Completely Replicable Functions

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Abstract. We find all completely replicable functions with integer coefficients, tabulate the new ones, and summarize the computations needed.

Monstrous moonshine. To each conjugacy class of cyclic subgroups, $\langle m\rangle$, of the Monster simple group, M, a modular function, $j_{\langle m\rangle}(z)$, was found empirically in [**CN**] for which the q-coefficients (Fourier coefficients) are the values of the trace in the so-called head representations. For the identity subgroup the function is the elliptic modular function $J(z) = j(z) - 744$. Here, and throughout, the computations are simplest to describe if we assume all our q-series to have constant term zero.

Replication. Replication enables us to associate with a formal q-series

$$f = \sum_{i=-1}^{\infty} a_i q^i, \quad a_{-1} = 1, \quad a_0 = 0, \tag{1}$$

$a_i \in \mathbf{C}$, certain functions of the same form, called the replicates of f. Although f is a formal q-series, it is useful to write $f = f(z)$, where $q = e^{2\pi i z}$, consistent with the properties of modular functions. We tacitly omit describing the Galois action [**N**], which is trivial when the q-series coefficients are rational integers.

The prototypical replication relation is that between the monstrous moonshine function $j_{\langle m\rangle}(z)$ for $\langle m\rangle \subset M$ and its p^{th} replicate $j_{\langle m^p\rangle}(z)$ for $\langle m^p\rangle$. Conway and Norton [**CN**] note that monstrous moonshine functions satisfy identities involving f and its replicates which they call *replication formulae*. A replicable function is a function with a q-expansion of the form (1) for which replicates exist. Such functions also satisfy the replication formulae.

Norton [**N**] has conjectured that a function $q^{-1} + \sum_{i=1}^{\infty} a_i q^i$, $a_i \in \mathbf{Z}, i \geq 1$, is replicable if and only if either $a_i = 0$ for all $i > 1$ or it is the canonical Hauptmodul for a group of genus zero, containing $\Gamma_0(N)$ for some N and containing $z \to z + k$ precisely when k is an integer.

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Hecke Operators. Motivation for introducing the twisted Hecke operator $\widehat{T}_n$ derives from the action of the standard Hecke operator, T_n, on $J(z) = j(z) - 744$, given by

$$J|T_n = \frac{1}{n} \sum_{\substack{ad=n \\ 0 \le b < d}} J((az+b)/d) = P_n(J(z))/n \tag{2}$$

which value is a polynomial in J since the sum is invariant under the modular group and J is a Hauptmodul holomorphic in the upper half-plane. Note that P_n is the unique polynomial such that $P_n(J(z))$ has a q-expansion q^{-n} mod $q\mathbf{Z}[q]$.

We introduce a twisted Hecke operator $\widehat{T}_n$ which, like T_n, acts linearly on q-coefficients yet takes certain functions $f(z)$ to $P_n(f(z))/n$.

More precisely, we call a function f replicable if there are replicate functions $\{f^{(a)}\}$ such that

$$P_n(f(z))/n = \frac{1}{n} \sum_{\substack{ad=n \\ 0 \le b < d}} f^{(a)}((az+b)/d), \tag{3}$$

with $P_n(f(z)) = q^{-n}$ mod $q\mathbf{Z}[q]$, and we define $f|\widehat{T}_n$ to be the right side of (3).

The monic polynomial $P_n(t) \in \mathbf{Z}[a_1, a_2, \dots, a_{n-1}][t]$ is unique and we shall abuse notation by using P_n to denote the polynomial in each case. This definition of $\widehat{T}_n$ is provisional since we have not yet incorporated the Galois action.

Note that $J(z)$ of level $N = 1$ is the sole normalized modular function on which the Hecke operators T_n act as in (2) for all n, since $(N, n) = 1$ for all n. Replicable functions are defined so as to share this property under the action of the twisted Hecke operator $\widehat{T}_n$. In this case, however, the sum involves both f and its replicates.

From Norton [**N**] it follows that

$$\sum_{n=1}^{\infty} \frac{P_n(t)}{n} q^n = -\ln(q(f(z) - t)), \tag{4}$$

and so $P_1(t) = t, P_2(t) = t^2 - 2a_1, P_3(t) = t^3 - 3a_1 t - 3a_2, \dots$.

We define coefficients $\{H_{m,n}\}$ by

$$f|\widehat{T}_n = \tfrac{1}{n} P_n(f(z)) = \tfrac{1}{n} q^{-n} + \sum_{m=1}^{\infty} H_{m,n} q^m, \qquad n \ge 1, \qquad ($$

so that $H_{m,n}$ is the coefficient of q^m in $f|\widehat{T}_n$ and $H_{m,1}$ is the coefficient of q^m in f (denoted H_m by Norton).

We find that $P_n(t)$ satisfies the recurrence relations:

$$P_0(t) = 1, \quad ra_{r-1} + \sum_{k=-1}^{r-2} a_k P_{r-k-1}(t) = tP_{r-1}(t), \quad r = 1, 2, \ldots \tag{6}$$

while $\widehat{H}_{r,s} = (r+s)H_{r,s}$ satisfies

$$\widehat{H}_{r,s} = (r+s)H_{r+s-1} + \sum_{m=1}^{r-1}\sum_{n=1}^{s-1} H_{m+n-1}\widehat{H}_{r-m,s-n}. \tag{7}$$

Norton has another definition of replicability that is somewhat easier to use in practice.

A function f is replicable if $H_{m,n} = H_{r,s}$ whenever $mn = rs$ and $\gcd(m,n) = \gcd(r,s)$.

This is equivalent to the definition given above: assume f is replicable in Norton's sense, then set

$$f^{(k)}(z) = \sum_{i=-1}^{\infty} a_i^{(k)} q^i \tag{8}$$

where

$$a_i^{(k)} = k\sum_{d|k} \mu(d) H_{\frac{k}{d},dki}\ , \ i > 0, \ a_{-1}^{(k)} = 1, \ a_0^{(k)} = 0 \tag{9}$$

and μ is the Möbius function. It follows that $f^{(1)} = f$. For any pair $r, s \in \mathbb{Z}^{>0}$, we find, by Möbius inversion, that

$$H_{r,rs} = \sum_{d|r} \frac{1}{d} a^{(d)}_{r^2 s/d^2} \tag{10}$$

and, since f is replicable under Norton's definition, this implies that

$$H_{m,n} = \sum_{d|(m,n)} \frac{1}{d} a^{(d)}_{mn/d^2} \tag{11}$$

which, from (5), gives (compare Serre [**S**, Chap.VII, §5.3])

$$f|\widehat{T}_n = \frac{1}{n} \sum_{\substack{ad=n \\ 0\le b<d}} f^{(a)}((az+b)/d). \tag{12}$$

Conversely if f has replicates which satisfy (12) it follows that the $H_{m,n}$ of (5) satisfy (11) and so f is replicable as defined by Norton.

When $n = p$, a prime, we see that

$$pf|\widehat{T}_p = f^{(p)}(pz) + \sum_{k=0}^{p-1} f((z+k)/p). \tag{13}$$

In terms of the standard operators U_p and V_p where

$$U_p : a_n q^n \to a_{pn} q^n,$$
$$V_p : a_n q^n \to a_n q^{pn},$$

we have

$$pf|\widehat{T}_p = f|(\Psi^p V_p + pU_p) = P_p(f(z)) \tag{14}$$

where Ψ^p acts as an Adams operator (see Mason [**Mas**]); equivalently we may compute $f^{(p)}$ from

$$f^{(p)}(pz) = P_p(f(z)) - pf|U_p. \tag{15}$$

Complete replicability. A function is completely replicable if it and all its replicates are replicable. One would expect properties of the monstrous moonshine functions to be shared by the completely replicable functions (and they are). At the end we tabulate all non-monstrous completely replicable functions with rational integer coefficients. Complementary monstrous data are found in [**CN**] and [**MS**].

Method of Calculation. To find all completely replicable functions, we computed the larger class of all completely 2-replicable functions. These are functions whose iterated duplicates are replicable. Table 1 of [**N**] contains a list of all completely 2-replicable functions satisfying $f^{(2)} = f$. We call g a replication p^{th} root of f if $g^{(p)} = f$. With a small prime π, the replication square roots of these functions are found by first testing all choices of a_1, a_2, a_3 and a_5 mod 2π for replicability using replication identities and identities derived from them (see [**CN**]). Solutions are then lifted by π-adic approximation using identities up to $H_{145} = H_{5,29}$ so that the solutions found mod 2π, for some prime π, lift uniquely to $2\pi^k, k > 1$.

These calculations require further coefficients which are computed from a_1, a_2, a_3, a_5 and the coefficients of $f^{(2)}$ via the generalized Mahler recurrence relations [**Mah**] (compare [**B**]) derived from:

$$f(\gamma_0 z) + f(\gamma_1 z) + f^{(2)}(\gamma_2 z) = f(z)^2 - 2a_1,$$
$$(f(\gamma_1 z) + f(\gamma_0 z))f^{(2)}(\gamma_2 z) + f(\gamma_0 z)f(\gamma_1 z) = 2a_2 f - f^{(2)} + 2(a_4 - a_1) \tag{16}$$

where

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix};$$

namely, for $k \geq 1$:

$$\begin{aligned}
a_{4k} &= a_{2k+1} + \sum_{j=1}^{k-1} a_j a_{2k-j} + \tfrac{1}{2}(a_k^2 - a_k^{(2)}), \\
a_{4k+1} &= a_{2k+3} + \sum_{j=1}^{k} a_j a_{2k+2-j} + \tfrac{1}{2}(a_{k+1}^2 - a_{k+1}^{(2)}) + \tfrac{1}{2}(a_{2k}^2 + a_{2k}^{(2)}) \\
&\qquad - a_2 a_{2k} + \sum_{j=1}^{k-1} a_j^{(2)} a_{4k-4j} + \sum_{j=1}^{2k-1} (-1)^j a_j a_{4k-j}, \\
a_{4k+2} &= a_{2k+2} + \sum_{j=1}^{k} a_j a_{2k+1-j}, \quad \text{and} \\
a_{4k+3} &= a_{2k+4} + \sum_{j=1}^{k+1} a_j a_{2k+3-j} - \tfrac{1}{2}(a_{2k+1}^2 - a_{2k+1}^{(2)}) \\
&\qquad - a_2 a_{2k+1} + \sum_{j=1}^{k} a_j^{(2)} a_{4k+2-4j} + \sum_{j=1}^{2k} (-1)^j a_j a_{4k+2-j}.
\end{aligned} \tag{17}$$

Replication square roots are repeatedly extracted until functions which have no replication square roots mod 2π are found. In addition the prime power maps are calculated. In each case enough coefficients of the p^{th} replicates of the non-monstrous functions are computed from (15) to reduce the number of candidate functions to at most one. A useful check is given by the congruence:

$$f^{(p)} \equiv f \pmod{p}.$$

Programs in Ford's language ALGEB [**F**] were written from procedures generated by Maple [**M**]. For the functions q^{-1} and $q^{-1} + q$ we found no prime for which the solutions mod 2π lifted uniquely to $2\pi^k$, $k > 1$. The function $q^{-1} - q$ is a root of $q^{-1} + q$ and we have assumed that no other roots of these functions exist. The recursive relations given here, together with the monstrous data in [**CN**] or [**MS**], determine the q-series.

Table. The table contains the initial coefficients a_1, a_2, a_3, and a_5 of 157 non-monstrous, completely replicable functions, which we believe to be the complete set. Each function is described by a number which is its "replication level", together with a small letter identifier; the prime power-maps follow. Capital letter identifiers indicate monstrous functions, for which

ATLAS notation is used as in [CN]. The ghosts [CN] 25Z, 49Z, and 50Z appear here as 25a, 49a, and 50a.

Non-monstrous completely replicable functions

f	Power maps		a_1	a_2	a_3	a_5
1a			1	0	0	0
1b			0	0	0	0
2a	1A		-492	0	-22590	-367400
2b	1a		-1	0	0	0
4a	2A		-76	0	-702	-5224
5a	1A		-6	20	15	0
6a	3A	2a	-33	0	-153	-713
6b	3A	2a	21	0	171	745
6c	3B	2a	-6	0	9	16
6d	3C	2A	16	-8	0	28
8a	4A		-20	0	-62	-216
8b	4B		8	0	-6	48
8c	4B		-8	0	-6	-48
9a	3A		0	14	0	65
9b	3A		9	-4	0	2
9c	3B		0	-4	0	2
9d	3C		-3	2	0	5
10a	5A	2a	8	0	35	100
10b	5a	2A	2	-4	7	0
10c	5a	2a	-2	0	-5	0
12a	6A	4B	-11	0	-21	-55
12b	6A	4a	5	0	27	41
12c	6C	4C	5	0	-5	9
12d	6C	4D	-3	0	3	-7
12e	6d	4B	4	0	0	-4
12f	6d	4a	-4	0	0	-4
14a	7A	2a	-9	0	-15	-33
14b	7B	2a	-2	0	-1	2
14c	7A	2a	5	0	13	37
15a	5A	3C	5	-2	0	-1
15b	5a	3A	3	2	-3	0
16a	8B		0	0	6	0
16b	8B		4	0	-2	8
16c	8B		-4	0	-2	-8
16d	8D		0	0	-2	0
16e	8C		2	0	-2	4
16f	8C		-2	0	-2	-4
16g	8b		2	0	2	-4
16h	8b		-2	0	2	4
18a	9b	6A	1	4	0	10
18b	9a	6b	0	0	0	7
18c	9A	6c	3	0	9	16
18d	9c	6B	0	4	0	10

18e	9a	6C		0	-2	0	1
18f	9b	6a		-3	0	0	-2
18g	9b	6b		3	0	0	-2
18h	9a	6A		4	-2	0	1
18i	9c	6c		0	0	0	-2
18j	9d	6d		1	-2	0	1
20a	10A	4a		-6	0	-7	-14
20b	10A	4a		4	0	3	16
20c	10C	4a		-1	0	-2	1
20d	10B	4C		0	0	3	-4
20e	10b	4B		2	0	-1	0
22a	11A	2a		3	0	4	11
24a	12A	8a		-5	0	-5	-9
24b	12A	8a		1	0	7	9
24c	12B	8a		-2	0	1	0
24d	12C	8b		-1	0	3	3
24e	12C	8c		1	0	3	-3
24f	12C	8C		-3	0	-1	-3
24g	12C	8C		3	0	-1	3
24h	12E	8E		1	0	-1	1
24i	12e	8b		2	0	0	0
24j	12e	8c		-2	0	0	0
25a	5B			-1	0	0	0
26a	13A	2a		2	0	4	6
27a	9A			3	-1	0	-1
27b	9A			0	2	0	5
27c	9B			0	-1	0	-1
27d	9b			0	2	0	-1
27e	9b			0	-1	0	2
28a	14A	4a		1	0	5	5
30a	15A	10a	6b	-4	0	-4	-5
30b	15a	10A	6d	1	2	0	3
30c	15B	10a	6c	-1	0	-1	1
30d	15A	10a	6a	2	0	2	7
30e	15b	10b	6A	-1	2	1	0
30f	15b	10c	6b	1	0	1	0
32a	16a			0	0	0	0
32b	16A			0	0	2	0
32c	16b			2	0	0	0
32d	16b			-2	0	0	0
32e	16d			0	0	0	0
34a	17A	2a		1	0	3	4
35a	7A	5a		1	-1	1	0
36a	18a	12b		-1	0	0	2
36b	18e	12A		0	2	0	1
36c	18h	12b		2	0	0	-1
36d	18a	12C		1	0	0	2
36e	18e	12d		0	0	0	-1
36f	18C	12I		-1	0	1	0
36g	18d	12F		0	0	0	2
36h	18h	12a		-2	0	0	-1

36i	18c	12f		-1	0	0	-1
38a	19A	2a		2	0	1	3
40a	20A	8a		0	0	3	4
40b	20B	8b		-2	0	-1	-2
40c	20B	8c		2	0	-1	2
40d	20D	8F		0	0	-1	0
40e	20e	8C		0	0	1	0
42a	21A	14a	6b	0	0	3	3
42b	21A	14c	6a	2	0	1	1
42c	21C	14A	6d	2	-1	0	0
42d	21D	14b	6c	1	0	2	2
44a	22A	4B		-3	0	-2	-3
44b	22B	4D		-1	0	0	-1
44c	22A	4a		1	0	2	1
45a	15A	9b		-1	1	0	2
45b	15C	9c		0	1	0	2
45c	15b	9a		0	-1	0	0
48a	24A	16b		1	0	1	-1
48b	24A	16c		-1	0	1	1
48c	24g	16e		-1	0	1	1
48d	24g	16f		1	0	1	-1
48e	24d	16g		-1	0	-1	-1
48f	24d	16h		1	0	-1	1
48g	24E	16a		0	0	0	0
48h	24H	16d		0	0	1	0
49a	7B			2	1	2	4
50a	25a	10E		1	2	2	4
52a	26A	4a		2	0	0	2
54a	27a	18B		1	1	0	1
54b	27c	18E		0	1	0	1
54c	27b	18c		0	0	0	1
54d	27c	18g		0	0	0	1
56a	28B	8a		1	0	1	1
56b	28A	8b		1	0	1	-1
56c	28A	8c		-1	0	1	1
58a	29A	2a		1	0	1	1
60a	30C	20d	12c	0	0	0	-1
60b	30B	20a	12b	0	0	2	1
60c	30b	20B	12e	-1	0	0	1
60d	30e	20e	12a	-1	0	-1	0
60e	30b	20b	12f	1	0	0	1
63a	21A	9a		0	0	0	2
64a	32b			0	0	0	0
66a	33B	22a	6a	0	0	1	2
70a	35A	14a	10a	1	0	0	2
72a	36A	24c		1	0	1	0
72b	36b	24A		0	0	0	1
72c	36d	24d		-1	0	0	0
72d	36d	24e		1	0	0	0
72e	36g	24F		0	0	0	0
76a	38A	4a		0	0	1	1

80a	40B	16a		0	0	1	0
82a	41A	2a		0	0	1	1
84a	42A	28a	12b	-2	0	-1	-1
90a	45a	30B	18a	1	-1	0	0
90b	45b	30A	18d	0	-1	0	0
96a	48g	32a		0	0	0	0
102a	51A	34a	6a	1	0	0	1
117a	39A	9a		0	1	0	0
120a	60B	40a	24a	0	0	0	1
126a	63a	42a	18b	0	0	0	0
132a	66A	44a	12a	0	0	1	0
140a	70A	28a	20a	1	0	0	0

We correct an error in [MS]: On page 265 class 29Z should read 25Z and signs should be inserted compatible with its sign pattern.

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