond p_k \wedge \mathtt{Cycle}^k \to (p_i \to f p_{|i+1|_k}),$$

i.e. $\vdash^0 p_i \wedge \Diamond p_k \wedge \mathtt{Cycle}^k \to f p_{|i+1|_k}$. From this we conclude that

$$\vdash^0 \mathtt{Tangle}^k \to \Diamond(f p_{|i+1|_k} \wedge f[f]\mathtt{Cycle}^k \wedge \mathtt{Tangle}^k),$$

and since it holds for all $i \in [1, k]$ we can use $\mathsf{Ind}_\Diamond$ to obtain

$$\vdash^0 \mathtt{Tangle}^k \to \bigotimes_{i \in [1,k]} (f p_{|i+1|_k} \wedge f[f]\mathtt{Cycle}^k),$$

which, rearranging indices and pulling out f, shows that

$$\vdash^0 \mathtt{Tangle}^k \to \bigotimes_{i \in [1,k]} f(p_i \wedge [f]\mathtt{Cycle}^k).$$

Now, we may use Cont^k to obtain

$$\vdash^k \mathtt{Tangle}^k \to f \bigotimes_{i \in [1,k]} (p_i \wedge [f]\mathtt{Cycle}^k);$$

by $\mathsf{Ind}_{[f]}$ this yields

$$\vdash^k \texttt{Tangle}^k \to [f] \bigdiamond_{i \in [1,k]} (p_i \wedge [f]\texttt{Cycle}^k),$$

i.e. $\vdash^k \texttt{Tangle}^k \to [f]\texttt{Tangle}^k$.

Putting this together with A.2 we see that

$$\vdash^k \texttt{Tangle}^k \to [f]\Diamond p_k, \tag{A.3}$$

which, together with (A.1) gives us

$$\vdash^k \texttt{Bundle}^k \to [f]\Diamond p_k,$$

i.e. $\vdash^k \texttt{Trouble}^k$, as claimed.   □

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