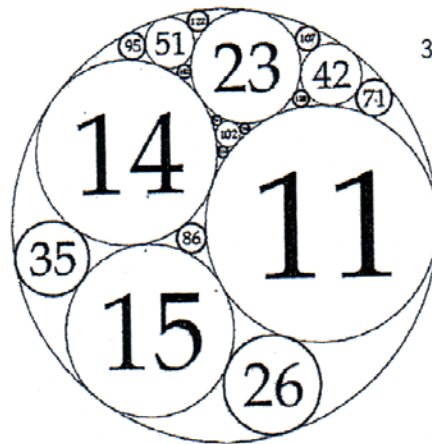# Letter to J. Lagarias about Integral Apollonian Packings
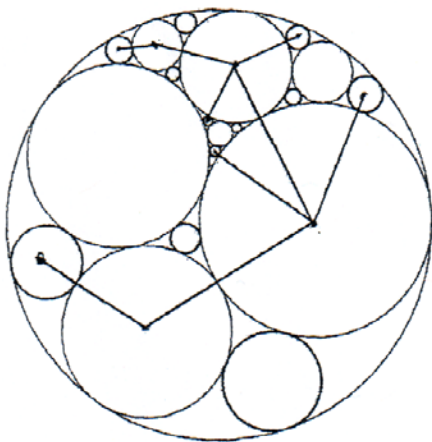
June, 2007

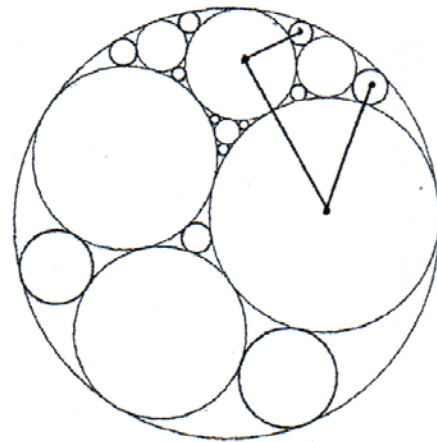from

Peter Sarnak

3rd generation of the Apollonian packing corresponding to the quadruple (-6,11,14,23)

Odd Subgraph O

Prime Subgraph P

Dear Jeff,

Thanks for your letter and for pointing out to me this lovely number theoretical side of Apollonian packings. I looked in some detail at your paper "Apollonian circle packings; number theory" *Journal of number theory*, **100**, (2003), 1-45 with Graham, Mallows Wilks and Yan. Here are some comments and answers to some of the open questions raised at the end of the paper, as well as my take on the diophantine properties of such packings.

Denote by F the Descartes form

$$F(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2 \tag{1}$$

and by O_F the corresponding orthogonal group of matrices preserving F. Let $A \leq O_F(\mathbb{Z})$ be your Apollonian packing group, that is the group generated by S_1, S_2, S_3, S_4 where the S_j's are the reflections

$$S_1 = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix} \tag{2}$$

Given four disjoint mutually tangent circles in the plane whose curvatures are $\xi = (a, b, c, d)^t$, $a, b, c, d \in \mathbb{Z}$, the set of curvatures of all four mutually tangent circles in the integral Apollonian packing determined by ξ, coincides with the orbit $A\xi \subset \mathbb{Z}^4 \cdot A\xi$ is contained in the affine cone C given by

$$C: \ F(x) = 0. \tag{3}$$

We restrict to ξ primitive (i.e. $(a, b, c, d) = 1$) which is preserved by A and yields a primitive integral Apollonian packing.

The salient features of A which are implicit in your paper are

(i) On the one hand A is small being of infinite index in $O_F(\mathbb{Z})$ and its limit set as a subgroup of $O_F(\mathbb{R}) = O_{\mathbb{R}}(3, 1)$ acting on hyperbolic 3-space $\mathbb{H}^3$, has Hausdorff dimension $\alpha = 1.30\ldots$ (This limit set is equal to the complement in the plane of the union of all the open disks of the packing). On the other hand A is Zariski dense in O_F.

(ii) A contains subgroups which are lattices in $O_{\mathbb{R}}(2, 1)$'s, that is arithmetic Fuchsian groups for which the theory of integral quadratic forms can be applied. These subgroups are conjugates of $A_1 = \langle S_2, S_3, S_4 \rangle$, $A_2 = \langle S_1, S_3, S_4 \rangle$, $A_3 = \langle S_1, S_2, S_4 \rangle$, $A_4 = \langle S_1, S_2, S_3 \rangle$.

(iii) A contains rank 1 unipotent subgroups (but not of rank 2). For example the element $S_2 S_1$ exploited in your proof of Theorem 6.2 is such an element.

The first problem that your paper addresses is the determination and counting of the number of distinct integral Apollonian packings. This is done with your root quadruples

which are the "smallest" members of a given integral packing $\mathcal{O} = L\xi$. The second problem concerns the patterns of integer curvatures (x_1, x_2, x_3, x_4) in a given packing $\mathcal{O}$.

The following refer to the open questions at the end of your paper.

Q.3: This asks for the number $N_r(T)$ of root quadruples of height at most T. That is the number of (a, b, c, d) with $F(a, b, c, d) = 0$ and satisfying

$$\begin{cases} a \leq 0 \leq b \leq c \leq d \\ a + b + c + d > 0 \\ a + b + c \geq d \\ a^2 + b^2 + c^2 + d^2 \leq T^2 \,. \end{cases} \tag{4}$$

Proceeding as you do in your Theorem 2.2, let $(w, x, y, z)^t = J_0(a, b, c, d)^t$ with

$$J_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Then (4) becomes that of counting integers satisfying

$$w^2 = x^2 + y^2 + z^2$$

$$0 < w \leq T/\sqrt{2}$$

$$1 + \tfrac{x}{w} + \tfrac{y}{w} + \tfrac{z}{w} \leq 0 \leq 1 + \tfrac{x}{w} - \tfrac{y}{w} - \tfrac{z}{w} \leq 1 - \tfrac{x}{w} + \tfrac{y}{w} - \tfrac{z}{w}$$

$$\leq 1 - \tfrac{x}{w} - \tfrac{y}{w} + \tfrac{z}{w}$$

and

$$1 + \frac{x}{w} + \frac{y}{w} - \frac{z}{w} \geq 0 \,.$$

Hence

$$N_r(T) = \sum_{0 < w < \frac{T}{\sqrt{2}}} \sum_{\substack{x^2 + y^2 + z^2 = w^2 \\ (\frac{x}{w}, \frac{y}{w}, \frac{z}{w}) \in \mathcal{F}}} 1$$

where $\mathcal{F}$ is the subset of S^2 cut out by

$$(\xi, \eta, \zeta) \,, \; \xi^2 + \eta^2 + \zeta^2 = 1$$

$$1 + \xi + \eta + \zeta \leq 0 \leq 1 + \xi - \eta - \zeta \leq 1 - \xi + \eta - \zeta \leq 1 - \xi - \eta + \zeta$$

$$1 + \xi + \eta - \zeta \geq 0.$$

Note that $\mathcal{F}$ has non-empty interior (for example $\left(\frac{-12}{17}, \frac{-9}{17}, \frac{-8}{17}\right) \in \mathcal{F}^0$). Apply Duke's equidistribution theorem (*Invent.*, **92**, (1988), 73-90) to the inner sum to get

$$N_r(T) = \sum_{0 < w < \frac{T}{\sqrt{2}}} \left(\frac{\mathsf{area}(\mathcal{F})}{4\pi} \sum_{x^2 + y^2 + z^2 = w^2} 1 + \circ(w^2) \right). \tag{5}$$

Hence,

$$N_r(T) \sim \frac{\mathsf{area}(\mathcal{F})}{4\pi} c_1 T^2 \tag{6}$$

where c_1 is the constant in your (2.9).

Concerning the counting function $N_A(T)$ of the number of elements of A with Frobenuis norm at most T (your Theorem 5.3). As suggested by Gamburd, one can use the Lax-Phillips result (*Journal Funct. Anal.*, **46**, (1982), 280-350) to give an asymptotic of the form

$$N_A(T) = cT^\alpha + O(T^{\alpha - \delta}) \tag{7}$$

for some $\delta > 0$.

Here c is given in terms of the base eigenfunction ϕ_0 of the Laplacian on the quotient $A \backslash \mathbb{H}^3$, where $\mathbb{H}^3 = SO_F(\mathbb{R})/SO_3(\mathbb{R})$. Given the form of (7) it is hard to imagine proving it without using the spectral theory of $A \backslash \mathbb{H}^3$.

The spectrum consists of numbers $0 < \lambda_0 < \lambda_1 \leq \lambda_r < 1$ and a continuous spectrum $[1, \infty)$ (Here $\lambda_0 = \alpha(2 - \alpha)$). I expect, and it would be nice to verify, that $\lambda_1(A \backslash \mathbb{H}^3) \geq 1$, that is there are no exceptional eigenvalues for A. The Lax-Phillips Theorem applies to counting the images in $\mathbb{H}^3$ under A of a point $w \in \mathbb{H}^3$ in a ball of radius R about $z \in \mathbb{H}^3$. With appropriate choices of w and z this is the same as $N_A(T)$ with T an explicit function of R. The remainder term in (7) depends on the gap between λ_1 and λ_0.

When doing a similar count but with congruence conditions on elements of A (such as in needed in the affine sieve for A in Bourgain-Gamburd-Sarnak [B-G-S 2007]), one needs a uniform spectral gap. Let $A(q) = \{\gamma \in A : \gamma \equiv I \mod q\}$ be the principal "congruence" subgroup of A of level q. Let $\lambda_0(q) = \lambda_0(A) < \lambda_1(q) \ldots \leq \lambda_{r_q}(q) < 1$ be its discrete spectrum. In [B-G-S] it shown that there is a fixed $\epsilon_0 > 0$ such that

$$\lambda_1(q) \geq \lambda_0(A) + \epsilon_0$$

$$(7')$$

for all square free $q \geq 1$.

This infinite volume spectral theory should also allow one to derive an asymptotics as in (7) for the counting function $N_{\mathcal{O}}(T)$ of the number of circles of curvature at most T in an Apollonian packing $\mathcal{O}$ (as in your Theorem 5.2). This requires developing the spectral counting method for an orbit of A on the cone C instead of the hyperboloid $\mathbb{H}^3$. One difficulty here is that the stabilizer in A of (a, b, c, d) on the cone is either a rank 0 or rank 1 parabolic subgroup (while in $O_F(\mathbb{Z})$ it is of rank 2). In the similar situation of infinite area quotients of $SO_{\mathbb{R}}(2, 1)$ this analysis on a cone is carried out in A. Kontorovich's thesis-Columbia 2007.

Q. 4: You ask for the biggest $\beta(\beta \leq 1)$ such that the number of *distinct* curvature of size at most T, in a given integral packing $\mathcal{O}$, is at least $T^{\beta + o(1)}$ as $T \longrightarrow \infty$. The following shows that $\beta = 1$.

Let $\xi = (a, b, c, d)$ be an element in $\mathcal{O}$ with $a \neq 0$ which we fix for the time being. Consider the curvatures that are gotten from the orbit $A_1 \xi$. All these points (a, x_2, x_3, x_4), are in the conic section of C by $x_1 = a$ and hence satisfy $F(a, x_2, x_3, x_4) = 0$. The point is that $A_1 \xi$ consists of a positive fraction of all the integral points (x_2, x_3, x_4) satisfying $F(a, x_2, x_3, x_4) = 0$. In terms of the action of A_1 in the plane, given the configuration of circles corresponding to ξ, the group A_1 is the Schottky group generated by the reflections in the three circles perpendicular to the circles corresponding to $\{a, x_2, x_4\}$, $\{a, x_3, x_4\}$ and $\{a, x_2, x_3\}$ respectively. This action preserves the circle corresponding to a and acts discontinuously on its interior and exterior. A fundamental domain for this action being a triangle bounded by circles perpendicular to the a circle and with angles $(0, 0, 0)$ at the vertices. That is A_1 acts as a Fuchsian triangle group with hyperbolic area equal to π. In order to examine the diophantine properties of $A_1 \xi$ we proceed as you do in the proof of your Theorem 4.2.

$$
\begin{aligned}
F(a, x_2, x_3, x_4) &= 2(a^2 + x_2^2 + x_3^2 + x_4^2) - (a + x_2 + x_3 + x_4)^2 \\
&= g(y) + 4a^2
\end{aligned}
$$

$$(8)$$

where

$$y = (y_2, y_3, y_4) = (x_2, x_3, x_4) + (a, a, a) \tag{9}$$

and

$$g(y) = y_2^2 + y_3^2 + y_4^2 - 2y_2 y_3 - 2y_2 y_4 - 2y_3 y_4. \tag{10}$$

Moreover the affine action of A_1 on the affine variables (x_2, x_3, x_4) is conjugated via (9) to a linear action Γ (independent of a) preserving g. That is $\Gamma \leq O_g(\mathbb{Z})$ and it is of small index in the latter since $Vol(\Gamma \backslash O_g(\mathbb{R})) = \pi$ with the hyperbolic metric normalization. By inspection Γ is generated by the reflections

$$
\begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix}. \tag{11}
$$

Moreover since $\xi = (a, b, c, d)^t$ is primitive $y^{(\xi)} = (b + a, c + a, d + a)$ is a primitive point on the quadric

$$g(y) = -4a^2 \tag{12}$$

The orbit $A_1 \xi^t$ is equal to

$$(a, \Gamma y^{(\xi)} - (a, a, a))^t . \tag{13}$$

To further examine the values that y_j assumes for $y \in \Gamma y^{(\xi)}$, it is convenient to change variables and bring the action to the more familiar one that you use in Theorem 4.2.

Let

$$y_2 = A, \ y_3 = A + C - 2B, \ y_4 = C \tag{14}$$

or

$$A = y_2, C = y_4, B = (y_4 + y_2 - y_3)/2 . \tag{15}$$

Note that for y integral and satisfying (12), B in (15) will be in $\mathbb{Z}$. Hence the change of variables (14), (15) preserves integrality as well as primitivity. The form g becomes the familiar one

$$4(B^2 - AC) = 4 \triangle (A, B, C) \tag{16}$$

and (12) becomes

$$\triangle (A, B, C) = -a^2 . \tag{17}$$

The action of Γ is conjugated to a subgroup $\widetilde{\Gamma}$ of $O_\triangle(\mathbb{Z})$ and it is generated by the reflections

$$\begin{bmatrix} 1 & -4 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} . \tag{18}$$

As was well known to Gauss, the spin double cover of $SO_\triangle(\mathbb{Z})$ is realized as the image of $GL_2(\mathbb{Z})$ and is realized as the image of $GL_2(\mathbb{Z})$ under the homomorphism

$$\rho : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longrightarrow \frac{1}{(\alpha\delta - \beta\delta)} \begin{bmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \delta^2 \end{bmatrix} \tag{19}$$

with kernel $\pm I$.

Hence $\rho^{-1}(SO_\triangle(\mathbb{Z}) \cap \widetilde{\Gamma})$ contains $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and hence contains the principal congruence subgroup $\Lambda(2)$ of $SL_2(\mathbb{Z})$. In fact $\rho^{-1}(SO_\triangle(\mathbb{Z}) \cap \widetilde{\Gamma}) = \Lambda(2)$ since the words of

even length in the generators (18) are in $SO_\triangle(\mathbb{Z})$ and hence the area of the quotient of $\mathbb{H}$ by this group is 2π (according to our previous remark that the group $\widetilde{\Gamma}$ is a Schottky group with fundamental domain of area π) which is the area of $\Lambda(2)\backslash\mathbb{H}$. We are interested in the values assumed by one of A, C or $A+C-2B$ as (A, B, C) varies over the orbit $\widetilde{\Gamma}(A_0, B_0, C_0)^t$. Using this extra choice of any one of the three, it is easy to see that these values are at least those of A where (A, B, C) ranges over the slightly bigger orbit $\rho(GL_2(\mathbb{Z}))\,(A_0, B_0, C_0)^t$. Hence from (19) we conclude that;

The set of values assumed by one of A, C or $A+C-2B$ as (A, B, C) varies over the orbit $\widetilde{\Gamma}(A_0, B_0, C_0)^t$ contains the set of primitive values of the binary quadratic form;

$$\phi_{A_0,B_0,C_0}\,(\xi, \eta) \;=\; A_0\xi^2 \;+\; 2B_0\,\xi\eta \;+\; C_0\eta^2 \;\; \text{with} \;\; (\xi, \eta), = 1\,. \tag{20}$$

Thus the set of values assumed by one of x_2, x_3, x_4 where $x = (a, x_2, x_3, x_4)$ varies over the orbit $A_1(a, b, c, d)^t$ contains the set of numbers

$$\phi_{A_0,B_0,C_0}\,(\xi, \eta) \;-\; a \;\; \text{with} \;\; (\xi, \eta) \;=\; 1\,,$$

where

$$A_0 = b + a,\; C_0 = d + a\,,\; B_0 \;=\; \frac{a + d - c}{2}\,. \tag{21}$$

As is well known (probably first due to Landau) the number of distinct primitive values assumed by such a ϕ_{A_0,B_0,C_0} (note that $B_0^2 - A_0C_0 = -a^2 < 0$ so is not a square) in the interval $[-T, T]$ is at least $c_1 T/\sqrt{\log T}$ as $T \longrightarrow \infty$ (here $c_1 > 0$ and depends on ϕ_{A_0,B_0,C_0}). Afortiori the number of distinct integers of size at most T which are the coordinates of some $x \in A\xi^t$ is at least $c_1 T/\sqrt{\log T}$ and hence $\theta = 1$ in your question 3.

This just falls short of your positive density conjecture (page 25). To prove that one needs to mix the actions of A_1, A_2, A_3 and A_4 more seriously and needs a further idea. Your strong density conjecture (page 31) asserting that for a given integral Apollonian packing, the set of curvatures that is achieved is what is predicted by congruence obstructions, possibly with finitely many exceptions, lies much deeper and is a very appealing problem!

Q.1: This concerns the congruences satisfied by the integers in an orbit $\mathcal{O} = A\xi^t$.

It is natural to first determine the images of A under reduction into $O_F(\mathbb{Z}/q\mathbb{Z})$, for $q \geq 1$ an integer. For this it is convenient to pass to the spin simply connected double cover of SO_F, or what is essentially the same thing, to work with the spinor norm 1 subgroup of SO_F and in particular the intersection of A with the spin norm 1 group in $SO_F(\mathbb{Z}_p)$ (here $\mathbb{Z}_p$ denotes the p-adic integers). All this is discussed in detail in Cassel's book "Rational Quadratic Forms". Once these images are known explicitly their orbits on the cone $C(\mathbb{Z}_p)$ are easy to determine. Since A is Zariski dense in O_F it follows from the general theorems of Matthews-Vaserstein

and Weisfeiler (*Proc. London Math. Soc.*, **48**, (1984), 514-532) that there is a finite set of primes $S = S(A)$ (which can be determined effectively and your Theorem 6.2 indicates that $S(A) = \{2, 3, 5\}$) such that for $p \notin S$ the image of A is dense in the spinor norm 1 subgroup of $SO_F(\mathbb{Z}_p)$ and is dense in $SO_F(\mathbb{Z}_p)$ iff the S_iS_j's generate the finite group $SO_F(\mathbb{Z}_p)/(\text{Spin}$ norm 1 $(\mathbb{Z}_p))$. The corresponding images of A in $SO_F(\mathbb{Z}/q\mathbb{Z})$, for q having its prime factors outside S, are the corresponding product groups over the primes dividing q. At the "ramified" primes $p \in S$ a further analysis needs to be done in order to see the stabilization. In any case, once one has the above information the corresponding orbits on the cones $C(\mathbb{Z}_p)$ and $C(\mathbb{Z}/q\mathbb{Z})$ can be determined. This analysis needs to be done explicitly and in detail for the group A in order to study the finer diophantine questions connected with integral Apollonian packings such as those about the prime factors of the curvatures which is what I discuss next.

Let $\mathcal{O} = A\xi^t$ be the orbit corresponding to an integral Apollonian packing and let $f \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ be an integral polynomial. Define the saturation number $r(\mathcal{O}, f)$ of the pair of $(\mathcal{O}, f)$ to be the least number r such that $\{x \in \mathcal{O} : f(x)$ has at most r prime factors$\}$ is Zariski dense in the affine cone C. The general saturation theorem of [B-G-S] applies here and asserts that the saturation number is finite. In particular for $f(x) = x_1x_2x_3x_4$ it follows from that paper that there is an $r < \infty$ such that for any Apollonian packing $\mathcal{O}$ the set of mutually tangent quadruples of circles all of whose curvatures have at most r-prime factors is Zariski dense in C. The general local to global conjectures in [B-G-S] will predict an exact value for $r(\mathcal{O}, x_1x_2x_3x_4)$ once the congruence analysis of the last paragraph is carried out. One cannot have all of x_1, x_2, x_3 and x_4 prime. Indeed for a Zariski dense set of such quadruples all the x_j's would then have to be odd. However the Descarte equation $F(x) = 0$ has no such solutions mod 16. Moreover parity considerations together with the primitivity of (x_1, x_2, x_3, x_4) show that for any Descarte quadruple, two of the coordinates are odd and two are even. Thus the best that one can do in terms of creating x's with x_j's prime, is to have two of them prime. Interestingly one can show that the set of such quadruples with two circles prime is Zariski dense in C. Or in a different terminology $r(\mathcal{O}, x_1x_2) = 2$. In particular any integral Apollonian packing has infinitely many circles whose curvatures are prime and also infinitely many "twin primes", i.e. pairs of tangent circles whose curvatures are primes.

To produce such circles in $\mathcal{O} = A\xi^t$ choose ξ_1 in $\mathcal{O}$ with $\xi_1 = (a, b, c, d)$ and $a \neq 0$. According the discussion in Q.4 leading to (21) the numbers x_2, x_3, x_4 occurring in $x^t = (a, x_2, x_3, x_4)^t \in A_1\xi_1^t$ contain the numbers of the form

$$A_0\xi^2 + 2B_0\xi\eta + C_0\eta^2 - a \tag{22}$$

where ξ, η vary through integers with $(\xi, \eta) = 1$ and $(A_0, B_0, C_0) = 1$ and $B_0^2 - A_0C_0 =, -a^2$ are fixed. It follows that $(A_0, 2B_0, C_0, a) = 1$ and we can apply Iwaniec's Theorem (*Acta Arithmetica* XXIV, (1974), 435-459) to conclude that there are infinitely many primes of the form (22). To be precise, Iwaniec proves this without the restriction $(\xi, \eta) = 1$ but one can modify the proof to accommodate this constraint. The number of primes produced whose size is at most T, is at least $C_1T/(\log T)^{3/2}$. It follows that the set of (ξ, η) for which (23) is prime, is Zariski dense in the affine plane (ξ, η). Hence the set of points $(a, x_2, x_3, x_4) \in A_1\xi_1^t$ for which one of x_2, x_3, x_4 is prime, is Zariski dense in the conic section $F(a, x_2, x_3, x_4) = 0$. This

establishes the first part, that there are infinitely many circles in $\mathcal{O}$ whose curvature is prime. To each fixed p of the infinitely many values for which (say) x_2 above is prime, we consider the orbit of $\xi_p = (a, p, c_p, d_p)$ under A_2. As above the points $x^t = (x_1, p, x_2, x_3) \in A_2 \xi_p^t$ with (say) x_1 prime is Zariski dense in the conic section $F(x_1, p, x_2, x_3) = 0$. This produces a Zariski dense set of points x in $\mathcal{O}$ with x_1 and x_2 both prime.

The above argument shows that every circle in an integral packing $\mathcal{O}$ is tangent to infinitely many circles whose curvatures are prime. By the nerve of $\mathcal{O}$, I mean the graph whose vertices are the circles in $\mathcal{O}$ and whose edges correspond to tangency. Consider the subgraph of the nerve whose vertices are the circles with odd curvatures. From the fact that each Descarte quadruple has two even and two odd circles, one sees that this odd subgraph O of the nerve has no cycles. It is easy to see that it is connected and is an infinite tree with every vertex having infinite degree. Now consider the subgroup P of the odd graph whose vertices are circles with prime curvatures.[*] By our arguments above P is an infinite union of its connected components and each component is isomorphic to the odd graph O. It would be interesting to investigate the densities and distributions of these components. The graphs O and P for the packing (-6,11,14,23) are depicted on the cover page. The pictures are copied from David Austin's feature column of the AMS March 2006.

Best regards,

Peter Sarnak

March 19, 2008: gpp.

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[*] We assume that 2 is a curvature in O or for this purpose that 2 is not a prime.