A Syntactic Realization Theorem for Justification Logics

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Abstract

Justification logics are refinements of modal logics where modalities are replaced by justification terms. They are connected to modal logics via so-called realization theorems. We present a syntactic proof of a single realization theorem that uniformly connects all the normal modal logics formed from the axioms d, t, b, 4, and 5 with their justification counterparts. The proof employs cut-free nested sequent systems together with Fitting's realization merging technique. We further strengthen the realization theorem for KB5 and S5 by showing that the positive introspection operator is superfluous.

Keywords: justification logic, modal logic, realization theorem, nested sequents, positive introspection

1 Introduction

Justification logic. The language of justification logic is a refinement of the language of modal logic. It replaces a single modality by a family of modalities, indexed by what are called *justification terms*. Given a modal formula such as $\Box A$, which can be read as *A is provable* or as *A is known*, a justification counterpart of this formula of the form $t : A$ can be read as *t is a proof of A* or as *A is known for reason t*.

The first justification logic, called the *Logic of Proofs* or LP, was introduced by Artemov [1,2] as a stepping stone for giving an arithmetical semantics for the modal logic S4. Justification logics are also interesting as epistemic logics. Justification terms have a structure and thus provide a measure of how hard it is to obtain knowledge of something. Because of that, justification logics avoid the well-known logical omniscience problem, as Artemov and Kuznets argue in [5].

The formal correspondence between S4 and LP is called a *realization theorem*. It has two directions. First, each provable formula of S4 can be turned into a provable formula of LP by realizing instances of modalities with justification terms. Second and vice versa, if all terms in a provable formula of LP are replaced with modalities, then the resulting modal formula is provable in S4.

Similar correspondences have been established for several other modal logics besides S4. An overview is given by Artemov in [3].

Methods for proving realization. There are two methods of establishing such correspondences: the syntactic method due to Artemov [1,2] and the semantic method due to Fitting [11]. The syntactic method makes use of cut-free Gentzen systems for modal logics, while the sematic method makes use of a Kripke-style semantics for justification logics. In contrast to the semantic method, the syntactic method is constructive. It provides an algorithm that, for each occurrence of a modality in a given provable modal formula, computes a justification term that realizes it.

The semantic method was used to prove several realization theorems: for S4, S5, K45, and KD45 [3,11,17]. Constructive realizations, via the syntactic method, are available for K, D, T, K4, D4, S4, and S5 [1,2,4,7,12,13]. In the case of S5, where no cut-free sequent system is available, two approaches have been used: first, a cut-free hypersequent system [4] and, second, an embedding of S5 into K45 [12]. This embedding also requires the use of a certain technique of *realization merging* developed by Fitting in [13]. However, neither approach applies to other modal logics that lack cut-free sequent systems, such as K5 and KB. The goal of this paper is to realize these logics and, in general, to provide a uniform constructive method of realizing all normal modal logics formed by the axioms d, t, b, 4, and 5.

Nested sequents. To that end, we use the cut-free proof systems given by Brünnler in [9], which are based on *nested sequents* and which capture all these modal logics. Nested sequents are essentially trees of sequents. They naturally generalize both sequents (which are nested sequents of depth zero) and hypersequents (which essentially are nested sequents of depth one). A crucial feature of these proof systems is *deep inference* [8,14], which is the ability to apply inference rules to formulas arbitrarily deep inside a nested sequent.

Outline. The paper is organized as follows. In Section 2 we introduce justification logics, in Section 3 we introduce nested sequent systems, and in Section 4 we recall Fitting's merging technique. We use them in Section 5 to prove our central result: the uniform realization theorem. In particular, this proves Pacuit's conjecture implicit in [16] that J5 is a justification counterpart of K5. It also creates justification counterparts for the modal logics D5, KB, DB, TB, and KB5, which, to our knowledge, did not have justification counterparts before. In Section 6 we go on to show that the operation of positive introspection is not necessary for the realization of KB5 and S5, which leads to new minimal realizations for them.

2 Justification Logic

Modal formulas. *Modal formulas* are given by the grammar

$$A ::= P_i \mid \neg P_i \mid (A \vee A) \mid (A \wedge A) \mid \Box A \mid \Diamond A \ ,$$

where i ranges over natural numbers, P_i denotes a *proposition*, and $\neg P_i$ denotes its *negation*. Negation of formulas is defined as usual by the De Morgan laws, with $\neg\neg P_i$ being P_i. Further, $A \to B$ denotes $\neg A \vee B$ and $\bot$ denotes $P_j \wedge \neg P_j$ for some fixed proposition P_j.

Justification formulas. *Justification terms*, or *terms* for short, are given by the grammar

$$t ::= c_i \mid x_i \mid (t \cdot t) \mid (t + t) \mid\, !\,t \mid\, ?\,t \quad .$$

The c_i are called *constants* and the x_i are called *variables*. The binary operators $\cdot$ and $+$ are called *application* and *sum* respectively. Application is left-associative. The unary operators ! and ? are called *positive introspection* (or *proof checker*) and *negative introspection* respectively. A sequence of n proof checker operators is denoted by $!^n$. Terms that do not contain variables are called *ground* and are denoted by p, p_1, p_2 and so on, whereas arbitrary terms are denoted by t, s, and q. We use the notation $t(x_1, \dots, x_n)$ for terms that do not contain variables other than $x_1, \dots, x_n$. *Justification formulas* are given by the grammar

$$A ::= P_i \mid \bot \mid (A \to A) \mid t : A \quad .$$

Negation, conjunction, and disjunction are defined as usual. Implication is right-associative and both conjunction and disjunction bind stronger than implication.

Axiom Systems. An axiom system for the modal logic K is assumed to be given. *Extensions* of system K are obtained by adding modal axioms from Figure 2 as described in Figure 3. The axiom system for the basic justification logic J consists of the axioms and rules given in Figure 1. The AN!-rule is called *axiom necessitation with embedded positive introspection*. *Extensions* of system J are obtained by adding justification axioms from Figure 2 as described in Figure 3. The justification axioms are mostly standard, except for jb, which is new. Observe that our choice of the jb-axiom does not increase the set of operations on terms but uses the well-known negative introspection operation. In Section 6 we will see that this is a natural choice. The reason why the zero-premise AN!-rule is defined as a rule and not as an axiom is to prevent it from referring to itself. We will often use the name of an axiom system to also denote its *logic*, which is its set of provable formulas.

From this point on by a *justification logic* we mean (the logic of) either system J or one of its extensions. Likewise, by a *modal logic* we mean either system K or one of its extensions. Each justification logic has a *corresponding* modal logic, and vice versa, as shown in Figure 3, with J corresponding to the modal logic K.

Remark 2.1 *Traditionally, the axiomatizations of justification logics that contain the* j4*-axiom had the following* axiom necessitation *rule, which is a simpler variant of the* AN!*-rule:*

$$\mathsf{AN}\ \frac{A \text{ is an axiom instance}}{c_i : A} \quad .$$

Since in these systems the AN!*-rule is derivable, our axiomatizations produce the same logics.*

taut: A fixed complete set of propositional axioms

app: $s:(A \to B) \to t:A \to (s \cdot t):B$

sum: $s:A \to (s+t):A$ and $s:A \to (t+s):A$

$$\mathsf{MP}\ \frac{A \quad A \to B}{B} \qquad \mathsf{AN!}\ \frac{A \text{ is an axiom instance}}{!^n\, c_i : !^{n-1}\, c_i : \ldots : !\,!\, c_i : !\, c_i : c_i : A}$$

Fig. 1. The axiom system for the basic justification logic J

d: $\Box\bot \to \bot$ t: $\Box A \to A$ b: $A \to \Box\neg\Box\neg A$

jd: $t:\bot \to \bot$ jt: $t:A \to A$ jb: $A \to ?t:(\neg t:\neg A)$

4: $\Box A \to \Box\Box A$ 5: $\neg\Box A \to \Box\neg\Box A$

j4: $t:A \to !t:t:A$ j5: $\neg t:A \to ?t:(\neg t:A)$

Fig. 2. Modal axioms and their corresponding justification axioms

D	T	KB	K4	K5	DB	D4	D5	TB	K45	S4	KB5	D45	S5
d	t	b	4	5	d, b	d, 4	d, 5	t, b	4, 5	t, 4	b, 4, 5	d, 4, 5	t, 4, 5
JD	JT	JB	J4	J5	JDB	JD4	JD5	JTB	J45	LP	JB45	JD45	JT45
jd	jt	jb	j4	j5	jd, jb	jd, j4	jd, j5	jt, jb	j4, j5	jt, j4	jb, j4, j5	jd, j4, j5	jt, j4, j5

Fig. 3. Axiom systems of modal logic and of justification logic

Clearly, we can turn justification formulas into modal formulas by replacing terms with boxes, which is made formal in the next definition.

Definition 2.2 (Forgetful projection) *Given a justification formula A, its* forgetful projection A° *is defined as:* $P_i^\circ := P_i$, $\bot^\circ := \bot$, $(A \to B)^\circ := A^\circ \to B^\circ$, *and* $(t{:}A)^\circ := \Box A^\circ$. *The forgetful projection of a set of justification formulas is defined in the obvious way.*

An important fact about justification logics is that they can internalize their own proofs, i.e. if A is provable, then so is $t:A$ for some term t. This is formally stated in the lemma below, originally proved by Artemov [2] for LP. A proof for most of our justification logics can be found in [15]; the remaining cases are similar.

Lemma 2.3 (Internalization) *For any justification logic* JL*, if*

$$\mathsf{JL} \vdash A_1 \to \ldots \to A_n \to B \ ,$$

then there exists a term $t(x_1, \ldots, x_n)$ such that for all terms $s_1, \ldots, s_n$

$$\mathsf{JL} \vdash s_1 : A_1 \to \ldots \to s_n : A_n \to t(s_1, \ldots, s_n) : B \ .$$

Note that t is ground if $n = 0$.

3 The Nested Sequent Calculus

Nested sequents. *Nested sequents*, or *sequents* for short, are inductively defined as

$$\mathsf{id}\ \frac{}{\Gamma\{P_i, \neg P_i\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\mathsf{ctr}\ \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \mathsf{exch}\ \frac{\Gamma\{\Delta, \Sigma\}}{\Gamma\{\Sigma, \Delta\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \mathsf{k}\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

$$\mathsf{d}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}\ \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}\ \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad \mathsf{4}\ \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

$$\mathsf{5a}\ \frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad \mathsf{5b}\ \frac{\Gamma\{[\Delta], [\Pi, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Pi]\}} \qquad \mathsf{5c}\ \frac{\Gamma\{[\Delta, [\Pi, \Diamond A]]\}}{\Gamma\{[\Delta, \Diamond A, [\Pi]]\}}$$

Fig. 4. Rules of the nested sequent calculus

follows: the empty sequence $\varnothing$ is a nested sequent; if Σ and Δ are nested sequents and A is a formula, then Σ, A and $\Sigma, [\Delta]$ are nested sequents, where the comma denotes concatenation of sequences. The brackets in the expression $[\Delta]$ are called *structural box.* The *corresponding formula* of a sequent Γ, denoted $\underline{\Gamma}$, is inductively defined by $\underline{\varnothing} := \bot$, $\underline{\Sigma, A} := \underline{\Sigma} \vee A$, and $\underline{\Sigma, [\Delta]} := \underline{\Sigma} \vee \Box\underline{\Delta}$. For simplicity we often do not explicitly distinguish between a sequent and its corresponding formula. We use the letters Γ, Δ, Λ, Ω, Π, and Σ to denote sequents.

Sequent contexts. A *sequent context*, or *context* for short, is a sequent with (exactly) one occurrence of the symbol $\{\ \}$, called a *hole*, which does not occur inside formulas. Contexts are denoted by $\Gamma\{\ \}$. An inductive definition can be given as follows: $\{\ \}$ is a context and if $\Sigma\{\ \}$ is a context, then so are $[\Sigma\{\ \}]$ and $\Delta, \Sigma\{\ \}, \Pi$, where Δ and Π are sequents. The sequent $\Gamma\{\Delta\}$ is obtained by replacing the hole in $\Gamma\{\ \}$ with Δ. For example, if $\Gamma\{\ \} = A, [[B], \{\ \}]$ and $\Delta = C, [D]$, then $\Gamma\{\Delta\} = A, [[B], C, [D]]$.

Sequent systems. Consider the inference rules in Figure 4. *System* SK consists of the rules id, $\vee$, $\wedge$, ctr, exch, $\Box$, and k. *Extensions of system* SK are obtained by adding further rules from Figure 4 according to Figure 3, where 5 means that all three rules 5a, 5b, and 5c are added. Note that a name in the first row of Figure 3 now denotes both a (Hilbert-style) axiom system and a sequent system.

These sequent systems are essentially the same as the ones in [9], where their completeness is proved, so we have the following theorem.

Theorem 3.1 (Completeness) *System* SK *and its extensions are sound and complete with respect to their corresponding modal logics (as defined by the corresponding axiom systems).*

4 Annotations and Realizations

Our goal is to turn provable formulas of a given modal logic into provable formulas of the corresponding justification logic by replacing boxes with terms and diamonds with variables. In order to do so we use *annotations*, which are indices on modalities.

Annotations have no semantical meaning but allow us to keep track of occurrences of modal operators. We adopt Fitting's notation from [13].

Definition 4.1 (Annotations) Annotated modal formulas*, or* annotated formulas *for short, are built according to the grammar*

$$A ::= P_i \mid \neg P_i \mid (A \vee A) \mid (A \wedge A) \mid \Box_{2k+1} A \mid \Diamond_{2l} A \ ,$$

where i, k, and l range over natural numbers. An annotated sequent (context) *is a sequent (context) in which only annotated formulas occur and all structural boxes are annotated by odd indices. The* corresponding annotated formula *of an annotated sequent Γ is defined in the obvious way, with $\underline{\Sigma, [\Delta]_k} := \underline{\Sigma} \vee \Box_k \underline{\Delta}$.*

If A is a modal formula that is obtained from an annotated formula A' by dropping all indices on its modalities, then we call A' an annotated version *of A, and likewise for sequents. An annotated formula or sequent is called* properly annotated *if no index occurs twice in it. From now on we will always assume that an annotated formula or sequent is properly annotated, unless stated otherwise.*

Remark 4.2 *Since our modal formulas are in negation normal form, in contrast to [13] every subformula of a properly annotated formula is itself properly annotated.*

Definition 4.3 (Annotated rule instance) *An* annotated rule instance *is any instance of a rule in Figure 5 provided that its conclusion and each of its premises are properly annotated sequents and, in case of the* ctr*-rule, additionally A_1, A_2, and A_3 do not share indices and are annotated versions of the same modal formula. An* annotated proof *is built as usual from annotated rule instances.*

Remark 4.4 *Note that we do not define the negation of an annotated formula. The obvious definition, where $\neg\Box_k A$ is $\Diamond_k \neg A$, does not work because it does not produce an annotated formula. In particular, this prevents us from even formulating a cut-rule for annotated sequents.*

Lemma 4.5 (Annotating Proofs) *For each sequent calculus proof $\mathcal{P}$ there exists an annotated proof $\mathcal{P}'$ that is an* annotated version *of $\mathcal{P}$, meaning that $\mathcal{P}$ can be obtained from $\mathcal{P}'$ by dropping all annotations.*

Proof. We take $\mathcal{P}$, replace the endsequent with a properly annotated version of it, and straightforwardly propagate the annotations upwards. □

Now we can define realizations as functions from natural numbers to terms, with the restriction that even numbers are mapped to variables. This restriction is often called the *normality condition.*

Definition 4.6 (Realization function) *A* realization function *r is a partial mapping from natural numbers to terms such that if $r(2i)$ is defined, then $r(2i) = x_i$. A* realization function on a given annotated formula (sequent) *is one that is defined on all indices of that formula (sequent).*

Definition 4.7 (Realization) *If A is an annotated formula and r is a partial mapping*

$$\text{id}\ \frac{}{\Gamma\{P_i, \neg P_i\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\text{ctr}\ \frac{\Gamma\{A_1, A_2\}}{\Gamma\{A_3\}} \qquad \text{exch}\ \frac{\Gamma\{\Delta, \Sigma\}}{\Gamma\{\Sigma, \Delta\}} \qquad \Box\ \frac{\Gamma\{[A]_k\}}{\Gamma\{\Box_k A\}} \qquad \text{k}\ \frac{\Gamma\{[A, \Delta]_k\}}{\Gamma\{\Diamond_{2m} A, [\Delta]_i\}}$$

$$\text{d}\ \frac{\Gamma\{[A]_k\}}{\Gamma\{\Diamond_{2m} A\}} \qquad \text{t}\ \frac{\Gamma\{A\}}{\Gamma\{\Diamond_{2m} A\}} \qquad \text{b}\ \frac{\Gamma\{[\Delta]_k, A\}}{\Gamma\{[\Delta, \Diamond_{2m} A]_i\}} \qquad \text{4}\ \frac{\Gamma\{[\Diamond_{2m} A, \Delta]_k\}}{\Gamma\{\Diamond_{2m} A, [\Delta]_i\}}$$

$$\text{5a}\ \frac{\Gamma\{[\Delta]_k, \Diamond_{2m} A\}}{\Gamma\{[\Delta, \Diamond_{2m} A]_i\}} \qquad \text{5b}\ \frac{\Gamma\{[\Delta]_k, [\Pi, \Diamond_{2m} A]_i\}}{\Gamma\{[\Delta, \Diamond_{2m} A]_l, [\Pi]_j\}} \qquad \text{5c}\ \frac{\Gamma\{[\Delta, [\Pi, \Diamond_{2m} A]_i]_k\}}{\Gamma\{[\Delta, \Diamond_{2m} A, [\Pi]_j]_l\}}$$

Fig. 5. Annotated rules of the nested sequent calculus

$$(P_i)^r := P_i \qquad (A \vee B)^r := A^r \vee B^r \qquad (\Diamond_{2l} A)^r := \neg r(2l) : \neg A^r$$

$$(\neg P_i)^r := \neg P_i \qquad (A \wedge B)^r := A^r \wedge B^r \qquad (\Box_{2k+1} A)^r := r(2k+1) : A^r$$

Fig. 6. Realization of a formula

from natural numbers to terms (not necessarily a realization function) that is defined on all indices of A, then the justification formula A^r is inductively defined as in Figure 6. Note that if r is a realization function, then $(\Diamond_{2l} A)^r = \neg x_l : \neg A^r$. Given an annotated sequent Γ, we define Γ^r as $(\underline{\Gamma})^r$.

We introduce some notation for stating restrictions on realization functions.

Definition 4.8 (diavars(A)**,** $r \restriction A$**)** *Given an annotated formula A, we define*

$$\text{diavars}(A) := \{x_k \mid \Diamond_{2k} \textit{ occurs in } A\}$$
$$r \restriction A := r \restriction \{i \mid \Box_i \textit{ or } \Diamond_i \textit{ occurs in } A\} \quad ,$$

where $f \restriction S$ is the restriction of the partial function f to the set S.

The next definition is mostly standard, see, e.g., Baader and Nipkow [6].

Definition 4.9 (Substitution) *A* substitution*, denoted by σ, is a total mapping from variables to terms. If σ is a substitution, then $\widetilde{\sigma}$ is the function that maps terms to terms and formulas to formulas by simultaneously replacing each occurrence of a variable x with the term $\sigma(x)$. The* domain of σ *is* $\text{dom}(\sigma) := \{x \mid \sigma(x) \neq x\}$*, the* range of σ *is* $\text{range}(\sigma) := \{\sigma(x) \mid x \in \text{dom}(\sigma)\}$*, and the* variable range of σ*, denoted by* $\text{vrange}(\sigma)$*, is the set of variables that occur in terms in* $\text{range}(\sigma)$*. We write $t\sigma$ and $A\sigma$ to denote $\widetilde{\sigma}(t)$ and $\widetilde{\sigma}(A)$ respectively. We also write $\sigma \circ r$ for $\widetilde{\sigma} \circ r$, where function composition is as usual, namely $(f_2 \circ f_1)(n) := f_2(f_1(n))$.*

The following lemma is standard, see, e.g., [15].

Lemma 4.10 (Substitution) *If $\mathsf{JL} \vdash A$ for a justification logic JL, then*

(i) $\mathsf{JL} \vdash A\sigma$ *for any substitution* σ *and*

(ii) $\mathsf{JL} \vdash A[P_i \mapsto B]$, *where* $A[P_i \mapsto B]$ *is the result of simultaneously replacing each occurrence of the proposition* P_i *in* A *with the formula* B.

The following immediate facts are used in many of the proofs that follow.

Lemma 4.11 (Facts about Substitutions and Realization Functions)

(i) $\sigma \circ r$ *is a realization function* *iff* $x_n \notin \mathrm{dom}(\sigma)$ *whenever* $r(2n)$ *is defined.*

(ii) $A^r\sigma = A^{\sigma \circ r}$.

(iii) *If* $\mathrm{dom}(r_1) \cap \mathrm{dom}(r_2) \subseteq \{n \mid n \text{ is even}\}$, *then* $r_1 \cup r_2$ *is a realization function.*

(iv) *If* $\mathrm{dom}(\sigma_1) \cap \mathrm{dom}(\sigma_2) = \emptyset$, *then* $\sigma_1 \cup \sigma_2$ *is a substitution.*

(v) *If* $\sigma \circ r$ *is a realization function, then* $\mathrm{dom}(\sigma \circ r) = \mathrm{dom}(r)$.

(vi) *If* $r_1 \cup r_2$ *is a realization function, then* $\mathrm{dom}(r_1 \cup r_2) = \mathrm{dom}(r_1) \cup \mathrm{dom}(r_2)$.

(vii) *If* $\sigma_1 \cup \sigma_2$ *is a substitution, then* $\mathrm{dom}(\sigma_1 \cup \sigma_2) = \mathrm{dom}(\sigma_1) \cup \mathrm{dom}(\sigma_2)$.

(viii) $\mathrm{dom}(\sigma_2 \circ \sigma_1) \subseteq \mathrm{dom}(\sigma_1) \cup \mathrm{dom}(\sigma_2)$.

The proof of our main result, the realization theorem in the next section, is by induction on the depth of a given proof. For branching rules, we need to merge realizations. The following theorem allows us to do that. It is essentially Theorem 8.2 in Fitting [13]. There it is formulated for LP but the proof only makes use of the operations $+$ and $\cdot$ and the Internalization Lemma. Hence, the theorem also holds for all justification logics we consider.

Theorem 4.12 (Realization Merging) *Let* JL *be a justification logic,* A *be a properly annotated formula, and* r_1 *and* r_2 *be realization functions on* A. *Then there exists a realization function* r *on* A *and a substitution* σ *such that: 1) for any* x *the term* $\sigma(x)$ *contains no variables other than* x, *2)* $\mathrm{dom}(\sigma) \subseteq \mathrm{diavars}(A)$,

$$\textit{3)}\quad \mathsf{JL} \vdash A^{r_1}\sigma \to A^r\ , \qquad \textit{and} \qquad \textit{4)}\quad \mathsf{JL} \vdash A^{r_2}\sigma \to A^r\ .$$

(Note that it is not assumed that A^{r_1} *or* A^{r_2} *is provable.)*

The next lemma easily follows from the associativity of Boolean disjunction. It is needed because in general the formula $\underline{\Gamma, \Sigma}$ does not coincide with the formula $\underline{\Gamma} \vee \underline{\Sigma}$.

Lemma 4.13 (Associativity of Disjunction) *For any sequents* Γ *and* Σ *and for any realization function* r, *we have* $\mathsf{J} \vdash (\underline{\Sigma, \Gamma})^r \leftrightarrow \underline{\Sigma}^r \vee \underline{\Gamma}^r$.

5 The Realization Theorem

We now prove the realization theorem. The argument is by induction on the depth of a given annotated proof. The following lemmas mostly correspond to the inductive cases for the various sequent calculus rules.

Lemma 5.1 (id-rule) *Given an annotated* id*-instance as in Figure 5, there exists a realization function* r *on its conclusion* Ω *such that* $\mathsf{J} \vdash \Omega^r$.

Proof. By induction on the structure of $\Gamma\{\ \}$.
Base case $\Gamma\{\ \} = \{\ \}$**.** The empty realization function suffices.
Induction step. By induction hypothesis, there exists a realization function r' on $\Sigma\{P_i, \neg P_i\}$ such that $\mathsf{J} \vdash \Sigma\{P_i, \neg P_i\}^{r'}$.
Case $\Gamma\{\ \} = [\Sigma\{\ \}]_k$**.** By the Internalization Lemma there exists a ground term p such that $\mathsf{J} \vdash p : \Sigma\{P_i, \neg P_i\}^{r'}$. Since the conclusion $\Omega = [\Sigma\{P_i, \neg P_i\}]_k$ is properly annotated, $r := (r' \restriction \Sigma\{P_i, \neg P_i\}) \cup \{(k, p)\}$ is a realization function on $[\Sigma\{P_i, \neg P_i\}]_k$ by Lemma 4.11. It follows that $\mathsf{J} \vdash ([\Sigma\{P_i, \neg P_i\}]_k)^r$.
Case $\Gamma\{\} = \Delta, \Sigma\{\}, \Pi$**.** Let r be a realization function on $\Delta, \Sigma\{P_i, \neg P_i\}, \Pi$ that extends $r' \restriction \Sigma\{P_i, \neg P_i\}$. Then $\mathsf{J} \vdash \Delta^r \vee \Sigma\{P_i, \neg P_i\}^r \vee \Pi^r$ follows by propositional reasoning. Therefore, by Lemma 4.13, $\mathsf{J} \vdash \big(\Delta, \Sigma\{P_i, \neg P_i\}, \Pi\big)^r$. □

Lemma 5.2 ($\wedge$**-rule)** *Given an annotated* $\wedge$*-instance as in Figure 5, let* r_1 *and* r_2 *be realization functions on its premises* Λ_1 *and* Λ_2 *respectively. Then there exists a substitution* σ *with* $\mathrm{dom}(\sigma) \subseteq \mathrm{diavars}(\Lambda_1) \cup \mathrm{diavars}(\Lambda_2) = \mathrm{diavars}(\Omega)$ *and a realization function* r *on the conclusion* Ω *such that* $\mathsf{J} \vdash (\Lambda_1)^{r_1}\sigma \to (\Lambda_2)^{r_2}\sigma \to \Omega^r$.

Proof. By induction on the structure of $\Gamma\{\ \}$.
Base case $\Gamma\{\} = \{\}$**.** Let $r := (r_1 \restriction A) \cup (r_2 \restriction B)$ and let σ be the identity substitution. The former is a realization function by Lemma 4.11 because $\Omega = A \wedge B$ is properly annotated. Thus, $A^{r_1} \wedge B^{r_2} = (A \wedge B)^r$ and $\mathsf{J} \vdash A^{r_1}\sigma \to B^{r_2}\sigma \to (A \wedge B)^r$ because it is a propositional tautology.
Induction step. By induction hypothesis there exists a substitution σ' with $\mathrm{dom}(\sigma') \subseteq \mathrm{diavars}(\Sigma\{A \wedge B\})$ and a realization function r' on $\Sigma\{A \wedge B\}$ such that

$$\mathsf{J} \vdash \Sigma\{A\}^{r_1}\sigma' \to \Sigma\{B\}^{r_2}\sigma' \to \Sigma\{A \wedge B\}^{r'} \quad . \tag{1}$$

Case $\Gamma\{\ \} = \big[\Sigma\{\ \}\big]_k$**.** By the Internalization Lemma,

$$\mathsf{J} \vdash r_1(k)\sigma' : (\Sigma\{A\}^{r_1}\sigma') \to r_2(k)\sigma' : (\Sigma\{B\}^{r_2}\sigma') \to\ t\big(r_1(k)\sigma', r_2(k)\sigma'\big) : \Sigma\{A \wedge B\}^{r'}$$

for some term $t(x, y)$. In other words,

$$\mathsf{J} \vdash ([\Sigma\{A\}]_k)^{r_1}\sigma' \to ([\Sigma\{B\}]_k)^{r_2}\sigma' \to ([\Sigma\{A \wedge B\}]_k)^r$$

for $r := (r' \restriction \Sigma\{A \wedge B\}) \cup \{(k, t(r_1(k)\sigma', r_2(k)\sigma')\}$, which by Lemma 4.11 is a realization function on the properly annotated sequent $\Omega = [\Sigma\{A \wedge B\}]_k$.
Case $\Gamma\{\ \} = \Delta, \Sigma\{\ \}, \Pi$**.** Since $\Omega = \Delta, \Sigma\{A \wedge B\}, \Pi$ is properly annotated, $\Sigma\{A \wedge B\}$ shares no indices with Δ, Π. Thus, by Lemma 4.11, both $\sigma' \circ (r_1 \restriction \Delta, \Pi)$ and $\sigma' \circ (r_2 \restriction \Delta, \Pi)$ are realization functions on Δ, Π. By Theorem 4.12 (Realization Merging) there exists a realization function r_m on Δ, Π and a substitution σ_m with $\mathrm{dom}(\sigma_m) \subseteq \mathrm{diavars}(\Delta, \Pi)$ such that

$$\mathsf{J} \vdash (\Delta, \Pi)^{\sigma' \circ (r_1 \restriction \Delta, \Pi)}\sigma_m \to (\Delta, \Pi)^{r_m} \quad , \tag{2}$$

$$\mathsf{J} \vdash (\Delta, \Pi)^{\sigma' \circ (r_2 \restriction \Delta, \Pi)}\sigma_m \to (\Delta, \Pi)^{r_m} \quad , \tag{3}$$

and x is the only variable in $\sigma_m(x)$, for any x. By Lemma 4.11, $(\Delta,\Pi)^{\sigma'\circ(r_1\restriction\Delta,\Pi)}\sigma_m$ is $(\Delta,\Pi)^{r_1}\sigma'\sigma_m$ and $(\Delta,\Pi)^{\sigma'\circ(r_2\restriction\Delta,\Pi)}\sigma_m$ is $(\Delta,\Pi)^{r_2}\sigma'\sigma_m$. Therefore, (2) and (3) are identical to

$$\mathsf{J}\vdash(\Delta,\Pi)^{r_1}\sigma'\sigma_m\to(\Delta,\Pi)^{r_m}\quad, \tag{4}$$
$$\mathsf{J}\vdash(\Delta,\Pi)^{r_2}\sigma'\sigma_m\to(\Delta,\Pi)^{r_m}\quad. \tag{5}$$

From (1) by Lemma 4.10 (Substitution) it follows that

$$\mathsf{J}\vdash\Sigma\{A\}^{r_1}\sigma'\sigma_m\to\Sigma\{B\}^{r_2}\sigma'\sigma_m\to\Sigma\{A\wedge B\}^{r'}\sigma_m\quad.$$

From this, (4), and (5) it follows by propositional reasoning that

$$\mathsf{J}\vdash\Sigma\{A\}^{r_1}\sigma'\sigma_m\vee(\Delta,\Pi)^{r_1}\sigma'\sigma_m\;\to\;\Sigma\{B\}^{r_2}\sigma'\sigma_m\vee(\Delta,\Pi)^{r_2}\sigma'\sigma_m$$
$$\to\;\Sigma\{A\wedge B\}^{r'}\sigma_m\vee(\Delta,\Pi)^{r_m}\quad. \tag{6}$$

Since $\mathrm{dom}(\sigma_m)\subseteq\mathrm{diavars}(\Delta,\Pi)$, it follows by Lemma 4.11 that $\sigma_m\circ(r'\restriction\Sigma\{A\wedge B\})$ is a realization function on $\Sigma\{A\wedge B\}$. Again by Lemma 4.11, we conclude that

$$r:=(\sigma_m\circ(r'\restriction\Sigma\{A\wedge B\}))\cup(r_m\restriction\Delta,\Pi)$$

is a realization function on $\Delta,\Sigma\{A\wedge B\},\Pi$. And since

$$\Sigma\{A\wedge B\}^{r'\restriction\Sigma\{A\wedge B\}}\sigma_m=\Sigma\{A\wedge B\}^{\sigma_m\circ(r'\restriction\Sigma\{A\wedge B\})}$$

by Lemma 4.11, we can rewrite (6) as

$$\mathsf{J}\vdash\big(\Sigma\{A\}\vee(\Delta,\Pi)\big)^{r_1}\sigma\;\to\;\big(\Sigma\{B\}\vee(\Delta,\Pi)\big)^{r_2}\sigma\;\to\;\big(\Sigma\{A\wedge B\}\vee(\Delta,\Pi)\big)^{r} \tag{7}$$

for $\sigma:=\sigma_m\circ\sigma'$ with $\mathrm{dom}(\sigma)\subseteq\mathrm{dom}(\sigma')\cup\mathrm{dom}(\sigma_m)\subseteq\mathrm{diavars}(\Delta,\Sigma\{A\wedge B\},\Pi)$. Finally, (7) is by Lemma 4.13 propositionally equivalent to

$$\mathsf{J}\vdash(\Delta,\Sigma\{A\},\Pi)^{r_1}\sigma\;\to\;(\Delta,\Sigma\{B\},\Pi)^{r_2}\sigma\;\to\;(\Delta,\Sigma\{A\wedge B\},\Pi)^{r}\quad.$$

□

The proof of the following lemma is in Appendix A.

Lemma 5.3 (ctr-rule) *Given an annotated* ctr*-instance as in Figure 5, let r_1 be a realization function on its premise Λ. Then there exists 1) a realization function r on its conclusion Ω and 2) a substitution σ with* $\mathrm{dom}(\sigma)\subseteq\mathrm{diavars}(\Lambda)$ *such that* $\mathsf{J}\vdash\Lambda^{r_1}\sigma\to\Omega^{r}$.

Lemma 5.4 ($\vee$- and exch-rule) *Given an annotated ρ-instance with $\rho\in\{\vee,\mathsf{exch}\}$ as in Figure 5, let r_1 be a realization function on its premise Λ. Then there exists a realization function r on its conclusion Ω such that* $\mathsf{J}\vdash\Lambda^{r_1}\to\Omega^{r}$.

Proof. By induction on the structure of $\Gamma\{\ \}$.
Base case $\Gamma\{\ \} = \{\ \}$**.** It suffices to take $r := r_1 \upharpoonright \Omega$ for either rule. Indeed, we have $\underline{A, B} = A \vee B = \underline{A \vee B}$ for $\rho = \vee$. For $\rho = \mathsf{exch}$, the desired statement follows from Lemma 4.13.
Induction step. The arguments are the same as in the proof of Lemma 5.3, given in Appendix A, except that here the substitution is the identity substitution. □

Lemma 5.5 (k-rule) *Given an annotated* k*-instance as in Figure 5, let* r_1 *be a realization function on its premise* Λ*. Then there exists a realization function* r *on its conclusion* Ω *such that* $\mathsf{J} \vdash \Lambda^{r_1} \to \Omega^r$*.*

Proof. By induction on the structure of $\Gamma\{\ \}$.
Base case $\Gamma\{\ \} = \{\ \}$**.** For the propositional tautology $(A, \Delta)^{r_1} \to \neg A^{r_1} \to \Delta^{r_1}$, by the Internalization Lemma, $\mathsf{J} \vdash r_1(k) : (A, \Delta)^{r_1} \to x_m : \neg A^{r_1} \to t(r_1(k), x_m) : \Delta^{r_1}$ for some term $t(x, y)$. It follows by propositional reasoning that

$$\mathsf{J} \vdash r_1(k) : (A, \Delta)^{r_1} \to \neg x_m : \neg A^{r_1} \vee t(r_1(k), x_m) : \Delta^{r_1} \quad , \quad \text{which is}$$
$$\mathsf{J} \vdash \quad \big([A, \Delta]_k\big)^{r_1} \to (\Diamond_{2m} A)^{r_1} \vee t(r_1(k), x_m) : \Delta^{r_1} \quad .$$

For $r := (r_1 \upharpoonright A, \Delta) \cup \{(i, t(r_1(k), x_m)), (2m, x_m)\}$ this is identical to

$$\mathsf{J} \vdash \big([A, \Delta]_k\big)^{r_1} \to (\Diamond_{2m} A, [\Delta]_i)^r \quad .$$

Induction step. The arguments are the same as in Lemma 5.4. □

In order to realize the modal rules 5b and 5c, we will use realizations of theorems $\Box(\Box A \to A)$ and $\Box(\neg\Box\Box A \to \neg\Box A)$ of K5. They are provided by the following two auxiliary lemmas. We have to omit the proofs for space reasons.

Lemma 5.6 (Internalized Factivity) *There is a term* $t(x)$ *such that for any term* s *and any formula* A *we have that* $\mathsf{J5} \vdash t(s) : (s : A \to A)$*.*

Lemma 5.7 (Internalized Positive Introspection) *There are terms* $t_1(x)$ *and* $t_2(x)$ *such that* $\mathsf{J5} \vdash t_1(t) : (\neg t_2(t) : t : A \to \neg t : A)$ *for any term* t *and any formula* A*.*

The following lemma covers the remaining rules.

Lemma 5.8 (Modal Rules) *Given an annotated* ρ*-instance with* $\rho \in \{$d, t, b, 4, 5a, 5b, 5c$\}$ *as given in Figure 5, let* r_1 *be a realization function on its premise* Λ*. Then there is a realization function* r *on its conclusion* Ω *such that* $\mathsf{J}\rho \vdash \Lambda^{r_1} \to \Omega^r$*, where by* Jd *we mean* JD*, and so on, except for* $\rho \in \{5a, 5b, 5c\}$ *where we mean* J5*.*

Proof. By induction on the structure of $\Gamma\{\ \}$.
Base case $\Gamma\{\ \} = \{\ \}$**.** We need to consider each rule ρ in turn.
Subcases $\rho = \mathsf{d}, \mathsf{t}, \mathsf{4}$**.** The proof is similar to the k-rule and is omitted for space reasons.

Subcase $\rho = \mathsf{b}$. Since $\Delta^{r_1} \to \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ is a propositional tautology, by the Internalization Lemma there exists a term $t_1(y)$ such that

$$\mathsf{JB} \vdash r_1(k) : \Delta^{r_1} \to t_1(r_1(k)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \ . \tag{8}$$

Similarly, for a propositional tautology $\neg x_m : \neg A^{r_1} \ \to \ \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$, there exists a term $t_2(x)$ such that

$$\mathsf{JB} \vdash ?\, x_m : \neg x_m : \neg A^{r_1} \ \to \ t_2(?\, x_m) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \ . \tag{9}$$

It follows from (8) and (9) by axiom sum and propositional reasoning that

$$\mathsf{JB} \vdash r_1(k) : \Delta^{r_1} \ \vee \ ?\, x_m : \neg x_m : \neg A^{r_1} \ \to \ t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$$

for $t := t_1(r_1(k)) + t_2(?\, x_m)$. Finally, from the instance $A^{r_1} \to ?\, x_m : \neg x_m : \neg A^{r_1}$ of axiom jb it follows that $\mathsf{JB} \vdash r_1(k) : \Delta^{r_1} \vee A^{r_1} \to t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$. Hence the desired realization function is $r := (r_1 \restriction \Delta, A) \cup \{(i, t), (2m, x_m)\}$.

Subcases $\rho = \mathsf{5a}, \mathsf{5c}$. The proof can be found in Appendix B.

Subcase $\rho = \mathsf{5b}$. By Lemma 5.7 there exist terms $t_1(x)$ and $t_2(x)$ that satisfy the condition $\mathsf{J5} \vdash t_1(x_m) : (\neg t_2(x_m) : x_m : \neg A^{r_1} \to \neg x_m : \neg A^{r_1})$. Thus, by app and MP,

$$\mathsf{J5} \vdash ?\, t_2(x_m) : \neg t_2(x_m) : x_m : \neg A^{r_1} \ \to \ (t_1(x_m) \cdot ?\, t_2(x_m)) : \neg x_m : \neg A^{r_1} \ .$$

From the instance $\neg t_2(x_m) : x_m : \neg A^{r_1} \to ?\, t_2(x_m) : \neg t_2(x_m) : x_m : \neg A^{r_1}$ of $\mathsf{j5}$ it follows:

$$\mathsf{J5} \vdash \neg t_2(x_m) : x_m : \neg A^{r_1} \ \to \ (t_1(x_m) \cdot ?\, t_2(x_m)) : \neg x_m : \neg A^{r_1} \ . \tag{10}$$

By a propositional tautology and the Internalization Lemma applied to it,

$$\mathsf{J5} \vdash p_1 : \big(x_m : \neg A^{r_1} \ \to \ \Pi^{r_1} \vee \neg x_m : \neg A^{r_1} \ \to \ \Pi^{r_1}\big)$$

for some ground term p_1. Thus, by app and MP,

$$\mathsf{J5} \vdash t_2(x_m) : x_m : \neg A^{r_1} \to (p_1 \cdot t_2(x_m)) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1} \to \Pi^{r_1}) \ .$$

Again by app and propositional reasoning,

$$\mathsf{J5} \vdash t_2(x_m) : x_m : \neg A^{r_1} \ \to \ r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \ \to \ (p_1 \cdot t_2(x_m) \cdot r_1(i)) : \Pi^{r_1} \ ,$$

which is propositionally equivalent to

$$\mathsf{J5} \vdash r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \ \to \ \neg t_2(x_m) : x_m : \neg A^{r_1} \vee s : \Pi^{r_1}$$

for $s := p_1 \cdot t_2(x_m) \cdot r_1(i)$. From this and (10) by propositional reasoning we obtain

$$\mathsf{J5} \vdash r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \ \to \ (t_1(x_m) \cdot ?\, t_2(x_m)) : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1} \ . \tag{11}$$

By the Internalization Lemma for the tautology $\neg x_m : \neg A^{r_1} \to \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ and propositional reasoning, there is a term $t_3(x)$ such that

$$\mathsf{J5} \vdash r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \;\to\; t_3(t_1(x_m) \cdot ?\, t_2(x_m)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \vee s : \Pi^{r_1} . \quad (12)$$

Since $\Delta^{r_1} \to \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ is a propositional tautology, by the Internalization Lemma there is a term $t_4(x)$ such that $\mathsf{J5} \vdash r_1(k) : \Delta^{r_1} \to t_4(r_1(k)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$. Therefore, by axiom sum,

$$\mathsf{J5} \vdash r_1(k) : \Delta^{r_1} \to t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \quad (13)$$

for $t := t_3(t_1(x_m) \cdot ?\, t_2(x_m)) + t_4(r_1(k))$. Similarly, by (12) and sum,

$$\mathsf{J5} \vdash r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \;\to\; t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \vee s : \Pi^{r_1} \quad . \quad (14)$$

Finally by propositional reasoning from (13) and (14),

$$\mathsf{J5} \vdash r_1(k) : \Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \to t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \vee s : \Pi^{r_1} \quad .$$

Hence the desired realization function is $r := (r_1 \restriction \Delta, \Diamond_{2m} A, \Pi) \cup \{(l,t),(j,s)\}$.

This completes the proof of the base case of the induction.

Induction step. The proof is the same as in Lemma 5.4. □

Now we are ready to prove our main result.

Theorem 5.9 (Realization) *For any modal logic* ML *and its corresponding justification logic* JL *we have that* ML = JL°.

Proof. The inclusion JL° ⊆ ML is easy since forgetful projections of axioms and rules of any justification logic can easily be derived in the corresponding modal logic. So we now turn to the more interesting opposite inclusion. It follows from Theorem 3.1 (Completeness), Lemma 4.5 (Annotating Proofs), and the following

Claim. Let S be the sequent system for a modal logic ML and let $\mathcal{P}$ be an annotated proof with the endsequent Δ such that the unannotated version of $\mathcal{P}$ is a sequent calculus proof in S. Then there exists a realization function r on Δ such that $\mathsf{JL} \vdash \Delta^r$ for the justification logic JL that corresponds to ML.

We prove the claim by induction on the depth of $\mathcal{P}$ by case analysis on the lowermost rule.

Case id. The claim follows from Lemma 5.1.

Cases ∨- and exch-rules. The claim follows from the induction hypothesis and Lemma 5.4.

Case $\wedge\dfrac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$**.** By induction hypothesis there exist realization functions r_1 and r_2 such that $\mathsf{JL} \vdash \Gamma\{A\}^{r_1}$ and $\mathsf{JL} \vdash \Gamma\{B\}^{r_2}$. By Lemma 5.2, there exists a realization function r on the conclusion $\Gamma\{A \wedge B\}$ and a substitution σ such that $\mathsf{J} \vdash \Gamma\{A\}^{r_1}\sigma \to \Gamma\{B\}^{r_2}\sigma \to \Gamma\{A \wedge B\}^r$. By Lemma 4.10, $\mathsf{JL} \vdash \Gamma\{A\}^{r_1}\sigma$ and $\mathsf{JL} \vdash \Gamma\{B\}^{r_2}\sigma$, hence, $\mathsf{JL} \vdash \Gamma\{A \wedge B\}^r$.

Case ctr-rule. The claim follows from the induction hypothesis, Lemma 5.3, and Lemma 4.10 (Substitution).

Case □-rule. The claim immediately follows from the induction hypothesis.

Case k-rule. The claim follows from the induction hypothesis and Lemma 5.5.

Cases for rules in $\{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5a}, \mathsf{5b}, \mathsf{5c}\}$. The claim follows from the induction hypothesis and Lemma 5.8. □

Remark 5.10 Fitting's Merging Theorem from [13] states a stronger result than used in this paper, namely that the proofs can be made *injective*. An injective proof uses each constant for only one axiom instance. We are confident that the results of this paper can also be extended to injective proofs.

6 A Strengthened Realization Theorem for S5 and KB5

We now introduce two new justification logics: JT5 and JB5. The axiom systems for them are obtained from the axiom systems for JT45 and JB45 respectively by removing the operator ! from the language and, therefore, dropping j4 and replacing AN! with AN from Remark 2.1. Note that, although S5 = KT5 = KT45 and KB5 = KB45, it is obvious that JT5 ≠ JT45 and JB5 ≠ JB45 simply because the languages are different. The proof of the Internalization Lemma relies on the AN!-rule, which is not admissible in either JT5 or JB5. Thus, we need to show the existence of a term $\mathrm{dpi}(x)$ that plays the role of $!\,x$ for these two logics, where dpi stands for *derived positive introspection*.

Lemma 6.1 (Positive Introspection in JB5 and JT5) *There is a term* $\mathrm{dpi}(x)$ *such that for any term* t *and any formula* A

$$\mathsf{JB5} \vdash t : A \rightarrow \mathrm{dpi}(t) : t : A \qquad \textit{and} \qquad \mathsf{JT5} \vdash t : A \rightarrow \mathrm{dpi}(t) : t : A \ .$$

Proof. Since j5 is an axiom of JB5, by AN there exists a constant c_i such that

$$\mathsf{JB5} \vdash c_i : \big(\neg y : P \rightarrow ?\,y : \neg y : P\big) \tag{15}$$

for some proposition P and variable y. It can be shown using AN, app, and propositional reasoning that there exists a ground term p such that

$$\mathsf{JB5} \vdash p : \big((\neg y : P \rightarrow ?\,y : \neg y : P) \ \rightarrow\ \neg\,?\,y : \neg y : P \ \rightarrow\ y : P\big) \ .$$

From this and (15) by app and MP, we have $\mathsf{JB5} \vdash (p \cdot c_i) : (\neg\,?\,y : \neg y : P \rightarrow y : P)$. Again by app and MP, we have $\mathsf{JB5} \vdash ?\,?\,y : \neg\,?\,y : \neg y : P \rightarrow (p \cdot c_i \cdot ?\,?\,y) : y : P$. Since

$$y : P \rightarrow ?\,?\,y : \neg\,?\,y : \neg y : P \tag{16}$$

is an instance of axiom jb, by propositional reasoning

$$\mathsf{JB5} \vdash y : P \rightarrow \mathrm{dpi}(y) : y : P \tag{17}$$

for $\mathrm{dpi}(y) := p \cdot c_i \cdot ??y$. We now show that (16) is provable in $\mathsf{JT5}$. Indeed, formula $\neg ?y : \neg y : P \to ??y : \neg ?y : \neg y : P$ is an instance of $\mathsf{j5}$. Hence, (16) follows by syllogism with $y : P \to \neg ?y : \neg y : P$, which is a contraposition of an instance of jt. Thus, (17) also holds if $\mathsf{JB5}$ is replaced with $\mathsf{JT5}$. The statement of the lemma for either logic now follows from (17) by the Substitution Lemma, which also holds for these logics. □

Because of Lemma 6.1, using $\mathrm{dpi}(t)$ instead of $!\,t$ we can adapt the standard proof of the Internalization Lemma to $\mathsf{JT5}$ and $\mathsf{JB5}$. As a consequence, versions of Theorem 4.12, as well as of Lemmas 5.1, 5.2, 5.3, 5.4, 5.5, and 5.8, for $\mathsf{JT5}$ and $\mathsf{JB5}$ also hold. The proofs apply literally except that in the case of the $\mathsf{4}$-rule in Lemma 5.8, we use Lemma 6.1 instead of axiom $\mathsf{j4}$.

It follows from the Realization Theorem for $\mathsf{JT45}$ and $\mathsf{JB45}$ that $\mathsf{JT5}^\circ \subseteq \mathsf{S5}$ and $\mathsf{JB5}^\circ \subseteq \mathsf{KB5}$. The opposite inclusions can be shown by literally repeating the proof of the Realization Theorem.

Theorem 6.2 (Strengthened Realization) $\mathsf{S5} = \mathsf{JT5}^\circ$ *and* $\mathsf{KB5} = \mathsf{JB5}^\circ$.

7 Conclusion

We have used cut-free nested sequent systems to constructively realize each of our 15 modal logics. In doing so, we have reproved in a uniform way several known realization theorems and have realized logics that did not have justification counterparts before. For two logics, we have also shown that the positive introspection operation is superfluous.

For now we have realized these *logics*. However, some of them have more than one *axiomatization*. Justification counterparts of different axiomatizations of the same modal logic can be different, e.g., $\mathsf{JT5}$ and $\mathsf{JT45}$ are both justification counterparts of $\mathsf{S5}$ but are based on different axiomatizations of it. Thus, it is a natural next step for us to try to obtain realizations for all the 32 different axiomatizations of these 15 logics. We believe that nested sequent systems with *structural modal rules* [9,10], which are *modular* in a certain sense, will allow us to do this.

Another direction for future research is to look for cut-free proof systems for all our justification logics. Currently many justification logics lack such proof systems, and the problems in obtaining them seem to be the same as for modal logics. Nested sequents have provided cut-free proof systems for all our modal logics, and thus we believe they can also provide cut-free proof systems for justification logics.

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A Proof of Lemma 5.3 (Contraction)

Proof. By induction on the structure of $\Gamma\{\ \}$.
Base case $\Gamma\{\ \} = \{\ \}$. In order to demonstrate the statement, a subinduction on the structure of the common unannotated version A of formulas A_1, A_2, and A_3 is employed. The statement proved by subinduction is the same as in the main induction with an extra restriction on σ, namely that $\mathrm{vrange}(\sigma) \subseteq \mathrm{diavars}(\Omega)$.
Subinduction base: $A = P_i$ or $A = \neg P_i$. The identity substitution σ and $r := \emptyset$ suffice.
Subinduction step. The following cases have to be considered:
Subinduction case $A = B \vee C$. The annotated formulas $A_1 = B_1 \vee C_1$, $A_2 = B_2 \vee C_2$, and $A_3 = B_3 \vee C_3$ do not share indices. By subinduction hypothesis, there exist realization functions r'_B on B_3 and r'_C on C_3, as well as substitutions σ_B with $\mathrm{dom}(\sigma_B) \subseteq \mathrm{diavars}(B_1 \vee B_2)$ and σ_C with $\mathrm{dom}(\sigma_C) \subseteq \mathrm{diavars}(C_1 \vee C_2)$ such that

$$\mathsf{J} \vdash (B_1 \vee B_2)^{r_1}\sigma_B \to (B_3)^{r'_B} \qquad \text{and} \qquad \mathsf{J} \vdash (C_1 \vee C_2)^{r_1}\sigma_C \to (C_3)^{r'_C} \ .$$

Also, we have that $\mathrm{vrange}(\sigma_B) \subseteq \mathrm{diavars}(B_3)$ and $\mathrm{vrange}(\sigma_C) \subseteq \mathrm{diavars}(C_3)$. By Lemma 4.11, $\sigma := \sigma_B \cup \sigma_C$ is a substitution with $\mathrm{dom}(\sigma) \subseteq \mathrm{diavars}(\Lambda)$. In addition, for restrictions $r_B := r'_B \restriction B_3$ and $r_C := r'_C \restriction C_3$, both $\sigma_C \circ r_B$ and $\sigma_B \circ r_C$ are realization functions on B_3 and C_3 respectively. Since $(B_3)^{r'_B} = (B_3)^{r_B}$ and $(C_3)^{r'_C} = (C_3)^{r_C}$, by Lemma 4.10

$$\mathsf{J} \vdash (B_1 \vee B_2)^{r_1}\sigma_B\sigma_C \to (B_3)^{r_B}\sigma_C \qquad \text{and} \qquad \mathsf{J} \vdash (C_1 \vee C_2)^{r_1}\sigma_C\sigma_B \to (C_3)^{r_C}\sigma_B \ .$$

Note that σ_C has no effect on any term $\sigma_B(x) \in \mathrm{range}(\sigma_B)$ because $\sigma_B(x)$ only contains variables from $\mathrm{diavars}(B_3)$, which is disjoint from $\mathrm{diavars}(C_1 \vee C_2) \supseteq \mathrm{dom}(\sigma_C)$. Thus, $(B_1 \vee B_2)^{r_1}\sigma_B\sigma_C = (B_1 \vee B_2)^{r_1}\sigma$. Similarly, $(C_1 \vee C_2)^{r_1}\sigma_C\sigma_B = (C_1 \vee C_2)^{r_1}\sigma$. From this and Lemma 4.11 it follows that

$$\mathsf{J} \vdash (B_1 \vee B_2)^{r_1}\sigma \to (B_3)^{\sigma_C \circ r_B} \qquad \text{and} \qquad \mathsf{J} \vdash (C_1 \vee C_2)^{r_1}\sigma \to (C_3)^{\sigma_B \circ r_C} \ .$$

Finally, by propositional reasoning,

$$\mathsf{J} \vdash \big((B_1 \vee C_1) \vee (B_2 \vee C_2)\big)^{r_1}\sigma \ \to\ (B_3)^{\sigma_C \circ r_B} \vee (C_3)^{\sigma_B \circ r_C} \ .$$

In other words, $\mathsf{J} \vdash \Lambda^{r_1}\sigma \to \Omega^r$ for $r := (\sigma_C \circ r_B) \cup (\sigma_B \circ r_C)$, which by Lemma 4.11 is a realization function on $\Omega = B_3 \vee C_3$.
Subinduction case $A = B \wedge C$. It is analogous to $B \vee C$.
Subinduction case $A = \Diamond B$. The annotated formulas $A_1 = \Diamond_{2k} B_1$, $A_2 = \Diamond_{2m} B_2$, $A_3 = \Diamond_{2n} B_3$ do not share indices. By induction hypothesis, there are a realization function r'_B on B_3 and a substitution σ_B with $\mathrm{dom}(\sigma_B) \subseteq \mathrm{diavars}(B_1 \vee B_2)$ such that $\mathsf{J} \vdash (B_1 \vee B_2)^{r_1}\sigma_B \to (B_3)^{r'_B}$. In addition, $\mathrm{vrange}(\sigma_B) \subseteq \mathrm{diavars}(B_3)$. By propositional reasoning,

$$\mathsf{J} \vdash \neg(B_3)^{r'_B} \ \to\ \neg(B_1)^{r_1}\sigma_B \qquad \text{and} \qquad \mathsf{J} \vdash \neg(B_3)^{r'_B} \ \to\ \neg(B_2)^{r_1}\sigma_B \ .$$

By the Internalization Lemma, there exist terms $t_1(y)$ and $t_2(y)$ such that

$$\mathsf{J} \vdash x_n : \neg(B_3)^{r'_B} \rightarrow t_1(x_n) : \neg(B_1)^{r_1}\sigma_B \quad \text{and} \quad \mathsf{J} \vdash x_n : \neg(B_3)^{r'_B} \rightarrow t_2(x_n) : \neg(B_2)^{r_1}\sigma_B \ .$$

It then follows by propositional reasoning that

$$\mathsf{J} \vdash \neg t_1(x_n) : \neg(B_1)^{r_1}\sigma_B \vee \neg t_2(x_n) : \neg(B_2)^{r_1}\sigma_B \rightarrow \neg x_n : \neg(B_3)^{r'_B} \ . \tag{A.1}$$

Since $\mathrm{dom}(\sigma_B) \subseteq \mathrm{diavars}(B_1 \vee B_2) \not\ni x_n$, the substitution σ_B does not affect x_n and, hence, (A.1) is identical to

$$\mathsf{J} \vdash \big(\neg t_1(x_n) : \neg(B_1)^{r_1} \vee \neg t_2(x_n) : \neg(B_2)^{r_1}\big)\sigma_B \rightarrow \neg x_n : \neg(B_3)^{r'_B} \ .$$

Let $\sigma' := \{(x_k, t_1(x_n)), (x_m, t_2(x_n))\} \cup \{(x_i, x_i) \mid i \notin \{k, m\}\}$. By Lemma 4.10,

$$\mathsf{J} \vdash \big(\neg t_1(x_n) : \neg(B_1)^{r_1} \vee \neg t_2(x_n) : \neg(B_2)^{r_1}\big)\sigma_B\sigma' \rightarrow \neg x_n : \neg(B_3)^{r'_B}\sigma' \ . \tag{A.2}$$

Since $2k$ and $2m$ do not occur in B_3, $\sigma' \circ (r'_B \restriction B_3)$ is a realization function on B_3 by Lemma 4.11. Let $r := (\sigma' \circ (r'_B \restriction B_3)) \cup \{(2n, x_n)\}$. Clearly, it is a realization function on $\Diamond_{2n}B_3$. Since σ_B affects neither x_k nor x_m, (A.2) becomes

$$\mathsf{J} \vdash (\Diamond_{2k}B_1 \vee \Diamond_{2m}B_2)^{r_1}\sigma \rightarrow (\Diamond_{2n}B_3)^r$$

for $\sigma := \sigma' \circ \sigma_B$. In other words, $\mathsf{J} \vdash \Lambda^{r_1}\sigma \rightarrow \Omega^r$. It remains to note that, by Lemma 4.11, $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\sigma_B) \cup \mathrm{dom}(\sigma') \subseteq \mathrm{diavars}(\Diamond_{2k}B_1 \vee \Diamond_{2m}B_2)$ and, in addition, we also have $\mathrm{vrange}(\sigma) \subseteq \mathrm{diavars}(B_3) \cup \{x_n\} = \mathrm{diavars}(\Diamond_{2n}B_3)$.

Subinduction case $A = \Box B$. The annotated formulas $A_1 = \Box_k B_1$, $A_2 = \Box_l B_2$, $A_3 = \Box_m B_3$ do not share indices. By induction hypothesis, there exists a realization function r_B on B_3 and a substitution σ with $\mathrm{dom}(\sigma) \subseteq \mathrm{diavars}(B_1 \vee B_2)$ such that $\mathsf{J} \vdash (B_1 \vee B_2)^{r_1}\sigma \rightarrow (B_3)^{r_B}$. In addition, $\mathrm{vrange}(\sigma) \subseteq \mathrm{diavars}(B_3)$. By propositional reasoning and the Internalization Lemma, there exist terms $t_1(y)$ and $t_2(y)$ such that

$$\mathsf{J} \vdash r_1(k)\sigma : (B_1)^{r_1}\sigma \rightarrow t_1(r_1(k)\sigma) : (B_3)^{r_B} \ ,$$
$$\mathsf{J} \vdash r_1(l)\sigma : (B_2)^{r_1}\sigma \rightarrow t_2(r_1(l)\sigma) : (B_3)^{r_B} \ .$$

By axiom sum, for $s := t_1(r_1(k)\sigma) + t_2(r_1(l)\sigma)$,

$$\mathsf{J} \vdash r_1(k)\sigma : (B_1)^{r_1}\sigma \rightarrow s : (B_3)^{r_B} \quad \text{and} \quad \mathsf{J} \vdash r_1(l)\sigma : (B_2)^{r_1}\sigma \rightarrow s : (B_3)^{r_B} \ .$$

Thus, by propositional reasoning,

$$\mathsf{J} \vdash (\Box_k B_1 \vee \Box_l B_2)^{r_1}\sigma \rightarrow (\Box_m B_3)^r$$

for $r := (r_B \restriction B_3) \cup \{(m, s)\}$. It is clear that r is a realization function on $\Box_m B_3$.

This completes the proof by subinduction of the base case $\Gamma\{\ \} = \{\ \}$.

Induction step. By induction hypothesis, there exists a realization function r' on $\Sigma\{A_3\}$ and a substitution σ with $\mathrm{dom}(\sigma) \subseteq \mathrm{diavars}(\Sigma\{A_1, A_2\})$ such that

$$\mathsf{J} \vdash \Sigma\{A_1, A_2\}^{r_1}\sigma \to \Sigma\{A_3\}^{r'} \ .$$

Case $\Gamma\{\ \} = [\Sigma\{\ \}]_k$. By the Internalization Lemma,

$$\mathsf{J} \vdash r_1(k)\sigma : \big(\Sigma\{A_1, A_2\}^{r_1}\sigma\big) \ \to\ t(r_1(k)\sigma) : \Sigma\{A_3\}^{r'}$$

for some term $t(x)$. In other words, the desired result

$$\mathsf{J} \vdash \big([\Sigma\{A_1, A_2\}]_k\big)^{r_1}\sigma \ \to\ \big([\Sigma\{A_3\}]_k\big)^{r} \ ,$$

is achieved for a realization function $r := (r' \restriction \Sigma\{A_3\}) \cup \{(k, t(r_1(k)\sigma))\}$ and the same substitution σ.

Case $\Gamma\{\ \} = \Delta, \Sigma\{\ \}, \Pi$. By propositional reasoning,

$$\mathsf{J} \vdash \Delta^{r_1}\sigma \vee \Sigma\{A_1, A_2\}^{r_1}\sigma \vee \Pi^{r_1}\sigma \ \to\ \Delta^{r_1}\sigma \vee \Sigma\{A_3\}^{r'} \vee \Pi^{r_1}\sigma \ .$$

Since $\mathrm{dom}(\sigma) \subseteq \mathrm{diavars}(\Sigma\{A_1, A_2\})$, by Lemma 4.11, $\sigma \circ (r_1 \restriction \Delta, \Pi)$ is a realization function on Δ, Π. Then for $r := \big(\sigma \circ (r_1 \restriction \Delta, \Pi)\big) \cup (r' \restriction \Sigma\{A_3\})$,

$$\mathsf{J} \vdash (\Delta \vee \Sigma\{A_1, A_2\} \vee \Pi)^{r_1}\sigma \ \to\ (\Delta \vee \Sigma\{A_3\} \vee \Pi)^{r} \ .$$

It remains to apply Lemma 4.13 to obtain the desired result

$$\mathsf{J} \vdash (\Delta, \Sigma\{A_1, A_2\}, \Pi)^{r_1}\sigma \ \to\ (\Delta, \Sigma\{A_3\}, \Pi)^{r}$$

for the realization function r and the same substitution σ.

Note that induction steps never alter σ. □

B Cases for Rules 5a and 5c in Lemma 5.8

Proof. Subcase $\rho = \mathsf{5a}$. By a propositional tautology $\Delta^{r_1} \to \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ and the Internalization Lemma there exists a term $t_1(x)$ such that

$$\mathsf{J5} \vdash r_1(k) : \Delta^{r_1} \to t_1(r_1(k)) : \big(\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}\big) \ . \tag{B.1}$$

Similarly, for a tautology $\neg x_m : \neg A^{r_1} \to \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ there is $t_2(y)$ such that

$$\mathsf{J5} \vdash ?\, x_m : \neg x_m : \neg A^{r_1} \to t_2(?\, x_m) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \ .$$

From the instance $\neg x_m : \neg A^{r_1} \to ?\, x_m : \neg x_m : \neg A^{r_1}$ of $\mathsf{j5}$ by propositional reasoning

$$\mathsf{J5} \vdash \neg x_m : \neg A^{r_1} \to t_2(?\, x_m) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \ . \tag{B.2}$$

It follows from (B.1) and (B.2) by axiom sum and propositional reasoning that

$$\mathsf{J5} \vdash r_1(k) : \Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \;\rightarrow\; t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}).$$

for $t := t_1(r_1(k)) + t_2(?\, x_m)$. In other words,

$$\mathsf{J5} \vdash ([\Delta]_k, \Diamond_{2m} A)^{r_1} \rightarrow ([\Delta, \Diamond_{2m} A]_i)^r$$

for $r := (r_1 \restriction \Delta, \Diamond_{2m} A) \cup \{(i,t)\}$.

Subcase $\rho = \mathsf{5c}$**.** The existence of terms $t_1(x_m)$, $t_2(x_m)$, and s that satisfy (11) follows as in the subcase of $\rho = \mathsf{5b}$. Thus, by propositional reasoning,

$$\mathsf{J5} \vdash \Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \;\rightarrow\; \Delta^{r_1} \vee (t_1(x_m) \cdot ?\, t_2(x_m)) : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1} \;.$$

By the Internalization Lemma there exists a term $s_1(x)$ such that

$$\mathsf{J5} \vdash r_1(k) : \big(\Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1})\big) \rightarrow s_1(r_1(k)) : \big(\Delta^{r_1} \vee q_1 : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}\big) \;, \tag{B.3}$$

where $q_1 := t_1(x_m) \cdot ?\, t_2(x_m)$ in the above formula. By Lemma 5.6 there exists a term $t(x)$ such that

$$\mathsf{J5} \vdash t(q_1) : \big(q_1 : \neg x_m : \neg A^{r_1} \rightarrow \neg x_m : \neg A^{r_1}\big) \;. \tag{B.4}$$

By a propositional tautology and the Internalization Lemma applied to it,

$$\mathsf{J5} \vdash p_2 : \Big(\big(q_1 : \neg x_m : \neg A^{r_1} \rightarrow \neg x_m : \neg A^{r_1}\big) \rightarrow \Delta^{r_1} \vee q_1 : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1} \;\rightarrow\; \Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}\Big)$$

for some ground term p_2. From this and (B.4) by app and MP it follows that

$$\mathsf{J5} \vdash (p_2 \cdot t(q_1)) : \big(\Delta^{r_1} \vee q_1 : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1} \;\rightarrow\; \Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}\big) \;.$$

It follows by app and MP that

$$\mathsf{J5} \vdash s_1(r_1(k)) : (\Delta^{r_1} \vee q_1 : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}) \rightarrow q_3 : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1})$$

for $q_3 := p_2 \cdot t(q_1) \cdot s_1(r_1(k))$. By propositional reasoning with (B.3) it follows that

$$\mathsf{J5} \vdash r_1(k) : \big(\Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1})\big) \;\rightarrow\; q_3 : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}) \;.$$

Hence the desired realization function is $r := (r_1 \restriction \Delta, \Diamond_{2m} A, \Pi) \cup \{(j,s),(l,q_3)\}$. □