A Monadic Logic of Ordered Abelian Groups

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Abstract

A many-valued modal logic with connectives interpreted in the ordered additive group of real numbers is introduced as a modal counterpart of the one-variable fragment of a (monadic) first-order real-valued logic. It is shown that the logic is decidable and admits an interpretation of the one-variable fragment of first-order Łukasiewicz logic. Completeness of an axiom system for the modal-multiplicative fragment is established via a Herbrand theorem for its first-order counterpart. A functional representation theorem is then proved for a class of monadic lattice-ordered abelian groups and used to establish completeness of an axiom system for the full logic.

Keywords: Modal Logic, Ordered Groups, Łukasiewicz Logic, Monadic Fragments.

1 Introduction

Many-valued modal logics with connectives interpreted in the ordered additive group of real numbers have been studied in a wide range of different settings. For example, modal logics based on the semantics of Łukasiewicz logic with truth values in the real unit interval have been considered as the basis for fuzzy description logics (see, e.g., [2, 21, 29]), logics for reasoning about belief and probabilities (see, e.g., [16–18, 20]), a Łukasiewicz mu-calculus [25], and as a fragment of continuous logic [4]. Such logics have also been studied from a purely algebraic perspective (see, e.g., [9, 11, 14, 23]) and appear in the guise of lattice-ordered groups (ℓ-groups, for short) with a (co-)nucleus in the study of semantics for substructural logics (see, e.g., [19, 26]). The appeal of these logics is clear: they make use of familiar arithmetical operations on the real numbers and well-studied computational methods (e.g., linear programming), and they relate to groups, arguably the most fundamental structures of classical algebra.

In [15], a minimal real-valued modal logic K(A) was defined as an extension of Abelian logic, the logic of abelian ℓ-groups, introduced independently in [24] as a relevant logic and [8] as a comparative logic. Among the advantages of focussing on modal extensions of Abelian logic are that the language is rich enough to interpret other logics (e.g., modal extensions of Łukasiewicz logic),

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$$
\begin{array}{rl}
\text{(B)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(C)} & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \\
\text{(I)} & \varphi \to \varphi \\
\text{(A)} & ((\varphi \to \psi) \to \psi) \to \varphi \\
(+1) & \varphi \to (\psi \to (\varphi + \psi)) \\
(+2) & (\varphi \to (\psi \to \chi)) \to ((\varphi + \psi) \to \chi) \\
(\overline{0}\,1) & \overline{0} \\
(\overline{0}\,2) & \varphi \to (\overline{0} \to \varphi) \\
(\wedge 1) & (\varphi \wedge \psi) \to \varphi \\
(\wedge 2) & (\varphi \wedge \psi) \to \psi \\
(\wedge 3) & ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)) \\
(\vee 1) & \varphi \to (\varphi \vee \psi) \\
(\vee 2) & \psi \to (\varphi \vee \psi) \\
(\vee 3) & ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)
\end{array}
$$

$$
\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(mp)} \qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi}\ \text{(adj)}
$$

Fig. 1. An Axiom System for Abelian Logic

the semantics are based directly on structures studied intensively in algebra and computer science, and there exists a natural separation between the group (multiplicative) and lattice (additive) fragments of the logics. Indeed, in [15], a sequent calculus, an axiom system, and a complexity result were obtained for the modal-multiplicative fragment of K(A) as first steps towards addressing the corresponding (much more challenging) problems for the full logic.

In this paper, we introduce a real-valued modal logic S5(A) as the modal counterpart of the one-variable fragment of (monadic) first-order Abelian logic. It is easily proved that S5(A) is decidable and admits an interpretation of the one-variable fragment of first-order Łukasiewicz logic axiomatized in [28]. The main contribution of the paper lies rather with the two distinct methods used to establish completeness results. First, we make use of a Herbrand theorem for the first-order counterpart of S5(A) and basic facts from linear programming to give a syntactic completeness proof for an axiomatization of the modal-multiplicative fragment. For an axiomatization of the full logic, we give an algebraic completeness proof using monadic abelian ℓ-groups, which, similarly to monadic Heyting algebras (see [5,7]) and MV-algebras (see [9,11,14]), may be viewed as abelian ℓ-groups with certain "relatively complete" subalgebras. We adapt a method used in [9] to prove a functional representation theorem for monadic MV-algebras to obtain a similar theorem for monadic abelian ℓ-groups, and then establish completeness with respect to the real-valued semantics via a partial embedding lemma for linearly ordered abelian groups.

2 A Real-Valued Monadic Logic

In this section, we introduce a many-valued modal logic defined over the ordered abelian group $\mathbf{R} = \langle \mathbb{R}, \min, \max, +, -, 0\rangle$ and prove a Herbrand theorem for the corresponding one-variable fragment of a (monadic) first-order Abelian logic.

Let $\mathcal{L}_{\mathrm{A}}$ be a propositional language with binary connectives $+$, $\to$, $\wedge$, and $\vee$, and a constant $\overline{0}$, where $\neg\varphi := \varphi \to \overline{0}$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, $0\varphi := \overline{0}$, and $(n+1)\varphi := \varphi + (n\varphi)$ $(n \in \mathbb{N})$. Let us also denote by $\mathrm{Fm}(\mathcal{L})$ the set of formulas for any propositional language $\mathcal{L}$ over a countably infinite set of variables $\{p_i \mid i \in \mathbb{N}\}$. An axiomatization of Abelian logic — a single-constant version of multiplicative additive intuitionistic linear logic extended with the axiom schema (A) — is presented in Fig. 1 that is complete with respect to both the logical matrix $\langle \mathbf{R}, \mathbb{R}^{\geq 0}\rangle$ and the variety of abelian ℓ-groups (defined in Section 5).

Now let $\mathcal{L}_{\mathrm{A}}^{\Box}$ be $\mathcal{L}_{\mathrm{A}}$ extended with a unary connective $\Box$, where $\Diamond\varphi := \neg\Box\neg\varphi$. An S5(A)-*model* is an ordered pair $\mathfrak{M} = \langle W, V\rangle$ consisting of a non-empty set W and a function $V\colon \{p_i \mid i \in \mathbb{N}\} \times W \to \mathbb{R}$ such that for each $i \in \mathbb{N}$ the function $V_i\colon W \to \mathbb{R};\ w \mapsto V(p_i, w)$ is bounded [2]; V is then extended to the function $V\colon \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box}) \times W \to \mathbb{R}$ as follows:

$$\begin{aligned}
V(\overline{0}, w) &= 0 & V(\varphi \wedge \psi, w) &= \min(V(\varphi, w), V(\psi, w))\\
V(\varphi + \psi, w) &= V(\varphi, w) + V(\psi, w) & V(\varphi \vee \psi, w) &= \max(V(\varphi, w), V(\psi, w))\\
V(\varphi \to \psi, w) &= V(\psi, w) - V(\varphi, w) & V(\Box\varphi, w) &= \bigwedge\{V(\varphi, u) \mid u \in W\}.
\end{aligned}$$

By calculation, also

$$V(\neg\varphi, w) = -V(\varphi, w) \quad \text{and} \quad V(\Diamond\varphi, w) = \bigvee\{V(\varphi, u) \mid u \in W\}.$$

A formula $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$ is said to be *valid* in $\mathfrak{M}$ if $V(\varphi, w) \geq 0$ for all $w \in W$. If φ is valid in all S5(A)-models, it is called S5(A)-*valid*, written $\models_{\mathrm{S5(A)}} \varphi$.

The logic S5(A) corresponds (as expected) to the one-variable fragment of a (monadic) first-order logic. Consider a first-order language with unary predicate symbols $P_0, P_1, \ldots$ and constants $c_0, c_1, \ldots$. We denote by Fm the set of first-order formulas for this language defined using the propositional connectives of $\mathcal{L}_{\mathrm{A}}$ and the universal quantifier $\forall$ over a countably infinite set of object variables, defining $(\exists x)\alpha := \neg(\forall x)\neg\alpha$. For convenience, we also often write $\bar{c}$ or $\bar{x}$ to denote an n-tuple of constants or variables, and, given $\bar{c} = c_1, \ldots, c_n$ and $\bar{d} = d_1, \ldots, d_m$, let $\bar{d} \subseteq \bar{c}$ stand for $\{d_1, \ldots, d_m\} \subseteq \{c_1, \ldots, c_n\}$.

A $\forall$A-*interpretation* $\mathcal{I} = \langle D_{\mathcal{I}}, v_{\mathcal{I}}\rangle$ consists of a non-empty set $D_{\mathcal{I}}$ and a function $v_{\mathcal{I}}$ that maps terms (constants and variables) to elements of $D_{\mathcal{I}}$, and each P_i $(i \in \mathbb{N})$ to a bounded function from $D_{\mathcal{I}}$ to $\mathbb{R}$. The function $v_{\mathcal{I}}$ is then extended to Fm by defining $v_{\mathcal{I}}(P_i(t)) = v_{\mathcal{I}}(P_i)(v_{\mathcal{I}}(t))$ for each $i \in \mathbb{N}$ and term t, and then inductively, where $v_{\mathcal{I}}[x \mapsto a]$ denotes the map that sends x to

[2] A function $f\colon A \to \mathbb{R}$ is *bounded* if there exists $r \in \mathbb{R}$ such that $|f(a)| \leq r$ for all $a \in A$.

a and coincides elsewhere with $v_{\mathcal{I}}$,

$$\begin{aligned} v_{\mathcal{I}}(\overline{0}) &= 0 & v_{\mathcal{I}}(\alpha \wedge \beta) &= \min(v_{\mathcal{I}}(\alpha), v_{\mathcal{I}}(\beta)) \\ v_{\mathcal{I}}(\alpha + \beta) &= v_{\mathcal{I}}(\alpha) + v_{\mathcal{I}}(\beta) & v_{\mathcal{I}}(\alpha \vee \beta) &= \max(v_{\mathcal{I}}(\alpha), v_{\mathcal{I}}(\beta)) \\ v_{\mathcal{I}}(\alpha \to \beta) &= v_{\mathcal{I}}(\beta) - v_{\mathcal{I}}(\alpha)) & v_{\mathcal{I}}((\forall x)\alpha) &= \bigwedge\{v_{\mathcal{I}}[x \mapsto a](\alpha) \mid a \in D\}. \end{aligned}$$

We say that $\mathcal{I}$ *satisfies* $\alpha \in Fm$ if $v_{\mathcal{I}}(\alpha) \geq 0$ and that α is $\forall$A-*valid*, written $\models_{\forall\mathrm{A}} \alpha$, if it is satisfied by all $\forall$A-interpretations.

Now let Fm_1 denote the set of formulas in Fm that contain at most one object variable x and no constants. For each $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$, let $\alpha_\varphi \in Fm_1$ be the result of replacing occurrences of $\Box$ by $(\forall x)$ and occurrences of p_i $(i \in \mathbb{N})$ by $P_i(x)$, and, conversely, for any $\alpha \in Fm_1$, let $\varphi_\alpha \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$ be the result of replacing occurrences of $(\forall x)$ by $\Box$ and occurrences of $P_i(x)$ $(i \in \mathbb{N})$ by p_i. Equivalences between S5(A)-validity and $\forall$A-validity then follow directly from the preceding definitions.

Proposition 2.1 *For any* $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$ *and* $\alpha \in Fm_1$,

$$\models_{\mathrm{S5(A)}} \varphi \iff \models_{\forall\mathrm{A}} \alpha_\varphi \quad \textit{and} \quad \models_{\forall\mathrm{A}} \alpha \iff \models_{\mathrm{S5(A)}} \varphi_\alpha.$$

It is not hard to check that $\forall$A-validity is preserved by all quantifier-shifts; that is, for any $\alpha, \beta \in Fm$, variable x that does not occur in β, and $\star \in \{+, \wedge, \vee\}$,

$$\begin{array}{ll} \models_{\forall\mathrm{A}} (\forall x)(\alpha \star \beta) \leftrightarrow ((\forall x)\alpha \star \beta) & \models_{\forall\mathrm{A}} (\exists x)(\alpha \star \beta) \leftrightarrow ((\exists x)\alpha \star \beta) \\ \models_{\forall\mathrm{A}} (\forall x)(\alpha \to \beta) \leftrightarrow ((\exists x)\alpha \to \beta) & \models_{\forall\mathrm{A}} (\exists x)(\alpha \to \beta) \leftrightarrow ((\forall x)\alpha \to \beta) \\ \models_{\forall\mathrm{A}} (\forall x)(\beta \to \alpha) \leftrightarrow (\beta \to (\forall x)\alpha) & \models_{\forall\mathrm{A}} (\exists x)(\beta \to \alpha) \leftrightarrow (\beta \to (\exists x)\alpha). \end{array}$$

Hence for any $\alpha \in Fm$, there exists a prenex $\beta \in Fm$ such that $\models_{\forall\mathrm{A}} \alpha \leftrightarrow \beta$. Moreover, the following Herbrand theorem holds for existential sentences. [3]

Theorem 2.2 *For any quantifier-free formula* $\alpha \in Fm$ *with free variables in* $\bar{x} = x_1, \ldots, x_m$ *and constants in* $\bar{c} = c_1, \ldots, c_n$,

$$\models_{\forall\mathrm{A}} (\exists \bar{x})\alpha \iff \models_{\forall\mathrm{A}} \bigvee\{\alpha(\bar{d}) \mid \bar{d} \subseteq \bar{c}\}.$$

Proof. The right-to-left direction follows using the easily-verified fact that $\models_{\forall\mathrm{A}} \beta(c) \to (\exists y)\beta(y)$ for any $\beta \in Fm$ and constant c. For the converse, we suppose contrapositively that $\not\models_{\forall\mathrm{A}} \bigvee\{\alpha(\bar{d}) \mid \bar{d} \subseteq \bar{c}\}$. Then there exists a $\forall$A-interpretation $\langle D_{\mathcal{I}}, v_{\mathcal{I}}\rangle$ such that $v_{\mathcal{I}}(\alpha(\bar{d})) < 0$ for all $\bar{d} \subseteq \bar{c}$. Consider now the $\forall$A-interpretation $\langle D'_{\mathcal{I}}, v'_{\mathcal{I}}\rangle$ such that $D'_{\mathcal{I}} = \{v_{\mathcal{I}}(c_1), \ldots, v_{\mathcal{I}}(c_n)\}$ and $v'_{\mathcal{I}}$ coincides on $c_1, \ldots, c_n$ with $v_{\mathcal{I}}$ and maps each P_i $(i \in \mathbb{N})$ to the restriction of $v_{\mathcal{I}}(P_i)$ to $D'_{\mathcal{I}}$. Then $v'_{\mathcal{I}}((\exists \bar{x})\alpha) = \bigvee\{v_{\mathcal{I}}(\alpha(\bar{d})) \mid \bar{d} \subseteq \bar{c}\} < 0$. So $\not\models_{\forall\mathrm{A}} (\exists \bar{x})\alpha$. $\Box$

[3] Note that if the logic $\forall$A is extended to allow non-constant function symbols and predicate symbols of arbitrary arity, it will admit Skolemization. However, the logic will then, as in the case of first-order Łukasiewicz logic (see [3, 12] for details), admit only an "approximate Herbrand theorem".

For any $\alpha \in Fm_1$, replacing any free variable x in α with a new constant, then iteratively replacing each positive occurrence of a subformula $(\forall x)\alpha'(x)$ with $\alpha'(c)$ for a new constant c, and finally shifting quantifiers, yields an existential sentence $\beta \in Fm$ such that $\models_{\forall \mathrm{A}} \alpha \iff \models_{\forall \mathrm{A}} \beta$. Theorem 2.2 now tells us that α is $\forall$A-valid if and only if a certain quantifier-free sentence is $\forall$A-valid. But validity of quantifier-free sentences can be checked in the ordered additive group $\mathbf{R}$ and is decidable [30], so we obtain the following result.

Corollary 2.3 S5(A)*-validity is decidable.*

3 The One-Variable Fragment of Łukasiewicz Logic

In this section, we prove that the one-variable fragment of first-order Łukasiewicz logic axiomatized as a many-valued modal logic by Rutledge in [28] (see also [9, 11, 14, 22]) may be viewed as a fragment of the logic S5(A).

Let $\mathcal{L}_{\text{Ł}}^{\Box}$ be a propositional language with a binary connective $\supset$ and unary connectives $\sim$ and $\Box$. An S5(Ł)*-model* is an ordered pair $\mathfrak{M} = \langle W, V\rangle$ consisting of a non-empty set W and a function $V\colon \{p_i \mid i \in \mathbb{N}\} \times W \to [0,1]$ that is extended to a function $V\colon \mathrm{Fm}(\mathcal{L}_{\text{Ł}}^{\Box}) \times W \to [0,1]$ by

$$V(\sim\varphi, w) = 1 - V(\varphi, w)$$
$$V(\varphi \supset \psi, x) = \min(1, 1 - V(\varphi, w) + V(\psi, w))$$
$$V(\Box\varphi, w) = \bigwedge\{V(\varphi, u) \mid u \in W\}.$$

An $\mathcal{L}_{\text{Ł}}^{\Box}$-formula φ is said to be *valid* in $\mathfrak{M}$ if $V(\varphi, w) = 1$ for all $w \in W$, and called S5(Ł)*-valid*, written $\models_{\mathrm{S5(Ł)}} \varphi$, if it is valid in all S5(Ł)-models. As in the case of S5(A) considered in Section 2, it is straightforward to prove that S5(Ł)-validity corresponds to validity in the one-variable fragment of first-order Łukasiewicz logic (see [28] for details).

Let us fix $\bot := \Box p_0 \wedge \neg\Box p_0$, noting that this constant is interpreted as the same nonpositive real number in all worlds of an S5(A)-model. We define the following map from the set $\mathrm{Fm}_0(\mathcal{L}_{\text{Ł}}^{\Box})$ of $\mathcal{L}_{\text{Ł}}^{\Box}$-formulas defined over $\{p_i \mid i \in \mathbb{N}^+\}$ to $\mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$:

$$p_i^* = (p_i \wedge \overline{0}) \vee \bot \qquad (\sim\varphi)^* = \varphi^* \to \bot$$
$$(\varphi \supset \psi)^* = (\varphi^* \to \psi^*) \wedge \overline{0} \qquad (\Box\varphi)^* = \Box\varphi^*.$$

We show that $*$ preserves validity between S5(Ł) and S5(A) by identifying the value of $\varphi \in \mathrm{Fm}_0(\mathcal{L}_{\text{Ł}}^{\Box})$ in $[0,1]$ with the value of $\varphi^* \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$ in $[\bot, 0]$.

Theorem 3.1 *Let* $\varphi \in \mathrm{Fm}_0(\mathcal{L}_{\text{Ł}}^{\Box})$*. Then* $\models_{\mathrm{S5(Ł)}} \varphi$ *if and only if* $\models_{\mathrm{S5(A)}} \varphi^*$*.*

Proof. Suppose first that φ is not valid in an S5(Ł)-model $\mathfrak{M} = \langle W, V\rangle$. Then $V(\varphi, x_0) < 1$ for some $x_0 \in W$. We consider the S5(A)-model $\mathfrak{M}' = \langle W, V'\rangle$ where $V'(p_0, x) = -1$ and $V'(p_i, x) = V(p_i, x) - 1$ $(i \in \mathbb{N}^+)$ for all $x \in W$, noting that $V'(\bot, x) = V'(\Box p_0 \wedge \neg\Box p_0, x) = -1$ for all $x \in W$. It suffices to prove that $V'(\psi^*, x) = V(\psi, x) - 1$ for all $x \in W$ and $\psi \in \mathrm{Fm}_0(\mathcal{L}_{\text{Ł}}^{\Box})$, since then $V'(\varphi^*, x_0) = V(\varphi, x_0) - 1 < 0$ and $\not\models_{\mathrm{S5(A)}} \varphi^*$. We proceed by induction on the size (number of symbols) of ψ. For the base case, we have $V'(p_i^*, x) = V'((p_i \wedge \overline{0}) \vee \bot) = V(p_i, x) - 1$ for each $i \in \mathbb{N}^+$. For the inductive step we obtain, using the induction hypothesis,

$$\begin{aligned}
V'((\psi_1 \supset \psi_2)^*, x) &= V'((\psi_1^* \to \psi_2^*) \wedge \overline{0}, x) \\
&= \min(V'(\psi_2^*, x) - V'(\psi_1^*, x), 0) \\
&= \min((V(\psi_2, x) - 1) - (V(\psi_1, x) - 1), 0) \\
&= \min(V(\psi_2, x) - V(\psi_1, x), 0) \\
&= \min(1 - V(\psi_1, x) + V(\psi_2, x), 1) - 1 \\
&= V(\psi_1 \supset \psi_2, x) - 1,
\end{aligned}$$

and, the case where ψ is $\sim\psi_1$ being very similar, for the modal case,

$$\begin{aligned}
V'((\Box\psi_1)^*, x) &= V'(\Box\psi_1^*, x) \\
&= \bigwedge\{V'(\psi_1^*, y) \mid y \in W\} \\
&= \bigwedge\{V(\psi_1, y) - 1 \mid y \in W\} \\
&= \bigwedge\{V(\psi_1, y) \mid y \in W\} - 1 \\
&= V(\Box\psi_1, x) - 1.
\end{aligned}$$

Suppose now conversely that φ^* is not valid in an S5(A)-model $\mathfrak{M} = \langle W, V\rangle$. That is, $V(\varphi^*, x_0) < 0$ for some $x_0 \in W$. Observe first that if $V(\Box p_0, x_0) = 0$, then, by a simple induction on the size of $\psi \in \mathrm{Fm}_0(\mathcal{L}_{\text{Ł}}^{\Box})$, we obtain $V(\psi^*, x) = 0$ for all $\psi \in \mathrm{Fm}_0(\mathcal{L}_{\text{Ł}}^{\Box})$ and $x \in W$, a contradiction. Hence $V(\Box p_0, x_0) \neq 0$. Moreover, by scaling (dividing $V(p_i, y)$ by $|V(\Box p_0, x_0)|$ for each $i \in \mathbb{N}^+$ and $x \in W$), we may assume that $V(\bot, x) = -1$ for all $x \in W$. We consider the S5(Ł)-model $\mathfrak{M}' = \langle W, V'\rangle$ where $V'(p_i, x) = \max(\min(V(p_i, x) + 1, 1), 0)$ for each $x \in W$ and $i \in \mathbb{N}^+$. It then suffices to prove that $V'(\psi, x) = V(\psi^*, x) + 1$ for all $\psi \in \mathrm{Fm}_0(\mathcal{L}_{\text{Ł}}^{\Box})$ and $x \in W$ by an easy induction on the size of ψ. $\Box$

The above proof can be extended to obtain an interpretation of the full first-order Łukasiewicz logic into a first-order Abelian logic. In particular, monadic first-order Łukasiewicz logic can be viewed as a fragment of the monadic logic $\forall$A defined in Section 2. Since the former has been shown by Bou in unpublished work to be undecidable, this is also the case for the latter.

4 The Modal-Multiplicative Fragment

In this section, we use the Herbrand theorem obtained in Section 2 to establish the completeness of an axiom system for the modal-multiplicative fragment of S5(A). [4] Let us consider first the axiom system $\mathcal{A}_{\mathrm{m}}$ defined over the language

[4] Note that we follow here standard terminology from the linear and substructural logic literature in referring to the *multiplicative* fragment of Abelian logic, even though the group multiplication for the real numbers is in fact addition.

$$
\begin{array}{rl}
\text{(K)} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\text{(T)} & \Box\varphi \to \varphi \\
\text{(5)} & \Diamond\varphi \to \Box\Diamond\varphi \\
\text{(M)} & \Box(\varphi + \varphi) \to (\Box\varphi + \Box\varphi)
\end{array}
$$

$$\frac{\varphi}{\Box\varphi}\ (\text{nec})$$

Fig. 2. Modal Axiom and Rule Schema

$\mathcal{L}_{\rm m}$ with connectives $+$, $\to$, and $\overline{0}$ by removing the axiom and rule schema for $\wedge$ and $\vee$ from those presented in Fig. 1 and adding

$$\frac{n\varphi}{\varphi}\ (\text{con}_n) \quad (n \geq 2).$$

It is not hard to show (see, e.g., [8, 24]) that $\mathcal{A}_{\rm m}$ is complete with respect to the multiplicative fragment of Abelian logic defined by the logical matrix $\langle\langle\mathbb{R}, +, -, 0\rangle, \mathbb{R}^{\geq 0}\rangle$.

Now let $\mathcal{L}_{\rm m}^{\Box}$ be the language extending $\mathcal{L}_{\rm m}$ with $\Box$ and let $\mathcal{S}5(\mathcal{A}_{\rm m})$ be the axiom system defined over $\mathcal{L}_{\rm m}^{\Box}$ by extending $\mathcal{A}_{\rm m}$ with the modal axiom and rule schema presented in Fig. 2. Soundness for this system is proved as usual by checking that the axioms are S5(A)-valid and the rules preserve S5(A)-validity.

Lemma 4.1 *Let $\varphi \in \mathrm{Fm}(\mathcal{L}_{\rm m}^{\Box})$. If $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \varphi$, then $\models_{\mathrm{S5(A)}} \varphi$.*

To prove completeness, we will make use of the fact that occurrences of $\Box$ can be shifted inwards and hence that every formula is provably equivalent in $\mathcal{S}5(\mathcal{A}_{\rm m})$ to a formula of modal depth at most one. For $\varphi, \psi \in \mathrm{Fm}(\mathcal{L}_{\rm m}^{\Box})$, let us write $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \varphi \equiv \psi$ to denote that $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \varphi \to \psi$ and $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \psi \to \varphi$.

Lemma 4.2 *For any $\varphi, \psi \in \mathrm{Fm}(\mathcal{L}_{\rm m}^{\Box})$,*

(i) $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \Box(\varphi + \Box\psi) \equiv \Box\varphi + \Box\psi$

(ii) $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \Box(\varphi + \neg\Box\psi) \equiv \Box\varphi + \neg\Box\psi$

(iii) $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \Box\Box\varphi \equiv \Box\varphi$

(iv) $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \Box\neg\Box\varphi \equiv \neg\Box\varphi$

(v) $\vdash_{\mathcal{S}5(\mathcal{A}_{\rm m})} \Box n\varphi \equiv n\Box\varphi$ *for all $n \in \mathbb{N}$.*

Proof. Derivations for (i)-(iv) are obtained, similarly to other "S5" logics, using the modal axiom schema (K), (T), and (5), and are omitted here. For (v), we note first that $n\Box\varphi \to \Box n\varphi$ is derivable in $\mathcal{S}5(\mathcal{A}_{\rm m})$ for $n \in \mathbb{N}$ using (nec) and (K) together with the axioms of $\mathcal{A}_{\rm m}$. For the converse, observe that $\Box(2^k)\varphi \to (2^k)\Box\varphi$ is derivable in $\mathcal{S}5(\mathcal{A}_{\rm m})$ for $k \in \mathbb{N}$ using repeated applications of (M), (mp), and the $\mathcal{A}_{\rm m}$-derivable formula $\psi_1 \to (\psi_2 \to (\psi_1 + \psi_2))$. But then also for any $n \geq 1$, we can choose $k \in \mathbb{N}$ such that $2^k \geq n$ and observe that $(\Box n\varphi + (2^k - n)\Box\varphi) \to \Box(2^k)\varphi$ and hence $(\Box n\varphi + (2^k - n)\Box\varphi) \to (2^k)\Box\varphi$ are derivable in $\mathcal{S}5(\mathcal{A}_{\rm m})$. Since $(((2^k - n)\Box\varphi) \to ((2^k - n)\Box\varphi)) \to \overline{0}$ is derivable in

$\mathcal{S}5(\mathcal{A}_{\mathrm{m}})$, also $\Box n\varphi \to n\Box\varphi$ is derivable in $\mathcal{S}5(\mathcal{A}_{\mathrm{m}})$ as required. Finally, for the case $n = 0$ just note that $\Box\overline{0} \to \overline{0}$ is an instance of (T). $\Box$

Let us write $\sum_{i=1}^{n} \varphi_i$ to denote $\varphi_1 + \ldots + \varphi_n$ for any $\varphi_1, \ldots, \varphi_n \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$. An easy induction on modal depth using Lemma 4.2 (i)-(iv) yields the following normal form property for modal-multiplicative formulas. [5]

Lemma 4.3 *For any modal-multiplicative formula $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{m}}^{\Box})$, there exist multiplicative formulas $\varphi_0, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \in \mathrm{Fm}(\mathcal{L}_{\mathrm{m}})$ such that*

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \varphi \;\equiv\; \varphi_0 \;+\; \sum_{i=1}^{n} \Box\varphi_i \;+\; \sum_{j=1}^{m} \neg\Box\psi_j.$$

Let Fm_m denote the set of first-order formulas in Fm not containing $\wedge$ or $\vee$. The following lemma is a consequence of a well-known duality principle for linear programming stating that either one or another linear system has a solution, but not both (see, e.g., [13]): more precisely, for any $M \in \mathbb{Z}^{m\times n}$, either $y^T M < \mathbf{0}$ for some $y \in \mathbb{R}^m$ or $Mx = \mathbf{0}$ for some $x \in \mathbb{N}^n \backslash \{\mathbf{0}\}$.

Lemma 4.4 *For any quantifier-free and variable-free $\alpha_1, \ldots, \alpha_n \in Fm_m$,*

$$\models_{\forall\mathrm{A}} \alpha_1 \vee \ldots \vee \alpha_n \quad\Longleftrightarrow\quad \models_{\forall\mathrm{A}} \sum_{i=1}^{n} \lambda_i \alpha_i \text{ for some } \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ not all } 0.$$

Proof. Let $\beta_1, \ldots, \beta_m$ denote the m ground atoms $P_i(c_j)$ that occur in $\alpha_1, \ldots, \alpha_n$. We may assume without loss of generality that $\alpha_j = \sum_{i=1}^{m} m_{ij}\beta_j$ for each $j \in \{1, \ldots, n\}$, where $M = (m_{ij}) \in \mathbb{Z}^{m\times n}$. Then $\models_{\forall\mathrm{A}} \alpha_1 \vee \ldots \vee \alpha_n$ if and only if there does not exist any $y \in \mathbb{R}^m$ such that $y^T M < \mathbf{0}$. Hence, by the duality principle mentioned above, $\models_{\forall\mathrm{A}} \alpha_1 \vee \ldots \vee \alpha_n$ if and only if $Mx = \mathbf{0}$ for some $x \in \mathbb{N}^n \backslash \{\mathbf{0}\}$, which is equivalent to the statement that $\models_{\forall\mathrm{A}} \sum_{i=1}^{n} \lambda_i \alpha_i$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all zero. $\Box$

We now have the tools required to prove our completeness theorem for $\mathcal{S}5(\mathcal{A}_{\mathrm{m}})$.

Theorem 4.5 *Let $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{m}}^{\Box})$. Then $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \varphi$ if and only if $\models_{\mathrm{S5(A)}} \varphi$.*

Proof. The left-to-right-direction is Lemma 4.1. For the converse, suppose that φ is S5(A)-valid. By Lemma 4.3, there exist $\varphi_0, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \in \mathrm{Fm}(\mathcal{L}_{\mathrm{m}})$ such that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \varphi \equiv \psi$, where

$$\psi \;=\; \varphi_0 \;+\; \sum_{i=1}^{n} \Box\varphi_i \;+\; \sum_{j=1}^{m} \neg\Box\psi_j.$$

By Lemma 4.1, also ψ is S5(A)-valid and it suffices to prove that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \psi$. Consider now the $\forall$A-valid (by Proposition 2.1) first-order formula

$$\alpha_\psi \;=\; \alpha_{\varphi_0}(x) \;+\; \sum_{i=1}^{n} (\forall x)\alpha_{\varphi_i}(x) \;+\; \sum_{j=1}^{m} \neg(\forall x)\alpha_{\psi_j}(x).$$

[5] It is not possible to obtain a similar normal form property for all $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$ simply by shifting boxes; e.g., $\Box(p \vee (q + \Box r))$ is not equivalent to any formula of modal depth ≤ 1.

Using generalization, renaming of variables, and quantifier shifts,

$$\models_{\forall\mathrm{A}} (\forall y_0)(\forall y_1)\ldots(\forall y_n)(\exists x_1)\ldots(\exists x_m)\big(\sum_{i=0}^{n}\alpha_{\varphi_i}(y_i) + \sum_{j=1}^{m}\neg\alpha_{\psi_j}(x_j)\big).$$

Hence also for constants $\bar{c} = c_0, c_1, \ldots, c_n$,

$$\models_{\forall\mathrm{A}} (\exists x_1)\ldots(\exists x_m)\big(\sum_{i=0}^{n}\alpha_{\varphi_i}(c_i) + \sum_{j=1}^{m}\neg\alpha_{\psi_j}(x_j)\big).$$

An application of Theorem 2.2 then yields, writing $\bar{d}$ for $d_1, \ldots, d_m$,

$$\models_{\forall\mathrm{A}} \bigvee\big\{\sum_{i=0}^{n}\alpha_{\varphi_i}(c_i) + \sum_{j=1}^{m}\neg\alpha_{\psi_j}(d_j) \mid \bar{d} \subseteq \bar{c}\big\}.$$

But then by Lemma 4.4, there exist $\lambda_{\bar{d}} \in \mathbb{N}$ for each $\bar{d} \subseteq \bar{c}$ not all 0 satisfying

$$\models_{\forall\mathrm{A}} \sum_{\bar{d}\subseteq\bar{c}}\lambda_{\bar{d}}\big(\sum_{i=0}^{n}\alpha_{\varphi_i}(c_i) + \sum_{j=1}^{m}\neg\alpha_{\psi_j}(d_j)\big).$$

Hence also, letting $\mu = \sum_{\bar{d}\subseteq\bar{c}}\lambda_{\bar{d}}$,

$$\models_{\forall\mathrm{A}} \sum_{i=0}^{n}\mu\alpha_{\varphi_i}(c_i) + \sum_{\bar{d}\subseteq\bar{c}}\lambda_{\bar{d}}\sum_{j=1}^{m}\neg\alpha_{\psi_j}(d_j).$$

Now let us rewrite the second part of this $\forall$A-valid formula to obtain

$$\models_{\forall\mathrm{A}} \sum_{i=0}^{n}\mu\alpha_{\varphi_i}(c_i) + \sum_{i=0}^{n}\sum_{j=1}^{m}\lambda_{ij}\neg\alpha_{\psi_j}(c_i),$$

for some λ_{ij} $(0 \le i \le n,\ 1 \le j \le m)$ such that $\sum_{i=0}^{n}\sum_{j=1}^{m}\lambda_{ij} = \sum_{\bar{d}\subseteq\bar{c}}\lambda_{\bar{d}} = \mu$.

Then for each $i \in \{0, 1, \ldots, n\}$, we must have

$$\models_{\forall\mathrm{A}} \mu\alpha_{\varphi_i}(c_i) + \sum_{j=1}^{m}\lambda_{ij}\neg\alpha_{\psi_j}(c_i).$$

So also, by Proposition 2.1,

$$\models_{\mathrm{S5(A)}} \mu\varphi_i + \sum_{j=1}^{m}\lambda_{ij}\neg\psi_j.$$

By the completeness of $\mathcal{A}_{\mathrm{m}}$ with respect to $\langle\langle\mathbb{R}, +, -, 0\rangle, \mathbb{R}^{\ge 0}\rangle$, it follows that for each $i \in \{0, 1, \ldots, n\}$,

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \mu\varphi_i + \sum_{j=1}^{m}\lambda_{ij}\neg\psi_j.$$

But then for each $i \in \{1,\ldots,n\}$, using (nec), (K), (T), and Lemma 4.2,

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \mu\Box\varphi_i + \sum_{j=1}^{m}\lambda_{ij}\neg\Box\psi_j \quad \text{and also} \quad \vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \mu\varphi_0 + \sum_{j=1}^{m}\lambda_{0j}\neg\Box\psi_j,$$

and using (mp) and the $\mathcal{A}_{\mathrm{m}}$-axiom (+1),

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \mu\varphi_0 \,+\, \sum_{i=1}^{n}\mu\Box\varphi_i \,+\, \mu\sum_{j=1}^{m}\neg\Box\psi_j.$$

Finally, an application of (con$_\mu$) yields $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathrm{m}})} \psi$ as required. $\Box$

In principle, this proof strategy can also be used to prove completeness for an axiom system for the full logic S5(A). New variables can be introduced to obtain a depth-one formula as in Lemma 4.3, and Theorem 2.2 can then be applied to the resulting existential sentence to obtain an S5(A)-valid disjunction of quantifier-free sentences. However, the presence of $\wedge$ and $\vee$ requires repeated applications of Lemma 4.4 and currently we are able only to prove completeness using this method for a system with a family of combinatorially defined axioms.

Let us remark finally that, as in the classical setting, the monadic logic $\forall$A restricted to Fm_m coincides (up to equivalence of sentences) with its one-variable fragment. Let $\alpha \in Fm_m$ be any sentence. Repeated applications of the quantifier-shift $\models_{\forall\mathrm{A}} (\forall x)(\alpha_1 + \alpha_2) \leftrightarrow ((\forall x)\alpha_1 + \alpha_2)$, where x is not free in α_2 yield a sentence $\beta \in Fm_m$ such that $\models_{\forall\mathrm{A}} \alpha \leftrightarrow \beta$ and no subformula $(\forall x)\beta'$ of β contains a free variable different to x. Hence we can rename all the bound variables in β to obtain a one-variable sentence $\chi \in Fm_m$ such that $\models_{\forall\mathrm{A}} \alpha \leftrightarrow \chi$. Since S5(A) is decidable (Corollary 2.3), first-order multiplicative Abelian logic provides a first interesting example (as far as we know) of a first-order infinite-valued logic that has a decidable monadic fragment.

5 Monadic Abelian ℓ-Groups

In this section, we introduce abelian ℓ-groups supplemented with a monadic operator as an algebraic semantics for S5(A). Following similar results for monadic Heyting algebras [5] and monadic MV-algebras [14], we describe a correspondence between these algebras and lattice-ordered abelian groups equipped with certain "relatively complete" subalgebras. We then use this correspondence to give a characterization of the "ideals" of these algebras.

An *abelian ℓ-group* is an algebraic structure $\mathbf{G} = \langle G, \wedge, \vee, +, -, 0\rangle$ such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, +, -, 0\rangle$ is an abelian group, and $+$ is compatible with the lattice order, i.e., $a \le b$ implies $a + c \le b + c$ for all $a, b, c \in G$. We call $\mathbf{G}$ an *abelian o-group* if the lattice order $\le$ is linear. A non-empty subset $H \subseteq G$ that is closed under the operations of $\mathbf{G}$ forms an *ℓ-subgroup* $\mathbf{H}$ of $\mathbf{G}$, where $\mathbf{H}$ is called an *ℓ-ideal* of $\mathbf{G}$ if it is also convex, i.e., if $a, b \in H$, $c \in G$, and $a \le c \le b$, then $c \in H$. For any ℓ-ideal $\mathbf{H}$ of $\mathbf{G}$, the set of right cosets of $\mathbf{H}$ in $\mathbf{G}$ forms an abelian ℓ-group $\mathbf{G}/\mathbf{H}$ with lattice order $H + a \le H + b :\Leftrightarrow a \le b + c$ for some $c \in H$. We refer to [1] for further details.

Example 5.1 The ordered additive group $\mathbf{R}$ encountered in Section 2 is an abelian o-group. Also important for our purposes are abelian ℓ-groups obtained as sets of functions from a set W to an abelian ℓ-group $\mathbf{G}$ with operations defined pointwise, denoted by $\mathbf{G}^W$. In particular, we will consider the case where $\mathbf{G}$ is an abelian o-group and the bounded functions from W to G form an ℓ-subgroup $\mathbf{B}(W, \mathbf{G})$ of $\mathbf{G}^W$.

A *monadic abelian ℓ-group* is an ordered pair $\langle \mathbf{G}, \Box \rangle$ consisting of an abelian ℓ-group $\mathbf{G}$ and a unary operator $\Box$ on G, with defined operator $\Diamond a := -\Box -a$, that satisfies for all $a, b \in G$,

(M1) $\Box(a+b) \leq \Box a + \Diamond b$	(M4) $\Box(a \wedge b) = \Box a \wedge \Box b$
(M2) $\Box a \leq a$	(M5) $\Diamond(a \wedge \Diamond b) = \Diamond a \wedge \Diamond b$
(M3) $\Diamond a = \Box \Diamond a$	(M6) $\Box(a+a) = \Box a + \Box a$.

A non-empty subset $H \subseteq G$ forms a *monadic ℓ-subgroup* $\langle \mathbf{H}, \Box \rangle$ of $\langle \mathbf{G}, \Box \rangle$ if $\mathbf{H}$ is an ℓ-subgroup of $\mathbf{G}$ that is closed under $\Box$.

Let $\mathsf{M\ell G}$ denote the variety of monadic abelian ℓ-groups. We call $\langle \mathbf{G}, \Box \rangle \in \mathsf{M\ell G}$ *functional* if $\mathbf{G}$ is an ℓ-subgroup of $\mathbf{B}(W, \mathbf{H})$ for a set W and abelian o-group $\mathbf{H}$, and for all $f \in G$, $x \in W$,

$$\Box f(x) = \bigwedge \{ f(y) \mid y \in W \}.$$

If $\Box f(x) = \min\{ f(y) \mid y \in W \}$ for all $f \in G$, $x \in W$, we call $\langle \mathbf{G}, \Box \rangle$ *witnessed*, and in the case where $\mathbf{H}$ is $\mathbf{R}$, we call $\langle \mathbf{G}, \Box \rangle$ *standard*.

Observe now that for any $\langle \mathbf{G}, \Box \rangle \in \mathsf{M\ell G}$, the set $\Box G := \{ \Box a \mid a \in G \} = \{ \Diamond a \mid a \in G \}$ forms an ℓ-subgroup $\Box \mathbf{G}$ of $\mathbf{G}$ satisfying for all $a \in G$,

$$\Box a = \bigvee \{ b \in \Box G \mid b \leq a \}.$$

More generally, an ℓ-subgroup $\mathbf{G}_0$ of an abelian ℓ-group $\mathbf{G}$ is *relatively complete* if $\bigvee \{ b \in \Box G \mid b \leq a \}$ exists for all $a \in G$, or, equivalently, the inclusion map of G_0 in G has a right adjoint $\Box_0 \colon G \to G_0$, i.e., for all $a \in G_0$ and $b \in G$,

$$a \leq \Box_0 b \iff a \leq b.$$

In this case, we obtain an algebraic structure $\langle \mathbf{G}, \Box_0 \rangle$ that satisfies conditions (M1)-(M4) in the definition of a monadic abelian ℓ-group. To ensure, however, that (M5) and (M6) are satisfied, we require also that for all $a, b \in G$,

$$\Box_0(a+a) = \Box_0 a + \Box_0 a \quad \text{and} \quad \Diamond_0(a \wedge \Diamond_0 b) = \Diamond_0 a \wedge \Diamond_0 b,$$

in which case $\langle \mathbf{G}, \Box_0 \rangle \in \mathsf{M\ell G}$ with $\Box_0 G = G_0$, and we call $\mathbf{G}_0$ *m-relatively complete*. Hence we obtain the following result.

Proposition 5.2 *There exists a one-to-one correspondence between monadic abelian ℓ-groups $\langle \mathbf{G}, \Box \rangle$ and ordered pairs $\langle \mathbf{G}, \mathbf{G_0} \rangle$ of abelian ℓ-groups such that $\mathbf{G_0}$ is an m-relatively complete ℓ-subgroup of $\mathbf{G}$.*

Example 5.3 The universe of any non-trivial relatively complete ℓ-subgroup of an abelian ℓ-group $\mathbf{B}(W, \mathbf{R})$ for some set W is a set of constant functions $\{f : W \to \{r\} \mid r \in H\}$, where $\mathbf{H}$ will be $\mathbf{R}$ if the ℓ-subgroup is m-relatively complete, and a one-generated ℓ-subgroup of $\mathbf{R}$ otherwise.

Given a monadic abelian ℓ-group $\langle \mathbf{G}, \Box \rangle$, we say that $\mathbf{K}$ is a *monadic ℓ-ideal* of $\langle \mathbf{G}, \Box \rangle$ if $\mathbf{K}$ is an ℓ-ideal of $\mathbf{G}$ and $a \in K$ implies $\Box a \in K$. It is straightforward to check that in this case, $\langle \mathbf{G}, \Box \rangle / \mathbf{K} := \langle \mathbf{G}/\mathbf{K}, \Box_K \rangle$ with $\Box_K(K+a) := K + \Box a$ is a monadic abelian ℓ-group.

Proposition 5.4 *The monadic ℓ-ideals of a monadic abelian ℓ-group $\langle \mathbf{G}, \Box \rangle$ and the ℓ-ideals of $\Box\mathbf{G}$ are in a one-to-one correspondence implemented by the maps $J \mapsto J \cap \Box G$ and $K \mapsto K^{\Box\Diamond} := \{a \in G \mid \Box a \in K$ and $\Diamond a \in K\}$.*

Proof. First consider any ℓ-ideal $\mathbf{K}$ of $\Box\mathbf{G}$. We show that $\mathbf{K}^{\Box\Diamond}$ is a monadic ℓ-ideal of $\mathbf{G}$. For closure under $-$, observe that if $a \in K^{\Box\Diamond}$ (i.e., $\Box a, \Diamond a \in K$), since $\mathbf{K}$ is an ℓ-ideal, $-\Box a = \Diamond{-a} \in K$ and $-\Diamond a = \Box{-a} \in K$, so $-a \in K^{\Box\Diamond}$. For closure under $+$, observe that if $a, b \in K^{\Box\Diamond}$ (i.e., $\Box a, \Box b, \Diamond a, \Diamond b \in K$), using property (M1) of monadic abelian ℓ-groups,

$$K \ni \Box a + \Box b \leq \Box(a+b) \leq \Diamond a + \Box b \in K$$
$$K \ni \Box a + \Diamond b \leq \Diamond(a+b) \leq \Diamond a + \Diamond b \in K,$$

so by convexity $\Box(a+b), \Diamond(a+b) \in K$ and hence $a + b \in K^{\Box\Diamond}$. Moreover, using properties (M4) and (M2),

$$K \ni \Box a \wedge \Box b = \Box(a \wedge b) \leq \Diamond(a \wedge b) \leq \Diamond a \wedge \Diamond b \in K,$$

so by convexity again, $a \wedge b \in K^{\Box\Diamond}$. Closure under $\Box$ is clear and convexity is a consequence of the monotonicity of $\Box$ and $\Diamond$. So $\mathbf{K}^{\Box\Diamond}$ is a monadic ℓ-ideal and, since $a = \Box a = \Diamond a$ for any $a \in K$, also $K = K^{\Box\Diamond} \cap \Box G$.

Now consider any monadic ℓ-ideal $\mathbf{J}$ of $\langle \mathbf{G}, \Box \rangle$. Since $\Box\mathbf{G}$ is an ℓ-subgroup of $\mathbf{G}$, it follows easily that $J \cap \Box G$ is the universe of an ℓ-ideal of $\Box\mathbf{G}$. Moreover, $\Box J \subseteq J \cap \Box G$, so $J \subseteq (J \cap \Box G)^{\Box\Diamond}$. Conversely, if $a \in (J \cap \Box G)^{\Box\Diamond}$, then $\Box a, \Diamond a \in J$ and, since $\Box a \leq a \leq \Diamond a$, by convexity, $a \in J$. So $J = (J \cap \Box G)^{\Box\Diamond}$ and we have shown that the maps implement a one-to-one correspondence. $\Box$

6 A Completeness Theorem

In this section, we prove the completeness with respect to S5(A)-validity of an axiom system $\mathcal{S}5(\mathcal{A})$ consisting of the axiom and rule schema for Abelian logic in Fig. 1, the modal axiom and rule schema in Fig. 2, and the axiom schema

$$(\wedge\Box)\ \ (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi) \qquad (\wedge\Diamond)\ \ (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \Diamond\psi).$$

First, a standard Lindenbaum-Tarski argument can be used to prove that $\mathcal{S}5(\mathcal{A})$ is complete with respect to the variety $\mathsf{M\ell G}$ of monadic abelian ℓ-groups.

Lemma 6.1 *Let $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathrm{A}}^{\Box})$. Then $\vdash_{\mathcal{S}5(\mathcal{A})} \varphi$ if and only if $\mathsf{M\ell G} \models \overline{0} \leq \varphi$.*

The remainder of this section is dedicated to proving the completeness of $\mathcal{S}5(\mathcal{A})$ with respect to first the functional members and then the standard members of $\mathsf{M}\ell\mathsf{G}$. As a first step towards these results, we show that it suffices to consider monadic abelian ℓ-groups $\langle \mathbf{G}, \Box \rangle$ such that $\Box \mathbf{G}$ is linearly ordered, which, for convenience, we call *chain-monadic* abelian ℓ-groups.

Recall (see e.g. [6]) that a monadic abelian ℓ-group $\langle \mathbf{G}, \Box \rangle$ is a *subdirect product* of a family of monadic abelian ℓ-groups $(\langle \mathbf{H}_j, \Box_j \rangle)_{j \in J}$ if it is a monadic ℓ-subgroup of the direct product $\prod_{j \in J} \langle \mathbf{H}_j, \Box_j \rangle$ such that each projection map $\pi_j \colon \prod_{k \in J} \langle \mathbf{H}_k, \Box_k \rangle \to \langle \mathbf{H}_j, \Box_j \rangle$; $(a_k)_{k \in J} \mapsto a_j$ is surjective. Crucially, if an equation fails in $\langle \mathbf{G}, \Box \rangle$, then it fails in some $\langle \mathbf{H}_j, \Box_j \rangle$. Let us also recall that an ℓ-ideal $\mathbf{K}$ of an abelian ℓ-group $\mathbf{G}$ is called *prime* if $\mathbf{G}/\mathbf{K}$ is linearly ordered.

Lemma 6.2 *Each monadic abelian ℓ-group is isomorphic to a subdirect product of chain-monadic abelian ℓ-groups.*

Proof. Let $\langle \mathbf{G}, \Box \rangle$ be a monadic abelian ℓ-group and let S be the set of all prime ℓ-ideals $\mathbf{P}$ of $\Box \mathbf{G}$. Then $\Box \mathbf{G}/\mathbf{P}$ is linearly ordered for each $\mathbf{P} \in S$ and $\bigcap \{P \mid \mathbf{P} \in S\} = \{0\}$ (see, e.g., [1, Proposition 1.2.9]). By Proposition 5.4, each $\mathbf{P} \in S$ corresponds to a monadic ℓ-ideal $\mathbf{P}^{\Box\Diamond}$ of $\langle \mathbf{G}, \Box \rangle$ such that $\Box \mathbf{G}/\mathbf{P}^{\Box\Diamond}$ is linearly ordered. Moreover, since $\Box a = \Diamond a = 0$ implies $a = 0$ for all $a \in G$, it follows that $\bigcap \{P^{\Box\Diamond} \mid \mathbf{P} \in S\} = \{0\}$ and the map $\sigma \colon \langle \mathbf{G}, \Box \rangle \to \prod_{\mathbf{P} \in S} \langle \mathbf{G}, \Box \rangle / \mathbf{P}^{\Box\Diamond}$; $a \mapsto (P^{\Box\Diamond} + a)_{\mathbf{P} \in S}$ is an embedding between monadic abelian ℓ-groups. Hence, $\langle \mathbf{G}, \Box \rangle$ is isomorphic to a subdirect product of the family of chain-monadic abelian ℓ-groups $(\langle \mathbf{G}, \Box \rangle / \mathbf{P}^{\Box\Diamond})_{\mathbf{P} \in S}$. $\Box$

Following a method used in [9] to characterize subdirectly irreducible monadic MV-algebras, we now show that each chain-monadic abelian ℓ-group $\langle \mathbf{G}, \Box \rangle$ admits a functional representation.

Lemma 6.3 *Let $\langle \mathbf{G}, \Box \rangle$ be a chain-monadic abelian ℓ-group and $a \in G$. Then there exists a prime ℓ-ideal $\mathbf{P}$ of $\mathbf{G}$ such that $P + a = P + \Box a$ and $P \cap \Box G = \{0\}$.*

Proof. Let $\langle \mathbf{G}, \Box \rangle$ be a chain-monadic abelian ℓ-group and $a \in G$. We apply Zorn's Lemma to the set $\mathcal{K}$ of all ℓ-ideals $\mathbf{K}$ of $\mathbf{G}$ such that $K \cap \Box G = \{0\}$ and $a - \Box a \in K$, ordered by inclusion. First, we check that $\mathcal{K}$ is non-empty. We show that the ℓ-ideal $\mathbf{K}(a - \Box a)$ of $\mathbf{G}$ generated by the element $a - \Box a$ is in $\mathcal{K}$. By, e.g., [1, Proposition 1.2.3], recalling that $|x| := x \vee -x$ for any $x \in G$,

$$K(a - \Box a) = \{b \in G \mid |b| \leq n|a - \Box a| \text{ for some } n \in \mathbb{N}\}.$$

Let $b \in K(a - \Box a) \cap \Box G$. Then for some $n \in \mathbb{N}$,

$$\begin{aligned} |b| = \Box |b| &\leq \Box(2^n |a - \Box a|) && \text{since } |b| \in \Box G,\ b \in K(a - \Box a) \\ &= 2^n \Box |a - \Box a| && \text{using (M6)} \\ &= 2^n \Box(a - \Box a) && \text{using (M2)} \\ &= 2^n (\Box a - \Box a) && \text{using (M1), (M2), and (M3)} \\ &= 0. \end{aligned}$$

So $b = 0$ and $\mathcal{K} \neq \emptyset$. Moreover, it is easy to see that $\mathcal{K}$ is closed under taking unions of chains, so Zorn's Lemma yields a maximal element $\mathbf{P} \in \mathcal{K}$.

Suppose for a contradiction that $\mathbf{P}$ is not prime. Then there exist $b, c \in G$ with $b \wedge c = 0$ but $b, c \notin P$ (see, e.g., [1, Theorem 1.2.10]). By the maximality of $\mathbf{P}$, there exist $r \in (P(b) \cap \Box G)\backslash\{0\}$ and $s \in (P(c) \cap \Box G)\backslash\{0\}$, where $\mathbf{P}(b)$ and $\mathbf{P}(c)$ are the ℓ-ideals generated by $P \cup \{b\}$ and $P \cup \{c\}$, respectively. Since $\Box\mathbf{G}$ is linearly ordered, we can assume without loss of generality that $|r| \leq |s|$. Convexity then implies that also $r \in P(c) \cap \Box G$. Hence $r \in P(b) \cap P(c) = P(b \wedge c) = P(0) = P$. But $P \cap \Box G = \{0\}$, so $r = 0$, a contradiction. That is, $\mathbf{P}$ is prime. Finally, note that since $a - \Box a \in P$, also $P + a = P + \Box a$. $\Box$

Lemma 6.4 *Let $\langle \mathbf{G}, \Box \rangle$ be a chain-monadic abelian ℓ-group and $a \in G\backslash\{0\}$. Then there exists a prime ℓ-ideal $\mathbf{P}$ of $\mathbf{G}$ such that $a \notin P$ and $P \cap \Box G = \{0\}$.*

Proof. Let $\langle \mathbf{G}, \Box \rangle$ be a chain-monadic abelian ℓ-group and $a \in G\backslash\{0\}$. We apply Zorn's Lemma to the set $\mathcal{K}$ of all proper ℓ-ideals $\mathbf{K}$ of $\mathbf{G}$ such that for all $r \in \Box G\backslash\{0\}$, $|a| \wedge |r| \notin K$, ordered by inclusion. To show that $\{0\} \in \mathcal{K}$, it suffices to show that for $a \in G$, $r \in \Box G$, $a \wedge r = 0$ implies that $a = 0$ or $r = 0$. If $a \wedge r = 0$, then also $\Box(a \wedge r) = \Box a \wedge r = 0$ and $\Diamond(a \wedge r) = \Diamond a \wedge r = 0$ using conditions (M4) and (M5), respectively. Since $\Box\mathbf{G}$ is linearly ordered, either $r = 0$ or $\Box a = \Diamond a = 0$, i.e. $r = 0$ or $a = 0$. Moreover, $\bigcup \mathcal{C} \in \mathcal{K}$ for any chain $\mathcal{C} \subseteq \mathcal{K}$, therefore $\mathcal{K}$ contains a maximal element $\mathbf{P}$.

We show next that $\mathbf{P}$ is prime. Consider $b, c \in G$ such that $b \wedge c = 0$ and suppose for a contradiction that $b, c \notin P$. By the maximality of $\mathbf{P}$, neither $\mathbf{P}(b)$ nor $\mathbf{P}(c)$ belongs to $\mathcal{K}$ and so there exist $p, q \in \Box G\backslash\{0\}$ such that $|a| \wedge |p| \in P(b)$ and $|a| \wedge |q| \in P(c)$. Since $\Box\mathbf{G}$ is linearly ordered, we can assume without loss of generality that $|p| \leq |q|$. Then $0 \leq |a| \wedge |p| \leq |a| \wedge |q|$, so by convexity, $|a| \wedge |p| \in P(b) \cap P(c) = P(b \wedge c) = P$, contradicting $\mathbf{P} \in \mathcal{K}$.

Lastly note that $\mathbf{P}$ satisfies the required properties. For, if $a \in P$, then $|a| \in P$ and so by convexity, $|a| \wedge |r| \in P$ for all $r \in \Box G$, contradicting $\mathbf{P} \in \mathcal{K}$. It follows similarly that $P \cap \Box G = \{0\}$. $\Box$

Theorem 6.5 *Any chain-monadic abelian ℓ-group $\langle \mathbf{G}, \Box \rangle$ is isomorphic to a witnessed functional monadic abelian ℓ-group.*

Proof. Let $\langle \mathbf{G}, \Box \rangle$ be a chain-monadic abelian ℓ-group, and let $\{\mathbf{P}_i\}_{i \in I}$ be the family of all prime ℓ-ideals $\mathbf{P}$ of $\mathbf{G}$ such that $P \cap \Box G = \{0\}$. It follows from Lemma 6.4 that $\bigcap\{P_i \mid i \in I\} = \{0\}$ and hence that $\sigma \colon \mathbf{G} \to \prod_{i \in I} \mathbf{G}/\mathbf{P}_i;\ a \mapsto (a + P_i)_{i \in I}$ is an embedding between abelian ℓ-groups. Moreover, for each $i \in I$, since $P_i \cap \Box G = \{0\}$, the map $\pi_i \circ \sigma|_{\Box G}$ is an ℓ-embedding, where π_i is the ith projection map.

We make use of a *generalized amalgamation property* for abelian o-groups: that is, for any abelian o-group $\mathbf{H}_0$, family of abelian o-groups $\{\mathbf{H}_j\}_{j \in J}$, and family of ℓ-embeddings $\{\gamma_j \colon \mathbf{H}_0 \to \mathbf{H}_j\}_{j \in J}$, there exists an abelian o-group $\mathbf{H}$ (called the *amalgam*) and family of ℓ-embeddings $\{\sigma_j \colon \mathbf{H}_j \to \mathbf{H}\}_{j \in J}$ such that $\sigma_{j_1} \circ \gamma_{j_1} = \sigma_{j_2} \circ \gamma_{j_2}$ for all $j_1, j_2 \in J$. This property was established by Pierce [27] for families of size 2 and extended to the generalized version in [9].

For the abelian o-group $\Box\mathbf{G}$, family of abelian o-groups $\{\mathbf{G}/\mathbf{P}_i\}_{i\in I}$ and family of ℓ-embeddings $\{\pi_i \circ \sigma|_{\Box G} \colon \Box\mathbf{G} \to \mathbf{G}/\mathbf{P}_i\}_{i\in I}$, we therefore obtain an amalgam $\mathbf{H}$ with ℓ-embeddings $\gamma_i \colon \mathbf{G}/\mathbf{P}_i \to \mathbf{H}$ for each $i \in I$. Defining $\gamma \coloneqq \prod_{i\in I}\gamma_i \colon \prod_{i\in I} G/P_i \to H^I$ yields an ℓ-embedding $\rho \coloneqq \gamma\circ\sigma \colon \mathbf{G} \to \mathbf{H}^I$. Observe now that for all $r \in \Box G$ and $i, j \in I$,

$$\rho(r)(i) = \gamma_i(\sigma(r)(i)) = \gamma_i(\pi_i(\sigma(r))) = \gamma_j(\pi_j(\sigma(r))) = \gamma_j(\sigma(r)(j)) = \rho(r)(j).$$

That is, $\rho(r)$ is a constant function. Moreover, for each $a \in G$, there exists, by Lemma 6.3, an $i \in I$ such that $P_i + a = P_i + \Box a$ and hence $\rho(\Box a)(i) = \rho(a)(i)$. So for any $a \in G$ and $i \in I$, we obtain $\rho(\Box a)(i) = \min\{\rho(a)(j) \mid j \in I\}$. $\Box$

To prove the promised completeness result for $\mathcal{S}5(\mathcal{A})$, we make use of the following folklore result from the theory of abelian ℓ-groups.

Lemma 6.6 (cf. [10]**)** *Let $\mathbf{G}$ be an abelian o-group. For each finite subset S of G, there exists a function $h\colon S \to \mathbb{R}$ satisfying for all $a, b, c \in S$,*

(i) *$a \le b$ if and only if $h(a) \le h(b)$;*

(ii) *if $0 \in S$, then $h(0) = 0$;*

(iii) *$a + b = c$ if and only if $h(a) + h(b) = h(c)$;*

(iv) *$b = -a$ if and only if $h(b) = -h(a)$.*

Theorem 6.7 *Let $\varphi \in \mathrm{Fm}(\mathcal{L}^{\Box}_{\mathrm{A}})$. Then $\vdash_{\mathcal{S}5(\mathcal{A})} \varphi$ if and only if $\models_{\mathrm{S5(A)}} \varphi$.*

Proof. For the left-to-right direction, it is easily checked that the axioms are S5(A)-valid and the rules preserve S5(A)-validity. For the converse, suppose that $\nvdash_{\mathcal{S}5(\mathcal{A})} \varphi$. By Lemmas 6.1 and 6.2, there exist a chain-monadic abelian ℓ-group $\langle \mathbf{G}, \Box\rangle$ and a valuation $e\colon \mathrm{Fm}(\mathcal{L}^{\Box}_{\mathrm{A}}) \to \langle \mathbf{G}, \Box\rangle$ such that $0 \not\le e(\varphi)$. By Theorem 6.5, we may assume that $\mathbf{G}$ is a witnessed ℓ-subgroup of $\mathbf{B}(W, \mathbf{H})$ for some non-empty set W and abelian o-group $\mathbf{H}$. Hence there exists $x_0 \in W$ such that $e(\varphi)(x_0) < 0$. Let Σ be the set of subformulas of φ. For each $\Box\psi \in \Sigma$, we choose $x_{\Box\psi} \in W$ such that

$$e(\Box\psi)(x_{\Box\psi}) = e(\psi)(x_{\Box\psi}).$$

Let $W' \coloneqq \{x_{\Box\psi} \in W \mid \Box\psi \in \Sigma\} \cup \{x_0\}$ and define

$$S \coloneqq \{e(\psi)(x) \mid x \in W',\ \psi \in \Sigma\} \cup \{-e(\psi)(x) \mid x \in W',\ \psi \in \Sigma\} \cup \{0\}.$$

Since both W' and Σ are finite, so is S. Using Lemma 6.6, we obtain a function $h\colon S \to \mathbb{R}$ satisfying the properties (i)-(iv). We consider the standard monadic abelian ℓ-group $\langle \mathbf{B}(W', \mathbf{R}), \Box\rangle$ and any valuation $e'\colon \mathrm{Fm}(\mathcal{L}^{\Box}_{\mathrm{A}}) \to \langle \mathbf{B}(W', \mathbf{R}), \Box\rangle$ such that for each $p \in \Sigma \cap \mathrm{Var}$ and $x \in W'$,

$$e'(p)(x) \coloneqq h(e(p)(x)).$$

A simple induction on formulas shows that $e'(\psi)(x) = h(e(\psi)(x))$ for all $\psi \in \Sigma$ and $x \in W'$, and in particular,

$$e'(\varphi)(x_0) = h(e(\varphi)(x_0)) < h(0) = 0.$$

Finally, consider the S5(A)-model $\langle W', V\rangle$ where $V(p,x) \coloneqq e'(p)(x)$ for each $x \in W'$ and observe that $V(\varphi, x_0) = e'(\varphi)(x_0) < 0$. Hence $\not\models_{\mathrm{S5(A)}} \varphi$. $\square$

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