Integral Apollonian Packings

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Section 1. An integral packing:

Figure 1

The quarter nickel and dime in the above figure are placed so that they are mutually tangent. This configuration is unique up to rigid motions. As far as I can tell there is no official exact size for these coins but the diameters of 24, 21 and 18 millemeters are accurate to the nearest mm and I assume henceforth that these are the actual diameters. Let C be the unique (see below) circle which is tangent to the three coins as shown in Figure 2. It is a small coincidence that its diameter is rational, as indicated.

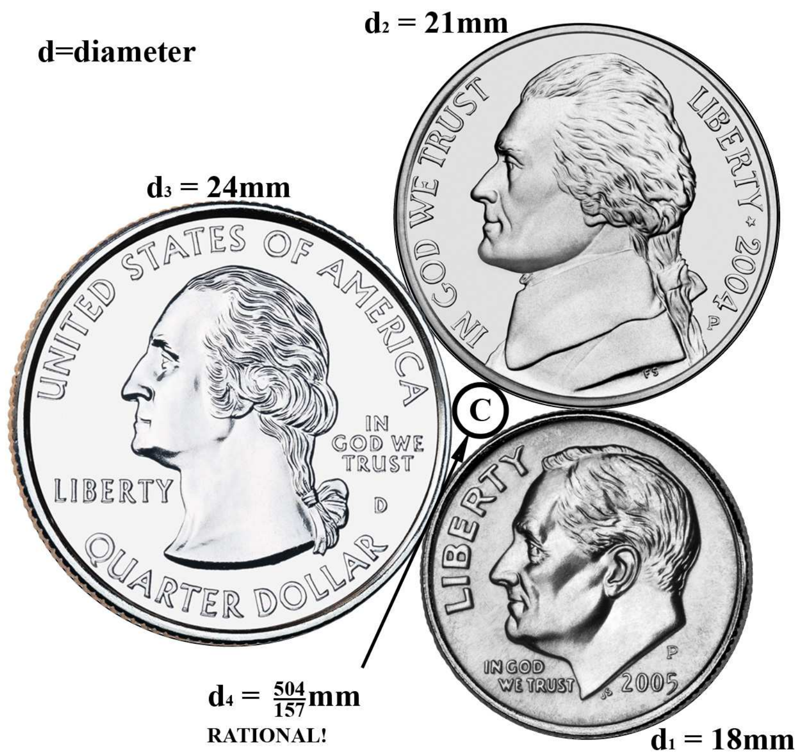

Figure 2

What is more remarkable is that if we continue to place circles in the resulting regions bounded by three circles as described next, that all the diameters are rational. Since the circles become very small so do their radii and it is more convenient to work with their curvatures; that is the reciprocal of the radius. In fact in this example it is natural to scale everything further by 252, so for C a circle as above let $a(C)$ be 252 times the curvature of C. In Figure 3 our three tangent circles are displayed together with the unique outer mutually tangent circle. Their $a(C)$'s are depicted inside the circle. Note that the outer circle has a negative sign indicating that the other circles are in its interior (it is the only circle with a negative sign).

Figure 3

At the next generation we place circles in each of the 4 lune regions obtaining the configuration in Figure 4 with the curvatures $a(C)$ as indicated.

Figure 4

For the 3^{rd} generation we fill in the 12 new lunes as in Figure 5.

Figure 5

Continuing in this way adinfinitum yields the integral Apollonian packing P_0 depicted in Figure 6.

Figure 6

My aim in this lecture is first to explain the elementary plane geometry behind the above construction and then to discuss the diophantine properties of the integers appearing as curvatures in integral apollonian packings such as P_0. As with many problems in number theory the basic questions here are easy to state but difficult to resolve. There are many papers in the literature dealing with apollonian packings and their generalizations. However the diophantine questions are quite recent and are raised in the lovely five author paper [G-L-M-W-Y]. The developments that we discuss below are contained in the letter and preprints, [Sa1], [K-O], [F], [F-S], [B-G-S] and [B-F] .

In Section 2 we review (with proofs) some theorems from Euclidian geometry that are central to understanding the construction of P_0. This requires no more than high school math. The proofs of the results in later sections involve some advanced concepts and so we only outline these proofs in general terms. However the notions involved in the statements of all the theorems are ones that are covered in basic undergraduate courses and it is my hope that someone with this background can follow the discussion to the end. In Section 3 we introduce the key object A which is the symmetry group of P_0. It is a group of 4×4 integer matrices which is deficient in a way that makes its study both interesting and challanging. Section 4 deals with the basic analytic question of counting the number of circles in P_0 when ordered by their curvatures. Sections 5 and 6 are concerned with diophantine questions such as which numbers are curvatures of circles in P_0, a possible local to global principle and the number of circles whose curvatures are prime numbers.

Section 2. Apollonius and Descartes′ Theorems

First some notation. P denotes an integral Apollonian packing and C a typical circle in P. Denote by $r(C)$ its radius and by $a(C) = 1/r(C)$ its curvature. Let $w(C)$ be the generation $n \geq 1$ at which C first appears in the packing. Thus there are 4.3^{n-1} circles at the n^{th} generation.

Apollonius' Theorem:

Given three mutually tangent circles C_1, C_2, C_3 there are exactly two circles C and C' tangent to all three.

Our proof is based on the use of motions of the plane that take circles to circles (we allow a straight line as a circle with "infinite" radius) preserve tangencies and angles. Specifically the operation of inversion in a circle E of radius r and center O as displayed in Figure 7 is such an operation. The transformation takes p to q as shown and one checks it satisfies the above properties.

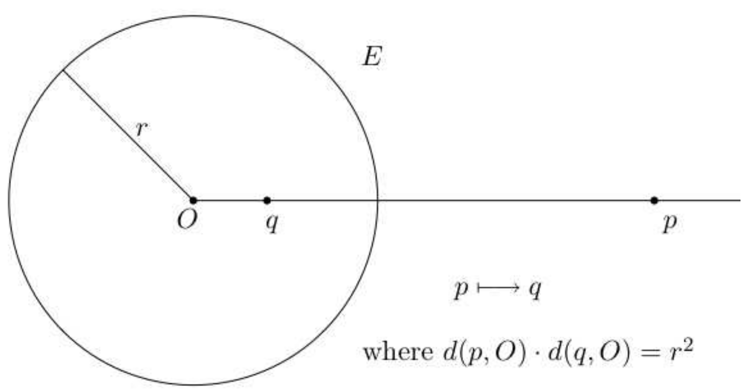

Figure 7

To prove Apollonius' Theorem, let C_1, C_2, C_3 be as shown in Figure 8 and let ξ be the point of tangency between C_1 and C_2.

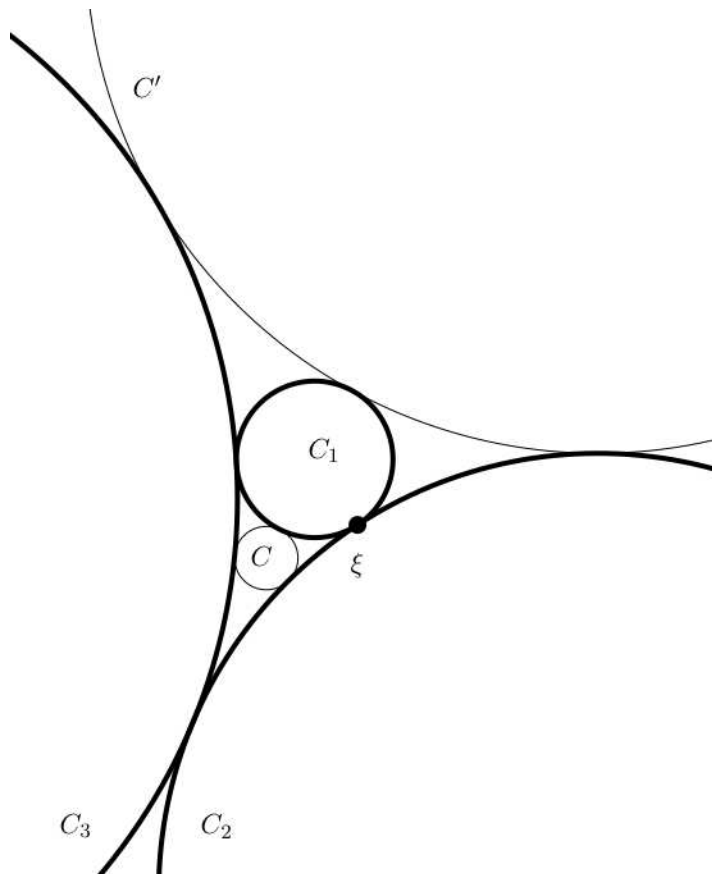

Figure 8

Invert Figure 8 in a circle E centered at ξ. Then C_1 and C_2 are mapped to circles through infinity, that is parallel straight lines $\tilde{C}_1$ and $\tilde{C}_2$, while C_3 is mapped to a circle $\tilde{C}_3$ tangent to both as indicated in Figure 9.

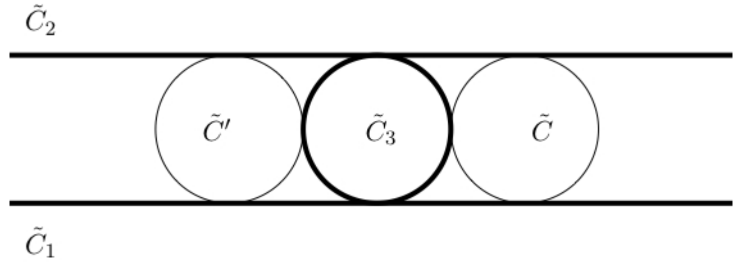

Figure 9

In this configuration where $\tilde{C}_1$ and $\tilde{C}_2$ are parallel lines it is clear that there are exactly two circles, $\tilde{C}'$ and $\tilde{C}$, which are mutually tangent to $\tilde{C}_1, \tilde{C}_2$ and $\tilde{C}_3$. Hence by inverting again Apollonius' Theorem follows.

Descartes' Theorem:

Given four mutually tangent circles whose curvatures are a_1, a_2, a_3, a_4 (with our sign convention) then

$$F(a_1, a_2, a_3, a_4) = 0$$

where F is the quadratic form

$$F(a) = 2a_1^2 + 2a_2^2 + 2a_3^2 + 2a_4^2 - (a_1 + a_2 + a_3 + a_4)^2 .$$

Proof: Again we employ inversion.

We need a couple of formulae relating the radius of a circle and its inversion in E of radius k.

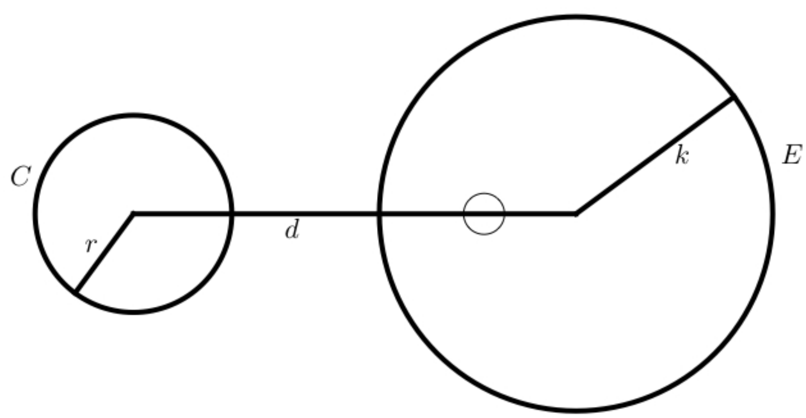

Figure 10

Inverting C in E yields a circle of radius

$$k^2r/(d^2 - r^2) \tag{1}$$

where r is the radius of C and d the distance between the centers of C and E.

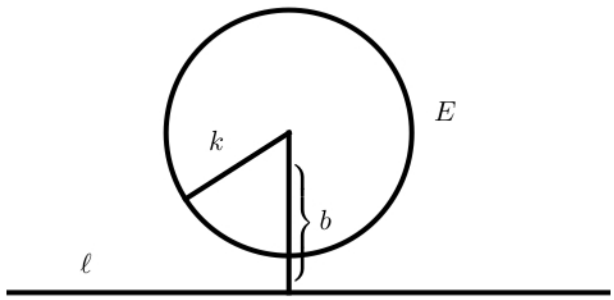

Figure 11

Inverting the straight line ℓ in E yields a circle of radius

$$k^2/2b \tag{2}$$

where b is the distance from the center of E to ℓ.

Now let C_1, C_2, C_3, C_4 be our four mutually tangent circles as shown in Figure 12. Let E be a circle centered at $\xi = (x_0, y_0)$ the point of tangency of C_1 and C_2. Inverting in E we arrive at the configuration $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4$ as shown (after further translation and rotation).

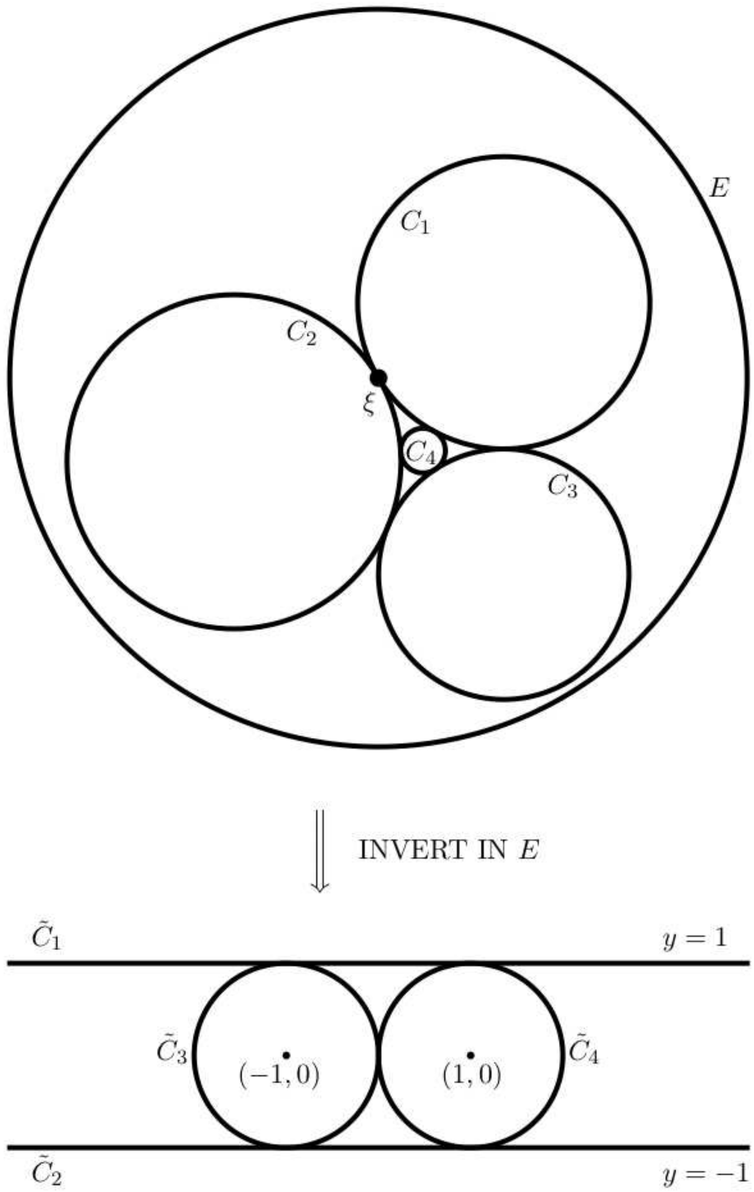
E
C_1
C_2
ξ
C_4
C_3
INVERT IN E
$\tilde{C}_1$
$y = 1$
$\tilde{C}_3$
$(-1, 0)$
$(1, 0)$
$\tilde{C}_4$
$\tilde{C}_2$
$y = -1$

Figure 12

Applying (1) and (2) above we find that

$$r(C_3) = \frac{k^2}{x_0^2 - 2x_0 + y_0^2}\,,\; r(C_4) = \frac{k^2}{x_0^2 + 2x_0 + y_0^2}$$

$$r(C_2) = \frac{k^2}{2(y_0 - 1)} \text{ and } r(C_1) = \frac{k^2}{2(y_0 + 1)}$$

where k is the radius of E. Substituting $a(C_i) = 1/r(C_i)$ in the Descartes form F and doing some algebraic manipulation yields $F(a_1, a_2, a_3, a_4) = 0$ (see Coxeter [C] for a further discussion). This proof of Descartes' Theorem is a little unsatisfying in that it requires some calculation at the end but it is conceptually simple. That is the proof is no more than inversion and keeping track of the quantities under this transformation.

We can now complete our discussion of the packing P_0 or any other integral packing. Firstly according to Apollonius' Theorem the placement of each circle in each lune region is unique once we have a starting configuration of three mutually tangent circles C_1, C_2, C_3. Thus these circles determine the entire packing. Now suppose that these starting circles have curvatures a_1, a_2, a_3, then according to Descartes' Theorem if C and C' are the two circles tangent to C_1, C_2, C_3 then their curvatures a_4 and a_4' satisfy

$$\left.\begin{aligned} F(a_1, a_2, a_3, a_4) &= 0 \\ \text{and} \qquad F(a_1, a_2, a_3, a_4') &= 0\,. \end{aligned}\right\} \tag{3}$$

Thus a_4 and a_4' are roots of the same quadratic equation and using the quadratic formula one finds that

$$a_4 + a_4' = 2a_1 + 2a_2 + 2a_3$$

and

$$a_4, a_4' = a_1 + a_2 + a_3 \pm 2\sqrt{\triangle} \tag{4}$$

where

$$\triangle = a_1 a_2 + a_1 a_3 + a_2 a_3\,. \tag{5}$$

So for our three coins with curvatures 21,24 and 28 in Figure 3 we have that $\triangle = 1764 = (42)^2$. Hence a_4 and a_4' are integers and this is the small coincindence that leads to P_0 being an integral packing. Indeed for a general packing with starting curvatures a_1, a_2, a_3; a_4 and a_4' involve a_1, a_2, a_3 and $\sqrt{\triangle}$. Now starting with these four circle C_1, C_2, C_3, C_4 we get all future circles in the packing by taking three circles at a time and using the existing fourth mutually tangent circle, to produce another such in the packing. In doing so we don't need to extract any further square roots. Thus the curvatures of the entire packing P are expressed as sums of the quantities $a_1, a_2, a_3, \sqrt{\triangle}$, with integer coefficients. In particular when a_1, a_2, a_3 are integers and $\triangle$ is a perfect square, the packing is an integral packing. In terms of the radii r_1, r_2, r_3 which we assume are rational numbers the further radii of circles in the packing lie in the field of rationals adjoin $\sqrt{(r_1 + r_2 + r_3)\, r_1 r_2 r_3}$.

Section 3. The Apollonian Group

A deeper study of an Apollonian packing is facilitated by introducing the symmetry group A which is called the Apollonian group. Given 4 mutually tangent circles in a packing whose curvatures are $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ we get 4 new such configurations by taking the 4 subsets of 3 of the four original circles and in each case introducing a new circle in the packing using Apollonius' Theorem. So if C_1, C_2, C_3, C_4 is our starting configuration and we take the subset C_1, C_2, C_3 and generate C_4' from C_4, we get a new configuration C_1, C_2, C_3, C_4' in the packing with C_4' being the new circle in the corresponding lune region. According to (4) the new 4-tuple of curvatures is $a' = (a_1, a_2, a_3, a_4')$ where in matrix notation

$$a' = aS_4 \tag{6}$$

and

$$S_4 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{7}$$

Taking the other subsets of C_1, C_2, C_3, C_4 yields

$$a' = aS_j \quad j = 1, 2, 3$$

with

$$S_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, S_3 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \tag{8}$$

Note that S_j have integer entries and that

$$S_j^2 = I\,. \tag{9}$$

<u>Definition:</u> *The Apollonian group A is the subgroup of the 4×4 integer matrices of determinant* ± 1 $(\mathrm{GL}_4(\mathbb{Z}))$ *generated by* S_1, S_2, S_3, S_4.

The transformations S_j as well as those generated by them, switch the roots of one coordinate as in equation (3). So the group A arises from Galois symmetries and it also acts as a symmetry of the packing. Indeed according to our discussion above the 4-tuples of curvatures of mutually tangent circles in a packing P are the orbits $\mathcal{O}_a = a \cdot A$ of A, where a is any such tuple in the packing. If a is integral then so is any x in $\mathcal{O}_a$ and if a is primitive (that is its coordinates have no common factor) then so is every $x \in \mathcal{O}_a$.

We assume that our packing P like P_0, is integral primitive and bounded (as in Figure 6). In this case any $x \in \mathcal{O}_a$ is a primitive integral point, which by Descartes' Theorem lies on the cone V given by

$$V = \{x : F(x) = 0\}\,. \tag{10}$$

It is clear and one can check it directly, that

$$F(xS_j) \,=\, F(x) \text{ for } x \in \mathbb{R}^4 \text{ and } j \,=\, 1,2,3,4. \tag{11}$$

Hence

$$F(x\gamma) \,=\, F(x) \text{ for any } x \text{ and } \gamma \in A\,. \tag{12}$$

Let O_F be the orthogonal group of F, that is

$$O_F \,=\, \{g \in \mathrm{GL}_4 : F(xg) \,=\, F(x)\}\,. \tag{13}$$

O_F is an "algebraic group" in that it is defined by algebraic (in this instance quadratic) equations in $(x_{ij})_{\substack{i=1,\ldots,4\\ j=1,\ldots 4}}$.
Explicitly it is given by:

$$X\,SX^t \,=\, S\,, \tag{13$'$}$$

where S is the matrix of F, that is

$$F(x) \,=\, xSx^t\,.$$

From (12) we have that

$$A \,\leq\, O_F(\mathbb{Z}) \tag{14}$$

where $O_F(\mathbb{Z})$ consists of the matrices in O_F whose entries are integers.

This brings us to the heart of the matter, at least as far diophantine properties of an integral packing. The group $O_F(\mathbb{Z})$ is a much studied and well understood group. It is an "arithmetic" group and as such is central in the arithmetic theory of quadratic forms (for example in connection with understanding which integers are represented by an integral quadratic form) and also in automorphic forms. It is also big as is demonstrated by the orbit of a primitive integral point $x \in V^{\mathsf{prim}}(\mathbb{Z})$ under $O_F(\mathbb{Z})$ being all of $V^{\mathsf{prim}}(\mathbb{Z})$. The salient features of the Apollonian group A are

(i) A is small; it is of infinite index in $O_F(\mathbb{Z})$.

(ii) A is not too small, it is Zariski dense in O_F.

> (i) makes the diophantine analysis of an integral packing nonstandard in that the familiar arithmetic tools don't apply.
>
> (ii) says that A is large in the algebraic geometric sense that any polynomial in the variables x_{ij} $i,j = 1,2,3,4$ of 4×4 matrices which vanishes on A must also vanish on the complex points of O_F. It is a modest condition on A and it plays a critical role in understanding what A looks like when reduced in arithmetic modulo q, for $q > 1$.

An instructive of way of seeing (i) is to consider the orbits of A on $V^{\mathsf{prime}}(\mathbb{Z})$. These correspond to the different integral primitive Apollonian packings and there are infinitely many of them. In [G-L-M-W-Y] it is shown how to use A to find a point v in each orbit aA called a "root quadruple" which is a reduced element. The definition of reduced being that $v = (a_1, a_2, a_3, a_4)$, is in $V^{\mathsf{prime}}(\mathbb{Z})$ and satisfies $a_1+a_2+a_3+a_4 > 0$, $a_1 \leq 0 \leq a_2 \leq a_3 \leq a_4$ and $a_1 + a_2 + a_3 \geq a_4$. For example for P_0, the reduced v is $(-11, 21, 24, 28)$. There are infinitely many root quadruples, in fact one can count their number asymptotically when ordered by the euclidian norm, see [Sa1].

Section 4. Counting Circles in a Packing

In order to investigate the diophantine properties of a packing P we need to count the circles in P. There are at least two useful ways to order the circles:

(α) By the size of the curvature, let

$$N_P(x) := |\{C \in P : a(C) \leq x\}| .$$

(β) Combinatorially by the generation $w(c)$. There are 4.3^{n-1} circles at generation n; what is their typical curvature?

The answer to these lie in noncommutative harmonic analysis. As to the first, let $\delta(P)$ be the exponent of convergence of the series

$$\sum_{C \in P} r(C)^s . \tag{15}$$

That is for $s > \delta$ the series converges while for $s < \delta$ it diverges. Clearly δ is at most 2 since $\pi \sum_{C \in P} r(C)^2$ is finite (it is the area of the circle enclosing P). On the other hand $\sum_{C \in P} r(C)$ is infinite, see [We] for an elegant proof and hence $1 \leq \delta(P) \leq 2$. Also δ doesn't depend on P since any two packings are equivalent by a Mobius transformation, that is a motion of the complex plane $\mathbb{C}$ by a conformal (angle preserving) transformation $z \to (\alpha z + \beta)(\gamma z + \delta)^{-1}$, $\alpha\delta - \beta\gamma = 1$. So $\delta = \delta(A)$ is an invariant of the Apollonian group and it is known to have many equivalent definitions. It can be estimated, [Mc] gives $\delta = 1.30568\ldots$. Using elementary methods Boyd [Bo] shows that

$$\lim_{x \to \infty} \frac{\log N_P(x)}{\log x} = \delta . \tag{16}$$

Very recently Kontorovich and Oh [K-O] have determined the asymptotics for $N_P(x)$. Their method uses ergodic properties of flows on $A \backslash O_F(\mathbb{R})$ and in particular the Lax-Phillips spectral theory for the Laplacian on the infinite volume hyperbolic three manifold $X = A \backslash O_F(\mathbb{R})/K$ where K is a maximal compact subgroup of $O_F(\mathbb{R})$, as well as the Patterson-Sullivan theory for the base eigenfunction on X.

Theorem [K-O]: *There is a positive, $b = b(P)$ such that*

$$N_P(x) \sim bx^\delta \, , \text{ as } x \to \infty \, .$$

Numerical calculations [F-S] indicate that $b(P_0) = 0.0458\ldots$.

As far as (β) goes the theory of random products of matrices, in this case $S_{j1}S_{j2}\cdots S_{jm}$ with $j_k \in \{1,2,3,4\}$ and $j_k \neq j_{k+1}$ for any k, of Furstenberg and in particular the positivity of the Lyapunov exponent γ associated with such products, dictates the distribution of the numbers $\log a(C)$ with $w(C) = m$. In fact there is a central limit theorem [L] which asserts that this distribution has mean γm and variance of size $\sqrt{m}$ as m tends to infinity. Here $\gamma = \gamma(A) \cong 0.9149\ldots$ according to Fuchs [F] who has done some numerical simulation.

Section 5. Diophantine Analysis

Which integers occur as curvatures of circles C in an integral packing P? According to the theorem in the last section, the number of $a(C)$'s less than x with $C \in P$ (counted with multiplicities) is about x^δ, hence one might expect that a positive proportion of all numbers occur as curvatures. This was conjectured in [G-L-M-W-Y]. An approach to this conjecture using the subgroups B_1, B_2, B_3, B_4 with $B_1 = \langle S_2, S_3, S_4\rangle$ etc., was introduced in [Sa1]. The point is that unlike A, if H_j is the Zariski closure of B_j then B_j is an arithmetic subgroup of $H_j(\mathbb{R})$. In this way the study of the integer orbits of Bj falls under the realms of the arithmetic theory of quadratic forms. In particular one finds that among the curvatures are the set of values at integers of various inhomogeneous binary quadratic forms. Very recently Bourgain and Fuchs have shown that the different forms are highly uncorrelated at certain scales and as a consequence they establish;

Theorem ([B -F]):

The positive density conjecture is true, that is the set of curvatures in an integral Apollonian packing has positive density in all the positive integers.

A much more ambitious conjecture about the set of numbers which are curvatures is that it should satisfy as local to global principle. According to the asymptotics for $N_P(x)$, we have that on average a large integer n is hit about $n^{\delta-1}$ times. So if n is large, one might hope that n is in fact hit unless there is some obvious reason that it shouldn't be. The obvious reason is that the n's that are curvatures satisfy congruence conditions and these can be studied in detail.

It is here that A being Zariski dense in O_F is relevant. There are general theorems ([M-V-W]) which assert that for groups such as A and $q \geq 1$ an integer having its prime factors outside a finite set of primes $S = S(A)$, the reduction of $A \bmod q$ in $\mathrm{GL}_4(\mathbb{Z}/q\mathbb{Z})$ is the same as that of $O_F(\mathbb{Z}) \bmod q$. While the description of the last is still a bit complicated because orthogonal groups don't quite satisfy strong approximation (see[Ca], one needs to pass to the spin double cover), it is nevertheless well understood. For the Apollonian group

A, Fuchs ([F]) has determined the precise image of A in $\mathrm{GL}_4(\mathbb{Z}/q\mathbb{Z})$ for every q. In particular the "ramified" set $S(A)$ consists only of 2 and 3.

From her characterization one obtains the following important product structure for the reduced orbits ([F-S]). Let $\mathcal{O}_a(q)$ be the reduction of $\mathcal{O}_a$ into $(\mathbb{Z}/q\mathbb{Z})^4$. If $q = q_1 q_2$ with $(q_1, q_2) = 1$ then $\mathcal{O}_a(q) = \mathcal{O}_a(q_1) \times \mathcal{O}_a(q_2)$ as subsets of $(\mathbb{Z}/q\mathbb{Z})^4 = (\mathbb{Z}/q_1\mathbb{Z})^4 \times (\mathbb{Z}/q_2\mathbb{Z})^4$ (the latter identification coming from the Chinese remainder theorem). Moreover for $p \geq 5$ a prime and $e \geq 1$ $\mathcal{O}_a(p^e) = V(\mathbb{Z}/p^e\mathbb{Z})/\{0\}$, that is the nonzero points on the Descarte cone in arithmetic mod p^e. For $p = 2$ and 3 this is not true but the description of $\mathcal{O}_a(p^e)$ stabilizes at $e = 8$ for $p = 2$ and at $e = 1$ for $p = 3$. With this the detailed information about the orbits mod q it is a simple matter to determine the exact congruence conditions that the curvatures in an integral packing must satisfy.

For example for the packing P_0, the reader might have noticed that

$$a(C) \equiv 0, 4, 12, 13, 16, 21 \pmod{24} \tag{17}$$

and this is the only congruence restriction. The local to global conjecture for P_0 (and a similar conjecture applies to any integral P) is then

Local to Global Conjecture ([G-L-M-W-Y], [F-S]):

Except for finitely many $m \geq 1$, every m satisfying (17) is the curvature of some $C \in P_0$.

If the conjecture is true and is proven effectively then one would have a completely satisfactory description of the set of curvatures. Fuchs and Sanden [F-S] have made a detailed numerical study of this local to global conjecture. For P_0 they list the $N_{P_0}(10^8)$ circles with curvatures at most 10^8 and they examine those with $10^7 \leq a(C) < 10^8$ grouping them into each of the six allowed progressions mod 24. For each progression the distribution of the frequencies with which the numbers are hit is calculated. The means of these distributions can be determined asymptotically using [Fu] and [K-O] and it is smallest for $m \equiv 0(24)$ and largest for $m \equiv 21(24)$, in fact the latter is double the former. The results of these calculations for these two progressions are displayed by the histograms in Figures 13 and 14. The number of exceptions that is numbers in $[10^7, 10^8)$ which satisfy the congruence but are not curvatures (which is the frequency of 0 in the histogram) is still sizable for $m \equiv 0(24)$ while for $m \equiv 21(24)$ it is the single number 11459805. The reason for the difference is that the mean for $m \equiv 0(24)$ is still quite small at 12.41 while for $m \equiv 21(24)$ it is 24.86. As x increases the means in each progression will be of order $x^{\delta-1}$ and the frequency of 0 will drop. The local to global principle asserts that from some point on this frequency count doesn't change and it appears to be quite plausible. From this data one might reasonably venture that every $m \equiv 21(24)$ bigger than 11459805 is a curvature of a circle in P_0.

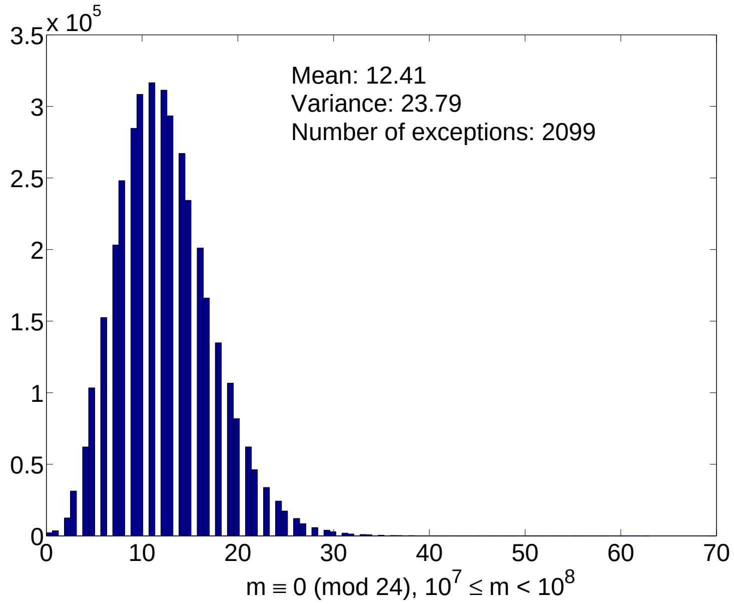

Figure 13

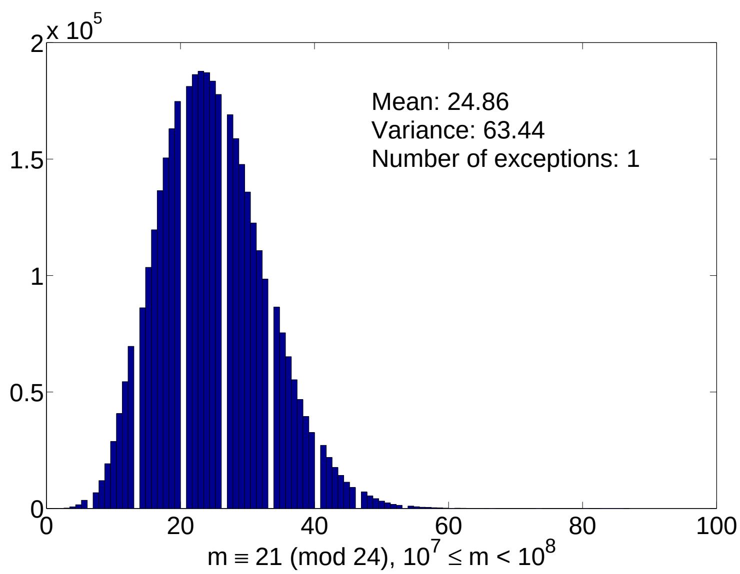

Figure 14

To put this local to global conjecture in perspective consider the same problem for $O_F(\mathbb{Z})$, rather than for A. That is the question of which numbers a are coordinates of points $x \in V^{\mathsf{prim}}(\mathbb{Z})$. For a given a this is a question of representing an integer by a ternary quadratic form. For the form at hand this is not a difficult problem (every a occurs) but if one were to change the form F the resulting form in three variables would be quite general. That is one is facing the question of a local to global principle (except for finitely many exceptions) for ternary quadratic forms. This is the most difficult case of Hilbert's 11^{th} problem and it has only recently been settled in general (see [D-SP], [Co]). Even there the solution is ineffective and the local to global principle needs to be modified beyond the naive congruence obstructions; there being obstructions coming from the spin group ([Ca pp 250]). Given this one should be cautious about a local to global conjecture in the context of the small group A, but my guess is something like this is true and to me this problem is a fundamental and attractive one.

One can ask which pairs of positive integers are curvatures of mutually tangent circles C_1, C_2 in P_0 (here and below the pairs C_1, C_2 are unordered). Again there are some congruence obstructions but this time there cannot be a stable (i.e. except for finitely many exceptions) local to global principle. The reason is that such pairs of circles are too sparse: Let

$$N_P^{(2)}(x) \,=\, |\{C_1, C_2 \in P | C_1 \text{ is tangent to } C_2 \text{ and } a(C_1) \leq a(C_2) \leq x\}|\,.$$

In generating the packing, a circle placed at generation $n > 1$ is tangent to exactly three circles from previous generations and its radius is no bigger than any of these three. From this it follows that

$$N_P^{(2)}(x) \,=\, 3\, N_P(x) \tag{18}$$

and this is too small to accommodate even infinitely many local congruence obstructions.

Section 6. Primes

If you are drawn to primes then on looking at Figure 6 you might have asked if there are infinitely many circles whose curvatures are prime? Are there infinitely many "twin primes", that is pairs of tangent circles both of whose curvatures are prime. The pair near the middle with curvatures 157 and 397 is such a twin. If these sets are infinite then can one count them asymptotically; is there a "prime number theorem"?

Theorem ([Sa1]):

In any primitive integral Apollonian packing there are infinitely many twin primes and in particular infinitely many circles whose curvatures are prime. In fact the set of points x in an orbit $\mathcal{O}_a = aA$ of a primitive integral point $a \in V^{\mathsf{prim}}(\mathbb{Z})$, for which at least two of x's coordinates are prime, is Zariski dense in V.

For congruence reasons (even-odd) P_0 contains no prime triples, that is circles C, C', C'' mutually tangent all of whose curvatures are prime. The proof of the above theorem uses the arithmetic groups B_1, B_2, B_3 and B_4 of A to place the problem in the ballpark of more standard problems concerning primes. Eventually the half dimensional sieve ([Iw]) is what is used to produce primes.

We turn to counting these primes and twin primes. What makes this feasible is the affine linear sieve introduced recently in [B-G-S]. This sieve applies to orbits of groups such as the Apollonian group and it achieves in this context roughly what the sieves of Brun and Selberg do in the classical setting of the integers. Let

$$\Pi_P(x) \,=\, |\{C \in P : \, a(C) \,\leq\, x, \, a(C)\text{prime}\}|$$

and the closely related weighted count

$$\psi_P(x) = \sum_{\substack{C\in P \\ a(C)\ \text{prime} \\ a(C)\leq x}} \log\, a(C)\,.$$

For twin primes set

$$\Pi_P^{(2)}\,(x) \,=\, |\{C, C' \in P|\, a(C) \,\leq\, a(C') \,\leq\, x, a(C), a(C')\ \text{prime}, C\ \text{tangent to}\ C'\}|$$

and the corresponding weighted count

$$\psi_P^{(2)}\,(x) = \sum_{\substack{C,C'\in P \\ a(C)\leq a(C')\leq x \\ a(C),a(C')\ \text{prime} \\ C\ \text{tangent to}\ C'}} \log a(C)\, \log a(C')\,.$$

The asymptotics of $\Pi_P(x)$ and $\psi_P(x)$ are related on summing by parts: $\Pi_P(x) \sim \psi_P(x)/\log x$ as $x \to \infty$. For $\Pi_P^{(2)}(x)$ and $\psi_P^{(2)}(x)$ the relation is less clear ($\gamma_1 \leq \frac{\Pi_P^{(2)}(x)(\log x)^2}{\psi_P^{(2)}(x)} \leq \gamma_2$ for $0 < \gamma_1 < \gamma_2 < \infty$ constants) and it is more natural to consider the weighted sum.

Using the affine sieve and standard heuristics concerning the randomness of the Mobius function $\mu(n)$ and a nontrivial calculation, [F-S] formulate a precise "prime number conjecture":

Conjecture ([F-S]): For any primitive integral packing P, as $x \to \infty$

$$\frac{\psi_P(x)}{N_P(x)} \longrightarrow L(2, \chi_4)$$

and

$$\frac{\psi_P^{(2)}(x)}{N_P^{(2)}(x)} \longrightarrow \beta\,,$$

where the numbers $L(2,\chi_4)$ and β are

$$L(2,\chi_4) \;=\; \prod_{p\equiv 1(4)} (1-p^{-2})^{-1} \cdot \prod_{p\equiv 3(4)} (1+p^{-2})^{-1} \;=\; 0.9159\ldots$$

$$\beta \;=\; \frac{2}{3} \cdot \prod_{p\equiv 1(4)} (1-p^{-2})^{-2} \cdot \prod_{p\equiv 3(4)} (1+p^{-2})^{-2} \cdot (1-2p(p-1)^{-2}) \;=\; 0.460\ldots$$

These numbers come from a detailed examination of the set $\mathcal{O}_a(q)$ and certain algebraically defined subsets therof which eventually leads to the product of the local densities over primes. It is a pleasant and unexpected feature that the prime and twin prime constants above don't depend on the packing P. A numerical check of these conjectures for P_0 with x up to 10^8 is given in the graphs in Figures 15 and 16. The graph in Figure 15 is that of $\psi_P(x)/N_P(x)$ against x and in Figure 16 of $\psi_P^{(2)}(x)/N_P^{(2)}(x)$. The agreement with the conjecture is good.

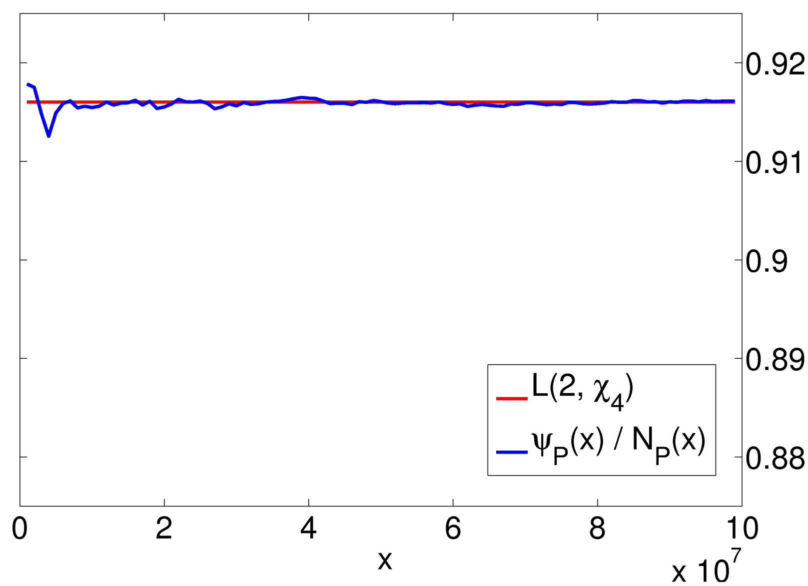

Figure 15

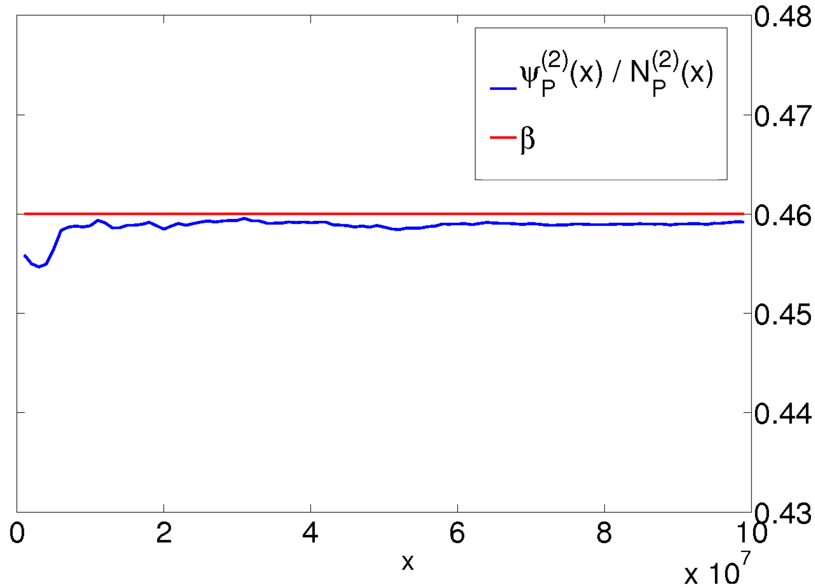

Figure 16

As with the classical sieve the affine linear sieve can be used to prove upper bounds which are of the "true" order of magnitude.

Theorem ([K-O])

$$\Pi_P(x) \underset{P}{\ll} \frac{N_P(x)}{\log x};$$

and

$$\Pi_P^{(2)}(x) \underset{P}{\ll} \frac{N_P^{(2)}(x)}{(\log x)^2}.$$

Note that from the refined asymptotics of [K-O] mentioned in Section 4, it follows that

$$\sum_{C \in P} a(C)^{-\delta} = \infty .$$

According to the conjectured "prime number theorem" above we should have that

$$\sum_{\substack{C \in P \\ a(C) \text{ prime}}} a(C)^{-\delta} = \infty.$$

On the other hand the upper bound above for twins implies that

$$\sum_{\substack{C,C'\in P \\ a(C)\leq a(C') \\ C,C' \text{ twin primes}}} a(C')^{-\delta} < \infty\,.$$

This is the analogue of Brun's Theorem for the usual twin primes, that the sum of their reciprocals converges.

There are many ingredients that go into the affine linear sieve and we end by mentioning one of them. For $q \geq 1$ the reduced orbit $\mathcal{O}_a(q)$ can be made into a 4-regular connected graph by joining ξ in $\mathcal{O}_a(q)$ to ξS_j for $j = 1, 2, 3, 4$. The key property proved in [B-G-S] is that for q-square-free these graphs are an expander family as $q \longrightarrow \infty$, (see [Sa2], and [H-L-W] for a definition and properties). This ensures that the random walk on $\mathcal{O}_a(q)$ gotten by moving with one of each S_j at each step is rapidly uniformly mixing and this is a critical ingredient in controlling remainder terms in the affine sieve.

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