Combinatorial Proofs for Constructive Modal Logic

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Abstract

Combinatorial proofs form a syntax-independent presentation of proofs, originally proposed by Hughes for classical propositional logic. In this paper we present a notion of combinatorial proofs for the constructive modal logics CK and CD, we show soundness and completeness of combinatorial proofs by translation from and to sequent calculus proofs, and we discuss the notion of proof equivalence enforced by these translations.

Keywords: combinatorial proofs, proof equivalence, arena nets, constructive modal logic.

1 Introduction

Combinatorial proofs have first been introduced by Hughes in order to give a "syntax-free" presentation of proof in classical propositional logic [19]. Their motivation is to capture the essence of a proof independently from any deductive proof system, such that we can speak about *proof equivalence* for proofs given in different formalisms [4]. Only recently it was possible to extend this idea to richer logics: (classical) modal logics [6], relevant logics [5,8], first order logic [20,21], and intuitionistic propositional logic [17]. In this paper we investigate combinatorial proofs for intuitionistic logic with modalities.

There are many different flavours of "intuitionistic modal logics" (see, e.g., [14,30,29,31,9,12]), depending on which additional variants of the classical k-axiom $\Box(A \supset B) \supset (\Box A \supset \Box B)$ are added. It is necessary to add more than just k, as k does not speak of the diamond modality $\Diamond$, which is in the intuitionistic case no longer the De Morgan dual of the box modality $\Box$.

We take here the minimal approach and only add $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ in addition to the k-axiom, leading to what is now called *constructive modal logics* in the literature [30,9,18,28,13,23]. We chose this setting because (1) we would like to make as few assumptions as possible, (2) these logics have a sequent calculus presentation, which makes it easier to show soundness and

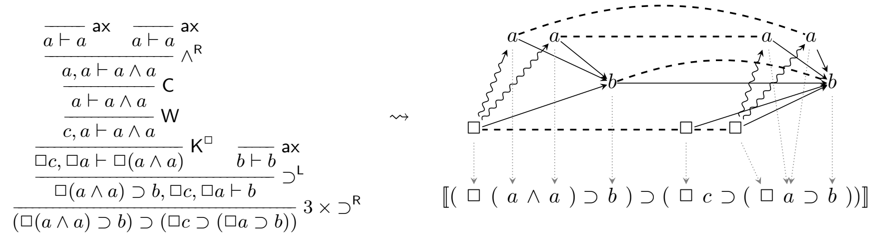


Fig. 1. A sequent calculus derivation of the formula $(\Box(a \wedge a) \supset b, \Box c, \Box a) \supset b$ and its corresponding combinatorial proof

completeness of combinatorial proofs, and (3) there is a close relation to game semantics for modalities [3], extending the work in [17], and lambda-calculus proof terms for constructive modal logics [9].

The main contribution of this paper is the definition of combinatorial proofs for the constructive modal logics CK and CD and to prove their soundness and completeness. We also show that they form a proof system in the sense of Cook and Reckhow [10], that is, checking the correctness of a combinatorial proof can be done in polynomial time in its size.

A combinatorial proof of a formula F is a certain kind of homomorphism $f \colon \mathcal{G} \to [\![F]\!]$ between two directed graphs. The directed graph $[\![F]\!]$ is a *modal arena* and encodes the formula F. Modal arenas are an extension of the arenas of [17], and are introduced in Section 4 of this paper. The directed graph $\mathcal{G}$ is a *modal arena net* and encodes the "linear part" of the proof. Modal arena nets (introduced in Section 5) are modal arenas equipped with a partition on their vertices, carrying the information of axiom linkings in proof nets. Finally, the homomorphism f is a *skew fibration* [19,17] and encodes the "resource management part" of the proof, i.e., it collects the information carried by the rule instances of contraction and weakening in the sequent calculus. We discuss skew fibrations in Section 6. Figure 1 shows an example of a combinatorial proof for the formula $F = (\Box(a \wedge a) \supset b) \supset (\Box c \supset (\Box a \supset b))$, where the solid and squiggly arrows are the edges of the arena $\mathcal{G}$ and the dashed edges represent the partition of $\mathcal{G}$ (encoding the ax and $\mathsf{K}^{\Box}$ rules). The dotted downwards directed arrows represent the skew fibration f (encoding the W and C rules).

In order to establish a close correspondence between combinatorial proofs and syntactic proofs in a deductive system, we need to have a *decomposition theorem* which allows to factorize proofs into a linear part, capturing the logic interactions between the components of the proof, and a resource management part, capturing resources duplication or erasing.

The second contribution of this paper is such a decomposition theorem for the logics CK and CD. To obtain this result, we use a combination of the sequent calculus and deep inference. More precisely, we use the cut-free sequent systems given in [24], and we add *deep* rules for contraction and weakening, in a similar way as it has been done in [6].

However, in an intuitionistic setting, we have to distinguish between the left-hand side and the right-hand side of a sequent, where contraction and

$$\frac{}{a \vdash a}\,\mathsf{ax} \qquad \frac{}{\vdash \top}\,\top \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset^{\mathsf{R}} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset^{\mathsf{L}} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge^{\mathsf{R}} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge^{\mathsf{L}}$$

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K}^{\Box} \qquad \frac{B, \Gamma \vdash A}{\Diamond B, \Box\Gamma \vdash \Diamond A}\,\mathsf{K}^{\Diamond} \qquad \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Diamond A}\,\mathsf{D}$$

$$\begin{array}{ll} \mathsf{IMLL} & = \{\mathsf{ax}, \top, \supset^{\mathsf{R}}, \supset^{\mathsf{L}}, \wedge^{\mathsf{L}}, \wedge^{\mathsf{R}}\} \\ \mathsf{LI} & = \mathsf{IMLL} \cup \{\mathsf{C}, \mathsf{W}\} \\ \mathsf{LCK} & = \mathsf{LI} \cup \{\mathsf{K}^{\Box}, \mathsf{K}^{\Diamond}\} \\ \mathsf{LCD} & = \mathsf{LCK} \cup \{\mathsf{D}\} \end{array}$$

Fig. 2. Sequent rules and sequent systems

weakening apply only to one side. Consequently, the *deep* versions of these rules need to have access to the information on which side a subformula will eventually occur. For this reason, we use polarities, in a similar way as done in [25,32]. The polarized system and the decomposition theorem are given in Section 3.

Finally, we discuss in Section 8 the proof equivalence (in terms of sequent calculus rule permutations) that is induced by our combinatorial proofs and compare it to the one induced by λ-terms [7] and the one induced by winning innocent strategies [3].

2 Preliminaries on Constructive Modal Logics

We consider the *(modal) formulas* generated by a countable set of (atomic) propositional variables $\mathcal{A} = \{a, b, \dots\}$ and the following grammar

$$A, B ::= a \mid \top \mid A \supset B \mid A \wedge B \mid \Box A \mid \Diamond A$$

We say that a formula is *modality-free* if it contains no occurrences of $\Box$ and $\Diamond$. A formula is a $\supset$-formula (resp. a $\wedge$-formula, $\Box$-formula, or $\Diamond$-formula) if it is a formula of the form $A \supset B$ (resp. $A \wedge B$, $\Box A$, or $\Diamond A$).

The constructive modal logic CK is obtained by extending the propositional intuitionistic logic with the *necessitation rule*: if F is provable then so is $\Box F$, and the two modal axiom schemes k_1 and k_2 shown below on the left:

$$\mathsf{k}_1 \colon \Box(A \supset B) \supset (\Box A \supset \Box B) \quad \mathsf{k}_2 \colon \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \quad \Big| \quad \mathsf{d} \colon \Box A \supset \Diamond A$$

The logic CD is obtained from CK by adding the axiom scheme d on the right above.

We are now recalling the sequent system for these logics. For this, we denote by capital Greek letters Γ or Δ a multiset of formulas, separated by comma. We write $\Box\Gamma$ (resp. $\Diamond\Gamma$) for any such multiset made only of $\Box$-formulas (resp. $\Diamond$-formulas). A *sequent* $\Gamma \vdash A$ is a pair of a multiset of formulas and a formula.

In Figure 2, we show the sequent system LI (e.g., given in [33]) for disjunction-free intuitionistic logic, its linear fragment IMLL [1], and the sequent systems LCK and LCD for the logics CK and CD, respectively, as presented in [24]. The cut rule

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\mathsf{cut}$$

is admissible for all four systems, and we have the following:

[1] Intuitionistic multiplicative linear logic (see e.g. [26])

Theorem 2.1 *Let* X *for* $\mathsf{X} \in \{\mathsf{CK}, \mathsf{CD}\}$*. The sequent system* LX *is a sound and complete proof system for the disjunction-free fragment of the logic* X*.*

Proof. Soundness follows from the observation that our systems are the disjunction-free versions of the calculi in [24]. Completeness follows from the cut elimination property. $\square$

If X is a set of rules, we write $F' \overset{\mathsf{X}}{\longmapsto} F$ if there is a derivation from $\vdash F'$ to $\vdash F$ using rules in X. Moreover, we write $\overset{\mathsf{X}}{\longmapsto} F$ if there is a proof of F in X, i.e., a derivation with empty premises of $\vdash F$ using rules in X.

Finally, we define the *formula isomorphism* as the equivalence relation $\overset{\mathsf{f}}{\sim}$ over formulas generated by the following relations:

$$\begin{array}{ccc} A \wedge \top \overset{\mathsf{f}}{\sim} A & A \supset \top \overset{\mathsf{f}}{\sim} \top & \top \supset A \overset{\mathsf{f}}{\sim} A \\ A \wedge B \overset{\mathsf{f}}{\sim} B \wedge A & A \wedge (B \wedge C) \overset{\mathsf{f}}{\sim} (A \wedge B) \wedge C & (A \wedge B) \supset C \overset{\mathsf{f}}{\sim} A \supset (B \supset C) \end{array} \tag{1}$$

3 Polarized System and Decomposition

We define the set of *polarized formulas* (or P*-formulas*) as the set generated by $\mathcal{A} = \{a, b, \dots\}$ using the following grammar

$$\begin{array}{l} A^\circ, B^\circ ::= a^\circ \mid \top^\circ \mid A^\circ \wedge B^\circ \mid A^\bullet \supset B^\circ \mid \Box A^\circ \mid \Diamond A^\circ \\ A^\bullet, B^\bullet ::= a^\bullet \mid \top^\bullet \mid A^\bullet \wedge B^\bullet \mid A^\circ \supset B^\bullet \mid \Box A^\bullet \mid \Diamond A^\bullet \end{array} \tag{2}$$

We say that formulas $A^\circ, B^\circ, \dots$ are of *even* polarity and formulas $A^\bullet, B^\bullet, \dots$ are of *odd* polarity. Note that the polarity of a formula determines the polarity of each subformula of that formula. For this reason we will omit the polarity markings for subformulas. A polarized formula is *clean* if it contains no subformulas of the shape $A \supset \top^\bullet$.

A *polarized sequent* is a sequent of P-formulas. We write $\Gamma^\bullet$ for a sequent containing only formulas of odd polarity. Then $\Gamma^\bullet, A^\circ$ is simply the polarized version of a sequent $\Gamma \vdash A$. A *context* is a (polarized) sequent $\Gamma\{\ \}$ in which an atom (or more generally a subformula) has been replaced by an hole $\{\ \}$. Then $\Gamma\{\Delta\}$ stands for the sequent obtained from $\Gamma\{\ \}$ by replacing $\{\ \}$ with Δ.

We can now define the polarized sequent rules given in Figure 3. Observe that the upper part and the two rules $\mathsf{c}^\bullet$ and $\mathsf{w}^\bullet$ on the lower left are just the polarized version of the rules in Figure 2.[2] The two rules $\mathsf{c}^\bullet_\downarrow$ and $\mathsf{w}^\bullet_\downarrow$ on the lower right are the *deep* version of $\mathsf{c}^\bullet$ and $\mathsf{w}^\bullet$, respectively. They can be applied deep inside any formula context. Note that they can only be applied to formulas of odd polarity. Figure 4 lists the various proof systems that are defined with these rules, and that we are using in this paper.

We can define the function $\lfloor \cdot \rfloor$ on polarized formulas that forgets the polarities. This function can be extended to polarized sequents with exactly one

[2] Note that the $\top^\bullet$-rule is just a special case of the $\mathsf{w}^\bullet$-rule. It is introduced here to simplify the presentation of some of the results in this paper.

$$\frac{}{a^\bullet, a^\circ}\,\mathsf{ax} \quad \frac{\Gamma^\bullet, A^\bullet, B^\circ}{\Gamma^\bullet, (A \supset B)^\circ}\,\supset^\circ \quad \frac{\Gamma^\bullet, A^\circ \quad \Delta^\bullet, B^\bullet, C^\circ}{\Gamma^\bullet, \Delta^\bullet, (A \supset B)^\bullet, C^\circ}\,\supset^\bullet \quad \frac{\Gamma^\bullet, A^\circ \quad \Delta^\bullet, B^\circ}{\Gamma^\bullet, \Delta^\bullet, (A \wedge B)^\circ}\,\wedge^\circ \quad \frac{\Gamma^\bullet, A^\bullet, B^\bullet, C^\circ}{\Gamma^\bullet, (A \wedge B)^\bullet, C^\circ}\,\wedge^\bullet$$

$$\frac{}{\top^\circ}\,\top^\circ \quad \frac{\Gamma^\bullet, A^\circ}{\top^\bullet, \Gamma^\bullet, A^\circ}\,\top^\bullet \quad \frac{\Gamma^\bullet, A^\circ}{\Box\Gamma^\bullet, \Box A^\circ}\,\mathsf{k}^\Box \quad \frac{A^\bullet, \Gamma^\bullet, B^\circ}{\Diamond A^\bullet, \Box\Gamma^\bullet, \Diamond B^\circ}\,\mathsf{k}^\Diamond \quad \frac{\Gamma^\bullet, A^\circ}{\Box\Gamma^\bullet, \Diamond A^\circ}\,\mathsf{d}$$

$$\frac{\Gamma^\bullet, A^\bullet, A^\bullet, B^\circ}{\Gamma^\bullet, A^\bullet, B^\circ}\,\mathsf{c}^\bullet \quad \frac{\Gamma^\bullet, A^\circ}{\Gamma^\bullet, B^\bullet, A^\circ}\,\mathsf{w}^\bullet \quad \Bigg| \quad \frac{\Gamma\{(A \wedge A)^\bullet\}}{\Gamma\{A^\bullet\}}\,\mathsf{c}^\bullet_\downarrow \quad \frac{\Gamma\{\top^\bullet\}}{\Gamma\{A^\bullet\}}\,\mathsf{w}^\bullet_\downarrow \text{ (for a } A \neq \top)$$

Fig. 3. Sequent rules and deep inference rules on P-formulas

$$\begin{array}{rlrl} \mathsf{LI}^\bullet & = \mathsf{LI}^\bullet_\ell \cup \{\mathsf{c}^\bullet, \mathsf{w}^\bullet\} & \mathsf{LI}^\bullet_\ell & = \{\mathsf{ax}, \top^\circ, \top^\bullet, \supset^\circ, \supset^\bullet, \wedge^\bullet, \wedge^\circ\} \\ \mathsf{LCK}^\bullet & = \mathsf{LI}^\bullet \cup \{\mathsf{k}^\Box, \mathsf{k}^\Diamond\} & \mathsf{LCK}^\bullet_\ell & = \mathsf{LI}^\bullet_\ell \cup \{\mathsf{k}^\Box, \mathsf{k}^\Diamond\} \\ \mathsf{LCD}^\bullet & = \mathsf{LI}^\bullet \cup \{\mathsf{k}^\Box, \mathsf{k}^\Diamond, \mathsf{d}\} & \mathsf{LCD}^\bullet_\ell & = \mathsf{LI}_\ell \cup \{\mathsf{k}^\Box, \mathsf{k}^\Diamond, \mathsf{d}\} \end{array}$$

Fig. 4. Rules systems for P-formulas

even formula via $\lfloor B_1^\bullet, \dots, B_n^\bullet, A^\circ \rfloor = \lfloor B_1 \rfloor, \dots, \lfloor B_n \rfloor \vdash \lfloor A \rfloor$. Since all systems defined in Figure 4 can only prove such sequents, we have an immediate one-to-one correspondence between derivations in the polarized and the corresponding unpolarized systems. However, the motivation for introducing the polarized systems is the following result.

Theorem 3.1 (Decomposition) *Let* $\mathsf{X} \in \{\mathsf{CK}, \mathsf{CD}\}$ *and* H *be a* P*-formula. The following are equivalent:* (i) $\overset{\mathsf{LX}}{\vdash} \lfloor H \rfloor$*;* (ii) $\overset{\mathsf{LX}^\bullet}{\vdash} H$*;* (iii) *there is a clean* P*-formula* H' *such that* $\overset{\mathsf{LX}^\bullet_\ell}{\vdash} H' \overset{\{\mathsf{c}^\bullet_\downarrow, \mathsf{w}^\bullet_\downarrow\}}{\vdash} H$*.*

Proof. (i) $\iff$ (ii) follows from the paragraph above. And (ii) $\iff$ (iii) can be obtained by a simple rule permutation argument and observing that every instance of $\mathsf{c}^\bullet$ can be decomposed into a $\wedge^\bullet$ followed by a $\mathsf{c}^\bullet_\downarrow$, and every instance of $\mathsf{w}^\bullet$ is a $\top^\bullet$ followed by a $\mathsf{w}^\bullet_\downarrow$. If a non-clean formula is introduced by a $\supset^\bullet$, we perform the following transformation.

$$\frac{\overset{\bigtriangledown \mathsf{LX}^\bullet}{\Gamma^\bullet, B^\circ} \qquad \dfrac{\overset{\vdots}{\Delta^\bullet, A^\circ}}{\top^\bullet, \Delta^\bullet, A^\circ}\,\top^\bullet}{\Gamma^\bullet, B \supset \top^\bullet, \Delta \vdash A}\,\supset^\bullet \quad \rightsquigarrow \quad \begin{array}{c} \vdots \\ \Delta^\bullet, A^\circ \\ \hline\hline \top^\bullet, \dots, \top^\bullet, \Delta^\bullet, A^\circ \\ \hline\hline \Gamma^\bullet, B \supset \top^\bullet, \Delta^\bullet, A^\circ \end{array} \begin{array}{l} \top^\bullet \times (|\Gamma| + 1) \\ \mathsf{w}^\bullet \times (|\Gamma| + 1) \end{array} \tag{3}$$

We conclude by permuting the rules $\mathsf{c}^\bullet_\downarrow$ and $\mathsf{w}^\bullet_\downarrow$ below all other rules, while applying the transformation above whenever a non-clean formula is introduced. □

4 Modal Arenas

A *directed graph* $\mathcal{G} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\to} \rangle$ is given by a set $V_\mathcal{G}$ of *vertices* and a set $\overset{\mathcal{G}}{\to} \subseteq V_\mathcal{G} \times V_\mathcal{G}$ of direct *edges*. A vertex v is a $\overset{\mathcal{G}}{\to}$*-root*, denoted $v \not\to$, if there is no vertex w such that $v \overset{\mathcal{G}}{\to} w$. We denote by $\vec{R}_\mathcal{G}$ the set of $\overset{\mathcal{G}}{\to}$-roots of $\mathcal{G}$. A *path* from v to w of length n is a sequence of vertices $x_0 \dots x_n$ such that $v = x_0$ and $w = x_n$ and $x_i \overset{\mathcal{G}}{\to} x_{i+1}$ for $i \in \{0, \dots, n-1\}$. We write $v \overset{\mathcal{G}}{\to}{}^n w$ if there is a path from v to w of length n. A *directed acyclic graph* (or dag for short) is a directed graph such that $v \overset{\mathcal{G}}{\to}{}^n v$ implies $n = 0$ for all $v \in V$. A *two-color directed acyclic graph* (or 2-dag for short) $\mathcal{G} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\to}, \overset{\mathcal{G}}{\rightsquigarrow} \rangle$ is given by a set of vertices $V_\mathcal{G}$ and two disjoint sets of edges $\overset{\mathcal{G}}{\to}$ and $\overset{\mathcal{G}}{\rightsquigarrow}$ such that the graph $\langle V_\mathcal{G}, \overset{\mathcal{G}}{\to} \cup \overset{\mathcal{G}}{\rightsquigarrow} \rangle$ is acyclic. We omit the superscript when clear from context and we denote by $\emptyset$ the empty 2-dag. We write $u \leftrightsquigarrow v$ if $u \rightsquigarrow v$ or $v \rightsquigarrow u$.

If $\mathcal{L}$ is a set, a 2-dag is $\mathcal{L}$*-labeled* if a *label* $\ell(v) \in \mathcal{L}$ is associated to each vertex $v \in V$. In this paper we fix the set of labels to be the set $\mathcal{L} = \mathcal{A} \cup \{\Box, \Diamond\}$, where $\mathcal{A}$ is the set of propositional variables occurring in formulas. We use the notation a, $\Box$ and $\Diamond$ to denote the graphs consisting of a single vertex labeled by a, $\Box$ and $\Diamond$, respectively.

Definition 4.1 Let $\mathcal{G}$, $\mathcal{H}$, $\mathcal{F}$ be 2-dags with $\mathcal{F} \neq \emptyset$. We write $R^\mathcal{G}_\mathcal{F}$ for the set of edges from the $\to$-roots of $\mathcal{G}$ to the $\to$-roots of $\mathcal{F}$, that is $R^\mathcal{G}_\mathcal{F} = \{(u,v) \mid u \in \vec{R}_\mathcal{G}, v \in \vec{R}_\mathcal{F}\}$. We define the following operations on 2-dags:

$$\begin{array}{ll}
\mathcal{G} + \mathcal{H} = \langle\, V_\mathcal{G} \cup V_\mathcal{H} \,,\, \overset{\mathcal{G}}{\to} \cup \overset{\mathcal{H}}{\to} \,,\, \overset{\mathcal{G}}{\rightsquigarrow} \cup \overset{\mathcal{H}}{\rightsquigarrow} \,\rangle & \\
\mathcal{G} \multimap\!\!\triangleright \mathcal{F} = \langle\, V_\mathcal{G} \cup V_\mathcal{F} \,,\, \overset{\mathcal{G}}{\to} \cup \overset{\mathcal{F}}{\to} \cup R^\mathcal{G}_\mathcal{F} \,,\, \overset{\mathcal{G}}{\rightsquigarrow} \cup \overset{\mathcal{F}}{\rightsquigarrow} \,\rangle & \text{and} \quad \mathcal{G} \multimap\!\!\triangleright \emptyset = \emptyset \\
\mathcal{G} \rightsquigarrow\!\!\triangleright \mathcal{H} = \langle\, V_\mathcal{G} \cup V_\mathcal{H} \,,\, \overset{\mathcal{G}}{\to} \cup \overset{\mathcal{H}}{\to} \,,\, \overset{\mathcal{G}}{\rightsquigarrow} \cup \overset{\mathcal{H}}{\rightsquigarrow} \cup R^\mathcal{G}_\mathcal{H} \,\rangle &
\end{array}$$

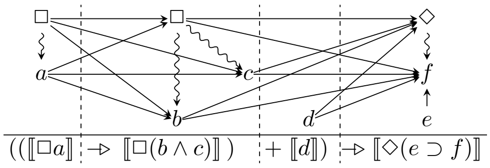
We associate to each formula F a $\mathcal{L}$-labeled 2-dag $[\![F]\!]$ as follows:

$$\begin{array}{lllllllll}
[\![a]\!] & = & a & [\![A \supset B]\!] & = & [\![A]\!] \multimap\!\!\triangleright [\![B]\!] & [\![\Box A]\!] & = & \Box \rightsquigarrow\!\!\triangleright [\![A]\!] \\
[\![\top]\!] & = & \emptyset & [\![A \wedge B]\!] & = & [\![A]\!] + [\![B]\!] & [\![\Diamond A]\!] & = & \Diamond \rightsquigarrow\!\!\triangleright [\![A]\!]
\end{array} \tag{4}$$

For a sequent $B_1, \ldots, B_n \vdash A$, we define $[\![B_1, \ldots, B_n \vdash A]\!]$ as $[\![(B_1 \wedge \cdots \wedge B_n) \supset A]\!]$.

Example 4.2 Consider the sequent $\Gamma \vdash A = \Box a \supset \Box(b \wedge c), d \vdash \Diamond(e \supset f)$. We have

$[\![\Box a \supset \Box(b \wedge c), d \vdash \Diamond(e \supset f)]\!] =$



In the following, we give a characterization of those 2-dags that are encodings of formulas.

Definition 4.3 A $\mathcal{L}$-labeled dag $\mathcal{G} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\to} \rangle$ is an *arena* if $\mathcal{G}$ is

- L-free: if $a \to u$ and $a \to w \to v$ then $u \to v$;
- Σ-free: if $a \to v$, $a \to w$, $b \to w$ and $b \to u$ then $a \to u$ or $b \to v$;

A *modal arena* $\mathcal{G} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\to}, \overset{\mathcal{G}}{\rightsquigarrow} \rangle$ is an $\mathcal{L}$-labeled 2-dag such that

- $\langle V_\mathcal{G}, \overset{\mathcal{G}}{\to} \rangle$ is an arena;
- $\mathcal{G}$ is *properly labeled*: if $v \overset{\mathcal{G}}{\rightsquigarrow} w$, then $\ell(v) \in \{\Box, \Diamond\}$;
- $\rightsquigarrow$ is *modal*, that is:

MA.1 if $v \rightsquigarrow w$ and $w \rightsquigarrow u$, then $v \rightsquigarrow u$; **MA**.4 if $v \rightsquigarrow w$ and $u \to v$, then $u \to w$;
MA.2 if $v \rightsquigarrow w$ and $u \rightsquigarrow w$, then $u \leftrightsquigarrow v$; **MA**.5 if $v \rightsquigarrow w$ and $v \to u$, then $w \to u$;
MA.3 if $v \rightsquigarrow w$ and $v \rightsquigarrow u$, then $u \not\to w$; **MA**.6 if $v \rightsquigarrow w$ and $w \to u$, then $v \to u$.

We write $V_{\mathcal{G}}^{\mathcal{A}}$ (resp. $V_{\mathcal{G}}^{\Box}$, $V_{\mathcal{G}}^{\Diamond}$) for the subsets of vertices of $\mathcal{G}$ with labels in $\mathcal{A}$ (resp. in $\{\Box\}$, in $\{\Diamond\}$). The vertices in $V_{\mathcal{G}}^{\mathcal{A}}$ are called *atomic*, and the vertices in $V_{\mathcal{G}}^{\Box\Diamond} = V_{\mathcal{G}}^{\Box} \cup V_{\mathcal{G}}^{\Diamond}$ are called *modal*.

The relation between arenas and modality-free clean formulas has been established in [17].

Lemma 4.4 ([17]) *In an arena, if $u \to^n w$ and $v \to^m w$, then*

$$\textit{either} \quad \{x \mid v \to^n x\} \subseteq \{x \mid w \to^m x\} \qquad \textit{or} \quad \{x \mid w \to^m x\} \subseteq \{x \mid v \to^n x\} \quad .$$

Theorem 4.5 ([17]) *An $\mathcal{L}$-labeled 2-dag $\mathcal{G}$ is an arena iff there is a modality-free clean formula F such that $\mathcal{G} = [\![F]\!]$.*

We now extend this result to modal formulas.

Lemma 4.6 *Let $\mathcal{G}$ be a modal arena, and let $u, v, w \in V_{\mathcal{G}}$. If $v \rightsquigarrow w$ then:*

(i) *v is a $\to$-root iff w is a $\to$-root;*

(ii) *$v \to^n u$ iff $w \to^n u$;*

(iii) *if $u \to^n v$ then $u \to^n w$.*

Proof. The first statement follows from the fact that in a modal arena, if $v \rightsquigarrow w$, then $v \to u$ iff $w \to u$. The second statement is proven using the same argument, proceeding by induction on n making use of Lemma 4.4. The third statement is also proven using Lemma 4.4 and the fact that in a modal arena if $v \rightsquigarrow w$ and $u \to v$, then $u \to w$. $\Box$

Lemma 4.7 *If F is a formula, then the $\mathcal{L}$-labeled 2-dag $[\![F]\!]$ is a modal arena.*

Proof. By induction over the number of connectives and modalities of a formula. It suffices to remark that the graph operations $+$ and $\multimap\!\!\triangleright$ cannot introduce forbidden modal arena configurations. Similarly, the operation $\rightsquigarrow\!\!\triangleright$ introduces no forbidden configurations whenever $\mathcal{G} = \mathcal{G}_1 \rightsquigarrow\!\!\triangleright \mathcal{G}_2$ with $\mathcal{G}_1$ a single vertex graph of the form $\Box$ or $\Diamond$. $\Box$

In order to prove the converse, we need the following definitions.

Definition 4.8 Let v be a modal vertex of a modal arena $\mathcal{G}$. The *scope* of v is the set

$$\mathsf{Scope}(v) = \{\ w \in V_{\mathcal{G}} \mid \text{there is a } u \in V_{\mathcal{G}} \text{ s.t. } v \rightsquigarrow u \text{ and } w \to^* u \text{ and } w \not\to^* v\ \}$$

Intuitively, the scope of a modal vertex v in $[\![F]\!]$ is the set of vertices corresponding to modalities and atoms in the scope of the corresponding modality in F. To give an example, consider the arena in Example 4.2. There, e is in the scope of the $\Diamond$ while d is not. In fact, despite the existence of f such that $\Diamond \rightsquigarrow f$ and $d \to f$ and $e \to f$, we have $e \in \mathsf{Scope}(\Diamond)$ since $e \to f$ and $e \not\to f$, while $d \notin \mathsf{Scope}(\Diamond)$ since $d \to f$ and $d \to \Diamond$.

Theorem 4.9 *An $\mathcal{L}$-labeled* 2-dag $\mathcal{G}$ *is a modal arena iff* $\mathcal{G} = [\![F]\!]$ *for some formula* F.

Proof. The "if" direction has been shown in Lemma 4.7. For the "only if" direction, we proceed by induction on the size of $\mathcal{G}$. If $\mathcal{G} = \emptyset$ then $F = \top$. If $|V_{\mathcal{G}}| = 1$ then if $\ell(v) \in \mathcal{A}$, then $F = a \in \mathcal{A}$, if $\ell(v) = \Diamond$ or $\ell(v) = \Box$ then $F = \Diamond\top$ or $F = \Box\top$, respectively. Otherwise, since $\langle V_{\mathcal{G}}, \stackrel{\mathcal{G}}{\rightarrow}\rangle$ is a arena, we conclude by Lemma 4.4 (see [17]) that

(i) either every vertex in $V_{\mathcal{G}} \setminus \vec{R}_{\mathcal{G}}$ has a $\rightarrow$-paths to all roots in $\vec{R}_{\mathcal{G}}$,

(ii) or $\vec{R}_{\mathcal{G}}$ admits a partition $\vec{R}_{\mathcal{G}} = R_1 \uplus R_2$ such that any vertex in $\mathcal{G}$ has $\rightarrow$-paths only to roots in one of the two sets.

If (i) holds, then we define $\mathcal{G}_2$ as the modal arena obtained from $\mathcal{G}$ taking the vertices in $V_2 = \vec{R}_{\mathcal{G}} \cup \left(\bigcup_{v\in\vec{R}_{\mathcal{G}}} \mathsf{Scope}(v)\right)$ and $\mathcal{G}_1$ as the modal arena over the remaining vertices $V_1 = V_{\mathcal{G}} \setminus V_2$. Since each vertex in $\mathcal{G}$ has a path to all the roots in $\vec{R}_{\mathcal{G}}$, then there is a $\rightarrow$ from any root of $\mathcal{G}_1$ to any root of $\mathcal{G}_2$. Since by definition $\vec{R}_{\mathcal{G}_2} = \vec{R}_{\mathcal{G}}$, then we have that $\mathcal{G} = \mathcal{G}_1 \rightarrow\!\triangleright \mathcal{G}_2$.

If (ii) holds and $\vec{R}_{\mathcal{G}} = R_1 \uplus R_2$ with R_1 and R_2 non-empty sets. Since $\rightsquigarrow$ is modal, we have the following possibilities:

(a) if $R_1 = \{v\}$ and $v \rightsquigarrow w$ for all $w \in R_2$, then there is no u such that $u \rightarrow v$. Otherwise $u \rightarrow v$ and $u \rightarrow w$ for all w such that $v \rightsquigarrow w$, that is for all $w \in R_2$. This implies that $u \rightsquigarrow w$ for all $w \in \vec{R}_{\mathcal{G}}$, which contradicts (ii). Thus we conclude that $\mathcal{G} = v \rightsquigarrow\!\triangleright \mathcal{G}'$ where $\mathcal{G}'$ is the modal arena with vertices $\mathsf{Scope}(v)$;

(b) if there are no $\rightsquigarrow$-edges between R_1 and R_2, then $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the modal arenas with vertices $V_1 = \{v \mid v \rightarrow^* w \text{ for a } w \in R_1\}$ and $V_2 = \{v \mid v \rightarrow^* w \text{ for a } w \in R_2\}$. In fact by definition there are no $\rightarrow$-edges between vertices in V_1 and V_2 otherwise by Lemma 4.4 we should have $R_1 = R_2$. Similarly there are no $\rightsquigarrow$-edges between vertices in V_1 and V_2 since there are no $\rightsquigarrow$-edges between R_1 and R_2 (by hypothesis) and if there is $v \in V_1 \setminus R_1$ and $w \in V_2$ such that $v \rightsquigarrow w$, then by Lemma 4.6 $w \notin R_2$ and we should have again $R_1 = R_2$;

(c) otherwise, we pick a $v \in \vec{R}_{\mathcal{G}} \cap \overset{\rightsquigarrow}{R}_{\mathcal{G}}$ and define $R_1 = \{v\} \cup \{w \mid v \rightsquigarrow w\}$ and $R_2 = \vec{R}_{\mathcal{G}} \setminus R_1$. If there is no $u \in \vec{R}_{\mathcal{G}}$ such that $v \not\rightsquigarrow u$, then $R_1 = \vec{R}_{\mathcal{G}}$ and we conclude by (a). If $R_2 \neq \emptyset$, then we define $V_1 = \{v \mid v \rightarrow^* w \text{ for a } w \in R_1\}$ and $V_2 = \{v \mid v \rightarrow^* w \text{ for a } w \in R_2\}$ and we conclude by (b). □

In light of this theorem, we may say that a vertex in $[\![F]\!]$ *corresponds* to an occurrence of an atom or a modality in the formula F.

We conclude this section by remarking that modal arenas identify formulas modulo the formula isomorphism $\stackrel{\mathsf{f}}{\sim}$ defined by the relations in Equation (1).

Proposition 4.10 *For any formulas* F *and* G *we have* $F \stackrel{\mathsf{f}}{\sim} G$ *iff* $[\![F]\!] = [\![G]\!]$.

Proof. This follows from the definition of the modal arena operations $+$, $\rightarrow\!\triangleright$ and $\rightsquigarrow\!\triangleright$. □

5 Modal Arena Nets

We introduce the notion of CK- and CD-arena nets, which are modal arenas equipped with an equivalence relation over vertices, satisfying certain conditions capturing the idea of "axiom links" in proof nets. We then show the correspondence between these modal arena nets and the linear proofs in $\mathsf{LCK}^{\mathsf{e}}_{\ell}$ and $\mathsf{LCD}^{\mathsf{e}}_{\ell}$, respectively.

Definition 5.1 A *partitioned modal arena* $\mathcal{G} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\to}, \overset{\mathcal{G}}{\rightsquigarrow}, \overset{\mathcal{G}}{\sim} \rangle$ is given by a modal arena $\langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\to}, \overset{\mathcal{G}}{\rightsquigarrow} \rangle$ together with an equivalence relation $\overset{\mathcal{G}}{\sim}$ over vertices such that:

- if $v \in V^{\mathcal{A}}_{\mathcal{G}}$ and $v \overset{\mathcal{G}}{\sim} w$, then $w \in V^{\mathcal{A}}_{\mathcal{G}}$ and $\ell(v) = \ell(w)$;
- if $v \in V^{\mathcal{A}}_{\mathcal{G}}$, then $v \overset{\mathcal{G}}{\sim} w$ for a unique $w \in V^{\mathcal{A}}_{\mathcal{G}}$.

In a partitioned modal arena we represent the equivalence relation $\sim$ by drawing a (dashed non-oriented blue) edge v- -w between two distinct vertices in the same $\sim$-class. For better readability, we only represent a minimal subset of these edges relying on the fact that $\sim$ is an equivalence relation. By means of example, if $\{u, v, w\}$ is an $\sim$-class, we may only draw u- -v- -w omitting the edge between u and w.

We say that a formula (or P-formula) F is *associated* to $\mathcal{G} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\to}, \overset{\mathcal{G}}{\rightsquigarrow}, \overset{\mathcal{G}}{\sim} \rangle$ if $[\![F]\!] = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\to}, \overset{\mathcal{G}}{\rightsquigarrow} \rangle$, and we denote by $\emptyset$ the empty arena net.

Remark 5.2 If v and w are vertices in a partitioned modal arena $\mathcal{G}$ such that $v \overset{\mathcal{G}}{\sim} w$, then $v \in V^{\Box\Diamond}_{\mathcal{G}}$ iff $w \in V^{\Box\Diamond}_{\mathcal{G}}$. If an $\overset{\mathcal{G}}{\sim}$ equivalence class contains more than two vertices then they are all labelled by $\Diamond$ or $\Box$.

If $\mathcal{G}$ is a modal arena and $v \in V_{\mathcal{G}}$, we define the *depth* of v (denoted $d(v)$) to be the length of the $\to$-paths from v to a $\to$-root $w \in \overrightarrow{R}_{\mathcal{G}}$. This is well-defined as all such paths have the same length (see [17, Lemma 9]). The *parity of a vertex* v is the parity of $d(v)$, which can be either *even* or *odd*. We write v° or $v^{\bullet}$ if the parity of v is respectively even or odd. Note that if F° is a P-formula, then the parity of the vertices in $[\![F^{\circ}]\!]$ are the same as the polarity of the corresponding atoms (and modal subformulas) in F°.

The *parity* of an $\to$-edge $v \to w$ is the parity of $d(w)$. We say that an edge $v \to w$ is a *chord* if there is a vertex u such that either $v \to u$ and $u \rightsquigarrow w$; or $u \to w$ and $u \rightsquigarrow v$. By means of example, in the following modal arenas the edges $a \to b$ are chords.

We write by $\overset{\mathcal{G}}{\to}_{\bullet}$ the set of odd $\to$-edges in $\mathcal{G}$ that are not chords. In the following, we may depict $\to$-edges which are not $\to_{\bullet}$-edges using dotted edges.

If v is a vertex in a modal arena $\mathcal{G} = [\![F]\!]$, we denote by $\hat{v}$ either v itself if there is no w such that $v \in \mathsf{Scope}(w)$, or the vertex $w = \hat{v}$ such that $v \in \mathsf{Scope}(w)$ and $w \in \mathsf{Scope}(u)$ for all $u \neq w$ with $v \in \mathsf{Scope}(u)$. That is, if $v \neq \hat{v}$, then $\hat{v}$ is the first modal vertex we encounter on in the path from v to the

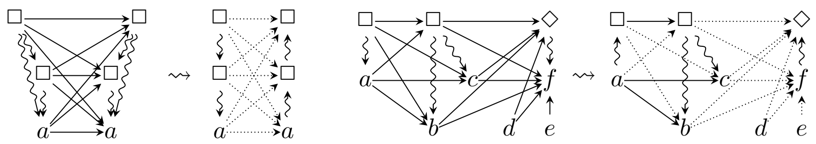


Fig. 5. The arenas of the sequents $\vdash \Box\Box a \supset \Box\Box a$ and $\Box a \supset \Box(b \wedge c), d \vdash \Diamond(e \supset f)$ and the corresponding graphs obtained by replacing the $\rightsquigarrow$-edges with $\rightsquigarrow_\partial$-edges.

root of the formula tree of F. By means of example, in Example 4.2 we have $\hat{f} = \hat{e} = \Diamond = \Diamond$ and $\hat{d} = d$.

We define the edge relation $\overset{\mathcal{G}}{\rightsquigarrow}_\partial$ as follows

$$v \overset{\mathcal{G}}{\rightsquigarrow}_\partial w \qquad \text{if either} \qquad w^\circ \text{ and } w = \hat{v} \neq v, \quad \text{or} \quad v^\bullet \text{ and } v = \hat{w} \neq w$$

Intuitively, the $\rightsquigarrow_\partial$-edges connect a modality to the root of the formula in its scope "one step at the time" (see Figure 5). Note that $v^\bullet \rightsquigarrow_\partial w^\bullet$ implies $v^\bullet \rightsquigarrow w^\bullet$, while $v^\circ \rightsquigarrow_\partial w^\circ$ implies $w^\circ \rightsquigarrow v^\circ$.

Definition 5.3 A partitioned modal arena $\mathcal{G}$ is *linked* if every $\overset{\mathcal{G}}{\sim}$-class is of the form $\{v_1^\bullet, \ldots, v_n^\bullet, w^\circ\}$. This induces the set directed edges $\overset{\mathcal{G}}{\rightharpoonup} = \{(v, w) \mid v^\bullet \overset{\mathcal{G}}{\sim} w^\circ\}$. The *linking graph* $\overset{\frown}{\mathcal{G}}$ of a modal arena is the directed graph with vertices $V_\mathcal{G}$ and edges $\overset{\mathcal{G}}{\to}_\bullet \cup \overset{\mathcal{G}}{\rightsquigarrow}_\partial \cup \overset{\mathcal{G}}{\rightharpoonup}$. We say that path in $\overset{\frown}{\mathcal{G}}$ is *checked* if it ends in a vertex in $\overrightarrow{R}_\mathcal{G} \cap \overset{\rightsquigarrow}{R}_\mathcal{G}$ and it contains no edge $v \rightharpoonup w$ with w a modal vertex with $\mathsf{Scope}(w) \neq \emptyset$.

A CK-*arena net* is a linked modal arena which satisfies Conditions (i)–(iv) below:

(i) $\overset{\frown}{\mathcal{G}}$ is acyclic: every checked path is acyclic;

(ii) $\overset{\frown}{\mathcal{G}}$ is *functional*: every checked path in $\overset{\frown}{\mathcal{G}}$ from a vertex $v^\bullet$ to a root includes a vertex w° such that $v \to w$;

(iii) $\mathcal{G}$ is *functorial*: if $v \rightsquigarrow w$ and $w \sim w'$ then there is v' such that $v \sim v'$ and $v' \rightsquigarrow w'$;

(iv) $\mathcal{G}$ is CK-*correct*: if $\{v_1^\bullet, v_2^\bullet, \ldots, v_n^\bullet, w^\circ\}$ is a $\sim$-class of modal vertices, then either $v_1, v_2, \ldots, v_n, w \in V_\mathcal{G}^\Box$ or there is a unique i such that $v_i, w \in V_\mathcal{G}^\Diamond$.

A linked modal arena is a CD-*arena net* if it satisfies Conditions (i)-(iii) above, plus the following:

(v) $\mathcal{G}$ is CD-*correct*: if $\{v_1^\bullet, v_2^\bullet, \ldots, v_n^\bullet, w^\circ\}$ is a $\sim$-class of modal vertices, then either $v_1, v_2, \ldots, v_n, w \in V_\mathcal{G}^\Box$ or $w \in V_\mathcal{G}^\Diamond$ there is at most one $i \in \{1, \ldots, n\}$ such that $v_i \in V_\mathcal{G}^\Diamond$.

A *modal arena net* is either a CK- or a CD-arena net. An *arena net* is a modal arena net with $V^{\Box\Diamond} = \emptyset$. Note that in this case Conditions (iii)-(v) are vacuous.

The intuition for Conditions (iv) and (v) is that $\sim$-classes represent either atoms paired by an ax, or the set of modalities introduced by a same instance

$$\frac{}{a \overset{\dashrightarrow}{\longrightarrow} a}\,\mathsf{ax} \qquad \frac{\mathcal{F},\mathcal{G}\vdash\mathcal{H}}{\mathcal{F}\vdash\mathcal{G}\mathrel{-\!\!\triangleright}\mathcal{H}}\supset^{\circ} \qquad \frac{\mathcal{F}\vdash\mathcal{G}\quad \mathcal{J},\mathcal{K}\vdash\mathcal{H}}{\mathcal{F},\mathcal{J},\mathcal{G}\mathrel{-\!\!\triangleright}\mathcal{K}\vdash\mathcal{H}}\supset^{\bullet} \qquad \frac{\mathcal{F}\vdash\mathcal{G}\quad \mathcal{I}\vdash\mathcal{K}}{\mathcal{F},\mathcal{I}\vdash\mathcal{G}+\mathcal{K}}\wedge^{\circ} \qquad \frac{\mathcal{F},\mathcal{G},\mathcal{H}\vdash\mathcal{K}}{\mathcal{F},\mathcal{G}+\mathcal{H}\vdash\mathcal{K}}\wedge^{\bullet}$$

$$\frac{\langle\mathcal{G}_1,\ldots,\mathcal{G}_n\vdash\mathcal{H}\mid\overset{\mathcal{G}}{\sim}\rangle}{\langle\Box\!\sim\!\!\triangleright\mathcal{G}_1,\ldots,\Box\!\sim\!\!\triangleright\mathcal{G}_n\vdash\Box\!\sim\!\!\triangleright\mathcal{H}\mid\overset{\mathcal{G}}{\sim}\cup\overset{\mathsf{k}}{\sim}\rangle}\,\mathsf{k}^{\Box} \qquad \frac{\langle\mathcal{G}_1,\ldots,\mathcal{G}_n\vdash\mathcal{H}\mid\overset{\mathcal{G}}{\sim}\rangle}{\langle\Box\!\sim\!\!\triangleright\mathcal{G}_1,\ldots,\Diamond\!\sim\!\!\triangleright\mathcal{G}_i,\ldots,\Box\!\sim\!\!\triangleright\mathcal{G}_n\vdash\Diamond\!\sim\!\!\triangleright\mathcal{H}\mid\overset{\mathcal{G}}{\sim}\cup\overset{\mathsf{k}}{\sim}\rangle}\,\mathsf{k}^{\Diamond}$$

$$\frac{}{\emptyset}\,\top^{\circ} \qquad \frac{\mathcal{F}\vdash\mathcal{G}}{\emptyset,\mathcal{F}\vdash\mathcal{G}}\,\top^{\bullet} \qquad \frac{\langle\mathcal{G}_1,\ldots,\mathcal{G}_n\vdash\mathcal{H}\mid\overset{\mathcal{G}}{\sim}\rangle}{\langle\Box\!\sim\!\!\triangleright\mathcal{G}_1,\ldots,\Box\!\sim\!\!\triangleright\mathcal{G}_n\vdash\Diamond\!\sim\!\!\triangleright\mathcal{H}\mid\overset{\mathcal{G}}{\sim}\cup\overset{\mathsf{k}}{\sim}\rangle}\,\mathsf{d}$$

Fig. 6. Translation of $\mathsf{LCK}^{\LEFTcircle}_{\ell}$ and $\mathsf{LCD}^{\LEFTcircle}_{\ell}$ sequent rules in modal arenas rules where $\overset{\mathsf{k}}{\sim}$ is the equivalence class containing all vertices in the conclusion which are not in the premise. Note that $\mathcal{G},\mathcal{H}$ and $\mathcal{K}$ have to be non-empty.

of a $\mathsf{K}^{\Box}$, $\mathsf{K}^{\Diamond}$ or D-rule. Following this intuition, if $\mathsf{c}=\{v_0,v_1,\ldots,v_n\}\subseteq V_{\mathcal{G}}^{\Box\Diamond}$ is a $\sim$-class, then the modal arena with vertices $\bigcup_{v\in\mathsf{c}}\mathsf{Scope}(v)$ corresponds to the sub-proof of the premise of any such rule.

Lemma 5.4 *Let* $\mathsf{X}\in\{\mathsf{CK},\mathsf{CD}\}$ *and* F *be a clean* P*-formula. If* $\overset{\mathsf{LX}^{\LEFTcircle}_{\ell}}{\vdash} F$*, then there is a* X*-arena net* $\mathcal{G}=\langle V_{\mathcal{G}},\overset{\mathcal{G}}{\to},\overset{\mathcal{G}}{\rightsquigarrow},\overset{\mathcal{G}}{\sim}\rangle$ *such that* $[\![F]\!]=\langle V_{\mathcal{G}},\overset{\mathcal{G}}{\to},\overset{\mathcal{G}}{\rightsquigarrow}\rangle$*.*

Proof. Let $\mathfrak{D}$ be a derivation of F in $\mathsf{LX}^{\LEFTcircle}_{\ell}$. We proceed by induction on $\mathfrak{D}$ translating it into a derivation of the desired modal arena net $\mathcal{G}$ via the rules in Figure 6. By definition, each rule in $\mathsf{LX}^{\LEFTcircle}_{\ell}$ preserves X-arena net conditions, that is, if the premises of a rule are X-arena nets, then the conclusion is. Note that Condition (iv) fails for the rule D, while Condition (v) holds. □

Lemma 5.5 *Let* $\mathsf{X}\in\{\mathsf{CK},\mathsf{CD}\}$ *and* F *be a clean* P*-formula. If* $\mathcal{G}$ *is an* X*-arena net with associated* P*-formula* F*, then* $\overset{\mathsf{LX}^{\LEFTcircle}_{\ell}}{\vdash} F$*.*

Proof. We prove the theorem for CK-arena nets since the proof for CD-arena nets is similar by considering also the rule D.

If $\mathcal{G}=\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$, we conclude since $[\![\top]\!]=\emptyset$ and $\overset{\mathsf{LCK}^{\LEFTcircle}_{\ell}}{\vdash}\top^{\circ}$. Otherwise to prove this theorem we define from the CK-arena net $\mathcal{G}$, with associated clean P-formula F, an arena net $\partial(\mathcal{G})$ with associated formula $\partial(F)$. We then use use of the result in [17] on (non-modal) arena nets to produce an $\mathsf{LX}^{\LEFTcircle}_{\ell}$-derivation of $\partial(F)$. Then we conclude by showing how to define a $\mathsf{LX}^{\LEFTcircle}_{\ell}$-derivation of F using the $\mathsf{LI}^{\LEFTcircle}_{\ell}$-derivation of $\partial(F)$.

Step 1: definition of $\partial(\mathcal{G})$*.* Let $\mathcal{G}=\langle V_{\mathcal{G}},\overset{\mathcal{G}}{\to},\overset{\mathcal{G}}{\rightsquigarrow},\overset{\mathcal{G}}{\sim}\rangle$ be a CK-arena net. We write $v\,\hat{\sim}\,w$ either if $\hat{v}\sim\hat{w}$, or if $v=\hat{v}$ and $w=\hat{w}$, that is, $v\,\hat{\sim}\,w$ iff either both v and w are not in the scope of any modality, or both v and w belong to the scope of modalities in the same $\sim$-class.

We define the arena $\partial(\mathcal{G})$ by removing all $\rightsquigarrow$-edges in $\mathcal{G}$ and keeping only the $\to$ between vertices $v,w\in V_{\mathcal{G}}$ such that $v\,\hat{\sim}\,w$. Then we replace each modal vertex v by a pair of $\rightharpoonup$-linked vertices $v^{\mathsf{in}},v^{\mathsf{out}}$ in such a way that the vertex v^{in} keeps track of the subformulas of the modality, while v^{out} is a placeholder to keep track of the interaction of the subformulas with the context.

Formally we define $\partial(\mathcal{G})=\langle\partial(V_{\mathcal{G}})\ ,\ \partial(\overset{\mathcal{G}}{\to}\cup\overset{\mathcal{G}}{\rightsquigarrow})\ ,\ \overset{\partial(\mathcal{G})}{\sim}\rangle$ by:

- $\partial(V_{\mathcal{G}})=V_{\mathcal{G}}^{\mathcal{A}}\cup\{v^{\mathsf{in}},v^{\mathsf{out}}\mid v\in V_{\mathcal{G}}^{\Box\Diamond}\}$;

- $\partial(\overset{\mathcal{G}}{\rightarrow} \cup \overset{\mathcal{G}}{\rightsquigarrow})$ is the union of the following five sets:

$$\begin{array}{rl} \{(l^{\mathsf{out}}, r^{\mathsf{out}}) \mid l \rightharpoonup r\} & \bigcup \{(u,v) \mid u \not\updownarrow v \text{ and } u \rightarrow v\} \\ \{(u, r^{\mathsf{in}}) \mid u \rightsquigarrow_{\partial} r\} & \bigcup \{(l^{\mathsf{in}}, u) \mid l \rightsquigarrow_{\partial} u\} \\ \{(u, m^{\mathsf{out}}) \mid u \rightarrow m \text{ and } u \not\updownarrow m\} & \bigcup \{(m^{\mathsf{out}}, v) \mid m \rightarrow v \text{ and } m \not\updownarrow v\} \\ \{(m^{\mathsf{out}}, n^{\mathsf{out}}) \mid m \rightarrow n \text{ and } m \not\updownarrow n\} & \bigcup \{(m^{\mathsf{out}}, n^{\mathsf{out}}) \mid m \rightarrow n \text{ and } m \not\updownarrow n\} \end{array}$$

where we assume $u, v \in V_{\mathcal{G}}^{\mathcal{A}}$ and $l^{\bullet}, r^{\circ}, m, n, p \in V_{\mathcal{G}}^{\Box\Diamond}$;

- $\overset{\partial(\mathcal{G})}{\sim}$ is defined as: $v \overset{\partial(\mathcal{G})}{\sim} w$ if $v \overset{\mathcal{G}}{\sim} w$ and as $v^{\mathsf{in}} \overset{\partial(\mathcal{G})}{\sim} v^{\mathsf{out}}$ for each $v \in V_{\mathcal{G}}^{\Box\Diamond}$.

See the first line of Figure 8for a running example.

We observe that if $\{v_0^{\circ}, v_1^{\bullet}, \ldots, v_n^{\bullet}\}$ is a $\overset{\mathcal{G}}{\sim}$-class of modal vertices, then a P-formula associated to $\mathcal{G}$ is of the form $H = H\{\ell(v_0)A_0^{\circ}\}\{\ell(v_1)A_1^{\bullet}\} \cdots \{\ell(v_n)A_n^{\bullet}\}$ for an $(n+1)$-ary context $H\{\ \} \cdots \{\ \}$. In this case, a P-formula associated to the arena $\partial(\mathcal{G})$ is of the form $\partial(H) = \partial(H)\{v_0^{\mathsf{out}\circ}\}\{v_1^{\mathsf{out}\bullet}\} \cdots \{v_n^{\mathsf{out}\bullet}\}\{H_{\mathsf{c}}^{\bullet}\}$ with $\partial(H)\{\ \} \cdots \{\ \}$ is an $(n+2)$-ary context, fresh propositional variables $v_i^{\mathsf{in}}, v_i^{\mathsf{out}}$ for all $i \in \{0, \ldots, n\}$ and

$$H_{\mathsf{c}}^{\bullet} = \Big(\Big(\big(v_1^{\mathsf{in}} \supset \partial(A_1^{\bullet}) \wedge \cdots \wedge v_n^{\mathsf{in}} \supset \partial(A_n^{\bullet})\big) \supset \partial(A_0^{\circ})\Big) \supset v_0^{\mathsf{in}}\Big)^{\bullet}$$

Step 2: proof that $\partial(\mathcal{G})$ is an arena net. We observe that, by definition of $\partial(\mathcal{G})$, every path $\partial(\mathsf{p})$ in $\widehat{\partial(\mathcal{G})}$ can be constructed from a checked path p in $\widehat{\mathcal{G}}$ by induction:

- the empty path is a path in both $\widehat{\mathcal{G}}$ and $\widehat{\partial(\mathcal{G})}$;
- if $\mathsf{p} = v \cdot \mathsf{p}'$
  - · if $v \in V_{\mathcal{G}}^{\mathcal{A}}$, then $\partial(\mathsf{p}) = v \cdot \partial(\mathsf{p})'$;
  - · if $v^{\circ} \in V_{\mathcal{G}}^{\Box\Diamond}$, then $\partial(\mathsf{p}) = v^{\mathsf{out}} \cdot v^{\mathsf{in}} \cdot \partial(\mathsf{p})'$;
  - · if $v^{\bullet} \in V_{\mathcal{G}}^{\Box\Diamond}$, then $\partial(\mathsf{p}) = v^{\mathsf{in}} \cdot v^{\mathsf{out}} \cdot \partial(\mathsf{p})'$;

We remark that the parity of atomic vertices is preserved by ∂, while the parity of a modal vertex $v \in V_{\mathcal{G}}$ is the same as the corresponding vertex $v^{\mathsf{out}} \in V_{\partial(\mathcal{G})}$. Since if $v^{\bullet}$ then $v^{\mathsf{out}} \rightharpoonup v^{\mathsf{in}}$, and if v° then $v^{\mathsf{in}} \rightharpoonup v^{\mathsf{out}}$, then we have that in $\partial(\mathcal{G})$ an even (odd) vertex may occur only in a even (odd) position in a path in $\widehat{\mathcal{G}}$. We conclude since from any path in $\widehat{\partial(\mathcal{G})}$ we obtain a path in $\widehat{\mathcal{G}}$ by replacing every subpath $v^{\mathsf{out}} \rightharpoonup v^{\mathsf{in}}$ and $v^{\mathsf{in}} \rightharpoonup v^{\mathsf{out}}$ by a the corresponding modal vertex v in $\mathcal{G}$.

By this correspondence between checked paths in $\widehat{\mathcal{G}}$ and paths in $\widehat{\partial(\mathcal{G})}$ we conclude that $\widehat{\partial(\mathcal{G})}$ is acyclic and functional. That is, $\partial(\mathcal{G})$ is an arena net.

Step 3: construct the derivation associated to $\partial(\mathcal{G})$. Since $\partial(\mathcal{G})$ is an arena net, then we apply the result in [17] to produce a derivation in $\mathsf{LI}_{\ell}^{\mathsf{c}}$ of the formula $\partial(F)$. In such a derivation, by functionality and functoriality of $\mathcal{G}$, whenever v and w are modal vertices such that $v \overset{\mathcal{G}}{\rightharpoonup} w$, then if a path in $\widehat{\partial(\mathcal{G})}$ contains v^{out}, then it also contains v^{in}, w^{in}, w^{out}. This means that if $\mathsf{c} = \{v_0^{\circ}, v_1^{\bullet}, \ldots, v_n^{\bullet}\}$

$$\begin{array}{c} \nabla\,\mathsf{CK}_\ell \\ A_1^\bullet,\ldots,A_n^\bullet,A_0^\circ \\ \partial(\mathfrak{D}')\,\Vert\,\mathsf{LI}_\ell \\ \Box_1^{\mathsf{out}\bullet},\ldots,\Box_n^{\mathsf{out}\bullet},F_{\mathsf{c}}^\bullet,\Box_0^{\mathsf{out}\circ} \\ \mathfrak{D}\{\Box_0^{\mathsf{out}\circ}\}\{\Box_1^{\mathsf{out}\bullet}\}\cdots\{\Box_n^{\mathsf{out}\bullet}\}\{F_{\mathsf{c}}^\bullet\}\,\Vert\,\mathsf{CK}_\ell \\ \partial(F)\{\Box_0^{\mathsf{out}\circ}\}\{\Box_1^{\mathsf{out}\bullet}\}\cdots\{\Box_n^{\mathsf{out}\bullet}\}\{F_{\mathsf{c}}^\bullet\} \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \nabla\,\mathsf{CK}_\ell \\ \dfrac{A_1^\bullet,\ldots,A_n^\bullet,A_0^\circ}{\Box A_1^\bullet,\ldots,\Box A_n^\bullet,\Box A_0^\circ}\,\mathsf{k}^\Box \\ \mathfrak{D}\{\Box A_0^\circ\}\{\Box A_1^\bullet\}\cdots\{\Box A_n^\bullet\}\{\emptyset\}\,\Vert\,\mathsf{CK}_\ell \\ F\{\Box_0 A_0^\circ\}\{\Box_1 A_1^\bullet\}\cdots\{\Box_n A_n^\bullet\} \end{array}$$

Fig. 7. An example of the construction of the derivation of F from the derivation of $\partial(F)$ assuming that in $\mathcal{G}$ there is only one $\sim$-class of the form $\{\Box_0,\ldots,\Box_n\}$

is an $\overset{\mathcal{G}}{\sim}$-class of vertices in $\overset{\mathcal{G}}{\rightsquigarrow}$, then any derivation of $\partial(F)$ in $\mathsf{LX}_\ell^\bullet$ contains a subderivation of the sequent $v_1^{\mathsf{out}\bullet},\ldots,v_n^{\mathsf{out}\bullet},H_c^\bullet,v_0^{\mathsf{out}\circ}$ of the following form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v_1^{\mathsf{out}\bullet},v_1^{\mathsf{in}\circ}}\mathsf{ax}\ \cdots\ \dfrac{}{v_n^{\mathsf{out}\bullet},v_n^{\mathsf{in}\circ}}\mathsf{ax}\qquad \begin{array}{c}\nabla\,\mathsf{LI}_\ell^\bullet\\ \partial(A_1)^\bullet,\ldots,\partial(A_n)^\bullet,\partial(A_0)^\circ\end{array}}{v_1^{\mathsf{out}\bullet},\ldots,v_n^{\mathsf{out}\bullet},v_1^{\mathsf{in}}\supset\partial(A_1)^\bullet,\ldots,v_n^{\mathsf{in}}\supset\partial(A_n)^\bullet,\partial(A_0)^\circ}\supset^{\mathsf{L}}}{v_1^{\mathsf{out}\bullet},\ldots,v_n^{\mathsf{out}\bullet},\bigwedge_{i=1}^n(v_i^{\mathsf{in}}\supset\partial(A_i))^\bullet,\partial(A_0)^\circ}\wedge^{\mathsf{L}}}{v_1^{\mathsf{out}\bullet},\ldots,v_n^{\mathsf{out}\bullet},\bigwedge_{i=1}^n(v_i^{\mathsf{in}}\supset\partial(A_i))\supset\partial(A_0)^\circ}\supset^{\mathsf{R}}\qquad \dfrac{}{v_0^{\mathsf{in}\bullet},v_0^{\mathsf{out}\circ}}\mathsf{ax}}{v_1^{\mathsf{out}\bullet},\ldots,v_n^{\mathsf{out}\bullet},\left(\bigwedge_{i=1}^n(v_i^{\mathsf{in}}\supset\partial(A_i))\supset\partial(A_0)\right)\supset v_0^{\mathsf{in}\bullet},v_0^{\mathsf{out}\circ}}\supset^{\mathsf{L}}$$

In order to construct a derivation in $\mathsf{LCK}_\ell^\bullet$ of the formula F it suffices to proceed by induction over the number of $\overset{\mathcal{G}}{\sim}$-classes of modal vertices. Starting from the top of the derivation, we replace every such subderivation in the derivation of $\partial(F)$ with an application of a $\mathsf{K}^\Box$- or a $\mathsf{K}^\Diamond$-rule, we remove all the occurrences of the formula $H_{\mathsf{c}}^\bullet = \left(\bigwedge_{i=1}^n(v_i^{\mathsf{in}}\supset\partial(A_i))\supset\partial(A_0)\right)\supset v_0^{\mathsf{in}\bullet}$ in the derivation, and we replace for each $i\in\{0,\ldots,n\}$ the atom v_i^{in} with the corresponding formula $\ell(v_i)A_i$ as shown in Figure 7. For a running example, refer to the lower line of Figure 8. □

By Lemma 5.4 and Lemma 5.5 we have the following theorem.

Theorem 5.6 *Let* $\mathsf{X}\in\{\mathsf{CK},\mathsf{CD}\}$ *and* F *be a clean* P*-formula. Then*

$$\overset{\mathsf{LX}_\ell^\bullet}{\vdash} F \iff \text{there is a } \mathsf{X}\text{-arena net } \mathcal{G} \text{ with } \mathcal{G}=[\![F]\!]$$

6 Skew Fibrations

After having characterized the linear part of a proof in CK or CD, we will now characterize the maps between modal arenas that characterize derivations built from the deep rules $\mathsf{w}_\downarrow^\bullet$ for weakenning and $\mathsf{c}_\downarrow^\bullet$ for contraction (shown in Figure 3).

Let u, v, and w be vertices in a modal arena. We say that u is a *meeting point* of v and w whenever $v\to^* u$ and $w\to^* u$, and there is no vertex $u'\neq u$ such that $v\to^* u'$ and $w\to^* u'$ and $u'\to^* u$. The *meeting depth* of v and w is the depth of their meeting point, or -1 if no meeting point exists. Note that this is well defined as all meeting points of v and w have the same depth (this follows from [17, Lemma 9]). Two distinct vertices v and w are *conjunct*, denoted $v \curlywedge w$ if their meeting depth is odd (or equal to -1).

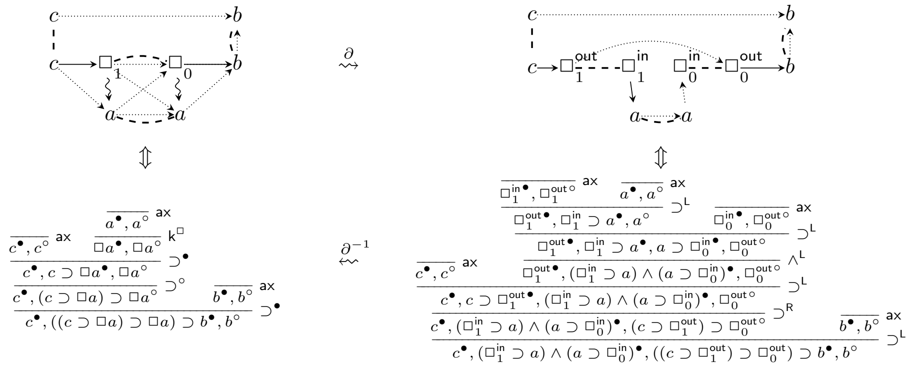


Fig. 8. A K-arena net $\mathcal{G}$ with associated formula $(c \wedge ((c \supset \Box a) \supset \Box a) \supset b) \supset b$, its corresponding arena $\partial(\mathcal{G})$, the $\mathsf{LCK}^{\bullet}_{\ell}$-derivation associated to $\mathcal{G}$ and the $\mathsf{LI}^{\bullet}_{\ell}$-derivation associated to $\partial(\mathcal{G})$

Definition 6.1 A *modal arena homomorphism* is either a map $\emptyset_{\mathcal{G}} \colon \emptyset \to \mathcal{G}$ from the empty 2-dag to a modal arena $\mathcal{G}$, or a structure preserving map $f \colon \mathcal{H} \to \mathcal{G}$ between two modal arenas, i.e., its a function $f \colon V_{\mathcal{H}} \to V_{\mathcal{G}}$ that preserves: [3]

$$\begin{array}{llll} \to : & \text{if } v \overset{\mathcal{H}}{\to} w \text{ then } f(v) \overset{\mathcal{G}}{\to} f(w) & d : & d(v) = d(f(v)) \\ \rightsquigarrow : & \text{if } v \overset{\mathcal{H}}{\rightsquigarrow} w \text{ then } f(v) \overset{\mathcal{G}}{\rightsquigarrow} f(w) & \ell : & \ell(v) = \ell(f(v)) \end{array}$$

A *(modal) skew fibration* is a modal arena homomorphism $f \colon \mathcal{H} \to \mathcal{G}$ which:

- preserves $\curlywedge$: if $v \curlywedge_{\mathcal{H}} w$ then $f(v) \curlywedge_{\mathcal{G}} f(w)$;
- is a *skew lifting*: if $f(v) \curlywedge_{\mathcal{G}} w$, then there exists u with $u \curlywedge_{\mathcal{H}} v$ and $f(u) \not\curlywedge_{\mathcal{G}} w$.
- is a *modal lifting*: if $f(v) \overset{\mathcal{G}}{\rightsquigarrow} f(w)$, then there exists u with $u \overset{\mathcal{H}}{\rightsquigarrow} w$ and $f(u) = f(v)$.

Lemma 6.2 *The composition of two skew fibrations is a skew fibration.*

Proof. By definition, the composition preserves $\to$, $\rightsquigarrow$, ℓ and d. Then the preservation of $\curlywedge$ and the skew lifting condition of the composition are guaranteed as consequence of the preservation of d and $\to$. Similarly, the modal lifting condition of the composition is guaranteed as consequence of the preservation of $\rightsquigarrow$ and the fact that source and target of a skew fibration are modal arenas. $\Box$

In order to prove the correspondence between $\{\mathsf{c}^{\bullet}_{\downarrow}, \mathsf{w}^{\bullet}_{\downarrow}\}$ derivations and skew fibrations, we provide the following definition.

[3] In [17] the definition of skew fibration only demands the weaker *root preserving* condition (that is, if $v \in \overrightarrow{R}_{\mathcal{H}}$ then $f(v) \in \overrightarrow{R}_{\mathcal{G}}$) instead of the depth preserving condition $d(v) = d(f(v))$ that we use here. However, in the same paper it is proven that root preservation is equivalent to depth preservation.

Definition 6.3 If $f_1 : \mathcal{H}_1 \to \mathcal{G}_1$ and $f_2 : \mathcal{H}_2 \to \mathcal{G}_2$ are modal arena homomorphisms such that $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint modal arenas, we define the following modal arena homomorphisms:

$$\begin{array}{lcll} f_1 + f_2 & = & f_1 \cup f_2 : \mathcal{H}_1 + \mathcal{H}_2 \to \mathcal{G}_1 + \mathcal{G}_2 & \\ f_1 \multimap\!\!\rhd f_2 & = & f_1 \cup f_2 : \mathcal{H}_1 \multimap\!\!\rhd \mathcal{H}_2 \to \mathcal{G}_1 \multimap\!\!\rhd \mathcal{G}_2 & (\mathcal{H}_2,\ \mathcal{G}_2 \neq \emptyset) \\ f_1 \rightsquigarrow\!\!\rhd f_2 & = & f_1 \cup f_2 : \mathcal{H}_1 \rightsquigarrow\!\!\rhd \mathcal{H}_2 \to \mathcal{G}_1 \rightsquigarrow\!\!\rhd \mathcal{G}_2 & \\ [f_1, f_2] & = & f_1 \cup f_2 : \mathcal{H}_1 + \mathcal{H}_2 \to \quad \mathcal{G} & (\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}) \end{array} \qquad (5)$$

Lemma 6.4 *The operations from Definition 6.3 preserve skew fibration properties.*

Proof. It suffices to check that if f_1 and f_2 are skew fibrations, then also $f_1 + f_2$, $f_1 \multimap\!\!\rhd f_2$, $f_1 \cup f_2$, $f_1 \rightsquigarrow\!\!\rhd f_2$ and $[f_1, f_2]$ are. □

Lemma 6.5 *Let H' and H be* P*-formulas. If* $H' \overset{\{\mathsf{c}_\downarrow^\bullet, \mathsf{w}_\downarrow^\bullet\}}{\vdash} H$*, then there is a skew fibration* $f\colon [\![H']\!] \to [\![H]\!]$.

Proof. After Lemma 6.2, it suffices to prove that if $\dfrac{H'}{H}\,\rho$ for a $\rho \in \{\mathsf{w}_\downarrow^\bullet, \mathsf{c}_\downarrow^\bullet\}$. then there is a skew fibration $f\colon [\![H']\!] \to [\![H]\!]$. This is immediate after Lemma 6.4 after remarking that any map $\emptyset_\mathcal{G}\colon \emptyset \to \mathcal{G}$ is a skew fibration. □

To prove the converse, we need some additional definitions and results.

Definition 6.6 Two distinct vertices v and w in a modal arena they are *disjunct*, denoted $v \curlyvee w$, if their meeting depth is even. An *odd skew fibration* is either a map $\emptyset_\mathcal{G}\colon \emptyset \to \mathcal{G}$, or a modal arena homomorphism $f\colon \mathcal{H} \to \mathcal{G}$ which:

- preserves $\curlyvee$: if $v \curlyvee_\mathcal{H} w$ then $f(v) \curlyvee_\mathcal{G} f(w)$;
- is a *odd skew lifting*: if $f(v) \curlyvee_\mathcal{G} w$, then there exists u with $v \curlyvee_\mathcal{H} u$ and $f(u) \not\curlyvee_\mathcal{G} w$.

Lemma 6.7 *If $f\colon \mathcal{H} \to \mathcal{G}$ be a modal arena homomorphism and $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, then $f = f_1 + f_2$ with $f_1\colon \mathcal{H}_1 \to \mathcal{G}_1$ and $f_2\colon \mathcal{H}_2 \to \mathcal{G}_2$ modal arena homomorphisms for some $\mathcal{H}_1, \mathcal{H}_2$ such that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$.*

Proof. Since f preserves $\to$, then if $v \to^* w$ for a $w \in \vec{R}_\mathcal{G}$ then $f(v) \to^* f(w)$. Thus if $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, then there is a partition $\vec{R}_\mathcal{G} = \vec{R}_{\mathcal{G}_1} \uplus \vec{R}_{\mathcal{G}_2}$. As remarked in the proof of Theorem 4.9, in construction such partition, because of $\rightsquigarrow$-coherence, whenever $v \rightsquigarrow w$ then v and w belong to the same subset. Then we can define $V_{\mathcal{H}_1}$ and $V_{\mathcal{H}_2}$ as the sets of vertices of $\mathcal{H}$ which images by f admit a $\to$-path to a vertex in $\vec{R}_{\mathcal{G}_1}$ and $\vec{R}_{\mathcal{G}_2}$ respectively. The modal arenas $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined from $\mathcal{H}$ by the sets $V_{\mathcal{H}_1}$ and $V_{\mathcal{H}_2}$ respectively. □

Lemma 6.8 *Let $f\colon \mathcal{H} \to \mathcal{G}$ be a skew fibration or an odd skew fibration, with $\mathcal{G} = \mathcal{G}_1 \multimap\!\!\rhd \mathcal{G}_2$ and $\mathcal{G}_1$ modal arenas. If there exist two modal arenas $\mathcal{H}'$ and $\mathcal{H}''$ such that $\mathcal{H} = \mathcal{H}' \multimap\!\!\rhd \mathcal{H}''$ and $\mathcal{H}''$ cannot be written as $\multimap\!\!\rhd$ of two modal arenas, then $f(v) \in V_{\mathcal{G}_2}$ for all $v \in V_{\mathcal{H}''}$.*

Proof. Let $v \in \mathcal{H}''$ such that $f(v) \in \mathcal{G}_1$. Since f preserves d, then $v \notin \vec{R}_\mathcal{H}$. Thus $\mathcal{H}''$ cannot be a single-vertex modal arena. If $\mathcal{H}''$ is a $+$ of two modal arenas, then there is $z \in \vec{R}_{\mathcal{H}''}$ such that $v \not\rightarrow^* z$, hence $v \curlywedge z$ in $\mathcal{H}$ but $f(v) \not\curlywedge f(z)$ in $\mathcal{G}$. Therefore f is not a skew fibration. Let $f(z) = w$. Then $f(v) \curlyvee w$ because $f(v) \in \mathcal{G}_1$ and $w \in \vec{R}_{\mathcal{G}}$. If there is a u with $v \curlyvee u$ in $\mathcal{H}$ then there is $x \in V_{\mathcal{H}}$ such that $u \rightarrow x^\circ$ and $v \rightarrow x^\circ$. Since $x \rightarrow^* w$ we have $f(u) \curlyvee w$, which means that f cannot be an odd skew fibration either. Then $\mathcal{H}''$ has to be of the shape $w \mathrel{\sim\!\!\rhd} \mathcal{H}''_2$ and $f(w) \in \mathcal{G}_2$ because $v \in \vec{R}_{\mathcal{H}}$. We can conclude as for the previous case that f is not an even or odd skew fibration. Contradiction. □

Lemma 6.9 *Let $f \colon \mathcal{H} \to \mathcal{G}$ is an odd skew fibration, with $\mathcal{G} = \mathcal{G}_1 \mathrel{-\!\!\rhd} \mathcal{G}_2$ for a modal arena $\mathcal{G}_1$. If there is a modal arena $\mathcal{H}'$ such that $\mathcal{H} = \mathcal{H}' \mathrel{-\!\!\rhd} \mathcal{H}''$, then there are $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H} = \mathcal{H}_1 \mathrel{-\!\!\rhd} \mathcal{H}_2$ and $f = f_1 \mathrel{-\!\!\rhd} f_2$ where $f_1 \colon \mathcal{H}_1 \to \mathcal{G}_1$ and $f_2 \colon \mathcal{H}_2 \to \mathcal{G}_2$ are modal arena homomorphisms.*

Proof. By hypothesis, we can assume that $\mathcal{H}$ is of the form $\mathcal{H} = \mathcal{H}' \mathrel{-\!\!\rhd} \mathcal{H}''$ where $\mathcal{H}''$ is not a $\mathrel{-\!\!\rhd}$ of two modal arenas. We conclude by Lemma 6.8 that $f(v) \in V_{\mathcal{G}_2}$ for any $v \in V_{\mathcal{H}''}$. If $V_{\mathcal{G}_2} = f(V_{\mathcal{H}''})$, then we conclude that $\mathcal{H}_1 = \mathcal{H}'$ and $\mathcal{H}_2 = \mathcal{H}''$. Otherwise, let $\mathcal{H}' = \mathcal{H}'_1 + \cdots + \mathcal{H}'_n$ such that $\mathcal{H}'_i$ is a $+$ of two modal arenas for no $i \in \{1, \ldots, n\}$. If $v, w \in V_{\mathcal{H}'}$, then there is a $(\leftrightarrow \cup \leftrightsquigarrow)$-path from v to w in $V_{\mathcal{H}'}$ iff there is $i \in \{1, \ldots, n\}$ such that $v, w \in V_{\mathcal{H}'_i}$. Since $\vec{R}_{\mathcal{G}} \subset f(V_{\mathcal{H}''})$, this implies that if there is $i \in \{1, \ldots, n\}$ such that $v, w \in V_{\mathcal{H}'_i}$, then there is $(\leftrightarrow \cup \leftrightsquigarrow)$-path from $f(v)$ to $f(w)$ in $V_{\mathcal{G}} \setminus \vec{R}_{\mathcal{G}}$. That is, $f(V_{\mathcal{H}'_i})$ is either a subset of $V_{\mathcal{G}_1}$ or a subset of $V_{\mathcal{G}_2}$ for all $i \in \{1, \ldots, n\}$. Without loss of generality we assume there is j such that that $f(V_{\mathcal{H}'_i}) \subset V_{\mathcal{G}_1}$ for all $i \leq j$. We conclude that $\mathcal{H}_1 = \mathcal{H}'_1 + \cdots + \mathcal{H}'_j$ and $\mathcal{H}_2 = (\mathcal{H}'_{j+1} + \cdots + \mathcal{H}'_n) \mathrel{-\!\!\rhd} \mathcal{H}''$. □

Lemma 6.10 *Let $f \colon \mathcal{H} \to \mathcal{G}$ be a modal arena homomorphism and $\mathcal{G} = v \mathrel{\sim\!\!\rhd} \mathcal{G}'$. If f is a skew fibration then, $\mathcal{H} = w \mathrel{\sim\!\!\rhd} \mathcal{H}'$ and $f = 1_w \mathrel{\sim\!\!\rhd} f'$ with $f' \colon \mathcal{H}' \to \mathcal{G}'$ a skew fibration. If f is odd skew fibration, then*

- *either $\mathcal{H} = w \mathrel{\sim\!\!\rhd} \mathcal{H}_2$ and $f = 1_w \mathrel{\sim\!\!\rhd} f_2$ with $f_2 \colon \mathcal{H}_2 \to \mathcal{G}_2$ an odd skew fibration;*
- *or $\mathcal{H} = (w \mathrel{\sim\!\!\rhd} \mathcal{H}_1) + \mathcal{H}_2$ and $f = [f_1, f_2]$ with $f_1 \colon (w \mathrel{\sim\!\!\rhd} \mathcal{H}_1) \to (v \mathrel{\sim\!\!\rhd} \mathcal{G}_2)$ and $f_2 \colon \mathcal{H}_2 \to (v \mathrel{\sim\!\!\rhd} \mathcal{G}_2)$.*

Proof. If f is a skew fibration, then to conclude it suffices to remark there is a unique w such that $f(w) = v$ since $v \in \vec{R}_{\mathcal{G}}$. If f is an odd skew fibration, let w such that $f(w) = v$. If $V_{\mathcal{H}} \setminus \{w\} = \mathsf{Scope}(w)$, then we can conclude. Otherwise we conclude with $\mathcal{H}_2$ be the modal arena with vertices in $V_{\mathcal{H}} \setminus (\{w\} \cup \mathsf{Scope}(w))$. □

Lemma 6.11 *Every skew fibration is of the form $1_{\mathcal{G}}$, $f^\circ + g^\circ$, $f^\bullet \mathrel{-\!\!\rhd} g^\circ$ or $1_v \mathrel{\sim\!\!\rhd} g^\circ$. Every odd skew fibration is of the form $1_{\mathcal{G}}$, $[f^\bullet, g^\bullet]$, $f^\bullet + g^\bullet$, $f^\circ \mathrel{-\!\!\rhd} g^\bullet$, $1_v \mathrel{\sim\!\!\rhd} g^\bullet$ or $\emptyset_{\mathcal{G}}$, where f° and g° are skew fibrations, $f^\bullet$ and $g^\bullet$ are odd skew fibrations, $v \in V^{\Box\Diamond}_{[\![H]\!]}$, and $\mathcal{G}$ can be any modal arena.*

Proof. By case analysis, let $f \colon \mathcal{H} \to \mathcal{G}$ be a modal arena homomorphism, remarking that for any modal arena $\mathcal{G}$, the identity map $1_{\mathcal{G}}$ is by definition an even and an odd skew fibration. If $f^\circ \colon \mathcal{H} \to \mathcal{G}$ is a skew fibration, then

- if $\mathcal{G}$ is a single-vertex modal arena, then $\mathcal{H}$ cannot be either of the shape $\mathcal{H}_1 + \mathcal{H}_2$ or $\mathcal{H}_1 \leadsto\!\triangleright \mathcal{H}_2$ otherwise f would not preserve $\curlywedge$, or of the shape $\mathcal{H}_1 \relbar\!\triangleright \mathcal{H}_2$ otherwise it would not preserve d. Then $f = 1_v$ with v the unique vertex in $V_{\mathcal{H}} = V_{\mathcal{G}}$;
- if $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, then by Lemma 6.7 we have that $f^\circ = f_1 + f_2$ with f_1 and f_2 arena homomorphisms. Since f° is an even skew fibration, it follows by definition of + that f_1 and f_2 are skew fibrations;
- if $\mathcal{G} = \mathcal{G}_1 \relbar\!\triangleright \mathcal{G}_2$, then we define $V_1 = \{v \in V_{\mathcal{H}} \mid f(v) \in \mathcal{G}_1\}$ and $V_2 = \{v \in V_{\mathcal{H}} \mid f(v) \in \mathcal{G}_2\}$. We have that $V_2 \neq \emptyset$ since f preserve d. If $V_1 = \emptyset$, then $f = \emptyset_{\mathcal{G}_1} \relbar\!\triangleright f_2$ with $f_2 \colon \mathcal{H} \to \mathcal{G}_2$. Otherwise, $V_1 \neq \emptyset$ and $\mathcal{H}$ cannot be a single vertex. Similarly, $\mathcal{H}$ cannot be of the shape $\mathcal{H}_1 + \mathcal{H}_2$ otherwise f would not preserve $\curlywedge$, nor of the shape $v \leadsto\!\triangleright \mathcal{H}_2$ otherwise f would not be modal. We conclude by Lemma 6.9 that $f = f_1 \relbar\!\triangleright f_2$. Moreover, since f is a skew fibration if follows that f_2 also preserves $\curlywedge$ and satisfies skew lifting while f_1 preserve $\curlyvee$ and satisfies odd skew lifting;
- if $\mathcal{G} = v \leadsto\!\triangleright \mathcal{G}_2$, we conclude by Lemma 6.10 .

If $f^\bullet \colon \mathcal{H} \to \mathcal{G}$ is an odd skew fibration, then we proceed similarly. If $\mathcal{G}$ is a single-vertex modal arena, then $\mathcal{H}$ cannot be of the shape $\mathcal{H}_1 \leadsto\!\triangleright \mathcal{H}_2$ otherwise f it would not be modal, or of the shape $\mathcal{H}_1 \relbar\!\triangleright \mathcal{H}_2$ otherwise it would not preserve d. Let $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ such that $\mathcal{H}_1 \neq \mathcal{H}_1' + \mathcal{H}_1''$. Since $f^\bullet$ preserve d and $\leadsto$, then $\mathcal{H}_1$ is a single-vertex modal arenas. Moreover, $f_2 \colon \mathcal{H}_2 \to \mathcal{G}_2$ is an odd skew fibration by definition. Then $f = [1_v, f_2]$ with v the unique vertex in $V_{\mathcal{H}} = V_{\mathcal{G}}$;

If $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, $\mathcal{G} = \mathcal{G}_1 \relbar\!\triangleright \mathcal{G}_2$ or $\mathcal{G} = v \leadsto\!\triangleright \mathcal{G}_2$ we apply a similar reasoning in the case of f skew fibration. □

Lemma 6.11 is now enough to complete the proof of Theorem 6.12: Given a skew fibration f, we can decompose f as an expression with the operations in Lemma 6.11, which can then be immediately transformed into a deep inference derivation using only $\mathsf{w}_{\downarrow}^{\bullet}$ and $\mathsf{c}_{\downarrow}^{\bullet}$. (This is a standard operation in deep inference, see e.g. [15].)

Theorem 6.12 *Let H and H' be* P*-formulas.*

$$H' \overset{\{\mathsf{c}_{\downarrow}^{\bullet}, \mathsf{w}_{\downarrow}^{\bullet}\}}{\vdash} H \iff \textit{there is a skew fibration } f \colon [\![H']\!] \to [\![H]\!]$$

Proof. To prove the "if" direction, it suffices to prove that if $\frac{H'}{H}\rho$ for a $\rho \in \{\mathsf{w}_{\downarrow}^{\bullet}, \mathsf{c}_{\downarrow}^{\bullet}\}$, then there is a skew fibration $f \colon [\![H']\!] \to [\![H]\!]$. Then we conclude by Lemma 6.2.

By Lemma 6.11, we have that any skew fibration can be written as composition from $1_v \colon v \to v$ and $\emptyset_{\mathcal{G}} \colon \emptyset \to \mathcal{G}$ via the operations in (5) above. In particular, each $\emptyset_{\mathcal{G}}$ occurring in the decomposition corresponds to an application of a $\mathsf{w}_{\downarrow}^{\bullet}$, while each occurrence of $[-,-]$ corresponds to an application of a $\mathsf{c}_{\downarrow}^{\bullet}$. We conclude by reconstructing a derivation in $\{\mathsf{c}_{\downarrow}^{\bullet}, \mathsf{w}_{\downarrow}^{\bullet}\}$ using this decomposition and the correspondence between P-formulas and modal arenas (Theorem 4.9). □

$$\dfrac{\dfrac{\Gamma, A, B, C \vdash D}{\Gamma, A, B \vdash C \supset D}\,{\supset^{\mathsf{R}}}}{\Gamma, A \wedge B \vdash C \supset D}\,{\wedge^{\mathsf{L}}} \quad \equiv \quad \dfrac{\dfrac{\Gamma, A, B, C \vdash D}{\Gamma, A \wedge B, C \vdash D}\,{\wedge^{\mathsf{L}}}}{\Gamma, A \wedge B \vdash C \supset D}\,{\supset^{\mathsf{R}}} \qquad \dfrac{\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,{\wedge^{\mathsf{R}}} \quad \Delta \vdash D}{\Gamma, \Delta, A \wedge B \vdash C \wedge D}\,{\wedge^{\mathsf{L}}} \quad \equiv \quad \dfrac{\dfrac{\Gamma, A, B \vdash C \quad \Delta \vdash D}{\Gamma, \Delta, A, B \vdash C \wedge D}\,{\wedge^{\mathsf{L}}}}{\Gamma, \Delta, A \wedge B \vdash C \wedge D}\,{\wedge^{\mathsf{R}}}$$

Fig. 9. Examples of independent rule permutations.

7 Combinatorial Proofs

We can now combine the results of the previous sections, to define combinatorial proofs for the logics CK and CD, and to prove their soundness and completeness.

Definition 7.1 A *modal intuitionistic combinatorial proof* is a skew fibration $f\colon \mathcal{G} \to [\![F]\!]$ from a modal arena net $\mathcal{G}$ to the modal arena of a formula F. We say f is a CK-*intuitionistic combinatorial proof*, or CK-ICP, (resp. CD-*intuitionistic combinatorial proof*, or CD-ICP) if $\mathcal{G}$ is a CK-arena net (resp. CD-arena net).

The *intuitionistic combinatorial proofs* (or ICPs) from [17] are the special cases where no modalities occur, that is, an ICP is a skew fibration $f\colon \mathcal{G} \to [\![F]\!]$ from an arena net $\mathcal{G}$ to the arena of a modality-free formula F.

Theorem 7.2 (Soundness and Completeness) *If F is a formula and $\mathsf{X} \in \{\mathsf{CK}, \mathsf{CD}\}$, then $\overset{\mathsf{LX}}{\vdash} F$ iff there is an X-ICP $f\colon \mathcal{G} \to [\![F]\!]$.*

Proof. By Theorem 3.1 there are P-formulas H and H' such that $F = \lfloor H \rfloor$ and H' is clean and $\overset{\mathsf{LX}}{\vdash} F \iff \overset{\mathsf{LX}^{\circ}_{\ell}}{\vdash} H' \overset{\{\mathsf{c}^{\bullet}_{\downarrow}, \mathsf{w}^{\bullet}_{\downarrow}\}}{\vdash} H$. In Theorem 5.6 we have shown that $\overset{\mathsf{LX}^{\circ}_{\ell}}{\vdash} H'$ iff there is an X-arena net $\mathcal{G}$ with H' the formula associated to $\mathcal{G}$. In Section 6 we have shown that $H' \overset{\{\mathsf{c}^{\bullet}_{\downarrow}, \mathsf{w}^{\bullet}_{\downarrow}\}}{\vdash} H$ iff there is a skew fibration $f : [\![\mathcal{H}']\!] \to [\![H]\!]$. This is equivalent to having an X-ICP $f : \mathcal{G} \to [\![F]\!]$, since $[\![H]\!] = [\![F]\!]$. □

Lemma 7.3 *Let $\mathsf{X} \in \{\mathsf{CK}, \mathsf{CD}\}$. If $\mathcal{H}$ and $\mathcal{G}$ are 2-dags and $f\colon V_{\mathcal{H}} \to V_{\mathcal{G}}$, then it can be checked in polynomial time (in the size of $\mathcal{H}$ and $\mathcal{G}$) whether f is an X-ICP.*

Proof. All the following checks can be done in polynomial time: that a 2-dag $\mathcal{G}$ is a modal arena; that a modal arena is an X-arena net; and that a map between two modal arenas is a skew fibration. □

Corollary 7.4 *Let $\mathsf{X} \in \{\mathsf{CK}, \mathsf{CD}\}$. Then the X-ICPs form a sound and complete proof system in the sense of Cook and Reckhow [10].*

8 On Proof Equivalence for Constructive Modal Logics

Let us now compare various notions of proof equivalence in constructive modal logics, building from the previous results in [17], where the authors show that intuitionistic combinatorial proofs capture a finer notion of proof equivalence than the one induced by the *simply typed lambda calculus* or by *winning innocent strategies* from games semantics [1,22,27].

In the following, we use $\equiv$ to denote the proof equivalence over derivations generated by *independent rule permutations*, that is, permutations of infer-

$$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}\,2\times\mathsf{C}}{\Gamma,A\wedge B\vdash C}\,\wedge^{\mathsf{L}} \quad\equiv_{\mathsf{c}}\quad \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}\,2\times\wedge^{\mathsf{L}}}{\Gamma,A\vdash B}\,\mathsf{C}$$

$$\dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}\,2\times\mathsf{W}}{\Gamma,A\wedge B\vdash C}\,\wedge^{\mathsf{L}} \quad\equiv_{\mathsf{c}}\quad \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\,\mathsf{W}$$

$$\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\,\mathsf{C}}{\Gamma,A,A\vdash B}\,\mathsf{W} \quad\equiv_{\mathsf{c}}\quad \Gamma,A,A\vdash B$$

$$\dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\,\mathsf{W}}{\Gamma,A\vdash B}\,\mathsf{C} \quad\equiv_{\mathsf{c}}\quad \Gamma,A\vdash B$$

$$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B\vdash C}{\Delta,B\vdash C}\,\mathsf{W}}{\Gamma,\Delta,A\supset B\vdash C}\,\supset^{\mathsf{L}} \quad\equiv_{\mathsf{e}}\quad \dfrac{\Delta,B\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\,\mathsf{W}$$

$$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\,\mathsf{C}}{\Gamma,A\supset B\vdash C}\,\supset^{\mathsf{L}} \quad\equiv_{\mathsf{u}}\quad \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\,\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\,\supset^{\mathsf{L}}}{\Gamma,\Delta,A\supset B\vdash C}\,\mathsf{C}$$

$$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\,\mathsf{W}}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{K}^{\Box} \quad\equiv_{\Box\mathsf{c}}\quad \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}\,\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{W}$$

$$\dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}\,\mathsf{C}}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{K}^{\Box} \quad\equiv_{\Box\mathsf{c}}\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}\,\mathsf{K}^{\Box}}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{C}$$

$$\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}\,\mathsf{W}}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}\,\mathsf{K}^{\Diamond} \quad\equiv_{\Box\mathsf{c}}\quad \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}\,\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}\,\mathsf{W}$$

$$\dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}\,\mathsf{C}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\,\mathsf{K}^{\Box} \quad\equiv_{\Box\mathsf{c}}\quad \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}\,\mathsf{K}^{\Box}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\,\mathsf{C}$$

$$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\,\mathsf{W}}{\Box\Gamma,\Box B\vdash\Diamond A}\,\mathsf{D} \quad\equiv_{\Box\mathsf{c}}\quad \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Diamond A}\,\mathsf{D}}{\Box\Gamma,\Box B\vdash\Diamond A}\,\mathsf{W}$$

$$\dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}\,\mathsf{C}}{\Box\Gamma,\Box B\vdash\Diamond A}\,\mathsf{D} \quad\equiv_{\Box\mathsf{c}}\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Diamond A}\,\mathsf{D}}{\Box\Gamma,\Box B\vdash\Diamond A}\,\mathsf{C}$$

$$\dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\,\mathsf{W}}{\Box\Gamma,\Diamond B\vdash\Diamond A}\,\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\,\mathsf{W} \quad\equiv_{\Diamond\mathsf{w}}\quad \dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,C\vdash A}\,\mathsf{W}}{\Box\Gamma,\Diamond C\vdash\Diamond A}\,\mathsf{K}^{\Diamond}}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\,\mathsf{W}$$

Fig. 10. Non independent rule permutations

ence rules whose active formulas and principal formulas are disjoint, as in the examples shown in Figure 9. Then, in Figure 10, we show examples of rule permutations which are non-independent. Based on these, we define the following proof equivalences:

$$\equiv_{\mathsf{ICP}} := (\equiv\cup\equiv_{\mathsf{e}}\cup\equiv_{\mathsf{c}}) \qquad \equiv_{\lambda} := (\equiv_{\mathsf{ICP}}\cup\equiv_{\mathsf{u}}) \qquad \equiv_{\mathsf{WIS}} := (\equiv_{\lambda}\cup\equiv_{\Box\mathsf{c}})$$

Note that $\equiv_{\mathsf{ICP}}$ and $\equiv_{\lambda}$ can be defined in the same way for the non-modal case. Indeed, it has been shown in [17] that $\equiv_{\mathsf{ICP}}$ is the proof equivalence induced by intuitionistic combinatorial proofs. We extend this result to the modal case in Theorem 8.1 below. In the modality-free case, $\equiv_{\lambda}$ corresponds to the proof identifications made by the simply-typed λ-calculus, and we conjecture that in the case with modalities, the proof equivalence $\equiv_{\lambda}$ is the same as the one induced by λ-terms/natural deduction proofs presented in [7] for constructive modal logics. We also conjecture that the proof equivalence $\equiv_{\mathsf{WIS}}$ is the same as the one induced by the winning strategies presented in [3]. However, it is worth remarking that even though $\equiv_{\Diamond\mathsf{w}}$ seems to be in the same spirit as $\equiv_{\Box\mathsf{c}}$, this permutation is even beyond the winning strategies of [3]. For this reason it is listed separately in Figure 10. Note that according to our conjecture, there is no one-to-one correspondence between winning innocent strategies and λ-terms (natural deduction proofs) for constructive modal logics. This is in contrast with the result in propositional intuitionistic logic [22], where $\equiv_\lambda$ and $\equiv_{\mathsf{WIS}}$ coincide.

Theorem 8.1 *Let* $\mathsf{X} \in \{\mathsf{CK}, \mathsf{CD}\}$ *and let* $\mathfrak{D}$ *and* $\mathfrak{D}'$ *be derivations in* LX. *Then we have* $\mathfrak{D} \equiv_{\mathsf{ICP}} \mathfrak{D}'$ *iff* $\mathfrak{D}$ *and* $\mathfrak{D}'$ *are represented by the same* X-ICP.

Proof. This is a direct consequence of the result on intuitionistic combinatorial proofs in [17]. It suffices to observe that if weakening and contraction rules could be permuted below/above K-rules, this would change the number of modalities handled by these rules. However, in CK-ICPs there is a one-to-one correspondence between $\sim$-classes and applications of K- and D-rules and at the same time a one-to-one correspondence between the number of modalities handled by each of these rules and the vertices in such $\sim$-equivalence class. Therefore the equivalence relation $\equiv_{\mathsf{ICP}}$ does not allow to permute weakening and contraction rules over K- and D-rules. □

The rule permutations in $\equiv_{\Box\mathsf{c}}$ are well-known in linear logic, as they correspond to the possibility of moving both weakening and contraction gates outside a !?-box in multiplicative exponential proof nets (see the notion of generalized ?-nodes introduced in [11] allowing to capture both the rule permutations in $\equiv_{\mathsf{c}}$ and $\equiv_{\Box\mathsf{c}}$). It has been observed before (see, e.g., [2]), that including the $\equiv_{\Box\mathsf{c}}$ rule permutations in the proof equivalence of classical linear logic makes proof equivalence **PSPACE**-complete. This immediately follows from the result in [16] about **PSPACE**-completeness of proof equivalence for multiplicative linear logic with units. More precisely, multiplicative linear logic proof nets require each $\bot$-gate to be attached to an axiom by a so-called "jump" in order to guarantee a polynomial proof equivalence.

A similar phenomenon occurs in the constructive modal logics studied in our paper since in combinatorial proof each $\top^\bullet$-rule instance is linked to a K- or a D-rule instance occurring below it. Since rule permutations in $\equiv_{\Box\mathsf{c}}$ or $\equiv_{\Diamond\mathsf{w}}$ trigger a "jump-rewiring" mechanism similar to the one observed in [16], we conjecture that proof equivalence including these permutations is **PSPACE**-hard.

9 Conclusions and Future Works

We have presented the syntax of combinatorial proofs for the disjunction-free fragment of the constructive modal logics CK and CD. We have proved that (1) this syntax is a sound and complete proof system in the sense of Cook and Reckhow [10], and that (2) it enforces a notion of proof equivalence which is finer than the one provided by natural deduction proofs, but still coarser than plain sequent calculus.

In future work we want to further investigate the various notions of proof equivalence for constructive modal logics, and in a next step study combinatorial proofs for other variants of intuitionistic modal logics.

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