Interpolation for Intermediate Logics via Hyper- and Linear Nested Sequents

Roman Kuznets[a 1] Björn Lellmann[a 2]

[a] *TU Wien, Austria*

Abstract

The goal of this paper is extending to intermediate logics the constructive proof-theoretic method of proving Craig and Lyndon interpolation via hypersequents and nested sequents developed earlier for classical modal logics. While both Jankov and Gödel logics possess hypersequent systems, we show that our method can only be applied to the former. To tackle the latter, we switch to linear nested sequents, demonstrate syntactic cut elimination for them, and use it to prove interpolation for Gödel logic. Thereby, we answer in the positive the open question of whether Gödel logic enjoys the Lyndon interpolation property.

Keywords: Intermediate logics, hypersequents, linear nested sequents, interpolation, cut elimination, Gödel logic, Lyndon interpolation

1 Introduction

The *Craig Interpolation Property* (CIP) is one of the fundamental properties of for logics, alongside decidability, compactness, etc. It states that for any theorem $A \to B$ of the logic, there must exist an *interpolant* C that only uses propositional variables common to A and B such that both $A \to C$ and $C \to B$ are theorems. The *Lyndon Interpolation Property* (LIP) strengthens the common language requirement by demanding that each variable in C occur in both A and B with the same polarity as it does in C. One of the more robust *constructive* methods of proving interpolation, which unlike many other methods, can yield both CIP and LIP, is the so-called *proof-theoretic method*, whereby an interpolant is constructed by induction on a given analytic sequent derivation of (a representation of) $A \to B$. Unfortunately, many modal and *intermediate logics*, including Jankov and Gödel logics (intermediate in the sense of being between intuitionistic and classical propositional logics, the former denoted Int), do not possess a known cut-free sequent calculus. Various extensions of sequent calculi were developed to address this situation, including *hypersequents* and nested sequents. Already hypersequents are sufficient to capture

[1] Funded by FWF projects S 11405 (RiSE) and Y 544-N23.
[2] Funded by WWTF project MA16-28

both Jankov and Gödel logics. A method for using such advanced calculi to prove interpolation was only recently developed and applied to classical-based modal logics in [9,12,14].

The goal of this paper is to finally extend the method to intermediate logics. It is a classical result [16] by Maksimova that exactly seven intermediate logics are interpolable (have CIP). It is also known that five of these logics, including the intuitionistic and Jankov logics, have LIP [18]. This paper is devoted to proving interpolation using intuitionistic hypersequents, i.e., hypersequents with at most one formula in the consequent of each sequent component. It turned out, however, that the existence of such a hypersequent calculus does not yet guarantee that the CIP for an interpolable intermediate logic can be proved using the method. Our counterexample for Gödel logic (see the proof of Theorem 4.6) demonstrates that the special interpolation property at the core of our method is strictly stronger that the CIP. In order to overcome this difficulty, we develop an alternative formalism for Gödel logic, that of *linear nested sequents*. In particular, we use it to solve the open problem [4,10,18] of Lyndon interpolation for Gödel logic.

The paper is structured as follows. After preliminaries in Section 2, we describe the method for hypersequents in Section 3 and use it in Section 4 to prove interpolation for Int and for Jankov logic LQ. In Section 5, we introduce linear nested sequents for Gödel logic G and provide a surprisingly intricate proof of syntactic cut elimination for them. In Section 6, we explain how to modify the method from hypersequents to linear nested sequents. Finally, in Section 7, we summarize the obtained results.

2 Intermediate Logics and Hypersequents

We consider the language $A ::= p \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \to A)$, where p is taken from a countably infinite set Prop of *propositional variables*. The Boolean constant $\top$ and connective $\neg$ are defined in the standard way. For all intermediate logics we consider, we use *intuitionistic Kripke models*, adapted to individual logics by frame conditions.

Definition 2.1 A *Kripke frame*, or simply a *frame*, is a pair $(W, \leqslant)$ of a set $W \neq \varnothing$ of *worlds* and a partial order $\leqslant$ on W.[3] A *Kripke model*, or simply a *model*, is a triple $(W, \leqslant, V)$ where $(W, \leqslant)$ is a frame and $V \colon W \to 2^{\mathsf{Prop}}$ is a *valuation* that is monotone w.r.t. $\leqslant$, i.e., $w \leqslant u$ implies $V(w) \subseteq V(u)$.

Definition 2.2 For a model $\mathcal{M} = (W, \leqslant, V)$, the *forcing relation* $\Vdash$ between $w \in W$ and formulas is defined by $\mathcal{M}, w \Vdash p$ iff $p \in V(w)$ for each $p \in \mathsf{Prop}$; $\mathcal{M}, w \nVdash \bot$; $\mathcal{M}, w \Vdash A \wedge B$ iff $\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$; $\mathcal{M}, w \Vdash A \vee B$ iff $\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$; $\mathcal{M}, w \Vdash A \to B$ iff $\mathcal{M}, v \nVdash A$ or $\mathcal{M}, v \Vdash B$ for all $v \geqslant w$. A formula A is *valid in* $\mathcal{M}$, written $\mathcal{M} \Vdash A$, if $(\forall w \in W)\, \mathcal{M}, w \Vdash A$. A formula A is *valid in a class* $\mathcal{C}$ *of models*, written $\mathcal{C} \Vdash A$, if $(\forall \mathcal{M} \in \mathcal{C})\, \mathcal{M} \Vdash A$.

[3] A *partial order* is a reflexive, transitive, and antisymmetric binary relation.

$$\mathsf{id}\,\frac{}{A \Rightarrow A} \qquad \mathsf{id}_\bot\,\frac{}{\bot \Rightarrow} \qquad \mathsf{Ex}\,\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Lambda \Rightarrow \Theta \mid \mathcal{H}}{\mathcal{G} \mid \Lambda \Rightarrow \Theta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \qquad \mathsf{EC}\,\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

$$\wedge\!\Rightarrow\frac{\mathcal{G} \mid \Gamma, A_i \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad \Rightarrow\!\wedge\,\frac{\mathcal{G} \mid \Gamma \Rightarrow A \quad \mathcal{G} \mid \Gamma \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \wedge B} \qquad \rightarrow\Rightarrow\frac{\mathcal{G} \mid \Gamma \Rightarrow A \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta}$$

$$\vee\!\Rightarrow\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \vee B \Rightarrow \Delta} \qquad \Rightarrow\!\vee\,\frac{\mathcal{G} \mid \Gamma \Rightarrow A_i}{\mathcal{G} \mid \Gamma \Rightarrow A_1 \vee A_2} \qquad \Rightarrow\rightarrow\frac{\mathcal{G} \mid \Gamma, A \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \rightarrow B}$$

$$\mathsf{W}\!\Rightarrow\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \qquad \Rightarrow\!\mathsf{W}\,\frac{\mathcal{G} \mid \Gamma \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow A} \qquad \mathsf{EW}\,\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \qquad \mathsf{C}\!\Rightarrow\frac{\mathcal{G} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}$$

Fig. 1. Hypersequent calculus HInt for Int ([6]).

Lemma 2.3 (Monotonicity [20]) *$\mathcal{M}, w \Vdash A$ implies $\mathcal{M}, v \Vdash A$ whenever $w \leqslant v$ for any model $\mathcal{M} = (W, \leqslant, V)$.*

Theorem 2.4 (Completeness [20,4,5]) *Intuitionistic logic* Int *is sound and complete w.r.t. the class of all models. Jankov logic* $\mathsf{LQ} = \mathsf{Int} + \neg A \vee \neg\neg A$ *and Gödel logic* $\mathsf{G} = \mathsf{Int} + (A \rightarrow B) \vee (B \rightarrow A)$ [4] *are sound and complete w.r.t. models with a maximum element and with linear $\leqslant$ respectively.*

Definition 2.5 A *sequent* is a figure $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite multisets of formulas. It is *single-conclusion* if $|\Delta| \leqslant 1$. A *hypersequent* $\mathcal{G}$ is a finite list $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ with $n > 0$ of single-conclusion sequents $\Gamma_i \Rightarrow \Delta_i$, called *components* of $\mathcal{G}$. We define the length $\|\mathcal{G}\| = n$ to be the number of components of $\mathcal{G}$. A *formula interpretation* $\iota(\mathcal{G}) := \bigvee_{i=1}^n (\bigwedge \Gamma_i \rightarrow \bigvee \Delta_i)$.

Definition 2.6 The hypersequent system HInt for Int is presented in Figure 1. Hypersequent systems HLQ and HG for Jankov logic LQ and Gödel logic G respectively are obtained by adding to HInt respectively the rules

$$\mathsf{lq}\,\frac{\mathcal{G} \mid \Gamma, \Lambda \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow \mid \Lambda \Rightarrow} \qquad \text{and} \qquad \mathsf{com}\,\frac{\mathcal{G} \mid \Gamma, \Gamma' \Rightarrow \Delta \quad \mathcal{G} \mid \Lambda, \Lambda' \Rightarrow \Delta'}{\mathcal{G} \mid \Gamma, \Lambda' \Rightarrow \Delta \mid \Lambda, \Gamma' \Rightarrow \Delta'}.$$

We write $\mathsf{HL} \vdash \mathcal{G}$ if the hypersequent $\mathcal{G}$ is derivable in HL for a logic L.

Theorem 2.7 (Hypersequent completeness [6]) *For* $\mathsf{L} \in \{\mathsf{Int}, \mathsf{LQ}, \mathsf{G}\}$ *we have* $\mathsf{HL} \vdash \mathcal{G}$ *iff* $\mathsf{L} \vdash \iota(\mathcal{G})$*. In particular,* $\mathsf{HL} \vdash A \Rightarrow B$ *iff* $\mathsf{L} \vdash A \rightarrow B$*.*

3 Interpolation via Hypersequents

The proof-theoretic method for proving interpolation constructively using hypersequents was first introduced in [12]. There it was applied to the classical-based modal logic S5 and later extended to S4.2 in [13]. In this paper, we adapt the method to intuitionistic propositional reasoning.

[4] Also known as KC and Dummett's logic LC respectively.

Definition 3.1 A *split multiset* $\widetilde{\Gamma} = \Gamma_l; \Gamma_r$ consists of the left and right multiset parts Γ_l and Γ_r (the semicolon can be omitted for the empty multiset). $|\Gamma_l; \Gamma_r| := |\Gamma_l| + |\Gamma_r|$. A split sequent $\widetilde{\Gamma} \Rightarrow \widetilde{\Delta}$ is obtained from split multisets $\widetilde{\Gamma}$ and $\widetilde{\Delta}$ the same way sequents are obtained from multisets. A *split hypersequent* $\widetilde{\mathcal{G}} = \Gamma_1; \Pi_1 \Rightarrow \Delta_1; \Sigma_1 \mid \ldots \mid \Gamma_n; \Pi_n \Rightarrow \Delta_n; \Sigma_n$ is built from split single-conclusion sequents $\Gamma_i; \Pi_i \Rightarrow \Delta_i; \Sigma_i$ the same way hypersequents are built from sequents. The left (right) side and the *conflation* of $\widetilde{\mathcal{G}}$ are obtained by dropping all right (left) formulas and by combining the two sides of $\widetilde{\mathcal{G}}$ respectively:

$$L\widetilde{\mathcal{G}} := \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \qquad R\widetilde{\mathcal{G}} := \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n$$
$$LR\widetilde{\mathcal{G}} := \Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1 \mid \ldots \mid \Gamma_n, \Pi_n \Rightarrow \Delta_n, \Sigma_n$$

As before, the length $\|\widetilde{\mathcal{G}}\| := n$. It is obvious that $\|\widetilde{\mathcal{G}}\| = \|L\widetilde{\mathcal{G}}\| = \|R\widetilde{\mathcal{G}}\| = \|LR\widetilde{\mathcal{G}}\|$.

Splits do not play any semantic role. They are used solely for interpolant construction. There is a standard way of turning a given sequent-like calculus HL into its split equivalent SHL. For each rule, one considers all possible splits of the conclusion and uses the corresponding splits of the premiss(es), i.e., side formulas remain on the same side and active formula(s) in the premiss(es) are on the same side as the principal formula. A logical rule typically produces two split variants depending on whether the principal formula is on the left or on the right. To save space, we do not present splits of the rules. Most of them can be read from Figure 2 by omitting interpolants.

Theorem 3.2 *For a logic* $\mathsf{L} \in \{\mathsf{Int}, \mathsf{LQ}, \mathsf{G}\}$ *we have* $\mathsf{SHL} \vdash \widetilde{\mathcal{G}}$ *iff* $\mathsf{HL} \vdash LR\widetilde{\mathcal{G}}$.

Proof. Both directions are by induction on derivation depth. The main observation is that each split rule of SHInt becomes a rule of HInt if one takes the union of left and right formulas separately in each antecedent and consequent. Vice versa, for each split of the conclusion of a rule of HInt, there is a split of the premiss(es) that turns this rule into a rule of SHInt. □

Corollary 3.3 *For* $\mathsf{L} \in \{\mathsf{Int}, \mathsf{LQ}, \mathsf{G}\}$ *we have* $\mathsf{SHL} \vdash A; \Rightarrow; B$ *iff* $\mathsf{L} \vdash A \to B$.

Proof. The statement follows from Theorems 2.7 and 3.2. □

We define an alternative interpolation property based on direct evaluation of hypersequents into models, as opposed to first translating them into a formula interpretation and evaluating the latter.

Definition 3.4 For a sequence $\boldsymbol{w}$ of worlds from a model $\mathcal{M} = (W, \leqslant, V)$ of length $\|\boldsymbol{w}\| = n > 0$, we denote its ith member by w_i, where $1 \leqslant i \leqslant n$. For a $v \in W$, the sequence $\boldsymbol{w}$ is *v-rooted in $\mathcal{M}$* if $v \leqslant w_i$ for all $1 \leqslant i \leqslant n$. The sequence $\boldsymbol{w}$ is *$\mathcal{M}$-rooted* if it is v-rooted for some $v \in W$.

Definition 3.5 A hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ *componentwise holds* at a sequence $\boldsymbol{w}$ of worlds from $\mathcal{M} = (W, \leqslant, V)$ of length $\|\boldsymbol{w}\| = n$, written $\mathcal{M}, \boldsymbol{w} \vDash \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, iff for some $i = 1, \ldots, n$

$$\mathcal{M}, w_i \nVdash A \text{ for some } A \in \Gamma_i \quad \text{ or } \quad \mathcal{M}, w_i \Vdash B \text{ for some } B \in \Delta_i.$$

A hypersequent $\mathcal{G}$ is *componentwise valid in a class $\mathcal{C}$ of models* iff $\mathcal{M}, \boldsymbol{w} \vDash \mathcal{G}$ for each $\mathcal{M} \in \mathcal{C}$ and each $\mathcal{M}$-rooted sequence $\boldsymbol{w}$ of length $\|\mathcal{G}\|$.

Lemma 3.6 *A hypersequent $\mathcal{G}$ is componentwise valid in a class $\mathcal{C}$ of models iff its formula interpretation is, i.e., iff $\mathcal{C} \Vdash \iota(\mathcal{G})$.*

Proof. Let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$. It is a simple exercise in intuitionistic Kripke semantics to prove that $\mathcal{G}$ is componentwise invalid iff $\iota(\mathcal{G})$ is. It is based on two observations: (1) if $\mathcal{M}, \boldsymbol{w} \nvDash \mathcal{G}$ for a v-rooted sequence $\boldsymbol{w}$, then $\mathcal{M}, v \nVdash \iota(\mathcal{G})$; (2) if $\mathcal{M}, v \nVdash \iota(\mathcal{G})$, then there exists a v-rooted sequence $\boldsymbol{w}$ such that $\mathcal{M}, \boldsymbol{w} \nvDash \mathcal{G}$. □

Compared to interpolants for classical-based modal logics ([12]), the components of interpolants need to be imbued with polarity.

Definition 3.7 A *uniformula* has the form $C^{(k)}$ or $\overline{C}^{(k)}$, where C is a propositional formula and $k \geqslant 1$ is an integer. Each uniformula is a *multiformula.* If $\mho_1$ and $\mho_2$ are multiformulas, then so are $(\mho_1 \otimes \mho_2)$ and $(\mho_1 \oslash \mho_2)$. The *arity* $\|\mho\|$ of $\mho$ is the largest number k such that either $C^{(k)}$ or $\overline{C}^{(k)}$ occurs in $\mho$.

Definition 3.8 Let $\mathcal{M} = (W, \leqslant, V)$ be a model and $\boldsymbol{w} = w_1, \ldots, w_n$ be a sequence of worlds from W of length $\|\boldsymbol{w}\| = n$. For a multiformula $\mho$ of arity $\|\mho\| \leqslant n$, we define its truth inductively: (a) $\mathcal{M}, \boldsymbol{w} \vDash C^{(k)}$ iff $\mathcal{M}, \boldsymbol{w} \nvDash \overline{C}^{(k)}$ iff $\mathcal{M}, w_k \Vdash C$; (b) $\mathcal{M}, \boldsymbol{w} \vDash \mho_1 \oslash \mho_2$ iff $\mathcal{M}, \boldsymbol{w} \vDash \mho_1$ and $\mathcal{M}, \boldsymbol{w} \vDash \mho_2$; (c) $\mathcal{M}, \boldsymbol{w} \vDash \mho_1 \otimes \mho_2$ iff $\mathcal{M}, \boldsymbol{w} \vDash \mho_1$ or $\mathcal{M}, \boldsymbol{w} \vDash \mho_2$.

Thus, $\oslash$ and $\otimes$ are componentwise analogs of $\wedge$ and $\vee$ respectively.

Definition 3.9 Multiformulas $\mho_1$ and $\mho_2$ are *componentwise equivalent*, written $\mho_1 \dashv\vDash \mho_2$, iff $\mathcal{M}, \boldsymbol{w} \vDash \mho_1 \iff \mathcal{M}, \boldsymbol{w} \vDash \mho_2$ for any model $\mathcal{M}$ and any sequence $\boldsymbol{w}$ of worlds from $\mathcal{M}$ of length $\|\boldsymbol{w}\| \geqslant \|\mho_1\|, \|\mho_2\|$.

Lemma 3.10 (Normal forms) *Each multiformula $\mho$ can be effectively transformed into either special DNF or CNF (SDNF or SCNF) that is componentwise equivalent to $\mho$, i.e., a DNF/CFN w.r.t. $\oslash$ and $\otimes$ such that for each $k = 1, \ldots, K \geqslant \|\mho\|$ each disjunct in the SDNF (conjunct in the SCNF) contains exactly one uniformula of type $C^{(k)}$ and one of type $\overline{D}^{(k)}$. Moreover, this can be done without changing the set of propositional variables occurring in $\mho$ or, indeed, without changing their polarities.*

Proof. The transformation to a DNF/CNF is standard. Uniformulas of the same type are merged using componentwise equivalences of the following types: $C^{(k)} \oslash D^{(k)} \dashv\vDash (C \wedge D)^{(k)}$, $C^{(k)} \otimes D^{(k)} \dashv\vDash (C \vee D)^{(k)}$, $\overline{C}^{(k)} \oslash \overline{D}^{(k)} \dashv\vDash \overline{C \vee D}^{(k)}$, $\overline{C}^{(k)} \otimes \overline{D}^{(k)} \dashv\vDash \overline{C \wedge D}^{(k)}$, $\overline{C}^{(k)} \dashv\vDash \overline{C}^{(k)} \oslash \top^{(k)}$, $C^{(k)} \dashv\vDash C^{(k)} \otimes \bot^{(k)}$, etc. □

Definition 3.11 A *componentwise interpolant of a split hypersequent* $\widetilde{\mathcal{G}}$ w.r.t. a class $\mathcal{C}$ of models, written $\widetilde{\mathcal{G}} \xleftarrow{\mathcal{C}} \mho$, is a multiformula $\mho$ of arity $\|\mho\| \leqslant \|\widetilde{\mathcal{G}}\|$ containing only propositional variables common to $L\widetilde{\mathcal{G}}$ and $R\widetilde{\mathcal{G}}$ and such that for each model $\mathcal{M} \in \mathcal{C}$ and each $\mathcal{M}$-rooted sequence $\boldsymbol{w}$ of length $\|\boldsymbol{w}\| = \|\widetilde{\mathcal{G}}\|$,

$$\mathcal{M}, \boldsymbol{w} \nvDash \mho \implies \mathcal{M}, \boldsymbol{w} \vDash L\widetilde{\mathcal{G}} \qquad \text{and} \qquad \mathcal{M}, \boldsymbol{w} \vDash \mho \implies \mathcal{M}, \boldsymbol{w} \vDash R\widetilde{\mathcal{G}}.$$

We call the existence of such $\mho$ for $\widetilde{\mathcal{G}}$ the *interpolation statement for* $\widetilde{\mathcal{G}}$.

Lemma 3.12 *Let a logic* L *be sound and complete w.r.t. a class* $\mathcal{C}$ *of models and* $\mho = \bigotimes_{i=1}^{n} \left(\overline{C_i}^{(1)} \otimes D_i^{(1)}\right)$*. If* $A; \Rightarrow ; B \overset{\mathcal{C}}{\leftarrow} \mho$*, then* $I := \bigwedge_{i=1}^{n}(C_i \to D_i)$ *is a Craig interpolant of* $A \to B$ *w.r.t.* L*.*

Proof. Since $L(A; \Rightarrow ; B)$ is $A \Rightarrow$ and $R(A; \Rightarrow ; B)$ is $\Rightarrow B$, the formulas C_i and D_i contain only propositional variables common to A and B.

Let $\mathcal{M} = (W, \leqslant, V) \in \mathcal{C}$ and $v \in W$ be arbitrary. We need to show that $\mathcal{M}, v \Vdash A \to I$ and $\mathcal{M}, v \Vdash I \to B$. For the former, assume $\mathcal{M}, w \Vdash A$ for an arbitrary $w \geqslant v$. For any $w' \geqslant w$, by monotonicity $\mathcal{M}, w' \Vdash A$, i.e., $\mathcal{M}, w' \not\models A \Rightarrow$. By the definition of componentwise interpolation $\mathcal{M}, w' \models \mho$. In particular, $\mathcal{M}, w' \models \overline{C_i}^{(1)} \otimes D_i^{(1)}$, in other words, $\mathcal{M}, w' \nVdash C_i$ or $\mathcal{M}, w' \Vdash D_i$ for $i = 1, \ldots, n$. Thus, $\mathcal{M}, w \Vdash C_i \to D_i$ for each $i = 1, \ldots, n$, yielding $\mathcal{M}, w \Vdash I$.

To show $\mathcal{M}, v \Vdash I \to B$, assume $\mathcal{M}, w \Vdash I$ for an arbitrary $w \geqslant v$. Then, for each $i = 1, \ldots, n$, $\mathcal{M}, w \Vdash C_i \to D_i$, implying $\mathcal{M}, w \nVdash C_i$ or $\mathcal{M}, w \Vdash D_i$ and $\mathcal{M}, w \models \overline{C_i}^{(1)} \otimes D_i^{(1)}$. Thus, $\mathcal{M}, w \models \mho$. By the definition of componentwise interpolation, $\mathcal{M}, w \models \Rightarrow B$, i.e., $\mathcal{M}, w \Vdash B$. □

4 Interpolation Algorithm for Int and LQ

Definition 4.1 We define operations on multiformulas $\mho^{n+1 \mapsto n}$ that changes every superscript $(n+1)$ into (n) and $\mho^{n \leftrightarrow n+1}$ that swaps (n) and $(n+1)$. In either case everything else is left intact.

Lemma 4.2 *Let* $\mathcal{M}$ *be a model,* $\mho$ *be a multiformula, and* $\boldsymbol{w}$ *be an* $\mathcal{M}$*-rooted sequence of worlds. For* $\|\boldsymbol{w}\| = n$ *and* $\|\mho\| \leqslant n+1$*, we have* $\mathcal{M}, \boldsymbol{w} \models \mho^{n+1 \mapsto n}$ *iff* $\mathcal{M}, \boldsymbol{w}, w_n \models \mho$*. For* $\|\boldsymbol{w}\| = n+k$ *and* $\|\mho\| \leqslant n+k$ *for some* $k \geqslant 1$*, we have* $\mathcal{M}, \boldsymbol{w} \models \mho^{n \leftrightarrow n+1}$ *iff* $\mathcal{M}, w_1, \ldots, w_{n-1}, w_{n+1}, w_n, w_{n+2}, \ldots, w_{n+k} \models \mho$*.*

An algorithm for constructing componentwise interpolants for split hypersequents derivable in SHInt is presented via interpolant transformations for each rule of SHInt. All non-identity transformations can be found in Figure 2. In order to apply the transformations for $(\Rightarrow \to^{\mathsf{l}})$ and $(\Rightarrow \to^{\mathsf{r}})$ the interpolant of the premiss must be first transformed to the relevant normal form by Lemma 3.10.

Lemma 4.3 *Let* $\mathcal{C}$ *be a class of models. For each rule from Figure 2, if the interpolation statement(s) in the premiss(es) of the rule hold(s), so does the interpolation statement in its conclusion. For each split hypersequent rule of* Int *not present in Figure 2, which are all unary, any componentwise interpolant of the premiss w.r.t.* $\mathcal{C}$ *is also a componentwise interpolant for its conclusion w.r.t.* $\mathcal{C}$*.*

Proof. We only consider several complex cases, the remaining ones being analogous and simpler. We consider an arbitrarily chosen model $\mathcal{M} = (W, \leqslant, V) \in \mathcal{C}$ and $\mathcal{M}$-rooted sequence $\boldsymbol{w}$ of appropriate length. We omit $\mathcal{M}$ from the $\Vdash$ statements about formulas and from $\models$ statements about multiformulas and hypersequents. We do not discuss the common variable conditions, which are always easy to verify, either.

$$\mathsf{id}^{\mathsf{ll}}\ \frac{}{A;\ \Rightarrow A;\ \overset{\mathcal{C}}{\longleftarrow}\ \bot^{(1)}} \qquad \mathsf{id}^{\mathsf{rl}}\ \frac{}{;A \Rightarrow A;\ \overset{\mathcal{C}}{\longleftarrow}\ \overline{A}^{(1)}} \qquad \mathsf{id}^{\mathsf{lr}}\ \frac{}{A;\ \Rightarrow\ ;A\ \overset{\mathcal{C}}{\longleftarrow}\ A^{(1)}}$$

$$\mathsf{id}^{\mathsf{rr}}\ \frac{}{;A \Rightarrow\ ;A\ \overset{\mathcal{C}}{\longleftarrow}\ \top^{(1)}} \qquad \mathsf{id}^{\mathsf{l}}_{\bot}\ \frac{}{\bot;\ \Rightarrow\ \ \overset{\mathcal{C}}{\longleftarrow}\ \bot^{(1)}} \qquad \mathsf{id}^{\mathsf{r}}_{\bot}\ \frac{}{;\bot \Rightarrow\ \ \overset{\mathcal{C}}{\longleftarrow}\ \top^{(1)}}$$

$$\Rightarrow\wedge^{\mathsf{l}}\ \frac{\mathcal{G} \mid \widetilde{\Gamma} \Rightarrow A;\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \quad \mathcal{G} \mid \widetilde{\Gamma} \Rightarrow B;\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_2}{\mathcal{G} \mid \widetilde{\Gamma} \Rightarrow A \wedge B;\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \ovee \mho_2} \qquad \Rightarrow\wedge^{\mathsf{r}}\ \frac{\mathcal{G} \mid \widetilde{\Gamma} \Rightarrow\ ;A\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \quad \mathcal{G} \mid \widetilde{\Gamma} \Rightarrow\ ;B\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_2}{\mathcal{G} \mid \widetilde{\Gamma} \Rightarrow\ ;A \wedge B\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \owedge \mho_2}$$

$$\vee^{\mathsf{l}}\Rightarrow\ \frac{\mathcal{G} \mid \Gamma, A; \Pi \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \quad \mathcal{G} \mid \Gamma, B; \Pi \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_2}{\mathcal{G} \mid \Gamma, A \vee B; \Pi \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \ovee \mho_2}$$

$$\vee^{\mathsf{r}}\Rightarrow\ \frac{\mathcal{G} \mid \Gamma; \Pi, A \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \quad \mathcal{G} \mid \Gamma; \Pi, B \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_2}{\mathcal{G} \mid \Gamma; \Pi, A \vee B \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \owedge \mho_2}$$

$$\to^{\mathsf{l}}\Rightarrow\ \frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow A;\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \quad \widetilde{\mathcal{G}} \mid \Gamma, B; \Pi \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_2}{\widetilde{\mathcal{G}} \mid \Gamma, A \to B; \Pi \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \ovee \mho_2}$$

$$\to^{\mathsf{r}}\Rightarrow\ \frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow\ ;A\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \quad \widetilde{\mathcal{G}} \mid \Gamma; \Pi, B \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_2}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi, A \to B \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho_1 \owedge \mho_2}$$

$$\Rightarrow\to^{\mathsf{l}}\ \frac{\widetilde{\mathcal{G}} \mid \Gamma, A; \Pi \Rightarrow B;\ \overset{\mathcal{C}}{\longleftarrow}\ \bigovee_{j=1}^{m}\Big(\overline{C_j}^{(n)} \owedge {D_j}^{(n)} \owedge \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)})\Big)}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow A \to B;\ \overset{\mathcal{C}}{\longleftarrow}\ \bigovee_{j=1}^{m}\Big(\overline{D_j \to C_j}^{(n)} \owedge \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)})\Big)}$$

$$\Rightarrow\to^{\mathsf{r}}\ \frac{\widetilde{\mathcal{G}} \mid \Gamma; \Pi, A \Rightarrow\ ;B\ \overset{\mathcal{C}}{\longleftarrow}\ \bigowedge_{j=1}^{m}\Big(\overline{C_j}^{(n)} \ovee {D_j}^{(n)} \ovee \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \ovee {F_{jl}}^{(l)})\Big)}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow\ ;A \to B\ \overset{\mathcal{C}}{\longleftarrow}\ \bigowedge_{j=1}^{m}\Big((C_j \to D_j)^{(n)} \ovee \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \ovee {F_{jl}}^{(l)})\Big)}$$

$$\mathsf{EC}\ \frac{\widetilde{\mathcal{G}} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho}{\widetilde{\mathcal{G}} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho^{n+1 \mapsto n}} \qquad \mathsf{Ex}\ \frac{\widetilde{\mathcal{G}} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta} \mid \widetilde{\Lambda} \Rightarrow \widetilde{\Theta} \mid \widetilde{\mathcal{H}}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho}{\widetilde{\mathcal{G}} \mid \widetilde{\Lambda} \Rightarrow \widetilde{\Theta} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta} \mid \widetilde{\mathcal{H}}\ \overset{\mathcal{C}}{\longleftarrow}\ \mho^{n \leftrightarrow n+1}}$$

Fig. 2. Interpolation transformations for SHInt. For the unary rules of SHInt and structural rules not depicted above, any interpolant for the premiss also interpolates the conclusion. It is required that (1) $\|\mho_i\| \leqslant \|\widetilde{\mathcal{G}}\| + 1$ for $i = 1, 2$ in $(\Rightarrow \wedge^{\mathsf{l}})$, $(\Rightarrow \wedge^{\mathsf{r}})$, $(\vee^{\mathsf{l}} \Rightarrow)$, $(\vee^{\mathsf{r}} \Rightarrow)$, $(\to^{\mathsf{l}} \Rightarrow)$, and $(\to^{\mathsf{r}} \Rightarrow)$; (2) $\|\widetilde{\mathcal{G}}\| = n - 1$ in $(\Rightarrow \to^{\mathsf{l}})$, $(\Rightarrow \to^{\mathsf{r}})$, (EC), and (Ex); (3) $\|\mho\| \leqslant n + 1$ in (EC); (4) $\|\mho\| \leqslant n + \|\widetilde{\mathcal{H}}\| + 1$ in (Ex).

Rule $(\Rightarrow \to^{\mathsf{l}})$. Let the componentwise interpolation statement for the premiss of $(\Rightarrow \to^{\mathsf{l}})$ in Figure 2 hold, in particular, $\|\mathcal{G}\| = n - 1$. Assume, for the left side, that $\boldsymbol{w} \not\models \bigovee_{j=1}^{m}\Big(\overline{D_j \to C_j}^{(n)} \owedge \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)})\Big)$. Let $\boldsymbol{v} := w_1, \ldots, w_{n-1}$. For each $j = 1, \ldots, m$ and each $w'_n \geqslant w_n$, either $\boldsymbol{v} \not\models \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)})$, or $w'_n \not\Vdash D_j$, or $w'_n \Vdash C_j$, i.e., for each $w'_n \geqslant w_n$

$$\mathsf{lqS}\ \frac{\widetilde{\mathcal{G}} \mid \Gamma, \Lambda; \Pi, \Phi \Rightarrow \overset{\mathcal{I}}{\longleftarrow} \bigowedge_{j=1}^{m}\Big(\overline{C_j}^{(n)} \varovee D_j{}^{(n)} \varovee \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})\Big)}{\widetilde{\mathcal{G}} \mid \Gamma; \Pi \Rightarrow \mid \Lambda; \Phi \Rightarrow \overset{\mathcal{I}}{\longleftarrow} \bigowedge_{j=1}^{m}\Big(\overline{\neg(C_j \to D_j)}^{(n)} \varovee \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})\Big)}$$

Fig. 3. Interpolation transformation for lqS. It is required that $\|\mathcal{G}\| = n - 1$.

$\boldsymbol{v}, w'_n \not\models \bigovee_{j=1}^{m}\Big(\overline{C_j}^{(n)} \varowedge D_j{}^{(n)} \varowedge \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varowedge F_{jl}{}^{(l)})\Big)$. Since $\boldsymbol{w} = \boldsymbol{v}, w_n$ is $\mathcal{M}$-rooted, so is $\boldsymbol{v}, w'_n$ for any $w'_n \geqslant w_n$. Thus, for each $w'_n \geqslant w_n$, it follows from the premiss interpolant that $\boldsymbol{v}, w'_n \models L\widetilde{\mathcal{G}} \mid \Gamma, A \Rightarrow B$. In other words, either $\boldsymbol{v} \models L\widetilde{\mathcal{G}}$ or, for each $w'_n \geqslant w_n$, (a) $w'_n \nVdash G$ for some $G \in \Gamma$ or (b) $w'_n \nVdash A$ or $w'_n \Vdash B$. By monotonicity, (a) implies $w_n \nVdash G$ for some $G \in \Gamma$. Thus, $\boldsymbol{v} \models L\widetilde{\mathcal{G}}$, or $w_n \nVdash G$ for some $G \in \Gamma$, or, for all $w'_n \geqslant w_n$, we have $w'_n \nVdash A$ or $w'_n \Vdash B$. In other words, $\boldsymbol{w} \models L\widetilde{\mathcal{G}} \mid \Gamma \Rightarrow A \to B$.

For the right side, let $\boldsymbol{w} \models \bigovee_{j=1}^{m}\Big(\overline{D_j \to C_j}^{(n)} \varowedge \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varowedge F_{jl}{}^{(l)})\Big)$. Then $\boldsymbol{v} \models \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varowedge F_{jl}{}^{(l)})$ and $w_n \nVdash D_j \to C_j$ for some $1 \leqslant j \leqslant m$. The latter implies that $w'_n \Vdash D_j$ and $w'_n \nVdash C_j$ for some $w'_n \geqslant w_n$. It follows that $\boldsymbol{v}, w'_n \models \bigovee_{j=1}^{m}\Big(\overline{C_j}^{(n)} \varowedge D_j{}^{(n)} \varowedge \bigowedge_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varowedge F_{jl}{}^{(l)})\Big)$ for this $w'_n \geqslant w_n$. It follows from the premiss interpolant that $\boldsymbol{v}, w'_n \models R\widetilde{\mathcal{G}} \mid \Pi \Rightarrow$ for this $w'_n \geqslant w_n$, and, by monotonicity, $\boldsymbol{w} \models R\widetilde{\mathcal{G}} \mid \Pi \Rightarrow$.

Rule (EC). Here $\|\widetilde{\mathcal{G}}\| + 1 = n$. By Lemma 4.2, $\boldsymbol{w} \models \mho^{n+1 \mapsto n}$ iff $\boldsymbol{w}, w_n \models \mho$. $\boldsymbol{w}, w_n \models S(\widetilde{\mathcal{G}} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta})$ implies $\boldsymbol{w} \models S(\widetilde{\mathcal{G}} \mid \widetilde{\Gamma} \Rightarrow \widetilde{\Delta})$ for $S \in \{L, R\}$. □

Lemma 4.4 *Let $\mathcal{J}$ be the class of models $\mathcal{M}$ such that each $\mathcal{M}$ has a largest element ∞, i.e., with $w \leqslant \infty$ for all worlds w from $\mathcal{M}$. The interpolation statement in the premiss of* (lqS) *implies that in the conclusion.*

Proof. $\|\mathcal{G}\| = n - 1$. Let $\boldsymbol{v} := w_1, \ldots, w_{n-1}$ for a given $\mathcal{M}$-rooted sequence $\boldsymbol{w}$ of length $\|\boldsymbol{w}\| = n + 1$ consisting of worlds from $\mathcal{M} = (W, \leqslant, V) \in \mathcal{J}$. For the left side, assume $\boldsymbol{w} \not\models \bigowedge_{j=1}^{m}\Big(\overline{\neg(C_j \to D_j)}^{(n)} \varovee \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})\Big)$. Then there is some $1 \leqslant j \leqslant m$ such that $\boldsymbol{v} \not\models \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})$ and, in addition, $w_n \Vdash \neg(C_j \to D_j)$. In particular, $\infty \nVdash C_j \to D_j$. Given that ∞ is the largest element, by monotonicity, $\infty \Vdash C_j$ and $\infty \nVdash D_j$. Therefore, for the $\mathcal{M}$-rooted sequence $\boldsymbol{v}, \infty$, we have $\boldsymbol{v}, \infty \not\models \bigowedge_{j=1}^{m}\Big(\overline{C_j}^{(n)} \varovee D_j{}^{(n)} \varovee \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})\Big)$. It follows from the premiss interpolant that $\boldsymbol{v}, \infty \models L\widetilde{\mathcal{G}} \mid \Gamma, \Lambda \Rightarrow$. Since both $w_n \leqslant \infty$ and $w_{n+1} \leqslant \infty$, by monotonicity, $\boldsymbol{w} \models L\widetilde{\mathcal{G}} \mid \Gamma \Rightarrow \mid \Lambda \Rightarrow$.

Assume now $\boldsymbol{w} \models \bigowedge_{j=1}^{m}\Big(\overline{\neg(C_j \to D_j)}^{(n)} \varovee \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})\Big)$ for the right side. Then $\boldsymbol{v} \models \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})$ or $w_n \nVdash \neg(C_j \to D_j)$ for each $j = 1, \ldots, m$. Thus, for each $j = 1, \ldots, m$, either $\boldsymbol{v} \models \bigovee_{l=1}^{n-1}(\overline{E_{jl}}^{(l)} \varovee F_{jl}{}^{(l)})$ or $z_j \Vdash C_j \to D_j$ for some $z_j \geqslant w_n$. The latter implies $\infty \Vdash C_j \to D_j$ by monotonicity. Thus, for each $j = 1, \dots, m$, $\boldsymbol{v} \vDash \bigovee_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \ovee F_{jl}^{(l)})$, or $\infty \nVdash C_j$, or $\infty \Vdash D_j$, i.e., $\boldsymbol{v}, \infty \vDash \bigowedge_{j=1}^{m} \left(\overline{C_j}^{(n)} \ovee D_j^{(n)} \ovee \bigovee_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \ovee F_{jl}^{(l)})\right)$. It follows from the premiss interpolant that $\boldsymbol{v}, \infty \vDash R\widetilde{\mathcal{G}} \mid \Pi, \Phi \Rightarrow$. As before, by monotonicity $\boldsymbol{w} \vDash R\widetilde{\mathcal{G}} \mid \Pi \Rightarrow \mid \Phi \Rightarrow$. □

Theorem 4.5 Int *and* LQ *enjoy the CIP.*

Proof. Let $\mathsf{L} \vdash A \rightarrow B$ for $\mathsf{L} \in \{\mathsf{Int}, \mathsf{LQ}\}$. By Corollary 3.3, $\mathsf{SHL} \vdash A; \Rightarrow ; B$. By Lemmata 4.3 and 4.4, we can construct a componentwise interpolant $\mho$ of $A; \Rightarrow ; B$. By Lemma 3.10, this $\mho$ can be efficiently transformed to another componentwise interpolant $\mho'$ of length 1 in SCNF. By Lemma 3.12, this $\mho'$ can be efficiently transformed to a formula interpolant C of A and B. □

Theorem 4.6 HSG *does not enjoy the componentwise interpolation property w.r.t. the class* $\mathcal{L}in$ *of all linear models, i.e., there is a derivable hypersequent for which the componentwise interpolation statement does not hold.*

Proof. The split sequent $;q \Rightarrow p; \mid p; \Rightarrow ;q$ is derivable in SHG from $p; \Rightarrow p;$ and $;q \Rightarrow ;q$ by the rule (comS). Since no propositional variable occurs on both sides, componentwise interpolants could only be constructed from $\bot$. Simple induction on the interpolant construction shows that any such interpolant would be componentwise equivalent to either $\top^{(1)}$ or $\bot^{(1)}$. Depending on which it were, either $\mathcal{M}, w_1, w_2 \vDash q \Rightarrow \mid \Rightarrow q$ for any linear model $\mathcal{M}$ and any $\mathcal{M}$-rooted sequence w_1, w_2 or $\mathcal{M}, w_1, w_2 \vDash \Rightarrow p \mid p \Rightarrow$ for any linear model $\mathcal{M}$ and any $\mathcal{M}$-rooted sequence w_1, w_2. However, both hypersequents can be easily refuted in componentwise semantics. This contradiction completes the proof. □

5 Linear Nested Sequents for G

Given that the standard hypersequent system HG for G turned out to be useless as far as interpolation proofs go, we consider a different calculus in the framework of *linear nested sequents* [15], essentially a reformulation of 2-sequents [19].

Definition 5.1 A *linear nested sequent* is a finite list of multi-conclusion sequents $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_n \Rightarrow \Delta_n$ with $n > 0$. Its *formula interpretation* is given by $\iota(\Gamma \Rightarrow \Delta) := \bigwedge \Gamma \rightarrow \bigvee \Delta$ and $\iota(\Gamma \Rightarrow \Delta /\!\!/ \mathcal{G}) := \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \iota(\mathcal{G})$.

While linear nested sequents resemble hypersequents, the formula interpretation is markedly different in that it introduces nested implications, in line with the linear structure of models from $\mathcal{L}in$. The presented calculi can be seen as building on the nested sequent calculus for intuitionistic logic from [8]. The rules capturing linearity of frames are inspired by the analogous treatment of linearity in the noncommutative hypersequent calculi for temporal logics presented in [11]. See also [7] for further discussion on other calculi for G.

Definition 5.2 The rules of the calculus LNG are given in Figure 4. In the notation $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}$, none, one, or both of $\mathcal{G}$ and $\mathcal{H}$ could be empty.

$$\frac{}{\mathcal{G}/\!\!/\Gamma, p \Rightarrow \Delta, p/\!\!/\mathcal{H}}\ \mathsf{init}_1 \qquad \frac{}{\mathcal{G}/\!\!/\Gamma, p \Rightarrow \Delta/\!\!/\mathcal{H}/\!\!/\Sigma \Rightarrow \Pi, p/\!\!/\mathcal{I}}\ \mathsf{init}_2$$

$$\frac{}{\mathcal{G}/\!\!/\Gamma, \bot \Rightarrow \Delta/\!\!/\mathcal{H}}\ \bot_L \qquad \frac{\mathcal{G}/\!\!/\Gamma, A \Rightarrow \Delta/\!\!/\Sigma, A \Rightarrow \Pi/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma, A \Rightarrow \Delta/\!\!/\Sigma \Rightarrow \Pi/\!\!/\mathcal{H}}\ \mathsf{Lift}$$

$$\frac{\mathcal{G}/\!\!/\Gamma, A, B \Rightarrow \Delta/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma, A \wedge B \Rightarrow \Delta/\!\!/\mathcal{H}}\ \wedge_L \qquad \frac{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A/\!\!/\mathcal{H} \quad \mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, B/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A \wedge B/\!\!/\mathcal{H}}\ \wedge_R$$

$$\frac{\mathcal{G}/\!\!/\Gamma, A \Rightarrow \Delta/\!\!/\mathcal{H} \quad \mathcal{G}/\!\!/\Gamma, B \Rightarrow \Delta/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma, A \vee B \Rightarrow \Delta/\!\!/\mathcal{H}}\ \vee_L \qquad \frac{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A, B/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A \vee B/\!\!/\mathcal{H}}\ \vee_R$$

$$\frac{\mathcal{G}/\!\!/\Gamma, B \Rightarrow \Delta/\!\!/\mathcal{H} \quad \mathcal{G}/\!\!/\Gamma, A \to B \Rightarrow \Delta, A/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma, A \to B \Rightarrow \Delta/\!\!/\mathcal{H}}\ \to_L \qquad \frac{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/A \Rightarrow B}{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A \to B}\ \to_R^1$$

$$\frac{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/A \Rightarrow B/\!\!/\Sigma \Rightarrow \Pi/\!\!/\mathcal{H} \qquad \mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/\Sigma \Rightarrow \Pi, A \to B/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A \to B/\!\!/\Sigma \Rightarrow \Pi/\!\!/\mathcal{H}}\ \to_R^2$$

Fig. 4. Linear nested sequent calculus LNG for G.

Note that the calculus contains two rules for the implication on the right hand side. This mirrors the two possibilities in attempting to construct a countermodel for the formula $A \to B$: either we have not created a successor to the current world yet, in which case we create such a successor satisfying A and falsifying B, or the current world already has a successor. In the latter case, the witness falsifying the implication $A \to B$ is either in between the current world and that successor, giving the first premiss, or beyond it, giving the second. We modified the $(\to_L)$ rule as usual to have contraction admissible.

Theorem 5.3 (Soundness of LNG) LNG $\vdash \mathcal{G}$ *implies* G $\vdash \iota(\mathcal{G})$.

Proof. By induction on the derivation depth, showing for every rule that whenever the formula interpretation of its conclusion is falsifiable, then so is that of one of its premisses. E.g., for an application of $(\to_R^2)$ with conclusion $\mathcal{G}$ being

$$\Gamma_1 \Rightarrow \Delta_1/\!\!/\ldots/\!\!/\Gamma_n \Rightarrow \Delta_n/\!\!/\Gamma_{n+1} \Rightarrow \Delta_{n+1}, A \to B/\!\!/\Gamma_{n+2} \Rightarrow \Delta_{n+2}/\!\!/\ldots/\!\!/\Gamma_m \Rightarrow \Delta_m$$

where the displayed $A \to B$ is principal, suppose that $\mathcal{M}, w_0 \not\Vdash \iota(\mathcal{G})$ for some $\mathcal{M} = (W, \leqslant, V)$ and $w_0 \in W$. (We omit the model whenever safe.) Then there are $w_1, \ldots, w_m \in W$ such that $w_i \leqslant w_{i+1}$, and $w_i \Vdash \bigwedge \Gamma_i$, and $w_i \not\Vdash \bigvee \Delta_i$ for $i \geqslant 1$. Further, $w_{n+1} \not\Vdash A \to B$, meaning there is a world $u \geqslant w_{n+1}$ such that $u \Vdash A$ and $u \not\Vdash B$. Since the model is linear, either $u < w_{n+2}$ or $w_{n+2} \leqslant u$. In the former case, the sequence $w_0, \ldots, w_{n+1}, u, w_{n+2}, \ldots, w_m$ witnesses that the formula interpretation of the first premiss is falsified at w_0. In the latter case, $w_{n+2} \not\Vdash A \to B$ and, hence, the sequence $w_0, \ldots, w_{n+1}, w_{n+2}, \ldots, w_m$ witnesses that the formula interpretation of the second premiss is falsified at w_0. The remaining cases are similar and simpler and essentially follow [8]. □

While we refrain from giving a cut rule explicitly, we will show completeness via a syntactic cut elimination proof (Theorem 5.16). In preparation for this

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \mathcal{H}} \; \mathsf{W} \qquad \frac{\mathcal{G} /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Rightarrow /\!\!/ \mathcal{H}} \; \mathsf{EW} \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi, A /\!\!/ \mathcal{H}} \; \mathsf{Lower}$$

$$\frac{\mathcal{G} /\!\!/ \Gamma, A, A \Rightarrow \Delta /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, A \Rightarrow \Delta /\!\!/ \mathcal{H}} \; \mathsf{ICL} \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A, A /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H}} \; \mathsf{ICR} \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \mathcal{H}} \; \mathsf{mrg}$$

Fig. 5. Structural rules: internal and external *weakening*, internal *contraction*, *lower*, and *merge*.

we prove a succession of lemmata concerning admissibility of certain structural rules and invertibility for the logical rules in the following sense.

Definition 5.4 A rule is *admissible* if derivability of the premiss(es) implies derivability of the conclusion. It is *depth-preserving admissible*, abbreviated *dp-admissible*, if some derivation of the conclusion has depth no greater than that of the derivations of all premiss(es). Finally, a rule is *invertible* if derivability of the conclusion implies derivability of the premisses.

Structural rules we consider are given in Figure 5. The proofs of the following lemmata are all by induction on the depth d of the derivation.

Lemma 5.5 *Weakening* (W) *is dp-admissible in* LNG.

Lemma 5.6 *External Weakening* (EW) *is admissible in* LNG.

Proof. The case of $d = 0$ is trivial. For $d = n + 1$ we distinguish cases according to the last applied rule. If the last rule was (Lift), we apply IH, followed either by one application of (Lift) or by Lemma 5.5 and two applications of (Lift). Let the last rule be $(\to_R^1)$:

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B} \; \to_R^1$$

If the new component is not introduced at the very end, we use IH followed by $(\to_R^1)$. Otherwise, by IH we obtain both $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B /\!\!/ \Rightarrow$ and $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Rightarrow /\!\!/ A \Rightarrow B$, and applying $(\to_R^1)$ to the latter followed by $(\to_R^2)$ we have the desired result. Finally, let the last rule be $(\to_R^2)$:

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H} \qquad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi, A \to B /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \; \to_R^2$$

If the new component is to be inserted anywhere apart from in between $\Gamma \Rightarrow \Delta, A \to B /\!\!/ \Sigma \Rightarrow \Pi$, we apply IH followed by $(\to_R^2)$. Otherwise, using IH thrice,

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Rightarrow /\!\!/ A \Rightarrow B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Rightarrow /\!\!/ \Sigma \Rightarrow \Pi, A \to B /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Rightarrow A \to B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \; \to_R^2$$

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B /\!\!/ \Rightarrow /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H} \qquad \vdots}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ \Rightarrow /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \; \to_R^2$$

For other rules, IH followed by the same rule suffices. □

Lemma 5.7 (Lower) *is dp-admissible in* LNG.

Proof. If $d = 0$, then the sequent is derived using (init_1), (init_2), or $(\bot_L)$. The last two cases are trivial; in the first case, the desired sequent is an instance of (init_1) or (init_2). Let $d = n + 1$. For the last rule $(\to^1_R)$, the lower-formula is a side formula, and applying IH and $(\to^1_R)$ suffices. For the last rule $(\to^2_R)$ with the lower-formula A a side formula, the most interesting case is

$$\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ C \Rightarrow D /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H} \qquad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \Sigma \Rightarrow \Pi, C \to D /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A, C \to D /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \to^2_R$$

Applying IH twice to the left premiss and once to the right premiss yields $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ C \Rightarrow D /\!\!/ \Sigma \Rightarrow \Pi, A /\!\!/ \mathcal{H}$ and $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi, A, C \to D /\!\!/ \mathcal{H}$. Now $(\to^2_R)$ produces the desired result. If the last rule was $(\to^2_R)$ with the principal formula being the lower-formula, then we take the right premiss. If the last rule was $(\wedge_R)$ or $(\vee_R)$ with the principal formula being the lower-formula, we apply IH to all active formulas and then the same rule. Note that we need dp-admissibility to be able to apply IH twice for $(\to^2_R)$ and $(\vee_R)$. Finally, in all remaining cases the lower-formula is a side formula of the last applied rule, and we apply IH followed by the same rule. □

Lemma 5.8 $(\wedge_R)$ *and* $(\vee_R)$ *are invertible in* LNG.

Since we absorb contraction into the (Lift) rule, we need to use a stronger form of invertibility, *m-invertibility*, for the rules $(\wedge_L)$, $(\vee_L)$, and $(\to_L)$, in a way reminiscent of the way cut is extended to multicut, as formulated in Lemma 5.9. As usual we write A^k for the multiset containing k copies of A.

Lemma 5.9 *For* $\sum_{i=1}^{n} k_i \geqslant 1$, (1) *implies* (2), *and* (3) *implies both* (4) *and* (5), *and* (6) *implies both* (7) *and* (8)*:*

$$\mathsf{LNG} \vdash \Gamma_1, (A \wedge B)^{k_1} \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n, (A \wedge B)^{k_n} \Rightarrow \Delta_n \tag{1}$$

$$\mathsf{LNG} \vdash \Gamma_1, A^{k_1}, B^{k_1} \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n, A^{k_n}, B^{k_n} \Rightarrow \Delta_n \tag{2}$$

$$\mathsf{LNG} \vdash \Gamma_1, (A \vee B)^{k_1} \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n, (A \vee B)^{k_n} \Rightarrow \Delta_n \tag{3}$$

$$\mathsf{LNG} \vdash \Gamma_1, A^{k_1} \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n, A^{k_n} \Rightarrow \Delta_n \tag{4}$$

$$\mathsf{LNG} \vdash \Gamma_1, B^{k_1} \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n, B^{k_n} \Rightarrow \Delta_n \tag{5}$$

$$\mathsf{LNG} \vdash \Gamma_1, (A \to B)^{k_1} \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n, (A \to B)^{k_n} \Rightarrow \Delta_n \tag{6}$$

$$\mathsf{LNG} \vdash \Gamma_1, B^{k_1} \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n, B^{k_n} \Rightarrow \Delta_n \tag{7}$$

$$\mathsf{LNG} \vdash \Gamma_1, (A \to B)^{k_1} \Rightarrow \Delta_1, A^{k_1} /\!\!/ \ldots /\!\!/ \Gamma_n, (A \to B)^{k_n} \Rightarrow \Delta_n, A^{k_n} \tag{8}$$

Proof. We prove that (1) implies (2) by induction on the derivation depth. The crucial case is when the last applied rule is (Lift). W.l.o.g. let the first 2 components be active and $k_1 > 0$:

$$\dfrac{\Gamma_1, (A \wedge B)^{k_1} \Rightarrow \Delta_1 /\!\!/ \Gamma_2, (A \wedge B)^{k_2+1} \Rightarrow \Delta_2 /\!\!/ \ldots /\!\!/ \Gamma_n, (A \wedge B)^{k_n} \Rightarrow \Delta_n}{\Gamma_1, (A \wedge B)^{k_1} \Rightarrow \Delta_1 /\!\!/ \Gamma_2, (A \wedge B)^{k_2} \Rightarrow \Delta_2 /\!\!/ \ldots /\!\!/ \Gamma_n, (A \wedge B)^{k_n} \Rightarrow \Delta_n} \mathsf{Lift}$$

$\Gamma_1, A^{k_1}, B^{k_1} \Rightarrow \Delta_1 /\!\!/ \Gamma_2, A^{k_2+1}, B^{k_2+1} \Rightarrow \Delta_2 /\!\!/ \dots /\!\!/ \Gamma_n, A^{k_n}, B^{k_n} \Rightarrow \Delta_n$ by IH for the premiss. Now apply (Lift) twice. Note that this breaks depth-preserving invertibility and shows why we need m-invertibility instead of standard invertibility. In all remaining cases, apply IH followed by the same rule. The proof of (4) and (5) from (3) is similar. (8) follows from (6) by Lemma 5.5. The proof of (7) from (6) is by induction on the depth of the derivation. If all of the implications are side formulas in the last applied rule, or principal in the (Lift) rule, apply IH and the same rule. If one of the implications is principal in $(\to_L)$ and it is not the only implication, then applying IH to the left premiss suffices. □

Lemma 5.10 *The rule* $(\to_R^2)$ *is invertible in* LNG.

Proof. Invertibility w.r.t. the right premiss follows immediately from admissibility of (Lower) (Lemma 5.7). Invertibility w.r.t. the left premiss is shown by induction on the derivation depth d. If $d = 0$, the same rule clearly applies. If $d = n + 1$, we distinguish cases according to the last applied rule. If it is one of $(\wedge_L)$, $(\wedge_R)$, $(\vee_L)$, $(\vee_R)$, $(\to_L)$, (Lift), or $(\to_R^1)$, then apply IH on the premiss(es) of that rule followed by the same rule, possibly twice together with admissibility of (W) in the case of (Lift). If the last rule is $(\to_R^2)$ and the relevant formula is principal, we are done. If the relevant formula is a side formula of a side component, we apply IH followed by $(\to_R^2)$. Otherwise,

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ C \Rightarrow D /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ \Sigma \Rightarrow \Pi, C \to D /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B, C \to D /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \to_R^2 \quad (9)$$

Using IH on the premisses of this we obtain the premisses of

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B /\!\!/ C \Rightarrow D /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B /\!\!/ \Sigma \Rightarrow \Pi, C \to D /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B, C \to D /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \to_R^2$$

Furthermore, by dp-admissibility of Lower (Lemma 5.7) and IH on the left premiss of (9) we obtain $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ C \Rightarrow D /\!\!/ A \Rightarrow B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}$ and applying $(\to_R^2)$ yields $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, C \to D /\!\!/ A \Rightarrow B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}$. □

Lemma 5.11 *The rule* $(\to_R^1)$ *is invertible in* LNG.

Proof. By induction on the derivation depth. The base case is easy. For the induction step we distinguish cases according to the last rule in the derivation of $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B$. If that rule was one of $(\wedge_L)$, $(\wedge_R)$, $(\vee_L)$, $(\vee_R)$, $(\to_L)$, (Lift), or $(\to_R^2)$, then we apply IH on its premiss(es) followed by the same rule. If the last rule was $(\to_R^1)$ with principal $A \to B$, we take the premiss and are done. Otherwise, the last rule was $(\to_R^1)$ with $A \to B$ a side formula, so

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ C \Rightarrow D}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B, C \to D} \to_R^1$$

Applying dp-admissibility of Lower (Lemma 5.7) and IH on the premiss yields $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ C \Rightarrow D /\!\!/ A \Rightarrow B$. Further, Lemma 5.10 and $(\to_R^1)$ on the same premiss gives $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B, C \to D$. Now applying $(\to_R^2)$ yields the result. □

Lemma 5.12 *Left Contraction* (ICL) *is admissible in* LNG.

Proof. By induction on the pairs $(|A|, d)$ in the lexicographic ordering, where d is the depth of the derivation and $|A|$ the complexity of the contraction formula. We distinguish cases according to the main connective of A. Let $\mathsf{LNG} \vdash \mathcal{G} /\!/ \Gamma, A, A \Rightarrow \Delta /\!/ \mathcal{H}$. Contracting initial sequents is easy. If A is a principal formula of (Lift) or a side formula, we apply IH to the premiss(es) of this rule, and then the same rule. If A is a principal conjunction/disjunction in the last rule, we apply Lemma 5.9, IH twice consecutively/in parallel and the rule $(\wedge_L)/(\vee_L)$. Finally, if $A = C \to D$ is principal in $(\to_L)$, then we have

$$\frac{\mathcal{G} /\!/ \Gamma, C \to D, D \Rightarrow \Delta /\!/ \mathcal{H} \qquad \mathcal{G} /\!/ \Gamma, C \to D, C \to D \Rightarrow \Delta, C /\!/ \mathcal{H}}{\mathcal{G} /\!/ \Gamma, C \to D, C \to D \Rightarrow \Delta /\!/ \mathcal{H}} \to_L$$

Applying Lemma 5.9 and IH for D to the left premiss gives $\mathcal{G} /\!/ \Gamma, D \Rightarrow \Delta /\!/ \mathcal{H}$. By IH for $C \to D$ in the right premiss, $\mathcal{G} /\!/ \Gamma, C \to D \Rightarrow \Delta, C /\!/ \mathcal{H}$. Now $(\to_L)$ yields the desired $\mathcal{G} /\!/ \Gamma, C \to D \Rightarrow \Delta /\!/ \mathcal{H}$. □

Lemma 5.13 *Merge* (mrg) *is admissible in* LNG.

Proof. By induction on the derivation depth. For the base case, the result of applying (mrg) to an initial sequent is still an initial sequent. For the induction step we distinguish cases according to the last rule. In most cases, including if it was one of $(\wedge_L)$, $(\wedge_R)$, $(\vee_L)$, $(\vee_R)$, $(\to_L)$, or $(\to^1_R)$, we apply IH and then the same rule. If (mrg) merges the principal components of (Lift), we apply IH and admissibility of (ICL) (Lemma 5.12). Finally, if (mrg) merges the principal components of $(\to^2_R)$, it is sufficient to apply IH to the right premiss. □

Lemma 5.14 *Right Contraction* (ICR) *is admissible in* LNG.

Proof. By induction on the pairs $(|A|, d)$ in the lexicographic ordering, where $|A|$ is the complexity of the contraction formula and d is the depth of the derivation. We distinguish cases according to the main connective of A and only show those where IH and the same rule cannot be used directly. If one of the occurrences of $A = C \wedge D / C \vee D$ is principal in $(\wedge_R)/(\vee_R)$, invert it by Lemma 5.9, and then use IH as needed. If $A = C \to D$ is principal in $(\to^1_R)$,

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, C \to D /\!/ C \Rightarrow D}{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, C \to D, C \to D} \to^1_R$$

Applying invertibility of $(\to^2_R)$ (Lemma 5.10) and admissibility of (mrg) (Lemma 5.13) to the premiss yields $\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ C, C \Rightarrow D, D$. Now applying IH for D and admissibility of (ICL) for C (Lemma 5.12), followed by $(\to^1_R)$ yields $\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, C \to D$. If $A = C \to D$ is principal in $(\to^2_R)$, we have

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, C \to D /\!/ C \Rightarrow D /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H} \quad \mathcal{G} /\!/ \Gamma \Rightarrow \Delta, C \to D /\!/ \Sigma \Rightarrow \Pi, C \to D /\!/ \mathcal{H}}{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, C \to D, C \to D /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H}} \to^2_R$$

From the left premiss, as before by the same Lemmata 5.10 and 5.13 we can obtain $\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ C, C \Rightarrow D, D /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H}$, and by IH for D and admissibility of (ICL) (Lemma 5.12) for C, we get $\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/C \Rightarrow D/\!\!/\Sigma \Rightarrow \Pi/\!\!/\mathcal{H}$. Further, from the right premiss above by dp-admissibility of (Lower) (Lemma 5.7) and IH we obtain $\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/\Sigma \Rightarrow \Pi, C \to D/\!\!/\mathcal{H}$. Now $(\to_R^2)$ gives the desired result. □

We can now prove the admissibility of the cut rule. As we absorbed contraction into (Lift), we use a more general kind of cuts similar to "one-sided multicuts." While we do not use it explicitly, the cut rule for linear nested sequents can be read off the statement of Theorem 5.16 for the case $n = k_1 = 1$.

Definition 5.15 We define the *splice* $\mathcal{G} \oplus \mathcal{H}$ of linear nested sequents $\mathcal{G}$ and $\mathcal{H}$:

$$
\begin{aligned}
(\Gamma \Rightarrow \Delta) \oplus (\Sigma \Rightarrow \Pi) &:= \Gamma, \Sigma \Rightarrow \Delta, \Pi \\
(\Gamma \Rightarrow \Delta) \oplus (\Sigma \Rightarrow \Pi /\!\!/ \Omega \Rightarrow \Theta /\!\!/ \mathcal{H}) &:= \Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \Omega \Rightarrow \Theta /\!\!/ \mathcal{H} \\
(\Gamma \Rightarrow \Delta /\!\!/ \Omega \Rightarrow \Theta /\!\!/ \mathcal{H}) \oplus (\Sigma \Rightarrow \Pi) &:= \Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \Omega \Rightarrow \Theta /\!\!/ \mathcal{H} \\
(\Gamma \Rightarrow \Delta /\!\!/ \Omega \Rightarrow \Theta /\!\!/ \mathcal{G}) \oplus (\Sigma \Rightarrow \Pi /\!\!/ \Xi \Rightarrow \Upsilon /\!\!/ \mathcal{H}) &:= \\
&\Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \big((\Omega \Rightarrow \Theta /\!\!/ \mathcal{G}) \oplus (\Xi \Rightarrow \Upsilon /\!\!/ \mathcal{H})\big)
\end{aligned}
$$

Theorem 5.16 (Cut elimination) *If* $\|\mathcal{G}\| = \|\mathcal{I}\|$, *and* $\sum_{i=1}^{n} k_i \geqslant 1$, *and* $\|\mathcal{H}\| = n - 1$, *then* (10) *and* (11) *imply* (12)*:*

$$\mathsf{LNG} \vdash \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H} \tag{10}$$

$$\mathsf{LNG} \vdash \mathcal{I} /\!\!/ A^{k_1}, \Sigma_1 \Rightarrow \Pi_1 /\!\!/ \ldots /\!\!/ A^{k_n}, \Sigma_n \Rightarrow \Pi_n \tag{11}$$

$$\mathsf{LNG} \vdash (\mathcal{G} \oplus \mathcal{I}) /\!\!/ \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 /\!\!/ \big(\mathcal{H} \oplus (\Sigma_2 \Rightarrow \Pi_2 /\!\!/ \ldots /\!\!/ \Sigma_n \Rightarrow \Pi_n)\big) \tag{12}$$

Proof. By induction on the pairs $(|A|, d)$ in the lexicographic ordering, where $|A|$ is the complexity of the cut formula and d is the depth of the derivation. If $d = 0$, then (11) is an instance of (init_1), (init_2), or $(\bot_L)$. If none of the displayed A's is principal in that rule, (12) is an instance of the same rule. If an occurrence of $A = p$ is principal in (init_1) or (init_2), then A occurs in Π_i for some $i \leqslant n$, and we obtain (12) by (cut-free) admissibility of (Lower) (Lemma 5.7) and (W) (Lemma 5.5) from $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H}$. If an occurrence of $A = \bot$ is principal in $(\bot_L)$, then we use the admissibility of

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \bot /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}}$$

which can be proved by induction on the depth of the derivation, together with admissibility of (W) (Lemma 5.5) to obtain (12) from $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \bot /\!\!/ \mathcal{H}$.

Let $d > 0$. If none of A's is principal in the last applied rule r, we apply IH to the premiss(es) of r, followed by r itself. If r is $(\to_R^1)$ or $(\to_R^2)$ we additionally use admissibility of (EW) (Lemma 5.6), e.g., if (11) is derived by

$$\frac{\mathcal{I} /\!\!/ A^{k_1}, \Sigma_1 \Rightarrow \Pi_1' /\!\!/ E \Rightarrow F /\!\!/ A^{k_2}, \Sigma_2 \Rightarrow \Pi_2 /\!\!/ \ldots /\!\!/ A^{k_n}, \Sigma_n \Rightarrow \Pi_n \qquad \mathcal{I} /\!\!/ A^{k_1}, \Sigma_1 \Rightarrow \Pi_1' /\!\!/ A^{k_2}, \Sigma_2 \Rightarrow \Pi_2, E \to F /\!\!/ \ldots /\!\!/ A^{k_n}, \Sigma_n \Rightarrow \Pi_n}{\mathcal{I} /\!\!/ A^{k_1}, \Sigma_1 \Rightarrow \Pi_1', E \to F /\!\!/ A^{k_2}, \Sigma_2 \Rightarrow \Pi_2 /\!\!/ \ldots /\!\!/ A^{k_n}, \Sigma_n \Rightarrow \Pi_n} \to_R^2$$

then admissibility of (EW) on $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H}$ yields $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \Rightarrow /\!\!/ \mathcal{H}$, and we get $\big(\mathcal{G} \oplus \mathcal{I}\big) /\!\!/ \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1' /\!\!/ E \Rightarrow F /\!\!/ \big(\mathcal{H} \oplus (\Sigma_2 \Rightarrow \Pi_2 /\!\!/ \ldots /\!\!/ \Sigma_n \Rightarrow \Pi_n\big)$ by IH on this and the top premiss above. By IH on $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H}$ and the bottom premiss, $\big(\mathcal{G} \oplus \mathcal{I}\big) /\!\!/ \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1' /\!\!/ \big(\mathcal{H} \oplus (\Sigma_2 \Rightarrow \Pi_2, E \to F /\!\!/ \ldots /\!\!/ \Sigma_n \Rightarrow \Pi_n\big)$. Applying $(\to_R^2)$ to these two sequents yields (12) because $\Pi_1 = \Pi_1', E \to F$.

If one of A's is a principal formula of (Lift), then IH suffices. Otherwise, we distinguish several cases according to the main connective in A. If $A = C \wedge D$, by invertibility of $(\wedge_L)$ (Lemma 5.9) and $(\wedge_R)$ (Lemma 5.8), we can derive:

$$\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, C /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, D /\!\!/ \mathcal{H} \quad \mathcal{I} /\!\!/ C^{k_1}, D^{k_1}, \Sigma_1 \Rightarrow \Pi_1 /\!\!/ \ldots /\!\!/ C^{k_n}, D^{k_n}, \Sigma_n \Rightarrow \Pi_n$$

Since $|C|, |D| < |A|$ we can apply IH twice to obtain a derivation of

$$(\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}) \oplus (\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}) \oplus (\mathcal{I} /\!\!/ \Sigma_1 \Rightarrow \Pi_1 /\!\!/ \ldots /\!\!/ \Sigma_n \Rightarrow \Pi_n)$$

and admissibility of (ICL) and (ICR) (Lemmata 5.12 and 5.14) yields (12). If $A = C \vee D$, we proceed analogously.

If $A = C \to D$ and none of the occurrences of A in the right premiss (11) of the cut is principal, we proceed as for the case where A is a propositional variable. If one of its occurrences in (11) is principal, we have

$$\frac{\begin{array}{c}\mathcal{I} /\!\!/ (C \to D)^{k_1}, \Sigma_1 \Rightarrow \Pi_1 /\!\!/ \ldots /\!\!/ (C \to D)^{k_m}, \Sigma_m \Rightarrow \Pi_m, C /\!\!/ \ldots /\!\!/ (C \to D)^{k_n}, \Sigma_n \Rightarrow \Pi_n \\ \mathcal{I} /\!\!/ (C \to D)^{k_1}, \Sigma_1 \Rightarrow \Pi_1 /\!\!/ \ldots /\!\!/ (C \to D)^{k_m - 1}, D, \Sigma_m \Rightarrow \Pi_m /\!\!/ \ldots /\!\!/ (C \to D)^{k_n}, \Sigma_n \Rightarrow \Pi_n\end{array}}{\mathcal{I} /\!\!/ (C \to D)^{k_1}, \Sigma_1 \Rightarrow \Pi_1 /\!\!/ \ldots /\!\!/ (C \to D)^{k_m}, \Sigma_m \Rightarrow \Pi_m /\!\!/ \ldots /\!\!/ (C \to D)^{k_n}, \Sigma_n \Rightarrow \Pi_n} \to_L$$

Suppose that $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, C \to D /\!\!/ \mathcal{H}$ is

$$\mathcal{G} /\!\!/ \Gamma_1 \Rightarrow \Delta_1, C \to D /\!\!/ \Gamma_2 \Rightarrow \Delta_2 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n. \tag{13}$$

First, we apply IH in "cross-cuts" to (13) and each premiss of $(\to_L)$ to eliminate all $C \to D$'s in the context (if $\sum_{i=1}^{n} k_i = k_m = 1$, then admissible (W) is used for the bottom premiss instead), resulting in derivations of

$$\big(\mathcal{G} \oplus \mathcal{I}\big) /\!\!/ \Gamma_1, \Sigma_1 \Rightarrow \Delta_1, \Pi_1 /\!\!/ \ldots /\!\!/ \Gamma_m, \Sigma_m, D \Rightarrow \Delta_m, \Pi_m /\!\!/ \ldots /\!\!/ \Gamma_n, \Sigma_n \Rightarrow \Delta_n, \Pi_n \tag{14}$$
$$\big(\mathcal{G} \oplus \mathcal{I}\big) /\!\!/ \Gamma_1, \Sigma_1 \Rightarrow \Delta_1, \Pi_1 /\!\!/ \ldots /\!\!/ \Gamma_m, \Sigma_m \Rightarrow \Delta_m, \Pi_m, C /\!\!/ \ldots /\!\!/ \Gamma_n, \Sigma_n \Rightarrow \Delta_n, \Pi_n \tag{15}$$

From (13), we obtain $\mathcal{G} /\!\!/ \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_m, C \Rightarrow \Delta_m, D /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$ by using admissibility of (Lower) (Lemma 5.7), invertibility of $(\to_R^2)$ (Lemma 5.10) or of $(\to_R^1)$ (Lemma 5.11) depending on whether $m < n$ or $m = n$ respectively, and admissibility of (mrg) (Lemma 5.13). Since $|C|, |D| < |C \to D|$, applying IH twice to the resulting sequent and the sequents (14) and (15) yields

$$\big(\mathcal{G} \oplus \mathcal{G} \oplus \mathcal{G} \oplus \mathcal{I} \oplus \mathcal{I}\big) /\!\!/ (\Gamma_1)^3, (\Sigma_1)^2 \Rightarrow (\Delta_1)^3, (\Pi_1)^2 /\!\!/ \ldots /\!\!/ (\Gamma_n)^3, (\Sigma_n)^2 \Rightarrow (\Delta_n)^3, (\Pi_n)^2$$

Finally, using admissibility of Contraction (Lemmata 5.12 and 5.14) we obtain the desired $\big(\mathcal{G} \oplus \mathcal{I}\big) /\!\!/ \Gamma_1, \Sigma_1 \Rightarrow \Delta_1, \Pi_1 /\!\!/ \ldots /\!\!/ \Gamma_n, \Sigma_n \Rightarrow \Delta_n, \Pi_n$. □

Corollary 5.17 (Completeness of LNG) $\mathsf{G} \vdash A$ *implies* $\mathsf{LNG} \vdash \Rightarrow A$.

Proof. It is easy to derive the axioms of G, including $\Rightarrow (A \to B) \vee (B \to A)$. Modus ponens is simulated using admissibility of cut as usual, by deriving $A, A \to B \Rightarrow B$ and applying Theorem 5.16 twice to this and the linear nested sequents $\Rightarrow A$ and $\Rightarrow A \to B$ respectively. □

6 Interpolation for G via Linear Nested Sequents

We now explain modifications to the construction of interpolants via hypersequents sufficient to adapt the method to linear nested sequents. If *validity* in Definition 3.5 is defined based on all *monotone sequences* $\boldsymbol{w}$, i.e., sequences with $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_{\|\boldsymbol{w}\|}$ instead of $\mathcal{M}$-rooted ones, then Lemma 3.6 can be proved by induction on $\|\boldsymbol{w}\|$ for linear nested sequents and the formula interpretation from Definition 5.1. More generally, all definitions and statements from Sections 3 and 4 that mention $\mathcal{M}$-rooted sequences must now use monotone sequences instead. Most proofs apply as is. E.g., Lemma 3.12 still holds because for singleton sequences, both $\mathcal{M}$-rootedness and monotonicity are trivial and, hence, equivalent. Definition 4.1 and Lemma 4.2 are omitted. Since Theorem 4.5 is a direct consequence of Lem 4.3, it remains to define all split versions of the rules from Figure 4, provide interpolant transformations for them, and prove their correctness.

It is clear that splits of initial linear nested sequents (init_1) and ($\bot_L$) can be interpolated the same way as the corresponding splits of initial hypersequents (id) and ($\mathsf{id}_\bot$) from Figure 1 respectively, except that the superscript should be $\|\widetilde{\mathcal{G}}\| + 1$ instead of 1, e.g., $\widetilde{\mathcal{G}} /\!\!/ \Gamma, p; \Pi \Rightarrow \Delta; \Lambda, p /\!\!/ \widetilde{\mathcal{H}} \xleftarrow{\mathcal{L}in} p^{(\|\widetilde{\mathcal{G}}\|+1)}$. It also works for ($\mathsf{init}_2$) split the same way, e.g.,

$$\widetilde{\mathcal{G}} /\!\!/ \Gamma; \Theta, p \Rightarrow \widetilde{\Delta} /\!\!/ \widetilde{\mathcal{H}} /\!\!/ \widetilde{\Sigma} \Rightarrow \Lambda, p; \Pi /\!\!/ \widetilde{\mathcal{I}} \xleftarrow{\mathcal{L}in} \overline{p}^{(\|\widetilde{\mathcal{G}}\|+1)}.$$

The transformations for splits of ($\wedge_L$), ($\wedge_R$), ($\vee_L$), ($\vee_R$), and ($\rightarrow_L$) are the same as for corresponding splits of ($\wedge \Rightarrow$), ($\Rightarrow \wedge$), ($\vee \Rightarrow$), ($\Rightarrow \vee$), and ($\rightarrow\Rightarrow$) respectively, and the proof is the same. It is also easy to see that, as with many unary rules, any interpolant for the premiss of (Lift) also works for its conclusion, mainly because $w_{k+1} \not\Vdash A$ implies $w_k \not\Vdash A$ for any monotone sequence $\boldsymbol{w}$. Finally, the transformations for ($\rightarrow_R^1$) and ($\rightarrow_R^2$) are presented in Figure 6.

Theorem 6.1 G *enjoys the CIP.*

Proof. As for hypersequents, it is sufficient to prove that all interpolant transformations in this section preserve componentwise interpolation w.r.t. $\mathcal{L}in$, which is tedious but not difficult. While, for the lack of space, we only provide an argument for ($\rightarrow_R^{2l}$) from Figure 6, it is worth mentioning that the ${D_j}^{(n)}$ and $\overline{C_j}^{(n)}$ terms in the conclusions of ($\rightarrow_R^{2l}$) and ($\rightarrow_R^{2r}$) respectively have to be added to make sure IH can be used for a world intermediate between the $(n-1)$th and nth worlds in a given sequence.

Rule ($\rightarrow_R^{2l}$) Assume the two interpolation statements in the premisses to be true w.r.t. $\mathcal{L}in$. Consider any linear model $\mathcal{M} = (W, \leqslant, V)$ and an arbitrary monotone sequence $\boldsymbol{w}$ of worlds of length $\|\boldsymbol{w}\| = n + k$, where $n = \|\widetilde{\mathcal{G}}\| + 2$ and $k = \|\widetilde{\mathcal{H}}\|$. It is clear that for any world u such that $w_{n-1} \leqslant u \leqslant w_n$ and for the sequence $\boldsymbol{v} := w_1, \ldots, w_{n-1}, u, w_n, \ldots, w_{n+k}$,

$$\boldsymbol{v} \vDash \bigotimes_{l \neq n} (\overline{E_{jl}}^{(l)} \otimes F_{jl}{}^{(l)}) \iff \boldsymbol{w} \vDash \bigotimes_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \otimes F_{jl}{}^{(l)}) \otimes \bigotimes_{l=n}^{n+k} (\overline{E_{j,l+1}}^{(l)} \otimes F_{j,l+1}{}^{(l)}) \quad (16)$$

$$\to_R^{1l}\ \frac{\widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Pi /\!\!/ A; \Rightarrow B; \ \leftarrow\ \bigovee_{j=1}^{m} \left(\overline{C_j}^{(n)} \owedge {D_j}^{(n)} \owedge \bigowedge_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)})\right)}{\widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta, A \to B; \Pi \ \leftarrow\ \bigovee_{j=1}^{m} \left(\overline{D_j \to C_j}^{(n-1)} \owedge \bigowedge_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)})\right)}$$

$$\to_R^{1r}\ \frac{\widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Pi /\!\!/ ; A \Rightarrow ; B \ \leftarrow\ \bigowedge_{j=1}^{m} \left(\overline{C_j}^{(n)} \ovee {D_j}^{(n)} \ovee \bigovee_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \ovee {F_{jl}}^{(l)})\right)}{\widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Pi, A \to B \ \leftarrow\ \bigowedge_{j=1}^{m} \left((C_j \to D_j)^{(n-1)} \ovee \bigovee_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \ovee {F_{jl}}^{(l)})\right)}$$

$$\to_R^{2l}\ \frac{\begin{array}{l}\widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Theta /\!\!/ \widetilde{\Sigma} \Rightarrow \Pi, A \to B; \Lambda /\!\!/ \widetilde{\mathcal{H}} \leftarrow \mho \\ \qquad \widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Theta /\!\!/ A; \Rightarrow B; /\!\!/ \widetilde{\Sigma} \Rightarrow \Pi; \Lambda /\!\!/ \widetilde{\mathcal{H}} \leftarrow \bigovee_{j=1}^{m} \left(\overline{C_j}^{(n)} \owedge {D_j}^{(n)} \owedge \bigowedge_{l \neq n} (\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)})\right)\end{array}}{\begin{array}{l}\longleftarrow \quad \mho \ovee \bigovee_{j=1}^{m} \left(\bigowedge_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)}) \owedge \overline{(D_j \to C_j)}^{(n-1)} \owedge {D_j}^{(n)} \owedge \bigowedge_{l=n}^{n+k} (\overline{E_{j,l+1}}^{(l)} \owedge {F_{j,l+1}}^{(l)})\right) \\ \widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta, A \to B; \Theta /\!\!/ \widetilde{\Sigma} \Rightarrow \Pi; \Lambda /\!\!/ \widetilde{\mathcal{H}} \quad \longleftarrow\end{array}}$$

$$\to_R^{2r}\ \frac{\begin{array}{l}\widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Theta /\!\!/ \widetilde{\Sigma} \Rightarrow \Pi; \Lambda, A \to B /\!\!/ \widetilde{\mathcal{H}} \leftarrow \mho \\ \qquad \widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Theta /\!\!/ ; A \Rightarrow ; B /\!\!/ \widetilde{\Sigma} \Rightarrow \Pi; \Lambda /\!\!/ \widetilde{\mathcal{H}} \leftarrow \bigowedge_{j=1}^{m} \left(\overline{C_j}^{(n)} \ovee {D_j}^{(n)} \ovee \bigovee_{l \neq n} (\overline{E_{jl}}^{(l)} \ovee {F_{jl}}^{(l)})\right)\end{array}}{\begin{array}{l}\longleftarrow \quad \mho \owedge \bigowedge_{j=1}^{m} \left(\bigovee_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \ovee {F_{jl}}^{(l)}) \ovee (C_j \to D_j)^{(n-1)} \ovee \overline{C_j}^{(n)} \ovee \bigovee_{l=n}^{n+k} (\overline{E_{j,l+1}}^{(l)} \ovee {F_{j,l+1}}^{(l)})\right) \\ \widetilde{\mathcal{G}} /\!\!/ \widetilde{\Gamma} \Rightarrow \Delta; \Theta, A \to B /\!\!/ \widetilde{\Sigma} \Rightarrow \Pi; \Lambda /\!\!/ \widetilde{\mathcal{H}} \quad \longleftarrow\end{array}}$$

Fig. 6. Transformations for component-creating rules of LNG. All interpolation statements are w.r.t. $\mathcal{L}in$. For all 4 rules, $\|\widetilde{\mathcal{G}}\| = n - 2$. For the last 2 rules, $\|\widetilde{\mathcal{H}}\| = k$.

We consider two possibilities depending on whether the interpolant in the conclusion holds or not. Assume first that it does not hold for $\boldsymbol{w}$. In particular $\boldsymbol{w} \not\models \mho$ and, for each $j = 1, \dots, m$, the following does not hold at $\boldsymbol{w}$:

$$\bigowedge_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \owedge {F_{jl}}^{(l)}) \owedge \overline{(D_j \to C_j)}^{(n-1)} \owedge {D_j}^{(n)} \owedge \bigowedge_{l=n}^{n+k} (\overline{E_{j,l+1}}^{(l)} \owedge {F_{j,l+1}}^{(l)}). \quad (17)$$

We need to prove that, whenever $\boldsymbol{w} \not\models L\widetilde{\mathcal{G}} /\!\!/ L\widetilde{\Gamma} \Rightarrow \Delta /\!\!/ L\widetilde{\Sigma} \Rightarrow \Pi /\!\!/ L\widetilde{\mathcal{H}}$, we have $w_{n-1} \Vdash A \to B$. Assuming the former, we conclude from $\boldsymbol{w} \not\models \mho$ and the interpolation statement for the top premiss that $w_n \Vdash A \to B$. We now use the interpolation statement for the bottom premiss to show that $u \not\Vdash A$ or $u \Vdash B$ for any $u \geqslant w_{n-1}$ such that $u \ngeqslant w_n$. By linearity of $\mathcal{M}$, it follows that $u < w_n$, making $\boldsymbol{v}$ above a monotone sequence. It remains to show that the interpolant of the bottom premiss, let us denote it $\mho'$, is false at $\boldsymbol{v}$. By (16), if the jth disjunct of the conclusion interpolant is false at $\boldsymbol{w}$ because of E's and F's, then the jth disjunct of $\mho'$ is false at $\boldsymbol{v}$. If $\boldsymbol{w} \not\models {D_j}^{(n)}$, i.e., if $w_n \not\Vdash D_j$, then $u \not\Vdash D_j$ by monotonicity, i.e., $\boldsymbol{v} \not\models {D_j}^{(n)}$. And if $\boldsymbol{w} \not\models \overline{(D_j \to C_j)}^{(n-1)}$, i.e., $w_{n-1} \Vdash D_j \to C_j$, then either $u \not\Vdash D_j$ or $u \Vdash C_j$, i.e., $\boldsymbol{v} \not\models \overline{C_j}^{(n)} \owedge {D_j}^{(n)}$. Thus, $\boldsymbol{v} \not\models \mho'$, completing the proof for the left side.

For the right side, assume that the conclusion interpolant is true at $\boldsymbol{w}$. If it is true because of $\mho$, then the right side $R\widetilde{\mathcal{G}} /\!\!/ R\widetilde{\Gamma} \Rightarrow \Theta /\!\!/ R\widetilde{\Sigma} \Rightarrow \Lambda /\!\!/ R\widetilde{\mathcal{H}}$ of the top premiss holds, which is the same as the right side of the conclusion. Otherwise, (17) is true at $\boldsymbol{w}$ for some $1 \leqslant j \leqslant m$, in particular, $w_{n-1} \nVdash D_j \to C_j$ and $w_n \Vdash D_j$. Thus, there exists some $u' \geqslant w_{n-1}$ such that $u' \Vdash D_j$ and $u' \nVdash C_j$. Let $u := u'$ if $u' \leqslant w_n$ or $u := w_n$ otherwise, i.e., if $u' > w_n$. In the latter case, $u \nVdash C_j$ by monotonicity. Hence, either way, $w_{n-1} \leqslant u \leqslant w_n$ and both $u \Vdash D_j$ and $u \nVdash C_j$. Therefore, $\boldsymbol{v}$ above is a monotone sequence and the jth disjunct makes the bottom premiss interpolant true at $\boldsymbol{v}$. By the interpolation statement for this premiss, $\boldsymbol{v} \models R\widetilde{\mathcal{G}} /\!\!/ R\widetilde{\Gamma} \Rightarrow \Theta /\!\!/ \Rightarrow /\!\!/ R\widetilde{\Sigma} \Rightarrow \Lambda /\!\!/ \widetilde{\mathcal{H}}$, which implies $\boldsymbol{w} \models R\widetilde{\mathcal{G}} /\!\!/ R\widetilde{\Gamma} \Rightarrow \Theta /\!\!/ R\widetilde{\Sigma} \Rightarrow \Lambda /\!\!/ R\widetilde{\mathcal{H}}$. □

Corollary 6.2 Int, LQ, *and* G *enjoy the LIP.*

Proof. All interpolant transformations preserve variable polarity. □

7 Conclusion and Discussion

We have provided constructive proofs of Craig and Lyndon interpolation using sequent-style calculi for Jankov logic and Gödel logic. In particular, Lyndon interpolation for Gödel logic was listed as an open problem in [4,10,18]. We answer this question in the positive. Its proof uses a novel calculus for G in the framework of linear nested sequents which is of independent interest.

We are grateful to one of the reviewers who suggested a possibility that there might be alternative constructive proofs of interpolation property for these logics via translation from modal logics or based on methods and calculi from [1,2,3]. While this is a very interesting direction for further research, doing so for LIP seems at the very least not trivial, as also confirmed by M. Baaz, one of the authors of the above three papers, in a private communication. As for using known results for modal logics, the Gödel translations of the three intermediate logics we considered are S4, S4.2, and S4.3. Thus, a method with a detour via modal logics could not produce any results for Gödel logic, due to S4.3 not having the CIP [17], in effect, leaving out the most interesting case. In particular, a linear nested calculus would not help prove interpolation in the modal case. The exact source of this disparity deserves further investigation.

While our method does not produce proofs of uniform interpolation, this property presents less interest for us than LIP because for intermediate logics uniform interpolation is known to be equivalent to CIP [18].

This work is part of the project of using proof-theoretic methods to show Craig and Lyndon interpolation for intermediate logics.

Acknowledgments. We would like to thank M. Baaz for many fruitful discussions, A. Ciabattoni for her general and generous support, and D. Gabbay for encouragement. We further thank the anonymous reviewers for their careful reading of the article and their useful and thought-provoking advice, which may lead to further publications, by us and/or by them.

References

[1] Baaz, M., A. Ciabattoni and C. G. Fermüller, *Cut-elimination in a sequents-of-relations calculus for Gödel logic*, in: *ISMVL 2001*, IEEE, 2001 pp. 181–186.

[2] Baaz, M., A. Ciabattoni and C. G. Fermüller, *Hypersequent calculi for Gödel logics—a survey*, Journal of Logic and Computation **13** (2003), pp. 835–861.

[3] Baaz, M. and H. Veith, *An axiomatization of quantified propositional Gödel logic using the Takeuti–Titani rule*, in: *Logic Colloquium 1998*, LNL **13**, ASL, 2000 pp. 91–104.

[4] Chagrov, A. and M. Zakharyaschev, "Modal Logic," Clarendon Press, 1997.

[5] Ciabattoni, A. and M. Ferrari, *Hypertableau and path-hypertableau calculi for some families of intermediate logics*, in: R. Dyckhoff, editor, *TABLEAUX 2000*, LNAI **1847**, Springer, 2000 pp. 160–174.

[6] Ciabattoni, A., R. Ramanayake and H. Wansing, *Hypersequent and display calculi—a unified perspective*, Studia Logica **102** (2014), pp. 1245–1294.

[7] Dyckhoff, R. and S. Negri, *Decision methods for linearly ordered Heyting algebras*, Archive for Mathematical Logic **45** (2006), pp. 411–422.

[8] Fitting, M., *Nested sequents for intuitionistic logics*, Notre Dame Journal of Formal Logic **55** (2014), pp. 41–61.

[9] Fitting, M. and R. Kuznets, *Modal interpolation via nested sequents*, Annals of Pure and Applied Logic **166** (2015), pp. 274–305.

[10] Gabbay, D. M. and L. Maksimova, "Interpolation and Definability: Modal and Intuitionistic Logic," Clarendon Press, 2005.

[11] Indrzejczak, A., *Linear time in hypersequent framework*, Bulletin of Symbolic Logic **22** (2016), pp. 121–144.

[12] Kuznets, R., *Craig interpolation via hypersequents*, in: D. Probst and P. Schuster, editors, *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, Ontos Mathematical Logic **6**, De Gruyter, 2016 pp. 193–214.

[13] Kuznets, R., *Multicomponent proof-theoretic method for proving interpolation property*, Annals of Pure and Applied Logic **To appear** (2018).

[14] Kuznets, R. and B. Lellmann, *Grafting hypersequents onto nested sequents*, Logic Journal of IGPL **24** (2016), pp. 375–423.

[15] Lellmann, B., *Linear nested sequents, 2-sequents and hypersequents*, in: H. De Nivelle, editor, *TABLEAUX 2015*, LNAI **9323**, Springer, 2015 pp. 135–150.

[16] Maksimova, L. L., *Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras*, Algebra and Logic **16** (1977), pp. 427–455.

[17] Maksimova, L. L., *Absence of the interpolation property in the consistent normal modal extensions of the Dummett logic*, Algebra and Logic **21** (1982), pp. 460–463.

[18] Maksimova, L. L., *The Lyndon property and uniform interpolation over the Grzegorczyk logic*, Siberian Mathematical Journal **55** (2014), pp. 118–124.

[19] Masini, A., *2-sequent calculus: A proof theory of modalities*, Annals of Pure and Applied Logic **58** (1992), pp. 229–246.

[20] Moschovakis, J., *Intuitionistic logic*, in: *Stanford Encyclopedia of Philosophy*, 2015.