Axiomatizing a Real-Valued Modal Logic

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Abstract

A many-valued modal logic is introduced that combines the standard (crisp) Kripke frame semantics of the modal logic K with connectives interpreted locally as abelian group operations over the real numbers. A labelled tableau system and a sequent calculus admitting cut elimination are then defined for this logic and used to establish completeness of an axiomatic extension of the multiplicative fragment of abelian logic.

Keywords: Modal Logics, Many-Valued Logics, Abelian Logic.

1 Introduction

Many-valued modal logics model modal notions such as necessity, belief, and spatio-temporal relations in the presence of multiple degrees of truth, certainty, or possibility. They are defined by extending the Kripke frames of classical modal logic with a many-valued semantics at each world, and have been used to model fuzzy belief [16,12], fuzzy similarity measures [13], many-valued tense logics [17,9], and spatial reasoning with vague predicates [28]. Such logics also provide the basis for fuzzy description logics, which, analogously to the classical case, can be viewed as many-valued multi-modal logics (see, e.g., [29,15,1]). General approaches to finite-valued modal logics are described in [10,11,3,27], while infinite-valued modal logics with propositional operations depending only on a given total order – in particular, Gödel modal logics – are investigated in [6,24,7,5,4].

Many-valued modal logics of "magnitude" typically involve reasoning about some form of addition over sets of real numbers, archetypal examples being

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Łukasiewicz modal logics, where propositional connectives are interpreted by continuous functions over the real unit interval [14,3,18,22] (see also [21,20,23] for related real-valued modal logics). Finite-valued (crisp) Łukasiewicz modal logics are axiomatized in [18], but the axiom system defined for the infinite-valued (crisp) Łukasiewicz modal logic includes a rule that has infinitely many premises. This matters because, although it is easy enough to define a many-valued modal logic semantically (simply decide on suitable sets of values and operations), studying such a logic when it lacks a finitary axiom system or algebraic semantics may be difficult; consider, for example, classical modal logic deprived of the theory of Boolean algebras with operators. Note also that, while validity in finite-valued Łukasiewicz modal logics is known to be PSPACE-complete [2], only a NEXPTIME upper bound is known for the infinite-valued case, as may be deduced from complexity results for Łukasiewicz description logics obtained in [20].

In this paper, we take a first step towards addressing these issues by defining and investigating a simple many-valued modal logic of magnitude $\mathrm{K}(\mathbb{R})$ with propositional connectives interpreted as the usual group operations over the real numbers. The next step would then be to interpret infinite-valued Łukasiewicz modal logic in an extension of $\mathrm{K}(\mathbb{R})$ with lattice connectives. The logic $\mathrm{K}(\mathbb{R})$ may be viewed as a minimal modal extension of the multiplicative fragment of abelian logic studied in [26,8,25]. We provide here a sound and complete axiom system for $\mathrm{K}(\mathbb{R})$, making use of both a labelled tableau system and a sequent calculus admitting cut elimination to establish the more difficult completeness result. We also obtain an EXPTIME upper bound for validity.

2 A Real-Valued Modal Logic

Let us fix Fm as the set of *formulas*, denoted by φ, ψ, χ, defined inductively for a language with a binary connective $\to$ and a modal connective $\Box$ over a countably infinite set Var of propositional variables, denoted by p, q. The *complexity* of a formula φ is defined as the number of occurrences of connectives in φ, and the *modal depth* of φ is defined as the deepest nesting of the modal connective $\Box$ in φ. Fixing some $p_0 \in \mathrm{Var}$, we define additional connectives

$$\overline{0} := p_0 \to p_0, \quad \neg\varphi := \varphi \to \overline{0}, \quad \varphi \& \psi := \neg\varphi \to \psi, \quad \text{and} \quad \Diamond\varphi := \neg\Box\neg\varphi.$$

We also define $0\varphi := \overline{0}$ and $(n+1)\varphi := \varphi \& (n\varphi)$ for $n \in \mathbb{N}$.

Let us remark that these (perhaps counter-intuitively) defined connectives arise as a natural feature of the multiplicative fragment of abelian logic [26,8,25] – an axiomatic extension of multiplicative linear logic that is complete with respect to the class of abelian groups with $x \to y$ interpreted as $y - x$, where a formula is valid if it is non-negative. Since the multiplicative conjunction and disjunction, and also the multiplicative constants, coincide in this logic, we can restrict to a language with just implication and define $\overline{0} := p_0 \to p_0$, where the "0" anticipates the interpretation in $\mathbb{R}$. Negation and multiplicative conjunction (equivalently, disjunction) connectives are then defined as usual.

A *frame* is a pair $\mathfrak{F} = \langle W, R \rangle$, where W is a non-empty set of *worlds* and $R \subseteq W \times W$ is an *accessibility relation*. $\mathfrak{F}$ is called *serial* if for all $x \in W$, there exists $y \in W$ such that Rxy. A K($\mathbb{R}$)*-model* $\mathfrak{M} = \langle W, R, V \rangle$ consists of a serial frame $\langle W, R \rangle$ and a map $V \colon \mathrm{Var} \times W \to [-r, r]$ for some $r \in \mathbb{R}^+$, called a *valuation*. This valuation is extended to $V \colon \mathrm{Fm} \times W \to \mathbb{R}$ by

$$V(\varphi \to \psi, x) = V(\psi, x) - V(\varphi, x)$$
$$V(\Box\varphi, x) = \bigwedge\{V(\varphi, y) : Rxy\}.$$

It follows also that

$$V(\overline{0}, x) = 0 \qquad V(\varphi \& \psi, x) = V(\varphi, x) + V(\psi, x)$$
$$V(\neg\varphi, x) = -V(\varphi, x) \qquad V(\Diamond\varphi, x) = \bigvee\{V(\varphi, y) : Rxy\}.$$

$\varphi \in \mathrm{Fm}$ is *valid* in a K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V \rangle$ if $V(\varphi, x) \geq 0$ for all $x \in W$. If φ is valid in all K($\mathbb{R}$)-models, then φ is K($\mathbb{R}$)*-valid*, written $\models_{\mathrm{K}(\mathbb{R})} \varphi$.

The restriction to serial frames is more or less imposed for this semantics by the fact that $\bigwedge \emptyset$ and $\bigvee \emptyset$ do not exist for $\mathbb{R}$. Note also that the seriality axiom $\Box\varphi \to \Diamond\varphi$ is derivable in any extension of the multiplicative fragment of abelian logic with the standard axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. Similarly, restricting the codomain of a valuation to a bounded subset of $\mathbb{R}$ circumvents problems with infima or suprema of unbounded sets of values and is justified to some extent by the following finite model property.

Lemma 2.1 $\models_{\mathrm{K}(\mathbb{R})} \varphi$ *if and only if* φ *is valid in all finite* K($\mathbb{R}$)*-models.*

Proof. It suffices to prove the following: for any K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V \rangle$, $x \in W$, finite set of formulas S, and $\varepsilon > 0$, there exists a finite K($\mathbb{R}$)-model $\mathfrak{M}' = \langle W', R', V' \rangle$ with $x \in W'$ such that $|V(\varphi, x) - V'(\varphi, x)| < \varepsilon$ for all $\varphi \in S$. We proceed by induction on the sum of the complexities of the formulas in S.

For the base case, S contains only variables and we let $\mathfrak{M}' = \langle W', R', V' \rangle$ with $W' = \{x\}$, $R' = \{(x, x)\}$, and $V'(p, x) = V(p, x)$ for each $p \in \mathrm{Var}$. For the inductive step, suppose first that $S = S' \cup \{\psi \to \chi\}$. Then we can apply the induction hypothesis with $\mathfrak{M}$, $x \in W$, $S'' = S' \cup \{\psi, \chi\}$, and $\frac{\varepsilon}{2} > 0$ to obtain a finite K($\mathbb{R}$)-model $\mathfrak{M}' = \langle W', R', V' \rangle$ with $x \in W'$ such that $|V(\varphi, x) - V'(\varphi, x)| < \frac{\varepsilon}{2}$ for all $\varphi \in S''$. It suffices then to observe that $|V(\psi \to \chi, x) - V'(\psi \to \chi, x)| = |V(\chi, x) - V(\psi, x) - V'(\chi, x) + V'(\psi, x)| \leq |V(\chi, x) - V'(\chi, x)| + |V(\psi, x) - V'(\psi, x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Now suppose that S consists of variables and boxed formulas $\Box\psi_1, \ldots, \Box\psi_n$ $(n \geq 1)$. Then for $1 \leq i \leq n$, there exists $y_i \in W$ such that Rxy_i and $|V(\Box\psi_i, x) - V(\psi_i, y_i)| < \frac{\varepsilon}{2}$. We apply the induction hypothesis to each submodel $\mathfrak{M}_i$ of $\mathfrak{M}$ generated by y_i (i.e., the restriction of $\mathfrak{M}$ to the smallest subset of W containing y_i and closed under R) with $S' = (S \setminus \{\Box\psi_1, \ldots, \Box\psi_n\}) \cup \{\psi_1, \ldots, \psi_n\}$, $y_i \in W_i$, and $\frac{\varepsilon}{2} > 0$ to obtain a finite K($\mathbb{R}$)-model $\mathfrak{M}'_i = \langle W'_i, R'_i, V'_i \rangle$ and $y_i \in W'_i$ such that $|V(\varphi, y_i) - V'(\varphi, y_i)| < \frac{\varepsilon}{2}$ for all $\varphi \in S'$. By renaming worlds, we may assume that these models are disjoint and do not include x. Now let $\mathfrak{M}' = \langle W', R', V' \rangle$ be the finite K($\mathbb{R}$)-model

$$
\begin{array}{rl}
\text{(B)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(C)} & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \\
\text{(I)} & \varphi \to \varphi \\
\text{(A)} & ((\varphi \to \psi) \to \psi) \to \varphi \\
\text{(K)} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
(\text{D}_n) & \Box(n\varphi) \to n\Box\varphi \qquad (n \geq 2)
\end{array}
$$

$$
\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(mp)} \qquad \frac{\varphi}{\Box\varphi}\ \text{(nec)} \qquad \frac{n\varphi}{\varphi}\ (\text{con}_n) \quad (n \geq 2)
$$

Fig. 1. The axiom system K($\mathbb{R}$)

with $W' = \{x\} \cup W'_1 \cup \ldots \cup W'_n$ such that for $u, v \in W'$,

$$
R'uv = \begin{cases} R'_i uv & \text{if } u, v \in W'_i \\ 1 & \text{if } u = x,\ v \in \{y_1, \ldots, y_n\} \\ 0 & \text{otherwise,} \end{cases} \qquad V'(p,u) = \begin{cases} V'_i(p,u) & \text{if } u \in W'_i \\ V(p,x) & \text{if } u = x. \end{cases}
$$

Clearly $V'(p,x) = V(p,x)$ for each propositional variable $p \in S$. For $1 \leq i \leq n$, recall that $|V(\Box\psi_i, x) - V(\psi_i, y_i)| < \frac{\varepsilon}{2}$ and also $|V(\psi_i, y_j) - V'(\psi_i, y_j)| < \frac{\varepsilon}{2}$ for $1 \leq j \leq n$, so $|V(\Box\psi_i, x) - V'(\Box\psi_i, x)| < \varepsilon$. $\Box$

The main goal of this paper will be to prove the following soundness and completeness theorem for the axiom system K($\mathbb{R}$) presented in Fig. 1.

Theorem 2.2 *For any* $\varphi \in \text{Fm}$, $\vdash_{\text{K}(\mathbb{R})} \varphi$ *if and only if* $\models_{\text{K}(\mathbb{R})} \varphi$.

Soundness (the left-to-right direction) is straightforward. It is easily checked that the axioms (B), (C), (I), (A), and (K) are valid in all K($\mathbb{R}$)-models. For the less standard axioms (D_n) ($n \geq 2$), it suffices to consider a K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V\rangle$ and $x \in W$, and to observe that

$$
\begin{aligned}
V(\Box(n\varphi), x) &= \textstyle\bigwedge\{V(n\varphi, y) : Rxy\} \\
&= \textstyle\bigwedge\{nV(\varphi, y) : Rxy\} \\
&= n \textstyle\bigwedge\{V(\varphi, y) : Rxy\} \\
&= V(n\Box\varphi, x).
\end{aligned}
$$

Clearly, (mp) and (nec) preserve validity in K($\mathbb{R}$)-models. For (con_n), note that if $V(n\varphi, x) \geq 0$ for a K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V\rangle$ and $x \in W$, also $V(\varphi, x) \geq 0$.

Proving completeness (the right-to-left direction) will be our main aim in the remainder of this paper. First, in Section 3, we define a labelled tableau system that is sound and complete with respect to the Kripke semantics. In Section 4, we then provide a sequent calculus that proves the same formulas as the axiom system K($\mathbb{R}$). In Section 5, we establish the soundness and completeness of all these systems with respect to the Kripke semantics by showing that formulas derivable in the labelled tableau calculus are derivable in the sequent calculus. Finally, in Section 6, we prove cut elimination for the sequent calculus.

3 A Labelled Tableau System

In this section we introduce a labelled tableau calculus LK($\mathbb{R}$) for checking K($\mathbb{R}$)-validity, based very closely on the Kripke semantics. Intuitively, to check whether a formula φ takes a value less than 0 in a world x, we decompose the propositional structure of φ to obtain an inequation between sums of formulas at x. Box formulas on the right of inequations generate new worlds accessible to x and new inequations between sums of formulas to be processed. Box formulas on the left are decomposed by considering accessible worlds and generating new inequations for those worlds. These inequations may involve formulas evaluated at different worlds; we therefore treat inequations between formulas labelled with integers representing worlds in Kripke frames. The formula φ will be valid if and only if the generated set of inequations (suitably interpreted) is unsatisfiable over the real numbers.

More precisely, we consider *(tableau) nodes* of the following forms:

(1) $(\Gamma)^{\boldsymbol{k}} \triangleright (\Delta)^{\boldsymbol{l}}$ such that $\triangleright \in \{>, \geq\}$ and $(\Gamma)^{\boldsymbol{k}} = [(\varphi_1)^{k_1}, \ldots, (\varphi_n)^{k_n}]$ and $(\Delta)^{\boldsymbol{l}} = [(\psi_1)^{l_1}, \ldots, (\psi_m)^{l_m}]$ are multisets of formulas $\Gamma = [\varphi_1, \ldots, \varphi_n]$ and $\Delta = [\psi_1, \ldots, \psi_m]$ labelled by $k_1, \ldots, k_n, l_1, \ldots, l_m \in \mathbb{Z}$;

(2) rij such that $i, j \in \mathbb{N}$.

Intuitively, (1) represents an inequation between sums of values of formulas evaluated at (possibly different) worlds of a K($\mathbb{R}$)-model (φ_i is evaluated at world $|k_i|$ on the left, and ψ_j at world $|k_j|$ on the right) and (2), the expression rij, denotes that world j is accessible from world i in this model.

We define the *complexity* of an inequation $(\Gamma)^{\boldsymbol{k}} \triangleright (\Delta)^{\boldsymbol{l}}$ to be the sum of the complexities of the formulas in Γ and Δ, where formulas of the form $\Box\varphi$ labelled by $-i$ for $i \in \mathbb{N}$ are treated as propositional variables.

A *tableau for a formula* φ is a finite sequence of nodes starting with $[] > [(\varphi)^1]$, $r12$ generated according to the inference rules of the system presented in Fig. 2; that is, if expressions above the line in an instance of a rule occur in the sequence, then the sequence can be extended with the expressions below the line. The tableau is called *complete* if the rules have been applied exhaustively, but only once to the same set of premises, and no application of (ex) is followed by another application of (ex). As labelled inequations occurring above the line in an instance of a rule have a higher complexity than those occurring below the line, there exists a complete tableau for every formula.

We call expressions of the form $(p)^i$ or $(\Box\varphi)^{-i}$ with $i \in \mathbb{N}$ *labelled variables.* The *system of inequations* associated to a tableau consists of all inequations over the labelled variables occurring in the tableau where the comma "," is interpreted as the usual addition over the real numbers. A tableau is *closed* if its associated system of inequations is inconsistent over $\mathbb{R}$; otherwise it is *open.* A formula $\varphi \in \mathrm{Fm}$ is *derivable* in the labelled tableau calculus LK($\mathbb{R}$), written $\vdash_{\mathrm{LK}(\mathbb{R})} \varphi$, if there exists a complete closed tableau for φ.

$$\frac{(\Gamma)^{\boldsymbol{k}}, (\varphi \to \psi)^i \rhd (\Delta)^{\boldsymbol{l}}}{(\Gamma)^{\boldsymbol{k}}, (\psi)^i \rhd (\varphi)^i, (\Delta)^{\boldsymbol{l}}}\ (\to\rhd) \qquad \frac{(\Gamma)^{\boldsymbol{k}} \rhd (\varphi \to \psi)^i, (\Delta)^{\boldsymbol{l}}}{(\Gamma)^{\boldsymbol{k}}, (\varphi)^i \rhd (\psi)^i, (\Delta)^{\boldsymbol{l}}}\ (\rhd\to)$$

$$\frac{\begin{array}{c} rij \\ (\Gamma)^{\boldsymbol{k}}, (\Box\varphi)^i \rhd (\Delta)^{\boldsymbol{l}} \end{array}}{\begin{array}{c} (\varphi)^j \geq (\Box\varphi)^{-i} \\ (\Gamma)^{\boldsymbol{k}}, (\Box\varphi)^{-i} \rhd (\Delta)^{\boldsymbol{l}} \end{array}}\ (\Box\rhd) \qquad \frac{(\Gamma)^{\boldsymbol{k}} \rhd (\Box\varphi)^i, (\Delta)^{\boldsymbol{l}}}{\begin{array}{c} (\Box\varphi)^{-i} \geq (\varphi)^j \\ (\Gamma)^{\boldsymbol{k}} \rhd (\Box\varphi)^{-i}, (\Delta)^{\boldsymbol{l}} \\ rij \end{array}}\ (\rhd\Box) \quad i, j \in \mathbb{N},\ j \text{ new}$$

$$\frac{rik}{rkj}\ (\text{ex}) \quad j \in \mathbb{N} \text{ new}$$

Fig. 2. The labelled tableau calculus LK($\mathbb{R}$)

Example 3.1 The seriality axiom is derivable in LK($\mathbb{R}$) using the following complete tableau for $\Box p \to \Diamond p = \Box p \to (\Box(p \to (p \to p)) \to (p \to p))$:

$$\begin{array}{rl}
1: & [] > (\Box p \to (\Box(p \to (p \to p)) \to (p \to p)))^1 \\
2: & r12 \\
3: & (\Box p)^1 > (\Box(p \to (p \to p)) \to (p \to p))^1 \\
4: & (\Box p)^1, (\Box(p \to (p \to p)))^1 > (p \to p)^1 \\
5: & (\Box p)^1, (\Box(p \to (p \to p)))^1, (p)^1 > (p)^1 \\
6: & (p)^2 \geq (\Box p)^{-1} \\
7: & (\Box p)^{-1}, (\Box(p \to (p \to p)))^1, (p)^1 > (p)^1 \\
8: & (p \to (p \to p))^2 \geq (\Box(p \to (p \to p)))^{-1} \\
9: & (\Box p)^{-1}, (\Box(p \to (p \to p)))^{-1}, (p)^1 > (p)^1 \\
10: & (p \to p)^2 \geq (p)^2, (\Box(p \to (p \to p)))^{-1} \\
11: & (p)^2 \geq (p)^2, (p)^2, (\Box(p \to (p \to p)))^{-1} \\
12: & r23
\end{array}$$

with the following inconsistent system of inequations over $\mathbb{R}$

$$\{y + u + v > v,\ x \geq y,\ x \geq 2x + u\}$$

where x, y, u, v stand for $(p)^2$, $(\Box p)^{-1}$, $(\Box(p \to (p \to p)))^{-1}$, $(p)^1$, respectively.

Let us call a K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V\rangle$ *faithful* to a tableau T if there is a map $f\colon \mathbb{N} \to W$ (said to *show* that $\mathfrak{M}$ is faithful to T) such that if rij occurs in T, then $Rf(i)f(j)$ is in $\mathfrak{M}$, and for every inequation $(\varphi_1)^{i_1}, \ldots, (\varphi_n)^{i_n} \rhd (\psi_1)^{j_1}, \ldots, (\psi_m)^{j_m}$ occurring in T,

$$V(\varphi_1, f(|i_1|)) + \ldots + V(\varphi_n, f(|i_n|)) \rhd V(\psi_1, f(|j_1|)) + \ldots + V(\psi_m, f(|j_m|)).$$

Note that whenever a K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V\rangle$ is faithful to a tableau T, the map defined by $e((p)^i) = V(p, i)$ and $e((\Box\varphi)^{-i}) = V(\Box\varphi, i)$ satisfies the system of inequations associated to T over $\mathbb{R}$, and hence T is open.

The following lemma establishes the soundness of the rules of LK($\mathbb{R}$).

Lemma 3.2 *Let* $\mathfrak{M} = \langle W, R, V\rangle$ *be a finite* $\mathrm{K}(\mathbb{R})$*-model faithful to a tableau* T*. If a rule of* $\mathrm{LK}(\mathbb{R})$ *is applied to* T *to obtain an extension* T'*, then* $\mathfrak{M}$ *is faithful to* T'*.*

Proof. Let f be a map showing that $\mathfrak{M} = \langle W, R, V\rangle$ is a finite $\mathrm{K}(\mathbb{R})$-model faithful to a tableau T. The cases of $(\to\rhd)$ and $(\rhd\to)$ follow easily. For $(\Box\rhd)$, suppose that $(\Gamma)^{\boldsymbol{k}}, (\Box\varphi)^i \rhd (\Delta)^{\boldsymbol{l}}$ and rij appear in T and we obtain an extension T' of T with $(\varphi)^j \geq (\Box\varphi)^{-i}$ and $(\Gamma)^{\boldsymbol{k}}, (\Box\varphi)^{-i} \rhd (\Delta)^{\boldsymbol{l}}$. Since $\mathfrak{M}$ is faithful to T, we have $Rf(i)f(j)$. But then $V(\varphi, f(|j|)) = V(\varphi, f(j)) \geq V(\Box\varphi, f(i)) = V(\Box\varphi, f(|-i|))$, so $\mathfrak{M}$ is faithful to T'.

For $(\rhd\Box)$, suppose that $(\Gamma)^{\boldsymbol{k}} \rhd (\Box\varphi)^i, (\Delta)^{\boldsymbol{l}}$ $(i \in \mathbb{N})$ appears in T and we obtain an extension T' of T with rij ($j \in \mathbb{N}$ new), $(\Box\varphi)^{-i} \geq (\varphi)^j$, and $(\Gamma)^{\boldsymbol{k}} \rhd (\Box\varphi)^{-i}, (\Delta)^{\boldsymbol{l}}$. Because $\mathfrak{M}$ is finite and serial, there exists $v \in W$ such that $Rf(i)v$ and $V(\Box\varphi, f(i)) = V(\varphi, v)$. Hence the map f' defined to be f but with $f'(j) = v$ shows that $\mathfrak{M}$ is faithful to T'.

Finally, for (ex) suppose that rik appears in T and we obtain an extension T' of T with rkj ($j \in \mathbb{N}$ new). Since rik is in T, we have $Rf(i)f(k)$. Because $\mathfrak{M}$ is serial, there exists $v \in W$ such that $Rf(k)v$. The map f' defined to be f but with $f'(j) = v$ shows that $\mathfrak{M}$ is faithful to T'. $\Box$

To establish the completeness of $\mathrm{LK}(\mathbb{R})$, we introduce the following notion. Let T be a complete open tableau and let e be a map satisfying the system of inequations associated to T. We say that $\mathfrak{M} = \langle W, R, V\rangle$ is an *e-induced model* of T if

- $W = \{w_i : i \in \mathbb{N}$ is a label occurring in $T\}$;
- Rw_iw_j if and only if rij occurs in T or $i = j$ and rik is not in T for any k;
- $V(p, w_i) = \begin{cases} e((p)^i) & \text{if } (p)^i \text{ occurs in } T \\ 0 & \text{otherwise.} \end{cases}$

Lemma 3.3 *Let* $\mathfrak{M} = \langle W, R, V\rangle$ *be an e-induced model of a complete open tableau* T*, and extend the map* e *by fixing* $e((\varphi)^i) = V(\varphi, w_i)$ *for each* $w_i \in W$*. If* $(\varphi_1)^{i_1}, \ldots, (\varphi_n)^{i_n} \rhd (\psi_1)^{j_1}, \ldots, (\psi_m)^{j_m}$ *appears in* T*, then*

$$e((\varphi_1)^{i_1}) + \ldots + e((\varphi_n)^{i_n}) \rhd e((\psi_1)^{j_1}) + \ldots + e((\psi_m)^{j_m}).$$

Proof. We proceed by induction on the complexity of the inequation. The base case follows using the definition of $\mathfrak{M}$ and the fact that e is a map satisfying the system of inequations associated to T, while the cases where $(\varphi_1)^{i_1}, \ldots, (\varphi_n)^{i_n} \rhd (\psi_1)^{j_1}, \ldots, (\psi_m)^{j_m}$ appears as a premise of an application of $(\to\rhd)$ or $(\rhd\to)$ in T follow directly using the induction hypothesis. Suppose that the inequation is of the form $(\Gamma)^{\boldsymbol{k}}, (\Box\varphi)^i \rhd (\Delta)^{\boldsymbol{l}}$ for $i \in \mathbb{N}$. Since $\mathfrak{M}$ is finite, there is a j such that rij occurs in T and $V(\Box\varphi, w_i) = V(\varphi, w_j)$. But also $(\varphi)^j \geq (\Box\varphi)^{-i}$ occurs in T, and hence, by the induction hypothesis, $V(\varphi, w_j) = e((\varphi)^j) \geq e((\Box\varphi)^{-i})$. We also have that $(\Gamma)^{\boldsymbol{k}}, (\Box\varphi)^{-i} \rhd (\Delta)^{\boldsymbol{l}}$ occurs in T, and the desired inequality follows by another application of the induction hypothesis. Finally, if the inequation is of the form $(\Gamma)^{\boldsymbol{k}} \rhd (\Box\varphi)^i, (\Delta)^{\boldsymbol{l}}$ with $i \in \mathbb{N}$, then by $(\rhd\Box)$ we must have in the tableau rij for some $j \in \mathbb{N}$, $(\Box\varphi)^{-i} \geq (\varphi)^j$, and $(\Gamma)^{\boldsymbol{k}} \rhd (\Box\varphi)^{-i}, (\Delta)^{\boldsymbol{l}}$. The desired inequality follows by applying the induction hypothesis to these two inequalities and observing that $V(\varphi, w_j) \geq V(\Box\varphi, w_i)$. $\Box$

Putting together these last two lemmas we obtain the following soundness and completeness theorem for LK($\mathbb{R}$).

Theorem 3.4 *For any* $\varphi \in \mathrm{Fm}$*,* $\vdash_{\mathrm{LK}(\mathbb{R})} \varphi$ *if and only if* $\models_{\mathrm{K}(\mathbb{R})} \varphi$*.*

Proof. For the left-to-right direction, assume $\not\models_{\mathrm{K}(\mathbb{R})} \varphi$. Then by Lemma 2.1, there is a finite K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V\rangle$ and some world $w_1 \in W$ such that $0 > V(\varphi, w_1)$. Let $f \colon \mathbb{N} \to W$ be any function such that $f(1) = w_1$ and $f(2) = w_2$, where Rw_1w_2. This function shows that $\mathfrak{M}$ is faithful to the tableau consisting just of $[] > [(\varphi)^1],\ r12$. Suppose that by applying the decomposition rules to this tableau, we obtain a complete tableau T. Applying Lemma 3.2 inductively, $\mathfrak{M}$ is faithful to T. So the system of inequations associated with T is consistent over $\mathbb{R}$, and T is open. Hence $\nvdash_{\mathrm{LK}(\mathbb{R})} \varphi$.

For the right-to-left direction, suppose that $\nvdash_{\mathrm{LK}(\mathbb{R})} \varphi$. Then there is a complete open tableau T beginning with $[] > [(\varphi)^1],\ r12$. Let e be a map satisfying the system of inequations associated to T and consider any e-induced model $\mathfrak{M} = \langle W, R, V\rangle$ of T. By Lemma 3.3, we obtain $0 > e((\varphi)^1) = V(\varphi, w_1)$. Hence $\not\models_{\mathrm{K}(\mathbb{R})} \varphi$. $\Box$

We also obtain an upper bound for the complexity of checking K($\mathbb{R}$)-validity.

Theorem 3.5 K($\mathbb{R}$)*-validity is in EXPTIME.*

Proof. Given a formula φ of modal depth d, we generate a complete tableau for φ. We do this stepwise, where after i steps, we are only left with nodes containing formulas of modal depth at most $d-i$. Step $i+1$ is then as follows. We first apply the $(\to\rhd)$ and $(\rhd\to)$ rules exhaustively. Inequations containing implicational formulas can then be removed, since they will not belong to the set of inequations associated to the tableau. Hence we obtain nodes containing only labelled variables and modal formulas. We then apply the rule $(\rhd\Box)$ for each boxed formula occurring on the right in one of these nodes and (ex) one time for each new label. We apply $(\Box\rhd)$ exhaustively and then remove all nodes containing formulas of modal depth $d-i$. After d steps we obtain a complete tableau for φ that contains exponentially (in d) many different nodes using exponentially many (in d) labels. Hence we obtain a linear programming problem of at most exponential size in d. The result follows from the fact that the linear programming problem is in P [19]. $\Box$

4 A Sequent Calculus

We define a *sequent* to be an ordered pair of finite multisets of formulas Γ and Δ, written $\Gamma \Rightarrow \Delta$. For multisets of formulas Γ and Δ, we write Γ, Δ to denote their multiset union, $n\Gamma$ for $\Gamma, \ldots, \Gamma$ (n times), and $\Box\Gamma$ for $[\Box\varphi : \varphi \in \Gamma]$.

We define a formula translation of sequents as follows:

$$\mathcal{I}(\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m) \ :=\ (\varphi_1\& \ldots \&\varphi_n) \to (\psi_1\& \ldots \&\psi_m),$$

$$\frac{}{\Delta \Rightarrow \Delta}\ (\textsc{id}) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Pi \Rightarrow \varphi, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{cut})$$

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{mix}) \qquad \frac{n\Gamma \Rightarrow n\Delta}{\Gamma \Rightarrow \Delta}\ (\textsc{sc}_n) \quad (n \geq 2)$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ ({\to}{\Rightarrow}) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ ({\Rightarrow}{\to})$$

$$\frac{\Gamma \Rightarrow n[\varphi]}{\Box\Gamma \Rightarrow n[\Box\varphi]}\ (\Box_n) \quad (n \geq 0)$$

Fig. 3. The sequent calculus GK($\mathbb{R}$)

where $\varphi_1 \& \ldots \& \varphi_n = \overline{0}$ for $n = 0$. We say that a sequent $\Gamma \Rightarrow \Delta$ is K($\mathbb{R}$)-*valid*, written $\models_{\mathrm{K}(\mathbb{R})} \Gamma \Rightarrow \Delta$, if $\models_{\mathrm{K}(\mathbb{R})} \mathcal{I}(\Gamma \Rightarrow \Delta)$.

A sequent calculus GK($\mathbb{R}$) is presented in Fig. 3. The following rules are derivable in this system.

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \& \psi \Rightarrow \Delta}\ (\&{\Rightarrow}) \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \& \psi, \Delta}\ ({\Rightarrow}\&)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\ (\neg{\Rightarrow}) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ ({\Rightarrow}\neg)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \overline{0} \Rightarrow \Delta}\ (\overline{0}{\Rightarrow}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \overline{0}, \Delta}\ ({\Rightarrow}\overline{0})$$

Example 4.1 The rule $(\Box_n)$ can be used to derive instances of (D_n) as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\varphi, \ldots, \varphi \Rightarrow \varphi, \ldots, \varphi}\ (\textsc{id})}{\vdots}\ (\&{\Rightarrow})}{n\varphi \Rightarrow \varphi, \ldots, \varphi}\ (\&{\Rightarrow})}{\Box(n\varphi) \Rightarrow \Box\varphi, \ldots, \Box\varphi}\ (\Box_n)}{\cfrac{\vdots}{\Box(n\varphi) \Rightarrow n\Box\varphi}\ ({\Rightarrow}\&)}\ ({\Rightarrow}\&)}{\Rightarrow \Box(n\varphi) \to n\Box\varphi}\ ({\Rightarrow}{\to})$$

We note also that the "cancellation" rule

$$\frac{\Gamma, \varphi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}\ (\textsc{can})$$

is both derivable in GK($\mathbb{R}$) and can be used, with (MIX), to derive (CUT):

$$\cfrac{\cfrac{\cfrac{}{\varphi \Rightarrow \varphi}\ (\textsc{id})}{\varphi \to \varphi \Rightarrow}\ ({\to}{\Rightarrow}) \quad \cfrac{\Gamma, \varphi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi \to \varphi, \Delta}\ ({\Rightarrow}{\to})}{\Gamma \Rightarrow \Delta}\ (\textsc{cut}) \qquad \cfrac{\cfrac{\Gamma, \varphi \Rightarrow \Delta \quad \Pi \Rightarrow \varphi, \Sigma}{\Gamma, \Pi, \varphi \Rightarrow \varphi, \Sigma, \Delta}\ (\textsc{mix})}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{can})$$

Moreover, the following rule (used in the proofs of Theorems 2.2 and 4.3)

$$\frac{\Gamma_0 \Rightarrow \quad \Gamma_1 \Rightarrow k[\varphi_1] \quad \dots \quad \Gamma_n \Rightarrow k[\varphi_n]}{\Delta, \Box\Gamma \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n, \Delta}\ (\Box_{k,n})$$

$$\text{where}\quad k \in \mathbb{N} \setminus \{0\},\ n \in \mathbb{N},\ k\Gamma = \Gamma_0 \uplus \Gamma_1 \uplus \dots \uplus \Gamma_n$$

is derivable in GK($\mathbb{R}$) as shown below:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$\Delta \Rightarrow \Delta$}
\AxiomC{$\Gamma_0 \Rightarrow$}
\RightLabel{$(\Box_0)$}
\UnaryInfC{$\Box\Gamma_0 \Rightarrow$}
\AxiomC{$\Gamma_1 \Rightarrow k[\varphi_1]$}
\RightLabel{$(\Box_k)$}
\UnaryInfC{$\Box\Gamma_1 \Rightarrow k[\Box\varphi_1]$}
\AxiomC{$\Gamma_n \Rightarrow k[\varphi_n]$}
\RightLabel{$(\Box_k)$}
\UnaryInfC{$\Box\Gamma_n \Rightarrow k[\Box\varphi_n]$}
\RightLabel{(MIX)}
\UnaryInfC{$\vdots$}
\RightLabel{(MIX)}
\BinaryInfC{$\Box(\Gamma_1 \uplus \dots \uplus \Gamma_n) \Rightarrow k[\Box\varphi_1], \dots, k[\Box\varphi_n]$}
\RightLabel{(MIX)}
\BinaryInfC{$\Box(\Gamma_0 \uplus \Gamma_1 \uplus \dots \uplus \Gamma_n) \Rightarrow k[\Box\varphi_1], \dots, k[\Box\varphi_n]$}
\RightLabel{$(\textsc{sc}_k)$}
\UnaryInfC{$\Box\Gamma \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n$}
\RightLabel{(MIX)}
\BinaryInfC{$\Delta, \Box\Gamma \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n, \Delta$}
\end{prooftree}
$$

We now establish the equivalence of GK($\mathbb{R}$) with the axiom system K($\mathbb{R}$):

Theorem 4.2 $\vdash_{\mathrm{GK}(\mathbb{R})} \Gamma \Rightarrow \Delta$ *if and only if* $\vdash_{\mathrm{K}(\mathbb{R})} \mathcal{I}(\Gamma \Rightarrow \Delta)$.

Proof. It suffices for the left-to-right direction to show that for any rule of GK($\mathbb{R}$) with premises $S_1, \dots, S_m$ and conclusion S, whenever $\vdash_{\mathrm{K}(\mathbb{R})} \mathcal{I}(S_i)$ for $i = 1 \dots m$, also $\vdash_{\mathrm{K}(\mathbb{R})} \mathcal{I}(S)$. For example, consider the rule $(\Box_n)$ and assume that $\vdash_{\mathrm{K}(\mathbb{R})} \mathcal{I}(\Gamma \Rightarrow n[\varphi])$. Suppose that $\Gamma = [\psi_1, \dots, \psi_m]$ and $\psi = \psi_1 \& \dots \& \psi_m$. We continue the derivation of $\mathcal{I}(\Gamma \Rightarrow n[\varphi]) = \psi \to n\varphi$ in K($\mathbb{R}$) to obtain a derivation of $\Box\psi \to n\Box\varphi$:

1.	$\psi \to n\varphi$	
2.	$\Box(\psi \to n\varphi)$	(nec)
3.	$\Box(\psi \to n\varphi) \to (\Box\psi \to \Box n\varphi)$	(K)
4.	$\Box\psi \to \Box n\varphi$	(mp) with 2,3
5.	$\Box n\varphi \to n\Box\varphi$	(D_n)
6.	$(\Box\psi \to \Box n\varphi) \to ((\Box n\varphi \to n\Box\varphi) \to (\Box\psi \to n\Box\varphi))$	(B)
7.	$(\Box n\varphi \to n\Box\varphi) \to (\Box\psi \to n\Box\varphi)$	(mp) with 4,6
8.	$\Box\psi \to n\Box\varphi$	(mp) with 5,7.

$(\Box\psi_1 \& \dots \& \Box\psi_m) \to \Box\psi$ is derivable using (B), (C), (I), and (K), so, using (B) and (mp), we obtain a derivation of $\mathcal{I}(\Box\Gamma \Rightarrow n[\Box\varphi]) = (\Box\psi_1 \& \dots \& \Box\psi_m) \to n\Box\varphi$ in GK($\mathbb{R}$).

For the right-to-left direction, it is straightforward to show that every axiom of K($\mathbb{R}$) is derivable in GK($\mathbb{R}$); see, e.g., Example 4.1 for derivations of instances of (D_n). Also, the rules of K($\mathbb{R}$) are derivable in GK($\mathbb{R}$). For example, for (con_n), starting with $\Rightarrow n\varphi$, we can apply (CUT) with the derivable sequent $n\varphi \Rightarrow n[\varphi]$ to obtain $\Rightarrow n[\varphi]$ and then, by an application of $(\textsc{sc}_n)$, obtain also $\Rightarrow \varphi$. Hence, if $\vdash_{\mathrm{K}(\mathbb{R})} \mathcal{I}(\Gamma \Rightarrow \Delta)$, then $\vdash_{\mathrm{GK}(\mathbb{R})} \Rightarrow \mathcal{I}(\Gamma \Rightarrow \Delta)$ and, applying (CUT) with the derivable sequent $\Gamma, \mathcal{I}(\Gamma \Rightarrow \Delta) \Rightarrow \Delta$, also $\vdash_{\mathrm{GK}(\mathbb{R})} \Gamma \Rightarrow \Delta$. $\Box$

The rule (CUT) is not really necessary for derivations in GK($\mathbb{R}$). That is, there exists an algorithm for constructively eliminating applications of the rule (CUT) from derivations in GK($\mathbb{R}$); this may be stated as follows:

Theorem 4.3 GK($\mathbb{R}$) *admits cut elimination.*

However, as this result is not required for the proof of the completeness theorem for K($\mathbb{R}$) (Theorem 2.2, proved in Section 5), we defer its proof to Section 6.

5 Completeness

This section is devoted to proving Theorem 2.2. We begin with a simple lemma establishing a separation of propositional variables and boxed formulas.

Lemma 5.1 *If* Γ *and* Δ *are multisets of propositional variables and* $\models_{\mathrm{K}(\mathbb{R})} \Gamma, \Box\Pi \Rightarrow \Delta, \Box\Sigma$*, then* $\Gamma = \Delta$ *and* $\models_{\mathrm{K}(\mathbb{R})} \Box\Pi \Rightarrow \Box\Sigma$*.*

Proof. Suppose that Γ and Δ are multisets of propositional variables and $\models_{\mathrm{K}(\mathbb{R})} \Gamma, \Box\Pi \Rightarrow \Delta, \Box\Sigma$. It suffices to show that $\Gamma = \Delta$ as then clearly also $\models_{\mathrm{K}(\mathbb{R})} \Box\Pi \Rightarrow \Box\Sigma$. Suppose for a contradiction that $\Gamma \neq \Delta$. Without loss of generality, some propositional variable p occurs strictly more times in Γ than Δ. Consider a K($\mathbb{R}$)-model with worlds x, y satisfying Rxy and Ryy where $V(p, x) = 1$, $V(p, y) = 0$, and $V(q, x) = V(q, y) = 0$ for $q \neq p$. Then $V(\Box\varphi, x) = 0$ for any $\varphi \in \mathrm{Fm}$, and $\not\models_{\mathrm{K}(\mathbb{R})} \Gamma, \Box\Pi \Rightarrow \Delta, \Box\Sigma$, a contradiction. $\Box$

Theorem 2.2 is a consequence of the following result and Theorem 4.2.

Theorem 5.2 *If* $\models_{\mathrm{K}(\mathbb{R})} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathrm{GK}(\mathbb{R})} \Gamma \Rightarrow \Delta$*.*

Proof. We prove the claim by induction on the lexicographically ordered pair consisting of the modal depth of $\mathcal{I}(\Gamma \Rightarrow \Delta)$ and the sum of the complexities of the formulas in $\Gamma \Rightarrow \Delta$. Assume $\models_{\mathrm{K}(\mathbb{R})} \Gamma \Rightarrow \Delta$. If $\Gamma = \Gamma' \uplus [\varphi \to \psi]$, then $\models_{\mathrm{K}(\mathbb{R})} \Gamma', \psi \Rightarrow \varphi, \Delta$ and, by the induction hypothesis, $\vdash_{\mathrm{GK}(\mathbb{R})} \Gamma', \psi \Rightarrow \varphi, \Delta$. Hence $\vdash_{\mathrm{GK}(\mathbb{R})} \Gamma', \varphi \to \psi \Rightarrow \Delta$. The case for $\Delta = \Delta' \uplus [\varphi \to \psi]$ is very similar.

If $\Gamma \Rightarrow \Delta$ has the form $\Gamma_1, \Box\Gamma_2 \Rightarrow \Delta_1, \Box\Delta_2$ where Γ_1 and Δ_1 contain only propositional variables, then, by Lemma 5.1, we obtain $\Gamma_1 = \Delta_1$ and $\models_{\mathrm{K}(\mathbb{R})} \Box\Gamma_2 \Rightarrow \Box\Delta_2$. Clearly $\vdash_{\mathrm{GK}(\mathbb{R})} \Gamma_1 \Rightarrow \Delta_1$. Hence it suffices, using (MIX), to prove that $\vdash_{\mathrm{GK}(\mathbb{R})} \Box\Gamma_2 \Rightarrow \Box\Delta_2$, where $\Box\Gamma_2 \Rightarrow \Box\Delta_2$ has the form

$$\Box\varphi_1, \ldots, \Box\varphi_n \Rightarrow \Box\psi_1, \ldots, \Box\psi_m \quad (m, n \in \mathbb{N}).$$

We know $\models_{\mathrm{K}(\mathbb{R})} \Box\varphi_1, \ldots, \Box\varphi_n \Rightarrow \Box\psi_1, \ldots, \Box\psi_m$. Hence, translating between sequents and formulas and using Theorem 3.4, there is a complete closed tableau T beginning with $r12$ and containing

$$(\Box\varphi_1)^{-1}, \ldots, (\Box\varphi_n)^{-1} > (\Box\psi_1)^{-1}, \ldots, (\Box\psi_m)^{-1}. \quad (1)$$

T must then also contain inequations for new labels $y_1, \ldots, y_m \in \mathbb{N}$

$$(\Box\psi_1)^{-1} \geq (\psi_1)^{y_1} \quad \ldots \quad (\Box\psi_m)^{-1} \geq (\psi_m)^{y_m} \quad (2)$$

and, fixing $y_0 = 2$ for convenience,

$$\begin{array}{ccc} (\varphi_1)^{y_0} \geq (\Box\varphi_1)^{-1} & \ldots & (\varphi_1)^{y_m} \geq (\Box\varphi_1)^{-1} \\ \vdots & & \vdots \\ (\varphi_n)^{y_0} \geq (\Box\varphi_n)^{-1} & \ldots & (\varphi_n)^{y_m} \geq (\Box\varphi_n)^{-1}. \end{array} \quad (3)$$

Since T is closed, the system $\mathcal{S}$ of inequations associated to T is inconsistent over $\mathbb{R}$. Note that an inequation $\Gamma \triangleright \Delta$ occurring in T may not occur in $\mathcal{S}$; however, there does always occur in $\mathcal{S}$ an inequation obtained by applying the tableau rules for $\to$ to $\Gamma \triangleright \Delta$ and then the tableau rules for $\Box$ switching each $i \in \mathbb{N}$ to $-i$; we call this inequation the *derived inequation* of $\Gamma \triangleright \Delta$.

Recall that a system of inequations of the form $f_i(\bar{x}) > g_i(\bar{x})$ $(1 \leq i \leq r)$ and $h_j(\bar{x}) \geq k_j(\bar{x})$ $(1 \leq j \leq s)$ where each f_i, g_i, h_j, k_j is a positive linear sum of variables in $\bar{x}$ containing no constants, is inconsistent over $\mathbb{R}$ if and only if there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{N}$ (not all zero) and $\mu_1, \ldots, \mu_s \in \mathbb{N}$ such that $\lambda_1 f_1 + \ldots + \lambda_r f_r + \mu_1 h_1 + \ldots + \mu_s h_s = \lambda_1 g_1 + \ldots + \lambda_r g_r + \mu_1 k_1 + \ldots + \mu_s k_s$. For convenience, we may say that the inequation $f_i(\bar{x}) > g_i(\bar{x})$ or $h_j(\bar{x}) \geq k_j(\bar{x})$ is "used" λ_i or μ_j times, respectively, in the linear combination.

We now consider a linear combination of the inequations in $\mathcal{S}$ that witnesses inconsistency over $\mathbb{R}$ and observe:

(i) The inequation (1) is the only strict inequation occurring in $\mathcal{S}$ and hence must be used in the linear combination some fixed $k > 0$ times.

(ii) The variables $(\Box\psi_1)^{-1}, \ldots, (\Box\psi_m)^{-1}$ occur in $\mathcal{S}$ only in (1) and in the inequations derived from (2); hence, using (i), each inequation derived from (2) must be used in the linear combination k times.

(iii) The variables $(\Box\varphi_1)^{-1}, \ldots, (\Box\varphi_n)^{-1}$ occur in $\mathcal{S}$ only in (1) and in the inequations derived from (3); hence, given that the derived inequation of $(\varphi_i)^{y_j} \geq (\Box\varphi_i)^{-j}$ is used in the linear combination $\lambda_{i,j} > 0$ times, we obtain $\lambda_{i,0} + \lambda_{i,1} + \ldots + \lambda_{i,m} = k$ for $1 \leq i \leq n$.

Let $\mathcal{S}'$ be the system of inequations obtained from $\mathcal{S}$ by replacing (1) and the inequations derived from (2) and (3) with the inequations derived from

$$\begin{array}{l} \lambda_{1,0}[(\varphi_1)^{y_0}], \ldots, \lambda_{n,0}[(\varphi_n)^{y_0}] > [] \\ \lambda_{1,j}[(\varphi_1)^{y_j}], \ldots, \lambda_{n,j}[(\varphi_n)^{y_j}] > k[(\psi_j)^{y_j}] \quad 1 \leq j \leq m \end{array} \quad (4).$$

Crucially, there is also a linear combination of the inequations in $\mathcal{S}'$ witnessing inconsistency over $\mathbb{R}$ that uses each inequation derived from one of the inequations in (4) exactly once. Moreover, all the inequations in $\mathcal{S}'$ are obtained by applying tableau rules to the inequations in (4). Observe now, however, that the different inequations in (4) contain different labels $y_0, y_1, \ldots, y_m$. Hence the inequations in $\mathcal{S}'$ obtained by applying tableau rules to different inequations in (4) will contain disjoint sets of variables. It follows that by applying tableau rules to any one particular inequation in (4) produces a subset of the inequations in $\mathcal{S}'$ that admits a linear combination witnessing inconsistency over $\mathbb{R}$.

But then, translating between sequents and formulas and using Theorem 3.4 again, we obtain

$$\begin{aligned}&\models_{\mathrm{K}(\mathbb{R})} \lambda_{1,0}[\varphi_1], \ldots, \lambda_{n,0}[\varphi_n] \Rightarrow\\ &\models_{\mathrm{K}(\mathbb{R})} \lambda_{1,j}[\varphi_1], \ldots, \lambda_{n,j}[\varphi_n] \Rightarrow k[\psi_j] \quad 1 \leq j \leq m.\end{aligned}$$

So, by the induction hypothesis,

$$\begin{aligned}&\vdash_{\mathrm{GK}(\mathbb{R})} \lambda_{1,0}[\varphi_1], \ldots, \lambda_{n,0}[\varphi_n] \Rightarrow\\ &\vdash_{\mathrm{GK}(\mathbb{R})} \lambda_{1,j}[\varphi_1], \ldots, \lambda_{n,j}[\varphi_n] \Rightarrow k[\psi_j] \quad 1 \leq j \leq m.\end{aligned}$$

But now we can apply the (derived) rule $(\Box_{k,n})$ to obtain the required derivation of $\Box\varphi_1, \ldots, \Box\varphi_n \Rightarrow \Box\psi_1, \ldots, \Box\psi_m$ in $\mathrm{GK}(\mathbb{R})$. □

6 Cut Elimination

This section is devoted to proving Theorem 4.3. Let $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$ be the sequent calculus consisting of the rules (ID), $(\to\Rightarrow)$, $(\Rightarrow\to)$, and $(\Box_{k,n})$. We show first that every cut-free derivation in $\mathrm{GK}(\mathbb{R})$ can be transformed algorithmically into a derivation in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$. Recall that for a sequent calculus C, a sequent rule is *admissible* for C if for any instance of the rule, whenever the premises are derivable in C, the conclusion is derivable in C; the rule is *invertible* for C if for any instance of the rule, whenever the conclusion is derivable in C, the premises are derivable in C.

We begin with two preparatory lemmas:

Lemma 6.1 *The rules $(\to\Rightarrow)$ and $(\Rightarrow\to)$ are invertible for* $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$.

Proof. Simple (constructive) inductions on the height of a derivation of the premise in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$ in each case. □

Lemma 6.2 *The rules* (MIX) *and* (SC_n) *are admissible in* $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$.

Proof. To show the admissibility of (MIX) in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$, we prove that whenever $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Gamma \Rightarrow \Delta$ and $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Pi \Rightarrow \Sigma$ with $r, s \in \mathbb{N}$, then $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} r\Gamma, s\Pi \Rightarrow s\Sigma, r\Delta$. We proceed by induction on the sum of the heights of derivations d_1 and d_2 of $\Gamma \Rightarrow \Delta$ and $\Pi \Rightarrow \Sigma$, respectively.

For the base case, if d_1 and d_2 have height 0, then $\Gamma \Rightarrow \Delta$ and $\Pi \Rightarrow \Sigma$ are instances of (ID), i.e., $\Gamma = \Delta$ and $\Pi = \Sigma$. Hence $r\Gamma \uplus s\Pi = r\Delta \uplus s\Sigma$ and $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} r\Gamma, s\Pi \Rightarrow s\Sigma, r\Delta$ by (ID). If the last application in d_1 is $(\Box_{k,n})$ and d_2 has height 0, then $\Pi = \Sigma$ and the result follows by an application of $(\Box_{k,rn})$. The case where d_1 has height 0 and d_2 ends with $(\Box_{k,n})$ is symmetrical.

If the last application of a rule in d_1 or d_2 is $(\to\Rightarrow)$ or $(\Rightarrow\to)$, then the result follows easily by an application of the induction hypothesis and further applications of the rule. Suppose then finally that d_1 ends with

$$\frac{\Gamma_0 \Rightarrow \quad \Gamma_1 \Rightarrow k[\varphi_1] \quad \ldots \quad \Gamma_n \Rightarrow k[\varphi_n]}{\Omega, \Box\Gamma' \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n, \Omega}\ (\Box_{k,n}) \quad \text{with } k\Gamma' = \Gamma_0 \uplus \Gamma_1 \uplus \ldots \uplus \Gamma_n$$

and that d_2 ends with an application of $(\Box_{l,m})$

$$\frac{\Pi_0 \Rightarrow \quad \Pi_1 \Rightarrow l[\psi_1] \quad \ldots \quad \Pi_m \Rightarrow l[\psi_m]}{\Theta, \Box\Pi' \Rightarrow \Box\psi_1, \ldots, \Box\psi_m, \Theta} \ (\Box_{l,m}) \text{with } l\Pi' = \Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_m.$$

Then we can complete our derivation as follows

$$\frac{rl\Gamma_0, sk\Pi_0 \Rightarrow \quad \{l\Gamma_i \Rightarrow kl[\varphi_i]\}_{1\leq i\leq n} \quad \{k\Pi_j \Rightarrow kl[\psi_j]\}_{1\leq j\leq m}}{r\Omega, s\Theta, r\Box\Gamma', s\Box\Pi' \Rightarrow r\Box\varphi_1, \ldots, r\Box\varphi_n, s\Box\psi_1, \ldots, s\Box\psi_m, r\Omega, s\Theta} \ (\Box_{kl,rn+sm})$$

where the premises are all derivable by the induction hypothesis.

We establish the admissibility of (SC$_n$) by proving that whenever $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} n\Gamma \Rightarrow n\Delta$, then $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Gamma \Rightarrow \Delta$, proceeding by induction on the sum of the complexities of the formulas in Γ, Δ. For the base case, if $n\Gamma = n\Delta$, in particular when Γ and Δ contain only propositional variables, then $\Gamma = \Delta$ and $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Gamma \Rightarrow \Delta$ by (ID). If Γ contains a formula $\varphi \to \psi$, then by the invertibility of the rule $(\to\Rightarrow)$ established in Lemma 6.1, $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} n(\Gamma - [\varphi \to \psi]), n\psi \Rightarrow n\varphi, n\Delta$. The induction hypothesis and an application of $(\to\Rightarrow)$ gives $\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Gamma \Rightarrow \Delta$. The case where Δ contains a formula $\varphi \to \psi$ is symmetrical. In the final case, the derivation of $n\Gamma \Rightarrow n\Delta$ must end with an application of $(\Box_{k,nl})$ where $\Gamma = \Pi \uplus [\Box\Sigma]$ and $\Delta = \Pi \uplus [\Box\varphi_1, \ldots, \Box\varphi_l]$. But then we obtain a derivation of $\Gamma \Rightarrow \Delta$ using $(\Box_{kn,l})$ and the admissibility of (MIX). $\Box$

Proof of Theorem 4.3. The rule $(\Box_n)$ is derivable in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$ using $(\Box_{k,n})$ with $k = n$ and $\varphi_1 = \ldots = \varphi_n = \varphi$ and $\Gamma_1 = \ldots = \Gamma_n = \Gamma$. Hence, using the proofs of Lemma 6.2, every cut-free derivation in $\mathrm{GK}(\mathbb{R})$ can be transformed algorithmically into a derivation in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$. To establish cut-elimination for $\mathrm{GK}(\mathbb{R})$, it suffices now to show that an uppermost application of (CUT) in a derivation in $\mathrm{GK}(\mathbb{R})$ can be eliminated. We will prove (constructively) that

$$\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Gamma, \varphi \Rightarrow \varphi, \Delta \quad \Longrightarrow \quad \vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Gamma \Rightarrow \Delta. \qquad (\star)$$

Suppose then that there are cut-free derivations in $\mathrm{GK}(\mathbb{R})$ of the premises of the uppermost application of $\Gamma, \varphi \Rightarrow \Delta$ and $\Pi \Rightarrow \varphi, \Sigma$. Clearly, by (MIX), we have a cut-free derivation of $\Gamma, \Pi, \varphi \Rightarrow \varphi, \Sigma, \Delta$ in $\mathrm{GK}(\mathbb{R})$, and hence a derivation of $\Gamma, \Pi, \varphi \Rightarrow \varphi, \Sigma, \Delta$ in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$. By $(\star)$, we obtain a derivation of $\Gamma, \Pi \Rightarrow \Sigma, \Delta$ in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$, which also gives the desired derivation in $\mathrm{GK}(\mathbb{R})$.

We prove $(\star)$ by induction on the lexicographically ordered pair consisting of the modal depth of φ and the sum of the complexities of the formulas in $\Gamma, \varphi \Rightarrow \varphi, \Delta$. If $\Gamma \uplus [\varphi] = [\varphi] \uplus \Delta$, in particular if the sequent contains only propositional variables, then $\Gamma = \Delta$ and $\Gamma \Rightarrow \Delta$ is derivable using (ID). If φ has the form $\psi \to \chi$, then we use the invertibility of $(\to\Rightarrow)$ and $(\Rightarrow\to)$ in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$ and apply the induction hypothesis twice. The cases where Γ or Δ includes a formula $\psi \to \chi$ are very similar. Lastly, suppose that $\Gamma, \varphi \Rightarrow \varphi, \Delta$ contains only propositional variables and box formulas. Then there is a derivation of the sequent ending with an application of $(\Box_{k,n})$. The case where $\Box\varphi$ does not appear in the premise is trivial, so just consider the case

$$\frac{\Pi_0, k_0[\varphi] \Rightarrow \quad \Pi_1, k_1[\varphi] \Rightarrow k[\varphi] \quad \{\Pi_i, k_i[\varphi] \Rightarrow k[\psi_i]\}_{i=2}^{n}}{\Sigma, \Box\Pi, \Box\varphi \Rightarrow \Box\varphi, \Box\psi_2, \ldots, \Box\psi_n, \Sigma}\ (\Box_{k,n})$$

where $k\Pi = \Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_n$ and $k = k_0 + k_1 + \ldots + k_n$. By the induction hypothesis, we obtain

$$\vdash_{\mathrm{GK}(\mathbb{R})^{\mathrm{r}}} \Pi_1 \Rightarrow (k - k_1)[\varphi].$$

By Lemma 6.2 (the admissibility of (MIX)), we have derivations in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$ of

$$k_0\Pi_1, (k - k_1)\Pi_0, (k - k_1)k_0[\varphi] \Rightarrow (k - k_1)k_0[\varphi]$$
$$k_i\Pi_1, (k - k_1)\Pi_i, (k - k_1)k_i[\varphi] \Rightarrow (k - k_1)k_i[\varphi], (k - k_1)k[\psi_i] \quad 2 \le i \le n.$$

So, by the induction hypothesis, we have derivations in $\mathrm{GK}(\mathbb{R})^{\mathrm{r}}$ of

$$k_0\Pi_1, (k - k_1)\Pi_0 \Rightarrow$$
$$k_i\Pi_1, (k - k_1)\Pi_i \Rightarrow (k - k_1)k[\psi_i] \quad 2 \le i \le n.$$

Now by an application of $(\Box_{(k-k_1)k,n-1})$, we have a derivation ending with

$$\frac{k_0\Pi_1, (k - k_1)\Pi_0 \Rightarrow \quad \{k_i\Pi_1, (k - k_1)\Pi_i \Rightarrow (k - k_1)k[\psi_i]\}_{i=2}^{n}}{\Sigma, \Box\Pi \Rightarrow \Box\psi_2, \ldots, \Box\psi_n, \Sigma}$$

where $(k - k_1)k\Pi = (k_0 + k_2 + \ldots + k_n)(\Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_n)$. $\Box$

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