Counting unlabelled topologies and transitive relations

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Abstract

We enumerate isomorphism classes of several types of transitive relations (equivalently, finite topologies) up to 15 or 16 points.

1 Introduction

Pfeiffer [3] presented a classification of various types of order relations and determined their numbers up to 12 points. In this paper, we extend the counts to 15 or 16 points. We defer to [3] for historical survey, and only give enough background to precisely define the objects we are counting.

We consider only directed graphs (digraphs) that do not have multiple edges but may have up to one loop per point. A *transitive relation digraph (TRD)* is a digraph such that presence of edges (x, y) and (y, z) implies that (x, z) is also present (even if x, y, z are not distinct). (We cannot use the more natural term "transitive digraph" since its most common definition does not allow loops; this has the unfortunate consequence that transitive digraphs don't correspond to transitive relations.)

A *strong component* of a digraph is a maximal set of points P such that there is a directed path within P from x to y for each pair $x, y \in P$. (This permits the zero-length path from $x \in P$ to itself.)

If a strong component of a TRD has only one point, there may or may not be a loop on that point. However, larger strong components of a TRD have loops on every point and edges in both directions between each pair of points. If the strong components of a TRD are contracted to single points, the result is a TRD that has no directed cycles apart from loops.

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Two digraphs are *isomorphic* if there is a bijection between their point-sets that induces a bijection between their edge-sets. Thus we can refer to *isomorphism classes* of digraphs.

Of course a digraph can be viewed as representing some kind of order relation $\leq$, where $x \leq y$ iff there is an edge (x, y). In the language and notation of Pfeiffer, we have the following correspondences:

- A *transitive relation* corresponds to an arbitrary TRD. Let $t(n)$ be the number of isomorphism classes of transitive relations.
- A *quasi-order* corresponds to a TRD such that every single-point strong component has a loop. Let $q(n)$ be the number of isomorphism classes of quasi-orders.
- A *soft order* corresponds to a TRD whose strong components each have only one point (with or without loop). Let $s(n)$ be the number of isomorphism classes of soft orders.
- A *partial order* corresponds to an TRD whose strong components are all single points with loops. Because we can remove and replace the loops in a unique fashion, the counts here are the same as for acyclic TRDs. Let $p(n)$ be the number of isomorphism classes of partial orders.

There are also relationships with finite topologies: general topologies are in bijective correspondence with quasi-orders, and T_0-topologies are in bijective correspondence with partial orders. These bijections preserve isomorphism, so we are also counting isomorphism classes of topologies.

For enumerative purposes, we will define some specialized numbers.

- $q(n, m)$ is the number of isomorphism classes of TRDs with n points and m strong components, such that each single-point strong component has a loop.
- $t(n, m)$ is the number of isomorphism classes of TRDs with n points and m strong components (with single-point strong components having a loop or not).

In terms of these specialized numbers, we have

$$q(n) = \sum_{m=1}^{n} q(n, m), \qquad t(n) = \sum_{m=1}^{n} t(n, m),$$

$$p(n) = q(n, n), \qquad s(n) = t(n, n).$$

2 Computing $t(n, m)$ and $q(n, m)$ for small n

Our main tool is a program which generates non-isomorphic partially ordered sets (*posets*). In our paper [1], we gave values of $p(n)$ up to $n = 16$ based on output from that program.

Given a poset P, we can make a TRD G by replacing each point by a directed clique (perhaps of a single point). Such directed cliques become the strong components of G. If points v, w of P become strong components c_v, c_w of G, then an edge (v, w) of P becomes the set of all possible edges from c_v to c_w in G. Clearly non-isomorphic posets lead to non-isomorphic TRDs, since G uniquely determines P. The only remaining issue is that some of the TRDs made from P may be isomorphic due to symmetries (automorphisms) of P.

Given a poset P, we can represent its expansion to a TRD by assigning a positive integer *weight* to each point. This weight corresponds to the size of the directed clique that the point will be expanded to. We also consider *extended weights* where there are two types of weight with value 1 (corresponding to loop and non-loop).

For any permutation γ, define

$$C_\gamma(x) = \prod_{i=1}^{k} (1 - x^{a_i})^{-1}$$
$$C'_\gamma(x) = \prod_{i=1}^{k} \bigl(1 + (1 - x^{a_i})^{-1}\bigr),$$

where $a_1, a_2, \ldots, a_k$ are the cycle lengths of γ.

Theorem 1. *Let $1 \le m \le n$. Then*

$$q(n,m) = [x^{n-m}] \sum_P \Bigl(|\mathrm{Aut}(P)|^{-1} \sum_{\gamma \in \mathrm{Aut}(P)} C_\gamma(x) \Bigr)$$
$$t(n,m) = [x^{n-m}] \sum_P \Bigl(|\mathrm{Aut}(P)|^{-1} \sum_{\gamma \in \mathrm{Aut}(P)} C'_\gamma(x) \Bigr),$$

where the first sum in each case is over isomorphism class representatives P of posets on m points, $\mathrm{Aut}(P)$ is the automorphism group of P, and $[x^{n-m}]$ denotes extraction of the coefficient of x^{n-m}.

Proof. We prove the formula for $q(n,m)$; the other is similar. Let P be a poset with m points. By the Frobenius-Burnside Lemma, the number of equivalence classes under $\mathrm{Aut}(P)$ of weight assignments with total weight n is the average over $\gamma \in \mathrm{Aut}(P)$ of the number $w_n(\gamma)$ of such weight assignments fixed by γ.

Consider one such element $\gamma \in \mathrm{Aut}(P)$ with cycles of length $a_1, a_2, \ldots, a_k$. A weight assignment is fixed by γ iff it is constant on each cycle of γ, so $w_n(\gamma)$ is the coefficient of x^n in

$$\prod_{i=1}^{k} (x^{a_i} + x^{2a_i} + x^{3a_i} + \cdots) = x^m C_\gamma(x).$$

The result now follows by averaging over γ. □

As mentioned earlier, $q(n,n) = p(n)$. We can also identify $q(n,n-1)$ as the total number of orbits of $\mathrm{Aut}(P)$ over all posets on $n-1$ points. Using our previously-computed value of $p(16)$, this enabled us to determine $q(n,m)$ for $n \le 16$ and $t(n,m)$ for $n \le 15$ by computing the automorphism groups of all the posets up to 15 points. Except in some simple (but common) situations where the poset generator had already determined Aut(P), this was computed using `nauty` [2].

The resulting values are given in Tables 1–3. The programs were run twice in case of machine errors. We also successfully recovered the numbers given by Pfeiffer [3] and the values of $p(n)$ up to $n = 15$ given by Brinkmann and McKay [1].

n	$q(n)$	$s(n)$	$t(n)$
1	1	2	2
2	3	7	8
3	9	32	39
4	33	192	242
5	139	1490	1895
6	718	15067	19051
7	4535	198296	246895
8	35979	3398105	4145108
9	363083	75734592	90325655
10	4717687	2191591226	2555630036
11	79501654	82178300654	93810648902
12	1744252509	3984499220967	4461086120602
13	49872339897	249298391641352	274339212258846
14	1856792610995	20089200308020179	21775814889230580
15	89847422244493	2081351202770089728	2226876304576948549
16	5637294117525695		

Table 1: Quasi-orders, soft orders, and transitive relations

References

[1] G. Brinkmann and B. D. McKay, Posets on up to 16 points, *Order*, **19** (2002) 147–179.

[2] B. McKay, `nauty` – a program for graph isomorphism and automorphism, available at `http://cs.anu.edu.au/~bdm/nauty/`.

[3] G. Pfeiffer, Counting transitive relations, *J. Integer Seq.*, **7** (2004) 11pp.

1	1	1
2	1	1
2	2	2
3	1	1
3	2	3
3	3	5
4	1	1
4	2	5
4	3	11
4	4	16
5	1	1
5	2	6
5	3	22
5	4	47
5	5	63
6	1	1
6	2	8
6	3	35
6	4	113
6	5	243
6	6	318
7	1	1
7	2	9
7	3	52
7	4	213
7	5	682
7	6	1533
7	7	2045
8	1	1
8	2	11
8	3	71
8	4	367
8	5	1503
8	6	4989
8	7	12038
8	8	16999
9	1	1
9	2	12
9	3	95
9	4	570
9	5	2923
9	6	12591
9	7	44842
9	8	118818
9	9	183231
10	1	1
10	2	14
10	3	120
10	4	849
10	5	5123
10	6	27439
10	7	127965
10	8	501591
10	9	1487301
10	10	2567284
11	1	1
11	2	15
11	3	150
11	4	1193
11	5	8406
11	6	53443
11	7	309719
11	8	1599822
11	9	7040921
11	10	23738557
11	11	46749427
12	1	1
12	2	17
12	3	182
12	4	1632
12	5	13011
12	6	96226
12	7	664467
12	8	4268404
12	9	24858756
12	10	124784466
12	11	484673601
12	12	1104891746
13	1	1
13	2	18
13	3	218
13	4	2154
13	5	19320
13	6	162404
13	7	1304373
13	8	10009358
13	9	72589838
13	10	483531684
13	11	2803234294
13	12	12677658783
13	13	33823827452
14	1	1
14	2	20
14	3	256
14	4	2790
14	5	27637
14	6	260840
14	7	2385741
14	8	21304106
14	9	184860968
14	10	1535230287
14	11	11832383054
14	12	80018898562
14	13	425004096962
14	14	1338193159771
15	1	1
15	2	21
15	3	299
15	4	3525
15	5	38420
15	6	401602
15	7	4124565
15	8	41997196
15	9	424381067
15	10	4221560826
15	11	40603719604
15	12	365485856203
15	13	2906408331804
15	14	18255153928204
15	15	68275077901156
16	1	1
16	2	23
16	3	343
16	4	4396
16	5	52033
16	6	597502
16	7	6804011
16	8	77823441
16	9	897440095
16	10	10402896209
16	11	119938485210
16	12	1348204100877
16	13	14281079724622
16	14	134410089884839
16	15	1003992754517006
16	16	4483130665195087

Table 2: Values of $q(n, m)$ for various n, m

1	1	2
2	1	1
2	2	7
3	1	1
3	2	6
3	3	32
4	1	1
4	2	8
4	3	41
4	4	192
5	1	1
5	2	9
5	3	63
5	4	332
5	5	1490
6	1	1
6	2	11
6	3	84
6	4	583
6	5	3305
6	6	15067
7	1	1
7	2	12
7	3	112
7	4	883
7	5	6537
7	6	41054
7	7	198296
8	1	1
8	2	14
8	3	139
8	4	1294
8	5	11096
8	6	90758
8	7	643701
8	8	3398105
9	1	1
9	2	15
9	3	174
9	4	1769
9	5	17694
9	6	170391
9	7	1577763
9	8	12823256
9	9	75734592
10	1	1
10	2	17
10	3	207
10	4	2380
10	5	26352
10	6	294156
10	7	3243880
10	8	34592661
10	9	325879156
10	10	2191591226
11	1	1
11	2	18
11	3	248
11	4	3068
11	5	37919
11	6	472519
11	7	6031290
11	8	77251333
11	9	960789368
11	10	10587762484
11	11	82178300654
12	1	1
12	2	20
12	3	288
12	4	3911
12	5	52415
12	6	724866
12	7	10377573
12	8	153952401
12	9	2314756589
12	10	33895893064
12	11	440211138507
12	12	3984499220967
13	1	1
13	2	21
13	3	335
13	4	4852
13	5	70797
13	6	1066041
13	7	16906476
13	8	282183725
13	9	4922404711
13	10	87597193530
13	11	1521294651297
13	12	23426706135708
13	13	249298391641352
14	1	1
14	2	23
14	3	381
14	4	5964
14	5	93159
14	6	1521101
14	7	26315265
14	8	486434324
14	9	9568317752
14	10	197959166598
14	11	4196507844123
14	12	86930341478767
14	13	1595279690032943
14	14	20089200308020179
15	1	1
15	2	24
15	3	435
15	4	7190
15	5	120514
15	6	2110489
15	7	39534004
15	8	798048843
15	9	17393487215
15	10	406531057397
15	11	10041016241522
15	12	254873608116034
15	13	6326335208572503
15	14	138933427209562650
15	15	2081351202770089728

Table 3: Values of $t(n, m)$ for various n, m