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Counting unrooted hypermaps on closed orientable surface

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Abstract. In this paper we derive an enumeration formula for the number of hypermaps of given genus g and given number of darts n in terms of the numbers of rooted hypermaps of genus $\gamma \leq g$ with m darts , where $m|\, n$.

Résumé. Dans ce travail on denombre les hypergraphes d'un genus g donné, et un nombr e de fleches n, selon le nombre de hyper cartes de genus $\gamma \leq g$, et av ec m fleches, ou $m|\, n$.

1. Introduction

In this paper we derive an enumeration formula for the number of hypermaps of given genus g and given number of darts n in terms of the numbers of rooted hypermaps of genus $\gamma \leq g$ with m darts , where $m|\, n$. Explicit expressions for the number of rooted hypermaps of genus g with n darts were derived by Walsh [**32**] for $g = 0$, and by Arques [**2**] for $g = 1$. We apply our general counting formula to derive explicit expressions for the number of unrooted spherical and toroidal hypermaps with given number of darts.

Oriented map is 2-cell decomposition of a closed orientable surface with a fixed global orientation. Generally, maps can be described combinatorially via graph embeddings. Oriented hypermaps are generalisations of oriented maps. While maps are 2-cell embeddings of graphs, hypermaps can be viewed as embeddings of hypermaps into closed orientable surfaces. Such a model was investigated by Walsh in [**32**], where the underlying hypergraph is described via the corresponding 2-coloured bipartite graph B, and the hypermap itself is determined by a 2-cell embedding $B \to S$.

Beginnings of the enumerative theory of maps are closely related with the enumeration of plane trees considered in 60-th by Tutte [**28**], Harary, Prins and Tutte [**6**], see [**7, 22**] as well. Later a lot of other distinguished classes of maps including triangulations, outerplanar, cubic, Eulerian, nonseparable, simple, looples, two-face maps and others were considered. Enumeration of maps on surfaces has attracted a lot of attention last decades [**23**]. Although there are more than 100 published papers on map enumeration most of them deal with the enumeration of rooted maps of given property. In particular, there is a lack of results on enumeration of unrooted maps of genus ≥ 1. Most of the results on map enumeration in the unrooted case restrict to planar maps [**17, 18, 33, 34, 20**]. A recent paper [**25**] presents a breakthrough in the enumeration problem for unrooted maps on closed oriented surface. In the presented paper we apply the methods employed in [**24**] and [**25**] to solve an analogous problem for hypermaps.

2. Hypermaps on surfaces and orbifolds

Hypermaps on surfaces. An *oriented combinatorial hypermap* is a triple $\mathcal{H} = (D; R, L)$, where D is a finite set of darts (called brins, blades, bits as well) and R, L are permutations of D such that $\langle R, L\rangle$ is transitive on D. Orbits of R are called hypervertices, orbits of L are called hyperedges and orbits of RL are called hyperfaces. The degree of a hypervertex (hyperedge, hyperface) is the size of the respective orbit.

Key words and phrases. Enumeration, Map, Surface, Orbifold, Rooted hypermap, Unrooted hypermap, Fuchsian group.

Let $|D| = n$. Denote by v, e and f the numbers of hypervertices, hyperedges and hyperfaces. Then genus g of $\mathcal{H}$ is given by Euler-Poincare formula as follows

$$v + e + f - n = 2 - 2g.$$

Given hypermaps $\mathcal{H}_i = (D_i; R_i, L_i)$, $i = 1, 2$ a mapping $\psi : D_1 \to D_2$ such that $R_2\psi = \psi R_1$ and $L_2\psi = \psi L_1$ is called a morphism (or a covering) $\mathcal{H}_1 \to \mathcal{H}_2$. Note that each morphism between hypermaps is by definition an epimorphism. If $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bijection, ψ is an isomorphism. Isomorphisms $\mathcal{H} \to \mathcal{H}$ form a group $\mathrm{Aut}(\mathcal{H})$ of automorphisms of $\mathcal{H}$. It is easily seen that $\mathrm{Aut}(\mathcal{H})$ acts semiregularly on D, equivalently, the stabiliser of a dart is trivial. A hypermap $\mathcal{H}$ is called *rooted* if one element x of D is chosen to be a root. Morphisms between rooted hypermaps take roots onto roots. It follows that a rooted hypermap admits no non-trivial automorphisms.

By *a surface* we mean a connected, orientable surface without boundary. A *topological map* is a 2-cell decomposition of a surface. Standardly, maps on surfaces are described as 2-cell embeddings of graphs. Oriented combinatorial maps are hypermaps $(D; R, L)$ such that L is a fixed-point-free involution. Walsh observed that oriented hypermaps can be viewed as particular maps. Namely, he proved a one-to-one correspondence [**32**, Lemma 1] between hypermaps and the set of (oriented) 2-coloured bipartite maps. That means that one of the two global orientations of the underlying surface is fixed, and moreover, we assume that a colouring of vertices, say by black and white colours, is preserved by morphisms between maps. The correspondence is given as follows. Let $\mathcal{M}$ 2-coloured bipartite map on an orientable surface S with a fixed global orientation. We set D to be the set of edges of $\mathcal{M}$. The orientation of S induces at each black vertex v of $\mathcal{M}$ a cyclic permutation R_v of edges incident with v. This way a permutation $R = \prod R_v$ of D is defined. Similarly, the orientation of S determines at each white vertex u a cyclic permutation L_u. Set $L = \prod L_u$. Hence we have a unique hypermap $(D; R, L)$ corresponding to $\mathcal{M}$. Conversely, given hypermap $(D; R, L)$ we first define a bipartite 2-colored graph X whose edges are elements of D, black vertices are orbits of R and white vertices are orbits of L. An edge $x \in D$ is incident to a (black or white) vertex u if $x \in u$. The permutation R and L induce local rotations of arcs outgoing from black and white vertices, respectively. It is well known (see Gross and Tucker [**5**, Section 3.2]) that the system of rotations determines a 2-cell embedding of X into an orientable surface.

Similarly as above, an oriented 2-coloured bipartite map is called *rooted* if one of the edges is selected to be a root. Morphisms between rooted 2-coloured bipartite maps take a root onto a root.

There is yet another way to describe hypermaps. Let $\mathcal{H} = (D; R, L)$ be a hypermap. Clearly, the permutation group $\langle R, L\rangle$ is an epimorphic image of the free product $\Delta^+ = C * C \cong \langle\rho\rangle * \langle\lambda\rangle$ of two infinite cyclic groups. The group Δ^+ acts on D via epimorphism taking $\rho \mapsto R$ and $\lambda \mapsto L$. Thus using some standard considerations in permutation group theory each hypermap can be described by a subgroup $F \le \Delta^+$ [**13, 30, 31, 9**]. The subgroup F, called a *hypermap subgroup*, can be identified with a stabiliser of a dart in the action of Δ^+ on D. Since the action of Δ^+ on D is transitive, the number of darts $|D| = n$ coincides with index $[\Delta^+ : F]$ of F in Δ^+. Given $F \le \Delta^+$ the corresponding hypermap can be constructed as an *algebraic hypermap* $\mathcal{H}(\Delta^+/F) = (D; R, L)$, where $D = \{xF | x \in \Delta^+\}$ is the set of left cosets, and the action of R, L on D is defined by $R(xF) = (\rho x)F$, $L(xF) = (\lambda x)F$. Note that the group Δ^+ is sometimes called a universal oriented triangle group. More precisely, Δ^+ is identified with the triangle group $T(\infty, \infty, \infty) =< x, y, z : x\,y\,z = 1 >$ acting on the hyperbolic plane $\mathbf{H}^2$ by orientation preserving isometries (see G.Jones, D.Singerman [**13**]). In this case $\mathbf{H}^2/\Delta^+$ is a trice punctured sphere and $\mathbf{H}^2/F$ is a punctured orientable surface, whose genus g coincides with the genus of the corresponding hypermap.

We summarise the above discussion in the following propositions.

Proposition 2.1. *The following objects are in one-to-one correspondence:*

(1) *rooted 2-coloured bipartite maps of genus g with n edges,*
(2) *rooted hypermaps $(D; R, L)$ of genus g with $|D| = n$,*
(3) *subgroups of the group $\Delta^+ = T(\infty, \infty, \infty)$ of index n and genus g .*

Part (1) ⇔ (2) follows from Walsh [**32**]. Part (2) ⇔ (3) is in ([**13, 4**]).

By definition isomorphic hypermaps have conjugated hypermap subgroups. Hence isomorphism classes of hypermaps correspond to conjugacy classes of subgroups.

Proposition 2.2. *The following objects are in one-to-one correspondence:*

(1) *isomorphism classes of 2-coloured bipartite maps of genus g with n edges,*

(2) *isomorphism classes of hypermaps* $(D;R,L)$ *of genus* g *with* $|D|=n$,
(3) *conjugacy classes subgroups of index* n *and genus* g *of the group* $\Delta^+ = T(\infty,\infty,\infty)$.

Regular coverings. Let $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ be a covering of hypermaps. The covering transformation group consists of automorphisms α of $\mathcal{H}_1$ satisfying the condition $\psi = \psi \circ \alpha$. A covering $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ will be called *regular* if the covering transformation group acts transitively on a fibre $\psi^{-1}(x)$ over a dart x of $\mathcal{H}_2$. Regular coverings can be constructed by taking a subgroup $G \le Aut(\mathcal{H}_1)$, $\mathcal{H}_1 = (D;R,L)$, and setting $\bar{D}$ to be the set of orbits of G, $\bar{R}[x] = [Rx]$, $\bar{L}[x] = [Lx]$. Then the natural projection $x \mapsto [x]$ defines a regular covering $M \to N$, where $\mathcal{H}_2 = (\bar{D},\bar{R},\bar{L})$.

Maps and hypermaps on orbifolds. Given regular covering $\psi : \mathcal{H} \to \mathcal{K}$, let be x be a hypervertex, hyperface or a hyperedge of $\mathcal{K}$. Let $\mathcal{H}$ be of genus g, $\mathcal{K}$ be of genus γ and let $G \le \mathrm{Aut}(\mathcal{H})$ be a covering transformation group. The ratio of degrees $b(x) = deg(\tilde{x})/deg(x)$, where $\tilde{x} \in \psi^{-1}(x)$ is a lift of x along ψ, will be called a *branch index* of x. By transitivity of the action of the group of covering transformations a branch index is a well-defined positive integer not depending on the choice of the lift $\tilde{x}$. Hence b is a well defined integer function defined on the union $V(\mathcal{K}) \cup E(\mathcal{K}) \cup F(\mathcal{K})$. Writing all the values $b(x)$, $b(x) \ge 2$ in a non-decreasing order we get an integer sequence $m_1, m_2, \ldots, m_r$. This way an orbifold S_g/G with signature $[\gamma; m_1, m_2, \ldots, m_r]$ is defined.

For our purposes we define a topological 2-dimensional orbifold $O = O[\gamma; m_1, \ldots, m_r]$ to be a closed orientable surface of genus γ with a distinguished set of points $\mathcal{B}$, called branch points, and an integer function assigning to each $x \in \mathcal{B}$ an integer $b(x) \ge 2$. A 2-coloured bipartite map of genus γ is a map on O provided the following two conditions are satisfied:

(1) no branch point $x \in \mathcal{B}$ lies on an edge,
(2) each face contains at most one branch point $x \in \mathcal{B}$.

The operation associating a 2-coloured bipartite map to a hypermap is functorial. In particular the signature of an orbifold associated with a regular covering of hypermaps coincides with the signature of an orbifold determined by the corresponding regular covering of Walsh 2-coloured bipartite maps. Note also that a regular covering $\psi : \mathcal{H} \to \mathcal{K}$, extends (uniquely) to a regular covering $S_g \to S_g/G$, where g is genus of $\mathcal{H}$ and G is the group of covering transformations.

Let O be an orbifold with signature $[\gamma; m_1, m_2, \ldots, m_r]$. The *orbifold fundamental group* $\pi_1(O)$ is an F-group

$$\pi_1(M,\sigma) = F[\gamma; m_1, m_2, \ldots, m_r] =$$
$$\langle a_1, b_1, a_2, b_2, \ldots, a_\gamma, b_\gamma, e_1, \ldots, e_r | \prod_{i=1}^{\gamma}[a_i,b_i] \prod_{j=1}^{r} e_j = 1, e_1^{m_1} = \ldots e_r^{m_r} = 1\rangle. \tag{2.1}$$

Let $\mathcal{H} \to \mathcal{H}/G = \mathcal{K}$ be a regular covering between hypermaps with a covering transformation group G, let $\mathcal{H}$ be finite. Let the the signarure of the orbifold $\mathcal{K} = \mathcal{H}/G$ be $[\gamma; m_1, m_2, \ldots, m_r]$. Then the Euler characteristic of the underlying surface of $\mathcal{H}$ is given by the Riemann-Hurwitz equation:

$$\chi = |G|\big(2 - 2\gamma - \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big). \tag{2.2}$$

3. General counting formula.

The following theorem is the main result of [**24**].

THEOREM 3.1. *Let* Γ *be a finitely generated group. Then the number of conjugacy classes of subgroups of index* n *in the group* Γ *is given by the formula*

$$N_\Gamma(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{K < \Gamma \\ [\Gamma:K]=m}} Epi(K, Z_\ell).$$

In fact, a little modification of the proof allows us to generalise the above statement to subsets of subgroups of given index closed under conjugacy. Let $\mathcal{P}$ be a set of subgroups of a finitely generated group Γ closed under conjugation. By $Epi_{\mathcal{P}}(K, Z_\ell)$ we denote the number of epimorphisms $K \to Z_\ell$ with the kernel in $\mathcal{P}$.

Hence we have the following

THEOREM 3.2. *Let Γ be a finitely generated group and $\mathcal{P}$ is a set of subgroups of Γ closed under conjugation. Then the number of conjugacy classes of subgroups of index n in $\mathcal{P}$ is given by the formula*

$$N_{\Gamma}^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{K < \Gamma \\ [\Gamma:K]=m}} Epi_{\mathcal{P}}(K, Z_\ell).$$

A group epimorphism is called *order preserving* if it preserves the orders of elements of finite order. Given closed orientable surface S_g of genus g and a cyclic orbifold $O = S_g/Z_\ell$ we denote by $Epi_0(\pi_1(O), Z_\ell)$ the number of order preserving epimorphisms $\pi_1(O) \to Z_\ell$.

The following result is the main tool to calculate the number of unrooted hypermaps on a closed oriented surface.

THEOREM 3.3. *Let S_g be a closed orientable surface of genus g. Denote by $h_O(m)$ be the number of rooted hypermaps with m darts on a cyclic orbifold $O = S_g/Z_\ell$.*

Then the number of unrooted hypermaps of genus g having n darts is

$$H_g(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{O \in Orb(S/Z_\ell)} h_O(m) Epi_0(\pi_1(O), Z_\ell),$$

where the second sum runs through all admissible cyclic orbifolds S_g/Z_ℓ.

Proof. Given $S = S_g$ let $\mathcal{P} = \mathcal{P}_g$ be the set subgroups of genus g of $\Delta^+ = T(\infty, \infty, \infty)$. By Propositions 2.1 and 2.2 rooted hypermaps on S correspond subgroups in $\mathcal{P}$, and isomorphism classes of unrooted hypermaps on S correspond to conjugacy classes of subgroups in $\mathcal{P}$. Setting $\Gamma = \Delta^+$ in Theorem 3.2 we get

$$H_g(n) = N_{\Delta^+}^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{K < \Delta^+ \\ [\Delta^+:K]=m}} Epi_{\mathcal{P}}(K, Z_\ell).$$

Given epimorphism $\psi : K \to Z_\ell$ with kernel $H \in \mathcal{P}$ determines a regular covering of algebraic hypermaps $\psi^* : \mathcal{H}(\Delta^+/H) \to \mathcal{H}(\Delta^+/K)$ induced by $H \trianglelefteq K$ with the group of covering transformations isomorphic to Z_ℓ. Let σ be the signature of the orbifold $O = O(\sigma) = S_g/Z_\ell$ determined by the covering of hypermaps. Hence the set of epimorphisms $\psi : K \to Z_\ell$ with $Ker(\psi) = H \in \mathcal{P}$ split into classes characterised by the signatures of the cyclic orbifolds $O = S/Z_\ell$. Denote by $Epi_\sigma(K, Z_\ell)$ the number of epimorphisms $K \to Z_\ell$ with kernel $H \in \mathcal{P}$ and quotient orbifold $O = S/Z_\ell$ with signature σ. We set $\mathcal{P}_\sigma = \{K | K < \Delta^+, Epi_\sigma(K, Z_\ell) \neq 0\}$.

It is well known that the group Δ^+ acts on the universal covering surface $\mathcal{H}^2$ as a discontinuous group of conformal automorphisms. This allows us to introduce the structure of Riemann surface (as well as the orbifold structure) on the hypermaps $\mathcal{H}(\Delta^+/H)$, $\mathcal{H}(\Delta^+/K)$, respectively. A regular covering of hypermaps $\psi : \mathcal{H}(\Delta^+/H) \to \mathcal{H}(\Delta^+/K)$ extends to a branched regular covering $S \to O$ of the orbifold $O = O(\sigma)$ by the closed surface S. By the Riemann Extension Theorem there is a one-to-one correspondence between coverings $\mathcal{H}^2/H \to \mathcal{H}^2/K$ and coverings of the compactified quotient spaces $S = \overline{\mathcal{H}^2/H} \to O = \overline{\mathcal{H}^2/K}$ (see [**12**] for a more detailed explanation). We want to show $Epi_\sigma(K, Z_\ell) = Epi_0(\Gamma(\sigma), Z_\ell)$. Given $K \in \mathcal{P}_\sigma$ we calculate the number of regular Z_ℓ-coverings $\mathcal{H}^2/H \to \mathcal{H}^2/K$ with $H \trianglelefteq K$ and $H \in \mathcal{P}$. By G. Jones [**11**] there are $Epi_\sigma(K, Z_\ell)/\varphi(\ell)$ such coverings. On the other hand, we have $Epi_0(\Gamma(\sigma), Z_\ell)/\varphi(\ell)$ of regular Z_ℓ-coverings $S = \overline{\mathcal{H}^2/H} \to O = \overline{\mathcal{H}^2/K}$ over the orbifold $O = O(\sigma)$ with the signature σ [**11**]. By virtue of the one-to-one correspondence these numbers coincide. Hence, we have $Epi_\sigma(K, Z_\ell) = Epi_0(\Gamma(\sigma), Z_\ell)$ as it was required. Given m, ℓ and σ denote by $\nu_\sigma(m)$ the number of subgroups $K < \Delta^+$ in $\mathcal{P}(\sigma)$ and by $Sign(S_g/Z_\ell)$ the set of signatures of cyclic g-admissible orbifolds. We have

$$H_g(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{K < \Delta^+ \\ [\Delta^+:K]=m}} Epi_{\mathcal{P}}(K, Z_\ell) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\sigma \in Sign(S_g/Z_\ell)} \nu_\sigma(m) Epi_\sigma(K, Z_\ell) =$$

$$\frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\sigma \in Sign(S_g/Z_\ell)} \nu_\sigma(m) Epi_0(\Gamma(\sigma), Z_\ell).$$

Taking into the account the correspondence between groups in $\mathcal{P}_\sigma$ and rooted hypermaps on the orbifold $O = O(\sigma)$ we get $\nu_\sigma(m) = h_O(m)$ and the proof is complete.

In what follows we derive a formula enumerating numbers of rooted hypermaps on orbifolds in terms of numbers of rooted hypermaps on surfaces. Let $\mathcal{H}$ be a rooted hypermap on an orbifold O such that $\mathcal{H} = \tilde{\mathcal{H}}/Z_\ell = (D; R, L)$ is a quotient of an ordinary finite map $\tilde{\mathcal{H}}$ on a surface S_g. Thus $O = S_g/G$ where $G \cong Z_\ell$ is a cyclic group of orientation preserving symmetries of S_g of order ℓ. It follows that each branch index of the branched covering $S_g \to O$ is a divisor of ℓ and can write $O = O[\gamma; 2^{q_2}, \ldots, \ell^{q_\ell}]$, where $q_i \geq 0$ denotes the number of branch points of index i, for $i = 2, \ldots, \ell$. In this case, genera γ and g are related by the Riemann-Hurwitz equation $2-2g = \ell(2-2\gamma-\sum_{j=2}^{\ell} q_j(1-1/j))$. We use the convention $h_\gamma(m) = \nu_{[\gamma;\,\emptyset]}(m)$ denoting the number of rooted hypermaps with m darts on a surface of genus g. Clearly, the exponential notation $O = O[\gamma; 2^{q_2}, \ldots, \ell^{q_\ell}]$ can be used for any oriented orbifold (not necessarily cyclic) provided the indexes of branch points are bounded by ℓ.

Given integers $x_1, x_2, \ldots, x_q$ and $y \geq x_1 + x_2 + \cdots + x_q$ we denote by

$$\binom{y}{x_1, x_2, \ldots, x_q} = \frac{y!}{x_1! x_2! \ldots x_q! (y - \sum_{j=1}^{q} x_j)!},$$

the multinomial coefficient.

Now we are able to determine the number of rooted hypermaps on an arbitrary orbifold.

Proposition 3.4. *The number of rooted hypermaps on an orbifold $O = O[\gamma; 2^{q_2}, \ldots, \ell^{q_\ell}]$ with m darts is*

$$h_O(m) = \binom{m+2-2\gamma}{q_2, q_3, \ldots, q_\ell} h_\gamma(m). \tag{5.1}$$

Proof. Let $\mathcal{H}$ be a rooted hypermap on S_γ with v hypervetices, e hyperedges and f hyperfaces. Then $\mathcal{H}$ gives rise to as many rooted hypermaps as is the number of partitions of the set $V(\mathcal{H}) \cup E(\mathcal{H}) \cup F(\mathcal{H})$ of cardinality $v + e + f = m + 2 - 2\gamma$ into disjoint subsets of cardinalities $q_1, q_2, \ldots, q_\ell$. This is exactly the number

$$\binom{m+2-2\gamma}{q_2, q_3, \ldots, q_\ell}.$$

Combining Proposition 3.4 and Theorem 3.3 we get our main theorem.

Theorem 3.5. *The number of unrooted hypermaps on a closed surface S_g of genus g with n darts is given by*

$$H_g(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{O \in Orb(S/Z_\ell) \\ O = O[\gamma; 2^{q_2}, 3^{q_3}, \ldots, \ell^{q_\ell}]}} Epi_0(\pi_1(O), Z_\ell) \binom{m+2-2\gamma}{q_2, q_3, \ldots, q_\ell} h_\gamma(m),$$

where the second sum runs through all cyclic orbifolds S_g/Z_ℓ.

Note that the numbers $Epi_0(\pi_1(O), Z_\ell)$ were computed by the authors in [**25**] in terms of some standard arithmetical functions. The following section surveys results on $Epi_0(\pi_1(O), Z_\ell)$.

4. Number of epimorphisms from an F-group onto a cyclic group

As one can see in Theorems 3.3 and 3.5 to derive an explicit formula for the number of unrooted hypermaps with given genus and given number of darts one needs to deal with the numbers $Epi_0(\pi_1(O), \mathtt{Z}_\ell)$ of order preserving epimorphisms from an F-group Γ onto a cyclic group $\mathtt{Z}_\ell$. These numbers are counted using some number theoretical machinery in [**25**]. In what follows we recall some relevant results used in later computations.

Denote by $\mu(n)$, $\phi(n)$ and $\Phi(x, n)$ the Möbius, Euler and von Sterneck functions, respectively. The relationship between them is given by the formula

$$\Phi(x, n) = \frac{\phi(n)}{\phi(\frac{n}{(x,n)})} \mu\left(\frac{n}{(x, n)}\right),$$

where (x, n) is the greatest common divisor of x and n. It was shown by O. Hölder that $\Phi(x, n)$ coincides with the Ramanujan sum $\sum_{\substack{1 \leq k \leq n \\ (k,\, n)=1}} \exp(\frac{2\,ikx}{n})$. For the proof, see Apolstol [**1**, p.164] and [**26**]. An arithmetic function, called by Liskovets *orbicyclic arithmetic function* [**21**], is a multivariate integer function defined by

$$E(m_1, m_2, \dots, m_r) = \frac{1}{m}\sum_{k=1}^{m} \Phi(k,\, m_1)\cdot\Phi(k,\, m_2)\dots\Phi(k,\, m_r).$$

Recall that the Jordan multiplicative function $\phi_k(n)$ of order k can be defined as follows:

$$\phi_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k.$$

The following proposition is proved in [**25**].

Proposition 4.1. *Let $\Gamma = F[g; m_1, \dots, m_r]$ be an $F-$group of signature $[g; m_1, \dots, m_r]$. Denote by $m = lcm\,(m_1, \dots, m_r)$ the least common multiple of $m_1, \dots, m_r$ and let $m|\ell$. Then the number of order-preserving epimorphisms of the group Γ onto a cyclic group Z_ℓ is given by the formula*

$$Epi_0(\Gamma, Z_\ell) = m^{2g}\phi_{2g}(\ell/m)E(m_1, m_2, \dots, m_r).$$

In particular, if $\Gamma = F[g; \emptyset] = F[g; 1]$ is a surface group of genus g we have

$$Epi_0(\Gamma, Z_\ell) = \phi_{2g}(\ell).$$

Let us note that the condition $m|\ell$ in the above proposition gives no principal restriction, since $Epi_0(\Gamma, Z_\ell) = 0$ by the definition provided m does not divide ℓ. An orbifold $O = O[g; m_1, \dots, m_r]$ will be called γ-*admissible* if it can be represented in the form $O = S_\gamma/Z_\ell$, where S_γ is an orientable surface of genus γ surface and Z_ℓ is a cyclic group of automorphisms of S_γ. There is an orbifold $O = S_\gamma/Z_\ell$ with signature $[g; m_1, m_2, \dots, m_r]$ if and only if there exists ℓ such that the number $Epi_0(\pi_1(O), Z_\ell) \neq 0$ and the numbers γ, g, $m_1, \dots, m_r$ and ℓ are related by the Riemann-Hurwitz equation $2 - 2\gamma = \ell(2 - 2g - \sum_{i=1}^{r}(1 - 1/m_i))$. The Wiman theorem makes us sure that $1 \le \ell \le 4\gamma + 2$ for $\gamma > 1$.

Using Proposition 4.1 and result by Harvey [**8**]we derive the following lists of γ-admissible orbifolds, for $\gamma = 0, 1$.

Corollary 4.2. *0-admissible orbifolds are $O = O[0; \ell^2]$, with $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$ for any positive integer ℓ.*

Corollary 4.3. *Let $O = O[g; m_1, m_2, \dots, m_r] = S_1/Z_\ell$ be a 1-admissible orbifold. Then one of the following cases happens:*

- $O = O[1; \emptyset]$, *with* $Epi_0(\pi_1(O), Z_\ell) = \sum_{k|\ell}\mu(\ell/k)k^2 = \phi_2(\ell)$ *for any* ℓ,
- $\ell = 2$ *and* $O = O[0; 2^4]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 1$,
- $\ell = 3$ *and* $O = O[0; 3^3]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2$,
- $\ell = 4$ *and* $O = O[0; 4^2, 2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2$,
- $\ell = 6$ *and* $O = O[0; 6, 3, 2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2$.

The lists of $2-$ and $3-$admissible orbifolds can be found in [**25**].

5. Counting unrooted hypermaps on the sphere and torus

In this section we apply the above results to calculate the number of unrooted hypermaps with given number of darts on the sphere and torus.

Theorem 5.1. *The number of spherical unrooted hypermaps with n darts is given by the formula*

$$H_0(n) = \frac{1}{n}\left(\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n} + \sum_{\substack{\ell|n,\, \ell>1\\ \ell m=n}} 3\cdot 2^{m-2}\binom{2m}{m}\phi(\ell)\right)$$

Proof. For $\ell > 1$ there is only one possible action of cyclic group Z_ℓ on the sphere S. The corresponding orbifold O has a signature $[0; \ell,\, \ell]$ and by Corollary 4.2 we have $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$. By Theorem 3.5 we obtain

$$H_0(n) = \frac{1}{n}\big(h_0(n) + \sum_{\substack{\ell|n,\, \ell>1\\ \ell m=n}} \phi(\ell)\binom{m+2}{2}h_0(m)\big).$$

To finish the proof we note that by T. Walsh [**32**]

$$h_0(m) = \frac{3 \cdot 2^{m-1}}{(m+1)(m+2)} \binom{2m}{m}. \tag{5.1}$$

The numbers of rooted and unrooted spherical hypermaps up to 30 darts is given in Table 1.
Table 1. Numbers of rooted and unrooted hypermaps on the sphere with at most 30 darts
No. of darts, rooted hypermaps, unrooted hypermaps

01, 1, 1
02, 3, 3
03, 12, 6
04, 56, 20
05, 288, 60
06, 1584, 291
07, 9152, 1310
08, 54912, 6975
09, 339456, 37746
10, 2149888, 215602
11, 13891584, 1262874
12, 91287552, 7611156
13, 608583680, 46814132
14, 4107939840, 293447817
15, 28030648320, 1868710728
16, 193100021760, 12068905911
17, 1341536993280, 78913940784
18, 9390758952960, 521709872895
19, 66182491668480, 3483289035186
20, 469294031831040, 23464708686960
21, 3346270487838720, 159346213738020
22, 23981605162844160, 1090073011199451
23, 172667557172477952, 7507285094455566
24, 1248519259554840576, 52021636161126702
25, 9063324995286990848, 362532999811480604
26, 66032796394233790464, 2539722940697502966
27, 482722511571640123392, 17878611539691757938
28, 3539965084858694238208, 126427324476844560112
29, 26035872237025235042304, 897788697828456380772
30, 192014557748061108436992, 6400485258395785352796

We note that the numbers $H_0(n)$ was determined in terms of unrooted planar 2-constellations formed by n polygons by M. Bosquet-Melon and G. Schaeffer [**3**].

Now we derive an explicit formula for counting unrooted maps on torus. Rooted toroidal maps were enumerated by D. Arquès in [**2**]. He proved that

$$h_1(n) = \frac{1}{3} \sum_{k=0}^{n-3} 2^k (4^{n-2-k} - 1) \binom{n+k}{k}. \tag{5.2}$$

Theorem 5.2. *The number of unrooted toroidal hypermaps $H_1(n)$ with n darts is equal to*

$$\tfrac{1}{n}\Big(\tbinom{\frac{n}{2}+2}{4} h_0\left(\tfrac{n}{2}\right) + 2\tbinom{\frac{n}{3}+2}{3} h_0\left(\tfrac{n}{3}\right) + 6\tbinom{\frac{n}{4}+2}{3} h_0\left(\tfrac{n}{4}\right) + 12\tbinom{\frac{n}{6}+2}{3} h_0\left(\tfrac{n}{6}\right) + \sum_{\substack{\ell | n \\ \ell m = n}} \phi_2(\ell) h_1(m)\Big),$$

where ϕ_2 is the Jordan function, and functions h_0 and h_1 are given by (5.1) and (5.2), respectively.

Proof. Following Theorem 3.5 and Corollary 4.3 we have

$$H_1(n) = \frac{1}{n}\big(h_{[0;2^4]}(n/2) + 2h_{[0;3^3]}(n/3) + 2h_{[0;2,4^2]}(n/4) +$$

$$2h_{[0;2,3,6]}(n/6) + \sum_{\substack{\ell|n \\ \ell m=n}} \sum_{k|\ell} \mu(\ell/k) k^2 h_1(n/\ell)\big). \tag{5.3}$$

It remains to calculate the numbers of rooted hypermaps on orbifolds $O[0;2^4]$, $O[0;3^3]$, $0[2;4^2]$ and $O[0;2,3,6]$.

By Proposition 3.4 we obtain

$$h_{[0;2^4]}(m) = \binom{m}{4} h_0(m),\ h_{[0;3^3]}(m) = \binom{m+2}{3} h_0(m),$$

$$h_{[0;2,3,6]}(m) = \binom{m+2}{1,1,1} h_0(m) = 6\binom{m+2}{3} h_0(m),$$

$$h_{[0;2,4^2]}(m) = \binom{m+2}{1,2} h_0(m) = 3\binom{m+2}{3} h_0(m).$$

Inserting the above numbers into (5.3) we get the theorem.

The following list containing the numbers of rooted and oriented unrooted maps of genus 1 up to 30 edges follows.

Table 2. Numbers of rooted and unrooted hypermaps on the torus with at most 30 darts

No. of darts, rooted hypermaps, unrooted hypermaps

03, 1, 1
04, 15, 6
05, 165, 33
06, 1611, 285
07, 14805, 2115
08, 131307, 16533
09, 1138261, 126501
10, 9713835, 972441
11, 81968469, 7451679
12, 685888171, 57167260
13, 5702382933, 438644841
14, 47168678571, 3369276867
15, 388580070741, 25905339483
16, 3190523226795, 199408447446
17, 26124382262613, 1536728368389
18, 213415462218411, 11856420991413
19, 1740019150443861, 91579955286519
20, 14162920013474475, 708146055343668
21, 115112250539595093, 5481535740059577
22, 934419385591442091, 42473608898628639
23, 7576722323539318101, 329422709719100787
24, 61375749135369153195, 2557322884534185500
25, 496747833856061953365, 19869913354242478293
26, 4017349254284543961771, 154513432889706455145
27, 32467023775647069984085, 1202482362061007078175
28, 262225359776626483309227, 9365191420865873023026
29, 2116714406654571321840981, 72990151953605907649689
30, 17077642118698511054318251, 569254737292213025378571

The above tables were computed using MATHEMATICA, Ver. 5. The input numbers of rooted maps come from [**2**].

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