APPLICATIONS OF A TECHNIQUE FOR LABELLED ENUMERATION*

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Abstract

A technique involving summation over roots of unity was used by Liskovec in 1971 to count labelled regular tournaments. The same method is used here to count regular tournaments to 21 vertices, Eulerian digraphs to 16 vertices, Eulerian oriented graphs to 15 vertices, regular graphs to 21 vertices, regular bipartite graphs to 40 vertices, and Eulerian circuits in complete graphs with up to 17 vertices. The last calculation was performed jointly with R. W. Robinson. All the objects counted are vertex-labelled.

1. Coefficient extraction for generating functions

Consider a multivariate generating function $f(x_1, \ldots, x_n) = \sum_{c_1=s_1}^{t_1} \cdots \sum_{c_n=s_n}^{t_n} \alpha(c_1, \ldots, c_n) x_1^{c_1} \ldots x_n^{c_n}$, where each $\alpha(c_1, \ldots, c_n)$ is a complex number.

<u>Lemma 1.1</u> *Let* $m_j > 0$ *and* k_j *be integers for* $1 \le j \le n$. *Define* $\omega_j = e^{2\pi i/m_j}$ $(1 \le j \le n)$. *Then*

$$\sum_{r_1=0}^{m_1-1} \cdots \sum_{r_n=0}^{m_n-1} \frac{f(\omega_1^{r_1}, \ldots, \omega_n^{r_n})}{\omega_1^{k_1 r_1} \cdots \omega_n^{k_n r_n}} = m_1 \ldots m_n \sum \alpha(c_1, \ldots, c_n),$$

where the sum on the right is over all $c_1, \ldots, c_n$ *such that* $m_j | c_j - k_j$ *for* $1 \le j \le n$.

<u>Proof</u>. This is an immediate consequence of the fact that

$$\sum_{r_j=0}^{m_j} \omega_j^{(c_j-k_j)r_j} = \begin{cases} m_j, & \text{if } m_j | c_j - k_j, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

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*This paper contains original research and will not be published elsewhere.

By judicious choice of the m_j we can extract any desired coefficient.

Lemma 1.2 *For* $1 \le j \le m$, *let* $m_j \ge 1 + \max\{t_j - k_j,\ k_j - s_j\}$. *Then*

$$\sum_{r_1=0}^{m_1-1} \cdots \sum_{r_n=0}^{m_n-1} \frac{f(\omega_1^{r_1}, \ldots, \omega_n^{r_n})}{\omega_1^{k_1 r_1} \cdots \omega_n^{k_n r_n}} = m_1 \cdots m_n\, \alpha(k_1, k_2, \ldots, k_n).$$

□

Lemma 1.2 requires the summation of $m_1 m_2 \ldots m_n$ terms. For many applications this number can be drastically reduced. For example, if f is a symmetric function and $k_1 = k_2 = \ldots = k_n = k$, Lemma 1.2 immediately yields the following result.

Lemma 1.3 *Suppose* f *is a symmetric function, and let* $s = s_1 = \ldots = s_n$ *and* $t = t_1 = \ldots = t_n$. *Choose an integer* m *such that* $m \ge 1 + \max\{t-k,\ k-s\}$ *and define* $\omega = e^{2\pi i/m}$. *Then*

$$\sum \binom{n}{u_0,\ u_1,\ \ldots,\ u_{m-1}} \frac{f(z_1, \ldots, z_n)}{\omega^{k(u_1+2u_2+\ \ldots\ +(m-1)u_{m-1})}} = m^n \alpha(k, k, \ldots, k).$$

The sum is over non-negative integers $u_0, \ldots, u_{m-1}$ *such that* $u_0 + u_1 + \ldots + u_{m-1} = n$. *The arguments to* f *are* ω^0 (u_0 *times*), ω^1 (u_1 *times*), ..., ω^{m-1} (u_{m-1} *times*).

Proof. For $0 \le j \le m-1$, interpret u_j as the number of r_i's (in Lemma 1.2) which are equal to j. The multinomial coefficient counts the number of times this occurs. □

Lemmas 1.1-1.3 are not very useful for computation in their stated forms because they employ complex arithmetic. Moreover, the amount of cancellation which occurs is so excessive that very high precision is required. Fortunately, there is an alternative approach, which we shall describe for Lemma 1.3. Let p be a prime number such that $m|p-1$ and $p > n$. Then there is a number $\omega \in Z_p$ whose order modulo p is m. Lemma 1.3 now holds modulo p, with the same proof. Thus we can obtain $\alpha(k, k, \ldots, k)$ mod p using only small integers. If this is repeated for a sufficient number of different primes, $\alpha(k, k, \ldots, k)$ itself can be found with the help of the Chinese Remainder Theorem. This assumes that some prior bound on $\alpha(k, k, \ldots, k)$ is available--no problem for our examples.

With a little care, the computation in Lemma 1.3 can often be arranged so that only a few machine operations per term are required on the average. Essentially, the terms are computed in such an order that each can be quickly computed from the preceding term. Another saving is obtained through the use of a table of logarithms to base z for computing powers, where z is a primitive root mod p.

2. Applications

We will describe each application using a standard format. The *object* subsection defines the number we wish to evaluate. The *method* subsection describes the generating function used, and the means by which the correct coefficient was extracted. The *bound* subsection gives the a priori upper bound needed as described in Section 1. The *checks* subsection gives one or more divisibility conditions which were applied as a check to the results. Anything else appears in the *comments* subsection. The numbers themselves appear in the Appendix.

(a) Regular Tournaments

Object. RT(n) is the number of labelled regular tournaments with n vertices. Clearly, n must be odd.

Method. Let $q = (n-1)/2$. Then RT(n) is the coefficient of $x_1^q \dots x_n^q$ in $\prod_{i<j} (x_i + x_j)$. This is symmetric in its arguments, so we can use Lemma 1.3 with $m = 1+q$. A little saving is possible by using $m = q$ instead. This counts tournaments with degrees in the set $\{0, q, n-1\}$. The numbers of regular tournaments are then easily extracted (see [6]).

Bound. $RT(n) \le \prod_{i=1}^{q} \binom{2i-1}{i}\binom{2i}{i}$. To see this, note that the neighbours of vertex 1 can be chosen in exactly $\binom{2q}{q}$ ways, then those of vertex 2 can be chosen in exactly $\binom{2q-1}{q}$ ways, etc.

Checks. $\binom{2q}{q}\binom{2q-1}{q} \mid RT(n)$, as proved in the previous subsection.

Comments. This case has been done before by Liskovec [6]. In fact, that paper is the principal source of our inspiration. Liskovec obtained RT(n) for $n \le 9$.

(b) Eulerian Digraphs

Object. ED(n) is the number of simple labelled Eulerian digraphs with n vertices.

ED(n) is the constant term in $\prod_{i=1}^{n} \prod_{j=1}^{n} (1 + x_i^{-1} x_j)$ so ED(n) can be found by using $m = n$ in Lemma 1.3. If n is odd we can do a little better with $m = n+1$, since many of the terms are then zero and the computation can be arranged to reject these *en masse*.

Bound. $ED(n) \leq \prod_{i=0}^{n-1} \binom{2i}{i}$. Consider constructing an Eulerian digraph by first choosing the vertices to be adjacent to and from vertex 1, then to and from vertex 2, etc. When we reach vertex i we find it already connected to or from some of the earlier vertices (say to d and d' of these respectively). To choose its other incident edges we have $\sum_{s=0}^{n-1} \binom{n-i}{s-d}\binom{n-i}{s-d'}$ possibilities, where s is the common value of the indegree and outdegree when we are finished. The sum is easily seen to be bounded above by $\binom{2n-2i}{n-i}$.

Checks. Let g_r be the gcd of the numbers $\binom{n-1}{i}\binom{n-1}{i}2^{n-2i-1}$ for $r \leq i \leq \lfloor(n-1)/2\rfloor$. Then $ED(n)$ is divisible by g_0 and $ED(n) - 2^{n-1}ED(n-1)$ is divisible by g_1. To prove this, define $\ell(n, i)$ to be the number of labelled simple digraphs with n vertices, of which i have $\delta = 1$, i have $\delta = -1$, and $n-2i$ have $\delta =$ indegree − outdegree. Clearly, $\ell(n, 0) = ED(n)$. By considering the edges incident with vertex 1, we see that $\sum \binom{n-1}{i}\binom{n-i-1}{i} 2^{n-2i-1} \ell(n-1, i)$. Both divisibility conditions now follow easily.

Comment. This case was also done by Liskovec [6], but without any actual calculations.

(c) Eulerian Oriented Graphs

Object. $EOG(n)$ is the number of simple labelled Eulerian oriented graphs with n vertices.

Method. $EOG(n)$ is the constant term in $\prod_{i<j}(1+x_i^{-1}x_j+x_ix_j^{-1})$.

Bound. $EOG(n) \leq ED(n)$, obviously.

Checks. Let g be the gcd of the numbers $\binom{n-1}{i}\binom{n-i-1}{i}$ for $1 \leq i \leq \lfloor(n-1)/2\rfloor$. Then $EOG(n) - EOG(n-1)$ is divisible by g.

(d) Regular Graphs

Object. $RG(n, k)$ is the number of k-regular simple labelled graphs of order n.

Method. $RG(n, k)$ is the coefficient of $x_1^k \ldots x_n^k$ in $\prod_{i<j}(1+x_ix_j)$, which can be extracted using $m = 1 + \max\{k, n-k-1\}$ in Lemma 1.3. If $k \gg n$

we can do better with the generating function $\prod_{i<j}(1+x_i x_j t)$, where t is an extra variable. Applying Lemma 1.3 with $m=k+1$ we obtain a generating function $\sum c_k t^i$, where c_i is the number of labelled simple graphs with n vertices, i edges, and all vertex degrees in the set $\{k, 2k+1, 3k+2, \ldots\}$. By summing t over suitable roots of unity, we can select $c_{kn/2}$, which is RG(n, k).

Bound. $RG(n, k) \le (nk)!/\big((nk/2)!2^{nk/2}(k!)^n\big)$. See [1], for example.

Checks. By considering the possible neighbours of vertex 1, we see that RG(n, k) is divisible by $\binom{n-1}{k}$.

Comments. Obviously, $RG(n, 0) = 1$ and $RG(n, 1) = n!/\big((n/2)!2^{n/2}\big)$. A recurrence for RG(n, 2) is found easily. Recurrences for RG(n, 3) have appeared in [13] and [16], for example, and recurrences for RG(n, 4) in [14] and [15]. A linear but very long recurrence for RG(n, 5) has been bound by Goulden, Jackson and Reilly [5]. General methods, which would probably yield linear recurrences for any fixed k, are discussed in [4] and [5].

(e) Semi-regular Bipartite Graphs

Object. $SRBG(n_1, n_2, k_1, k_2)$ is the number of simple labelled bicoloured graphs, where one colour class has n_1 vertices of degree k_1 and the other has n_2 vertices of degree k_2. Clearly, $n_1k_1 = n_2k_2$ or no such graphs exist. Equivalently, $SRBG(n_1, n_2, k_1, k_2)$ is the number of $n_1 \times n_2$ 0-1 matrices with all rows summing to k_1 and all columns summing to k_2.

Method. $SRBG(n_1, n_2, k_1, k_2)$ is the coefficient of $x_1^{k_1} \ldots x_{n_1}^{k_1} y_1^{k_2} \ldots y_{n_2}^{k_2}$ in $\prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1+x_i y_j)$. This is a symmetric function independently in the x's and the y's. We can extract the coefficient by summing each x_i over the m_1-th roots of unity and each y_j over the m_2-th roots of unity, where $m_1 = 1+\max\{k_1, n_1-k_1\}$ and $m_2 = 1+ \max\{k_2, n_2-k_1\}$. Actually, if $k_1 \le n_1-k_1$, it suffices to use $m_1 = 1+k_1$ and the same value of m_2. By Lemma 1.1, the graphs counted have n_2 vertices of degree k_2 and n_1 vertices with degrees in the set $\{k_1, 2k_1+1, 3k_1+2, \ldots\}$. Since the number of edges is n_2k_2, all these n_1 vertices must in fact have degree k_1. An additional major improvement comes on noticing that each of the y's occurs independently, in the sense that $\sum_{y_1 \in S} \cdots \sum_{y_{n_2} \in S} \prod_i \prod_j (1+x_i y_j) = \left(\sum_{y \in S} \prod_i (1+x_i y) \right)^{n_2}$ any set S.

Bound. $SRBG(n_1, n_2, k_1, k_2) \le (n_1k_1)!/(k_1!^{n_1} k_2!^{n_2})$. See [10], for example.

Checks. By considering the neighbours of one vertex, then those of a vertex adjacent to the first, we see that $SRBG(n_1, n_2, k_1, k_2)$ is divisible by $\binom{n_2}{k_1}\binom{n_1-1}{k_2}$.

Comments. We have only done computations for the case where $n_1 = n_2$ and $k_1 = k_2$. SRBG(n, n, 0, 0), SRBG(n, n, 1, 1) and SRBG(n, n, 2, 2) are easily found. A formula for SRBG(n, n, 3, 3) was found by Read [12].

In the tables, RBG(n, k) means SRBG(n, n, k, k).

(f) Eulerian Circuits in Complete Graphs

The details of this case will appear in a forthcoming paper (jointly with R. W. Robinson). Let $Eul(K_n)$ be the number of Eulerian circuits in K_n, counted without regard to starting point. The tables give $EK(n) = Eul(K_n)/\left((n-3)/2\right)!^n$. A proof that EK(n) is an integer, and values up to $n = 11$, can be found in [7].

3. Wishful Thinking

Asymptotic calculations have only been performed for two of the six categories listed in Section 2. Specifically, we have

(a) $RG(n, k) \sim \dfrac{(nk)!}{(nk/2)!\,2^{nk/2}(k!)^n} \exp\left(\dfrac{1-k^2}{4}\right)$, and

(b) $SRBG(n_1, n_2, k_1, k_2) \sim \dfrac{(n_1k_1)!}{(k_1!)^{n_1}(k_2!)^{n_2}} \exp\left(-\dfrac{(k_1-1)(k_2-1)}{2}\right)$. Formula

(a) was proved for fixed k by Bender and Canfield [1], for $k = O\left((\log n)^{\frac{1}{2}}\right)$

by Bollobás [3], and for $k = o(n^{1/3})$ by McKay [9]. Formula (b) was proved for $\max\{k_1, k_2\} \le (\log(n_1 + n_2))^{1/4 - \varepsilon}$ by O'Neil [1], somewhat more accurately by Mineev and Pavlor [10], and for $\max\{k_1, k_2\} = o((n_1 + n_2)^{1/3})$ by McKay [8].

The following conjectures are based on a careful analysis of the number given in the Appendix using a technique for numerical extrapolation [2]. We have not specified how k is to vary with n, but in each case we would expect the result to hold certainly for fixed k, probably for $k = O(n^{1-\varepsilon})$, and possibly for $k \le cn$ for some suitable constant c. Conjecture 2 appeared previously in [8].

Conjecture 1.

$$RG(n, k) = \frac{(nk)!}{(nk/2)!2^{nk/2}(k!)^n} \exp\left\{-\frac{(k-1)(k+1)}{4} - \frac{(k-1)(k+2)(k^2-k+1)}{12kn} - \frac{(k-1)^4(k^2+4k+6)}{24k^2n^2} + O\left(\frac{k^5}{n^3}\right)\right\}.$$

Conjecture 2.

$$RBG(n, k) = \frac{(nk)!}{(k!)^n} \exp\left[-\frac{(k-1)^2}{2} - \frac{(k-1)^2(k^2-k+1)}{6kn} - \frac{(k-1)^5(k+1)}{12k^2n^2} + O\left(\frac{k^5}{n^3}\right)\right].$$

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Appendix — the numbers

(a) labelled regular tournaments

RT(1) = 1
RT(3) = 2
RT(5) = 24
RT(7) = 2640
RT(9) = 32 30080
RT(11) = 4 82515 08480
RT(13) = 9 30770 06112 92160
RT(15) = 240 61983 49824 94283 79648
RT(17) = 85584 72055 41481 49511 79758 79680
RT(19) = 4271 02683 12628 45202 01657 80015 93666 76480
RT(21) = 3035 99177 67255 01434 06909 90026 40396 04333 20198 14400

(b) labelled eulerian digraphs

ED(1) = 1
ED(2) = 2
ED(3) = 10
ED(4) = 152
ED(5) = 7736
ED(6) = 13 75952
ED(7) = 8779 01648
ED(8) = 204 63203 73120
ED(9) = 17 65822 17023 61472
ED(10) = 5 69773 21983 69652 65152
ED(11) = 6 92800 70663 38878 38902 48448
ED(12) = 31 94140 76928 47758 20130 37245 06112
ED(13) = 561 21720 93887 11105 02272 39130 00322 61120
ED(14) = 37736 24389 96731 35332 92562 82026 36271 68278 87616
ED(15) = 97 44754 03179 97541 69218 00337 62069 41877 94308 61883 08480
ED(16) = 96934 27419 43194 32347 65129 25742 87605 35010 22995 32573 44779 87840

(c) labelled eulerian oriented graphs

EOG(1) = 1
EOG(2) = 1
EOG(3) = 3
EOG(4) = 15
EOG(5) = 219
EOG(6) = 7839
EOG(7) = 7 77069
EOG(8) = 2088 36207
EOG(9) = 15 64583 82975
EOG(10) = 32820 80160 21561
EOG(11) = 1946 87965 62657 10431
EOG(12) = 328 34193 09869 77413 59313
EOG(13) = 158 28097 85794 57849 90632 05301
EOG(14) = 218 98960 75577 09869 78834 04184 32175
EOG(15) = 872 69441 10689 80079 02526 09985 08645 17077

(d) labelled regular graphs

RG(4, 1) = 3
RG(5, 2) = 12
RG(6, 1) = 15
RG(6, 2) = 70
RG(7, 2) = 465
RG(8, 1) = 105
RG(8, 2) = 3507
RG(8, 3) = 19355
RG(9, 2) = 30016
RG(9, 4) = 10 24380
RG(10, 1) = 945
RG(10, 2) = 2 86884
RG(10, 3) = 111 80820
RG(10, 4) = 664 62606
RG(11, 2) = 30 26655
RG(11, 4) = 51884 53830
RG(12, 1) = 10395
RG(12, 2) = 349 44085
RG(12, 3) = 1 15552 72575
RG(12, 4) = 48 04139 21130
RG(12, 5) = 297 76351 37862
RG(13, 2) = 4382 63364
RG(13, 4) = 5211 33763 10985

RG(13, 6) = 2 09913 28709 73600

RG(14, 1) = 1 35135

RG(14, 2) = 59335 02822

RG(14, 3) = 1950 66318 14670

RG(14, 4) = 6 55124 65965 01035

RG(14, 5) = 283 09726 01841 59421

RG(14, 6) = 1803 59535 89647 73088

RG(15, 2) = 8 62489 51243

RG(15, 4) = 945 31390 72536 06891

RG(15, 6) = 18 72726 69012 71816 63775

RG(16, 1) = 20 27025

RG(16, 2) = 133 97519 21865

RG(16, 3) = 50 26295 87137 92825

RG(16, 4) = 1 55243 72224 85240 67795

RG(16, 5) = 524 69332 40770 03653 20163

RG(16, 6) = 23296 76580 69802 21975 16875

RG(16, 7) = 1 51385 92322 75324 22353 38875

RG(17, 2) = 2214 80510 88480

RG(17, 4) = 287 97220 46058 68264 22720

RG(17, 6) = 344 30864 02825 29972 04036 73760

RG(17, 8) = 14939 08809 73211 82119 40442 93500

RG(18, 1) = 344 59425

RG(18, 2) = 38824 67258 73208

RG(18, 3) = 1 87747 83788 96998 87800

RG(18, 4) = 59930 02310 42715 04940 60340

RG(18, 5) = 1764 78838 28569 85865 99722 68092

RG(18, 6) = 5 99722 97699 47050 27153 59174 22040

RG(18, 7) = 271 84977 22059 48458 08509 08045 26392

RG(18, 8) = 1793 19666 50258 85172 29050 89715 92750

RG(19, 2) = 7 19342 31097 63089

RG(19, 4) = 139 07595 61507 55900 18236 65540

RG(19, 6) = 12218 90111 37527 12984 45845 84754 80428

RG(19, 8) = 260 51341 89838 73007 07368 58744 50318 27810

RG(20, 1) = 6547 29075

RG(20, 2) = 140 46235 58216 28771

RG(20, 3) = 9762 73961 16036 31721 31825

RG(20, 4) = 35792 05185 12934 32427 84678 20756

RG(20, 5) = 10148 61308 10401 17624 31953 69019 32188

RG(20, 6) = 289 16028 25271 77828 14901 37004 21235 35900

RG(20, 7) = 98830 18890 80323 33162 33360 72448 97992 27748

RG(20, 8) = 45 54732 95761 28860 30974 46621 93995 60781 33650

RG(20, 9) = 304 00592 81615 70414 70070 85764 67967 97406 91838

RG(21, 2) = 2883 01399 43484 84940

RG(21, 4) = 101 64451 01548 76155 55025 02930 31135
RG(21, 6) = 7 89642 45348 87112 95542 07357 67585 13220 40800
RG(21, 8) = 9 52659 09036 62623 49618 60731 83718 44302 86563 11550
RG(21, 10) = 431 99239 38801 24746 08477 96806 54972 03596 19669 98592
RG(22, 10) = 745 97015 24698 60833 84362 42835 75087 30776 06371 61906 67288

(e) labelled regular bipartite graphs

RBG(2, 1) = 2
RBG(3, 1) = 6
RBG(4, 1) = 24
RBG(4, 2) = 90
RBG(5, 1) = 120
RBG(5, 2) = 2040
RBG(6, 1) = 720
RBG(6, 2) = 67950
RBG(6, 3) = 2 97200
RBG(7, 1) = 5040
RBG(7, 2) = 31 10940
RBG(7, 3) = 689 38800
RBG(8, 1) = 40320
RBG(8, 2) = 1875 30840
RBG(8, 3) = 2 40461 89440
RBG(8, 4) = 11 69637 96250
RBG(9, 1) = 3 62880
RBG(9, 2) = 1 43981 71200
RBG(9, 3) = 1202 57808 92160
RBG(9, 4) = 31503 14008 02720
RBG(10, 1) = 36 28800
RBG(10, 2) = 137 17853 98200
RBG(10, 3) = 8 30281 64994 43200
RBG(10, 4) = 1289 14458 41435 23800
RBG(10, 5) = 6736 21828 74304 60752
RBG(11, 1) = 399 16800
RBG(11, 2) = 15881 53879 62000
RBG(11, 3) = 7673 68877 74636 32000
RBG(11, 4) = 77 22015 01701 39844 56000
RBG(11, 5) = 2268 85231 70021 57135 35680
RBG(12, 1) = 4790 01600
RBG(12, 2) = 21 95954 74100 77200
RBG(12, 3) = 92 54768 77016 01242 88000
RBG(12, 4) = 6 55998 39591 25190 89827 12750
RBG(12, 5) = 1164 93371 08041 07898 07329 43360

RBG(12, 6) = 6405 13758 89927 38003 55498 04336

RBG(13, 1) = 62270 20800

RBG(13, 2) = 3574 34059 91044 75200

RBG(13, 3) = 1 42556 16537 57873 59868 67200

RBG(13, 4) = 76923 70719 09157 57910 85711 90000

RBG(13, 5) = 885 28277 62101 20715 08671 56197 24160

RBG(13, 6) = 28278 44745 41650 11203 55173 45844 21120

RBG(14, 1) = 8 71782 91200

RBG(14, 2) = 6 76508 13362 31358 14000

RBG(14, 3) = 2753 71524 49960 68059 77394 68800

RBG(14, 4) = 12163 52574 13474 97524 17830 77409 04300

RBG(14, 5) = 969 86285 29415 10660 94112 97026 27979 53280

RBG(14, 6) = 1 90404 19266 27879 97666 31032 46184 91390 13040

RBG(14, 7) = 10 87381 82111 44649 86147 05217 75461 49763 71200

RBG(15, 1) = 130 76743 68000

RBG(15, 2) = 1473 20988 74154 20994 84000

RBG(15, 3) = 65 66204 06980 02721 81065 90051 84000

RBG(15, 4) = 2541 43667 82268 66358 50590 66155 50954 68000

RBG(15, 5) = 1496 26288 16774 97094 07727 77740 08499 85217 38256

RBG(15, 6) = 19 09217 49838 17380 04722 91626 51397 27069 77650 56000

RBG(15, 7) = 649 92060 69716 25785 99383 34721 82540 93312 18314 48000

RBG(16, 1) = 2092 27898 88000

RBG(16, 2) = 3 65747 51938 49174 83413 60000

RBG(16, 3) = 1 90637 22850 65358 83540 30203 83641 60000

RBG(16, 4) = 689 26927 35539 27875 30584 56514 22173 77622 15000

RBG(16, 5) = 3183 52962 46458 47695 37507 81437 69686 74106 56203 16160

RBG(16, 6) = 278 87419 38278 12806 74144 42355 62382 83796 79491 75021 53600

RBG(16, 7) = 59195 05849 16356 81493 50844 92609 53867 86290 52742 02726 40000

RBG(16, 8) = 3 48122 90428 17629 82853 94893 93677 37079 51192 22412 42397 96250

RBG(17, 1) = 35568 74280 96000

RBG(17, 2) = 1026 89029 98771 35115 73271 04000

RBG(17, 3) = 6658 25560 53277 22511 75492 20297 29382 40000

RBG(17, 4) = 238 87159 61292 85108 31568 47890 88803 52576 29425 60000

RBG(17, 5) = 9168 86244 09579 91340 21184 73210 01642 08874 60056 79011 12320

RBG(17, 6) = 5814 83639 87656 80061 78350 59113 77029 77810 73474 79697 64240 79360

RBG(17, 7) = 80 57126 57304 85075 54350 27526 80373 84983 17410 00393 05713 37498 62400

RBG(17, 8) = 2883 59745 10435 13013 53483 15975 14855 70644 70507 35254 64474 82832 00000

RBG(18, 1) = 6 40237 37057 28000

RBG(18, 2) = 3 23741 52474 16050 49157 79711 84000

RBG(18, 3) = 276 78064 80542 21157 16515 50187 27922 24727 04000

RBG(18, 4) = 104 31401 63479 38179 06193 87316 37674 79249 71079 75682 00000

RBG(18, 5) = 35153 42842 78878 21891 66618 02093 88087 91403 55655 30999 24443 95520

RBG(18,6) = 1 69902 96408 43749 25521 22113 93198 75383 12068 34714 11242 51843 10626 98496

RBG(18,7) = 16085 61169 52745 41217 70507 56300 36200 29191 00798 86551 64134 37889 41131 77600

RBG(18,8) = 36 35049 05535 87197 69285 33373 95779 86374 44106 30573 95414 70067 07586 25750 50400

RBG(18,9) = 218 82630 32066 76892 25357 10968 72403 64487 59525 15497 73489 44382 85330 14608 50000

RBG(19,1) = 121 64510 04088 32000

RBG(19,2) = 1138 80369 80465 07486 98191 89710 40000

RBG(19,3) = 13 56440 63609 15457 77172 03991 43711 43095 22677 76000

RBG(19,4) = 56 68952 42033 46063 74610 17821 03742 97911 82821 01595 34954 88000

RBG(19,5) = 1 76738 29490 59962 13547 52959 24031 59232 11049 90709 04289 49362 97904 02560

RBG(19,6) = 68 39994 13471 98950 14161 42783 54058 12008 45617 29294 45342 88003 77070 00536 47360

RBG(19,7) = 46 28330 74905 16592 14753 95804 92567 77549 15117 64842 78047 72565 64810 46207 79892 73600

RBG(19,8) = 68612 67651 67803 58177 39055 83926 29673 61238 64947 79212 03431 41661 26668 95590 47174 40000

RBG(19,9) = 25 56950 74974 13510 95377 56379 30189 11334 91302 77163 95253 81023 18214 58496 11646 29973 05600

RBG(20,1) = 2432 90200 81766 40000

RBG(20,2) = 4 44432 47430 08447 87327 72568 49694 40000

RBG(20,3) = 77705 10468 93402 39554 38806 16451 33412 62150 71334 40000

RBG(20,4) = 37 91158 96134 25952 73339 37182 64069 14767 88778 77626 16902 20245 15000

RBG(20,5) = 11 49470 69528 37749 55905 66828 13209 87092 83275 36743 80465 80548 62081 24926 73024

RBG(20,6) = 3735 88018 43970 43563 78290 38407 62733 51873 54506 30512 97912 08448 13042 47703 03792 00000

RBG(20,7) = 18880 43648 94220 46737 94518 29831 85350 75165 35678 19041 66261 14575 17750 16381 89936 23449 60000

RBG(20,8) = 1908 47751 25745 21314 93459 20586 81239 88140 51323 24595 15599 80401 96602 54318 22577 35472 85125 35000

RBG(20,9) = 4 54023 31876 50987 86174 37844 66977 64598 85762 04689 01278 65489 65759 92995 91260 69068 17792 03609 60000

RBG(20,10) = 27 85672 66191 48825 49478 40785 80561 90269 80295 61072 47731 39219 03857 59473 98137 13335 06957 28357 19440

RBG(21,3) = 5163 44916 59246 59859 16193 13295 56346 68149 85331 42528 00000

RBG(21,4) = 30 88496 93867 80026 62586 22352 50047 09655 84171 50767 55052 99965 88866 00000

RBG(21,5) = 95 52333 76645 53264 78146 00896 22799 78720 90794 24811 30861 19762 31773 74095 20222 38720

RBG(21,6) = 2 72943 86329 93987 39582 40923 08055 84611 70115 40182 32073 61888 63349 12640 80340 76447 66351 36000

RBG(22,3) = 395 22359 63968 63623 53739 20011 83776 40822 75323 31216 56832 00000

RBG(22, 4) = 30 36905 52732 35097 17071 02530 20386 08437 20388 55938 31697 81279 41004 52216 50000

RBG(22, 5) = 1002 95780 68852 10288 65434 81690 81933 91633 29022 63902 49845 04722 50331 71927 62732 14244 45440

RBG(22, 6) = 263 30984 84807 14038 14616 04127 79422 95198 84701 14580 77593 27493 19435 77046 64198 62807 53259 80717 85600

RBG(23, 3) = 34 62586 57510 56601 78604 09390 91232 06792 50540 65762 03499 20870 40000

RBG(23, 4) = 35 74210 97188 85426 83805 46529 17177 75117 19323 00422 06859 35977 99569 49256 51220 00000

RBG(23, 5) = 13169 00713 84232 47912 69779 02614 82287 52605 98137 85932 69903 81173 78417 68750 80907 31520 42843 75040

RBG(24, 3) = 3 45216 77003 53276 87605 58513 86158 97466 61018 46820 80602 69335 41273 60000

RBG(24, 4) = 49 96413 67549 25549 94842 75577 04072 25675 62912 75102 47792 71005 27069 84849 71227 46036 90000

(f) eulerian circuits in the complete graph

EK(3) = 2

EK(5) = 264

EK(7) = 10 15440

EK(9) = 9 04492 51200

EK(11) = 169 10704 34783 65440

EK(13) = 62674 16821 16507 92035 99360

EK(15) = 4435 71127 63059 05572 69512 76764 67200

EK(17) = 5839 30527 51308 54565 39291 38771 58038 68245 19680

Appendix — the numbers

(a) labelled regular tournaments

RT(1) = 1
RT(3) = 2
RT(5) = 24
RT(7) = 2640
RT(9) = 32 30080
RT(11) = 4 82515 08480
RT(13) = 9 30770 06112 92160
RT(15) = 240 61983 49824 94283 79648
RT(17) = 85584 72055 41481 49511 79758 79680
RT(19) = 4271 02683 12628 45202 01657 80015 93666 76480
RT(21) = 3035 99177 67255 01434 06909 90026 40396 04333 20198 14400

(b) labelled eulerian digraphs

ED(1) = 1
ED(2) = 2
ED(3) = 10
ED(4) = 152
ED(5) = 7736
ED(6) = 13 75952
ED(7) = 8779 01648
ED(8) = 204 63203 73120
ED(9) = 17 65822 17023 61472
ED(10) = 5 69773 21983 69652 65152
ED(11) = 6 92800 70663 38878 38902 48448
ED(12) = 31 94140 76928 47758 20130 37245 06112
ED(13) = 561 21720 93887 11105 02272 39130 00322 61120
ED(14) = 37736 24389 96731 35332 92562 82026 36271 68278 87616
ED(15) = 97 44754 03179 97541 69218 00337 62069 41877 94308 61883 08480
ED(16) = 96934 27419 43194 32347 65129 25742 87605 35010 22995 32573 44779 87840

(c) labelled eulerian oriented graphs

EOG(1) = 1
EOG(2) = 1
EOG(3) = 3
EOG(4) = 15
EOG(5) = 219
EOG(6) = 7839
EOG(7) = 7 77069
EOG(8) = 2088 36207
EOG(9) = 15 64583 82975
EOG(10) = 32820 80160 21561
EOG(11) = 1946 87965 62657 10431
EOG(12) = 328 34193 09869 77413 59313
EOG(13) = 158 28097 85794 57849 90632 05301
EOG(14) = 218 98960 75577 09869 78834 04184 32175
EOG(15) = 872 69441 10689 80079 02526 09985 08645 17077

(d) labelled regular graphs

RG(4, 1) = 3
RG(5, 2) = 12
RG(6, 1) = 15
RG(6, 2) = 70
RG(7, 2) = 465
RG(8, 1) = 105
RG(8, 2) = 3507
RG(8, 3) = 19355
RG(9, 2) = 30016
RG(9, 4) = 10 24380
RG(10, 1) = 945
RG(10, 2) = 2 86884
RG(10, 3) = 111 80820
RG(10, 4) = 664 62606
RG(11, 2) = 30 26655
RG(11, 4) = 51884 53830
RG(12, 1) = 10395
RG(12, 2) = 349 44085
RG(12, 3) = 1 15552 72575
RG(12, 4) = 48 04139 21130
RG(12, 5) = 297 76351 37862
RG(13, 2) = 4382 63364
RG(13, 4) = 5211 33763 10985

RG(13, 6) = 2 09913 28709 73600

RG(14, 1) = 1 35135

RG(14, 2) = 59335 02822

RG(14, 3) = 1950 66318 14670

RG(14, 4) = 6 55124 65965 01035

RG(14, 5) = 283 09726 01841 59421

RG(14, 6) = 1803 59535 89647 73088

RG(15, 2) = 8 62489 51243

RG(15, 4) = 945 31390 72536 06891

RG(15, 6) = 18 72726 69012 71816 63775

RG(16, 1) = 20 27025

RG(16, 2) = 133 97519 21865

RG(16, 3) = 50 26295 87137 92825

RG(16, 4) = 1 55243 72224 85240 67795

RG(16, 5) = 524 69332 40770 03653 20163

RG(16, 6) = 23296 76580 69802 21975 16875

RG(16, 7) = 1 51385 92322 75324 22353 38875

RG(17, 2) = 2214 80510 88480

RG(17, 4) = 287 97220 46058 68264 22720

RG(17, 6) = 344 30864 02825 29972 04036 73760

RG(17, 8) = 14939 08809 73211 82119 40442 93500

RG(18, 1) = 344 59425

RG(18, 2) = 38824 67258 73208

RG(18, 3) = 1 87747 83788 96998 87800

RG(18, 4) = 59930 02310 42715 04940 60340

RG(18, 5) = 1764 78838 28569 85865 99722 68092

RG(18, 6) = 5 99722 97699 47050 27153 59174 22040

RG(18, 7) = 271 84977 22059 48458 08509 08045 26392

RG(18, 8) = 1793 19666 50258 85172 29050 89715 92750

RG(19, 2) = 7 19342 31097 63089

RG(19, 4) = 139 07595 61507 55900 18236 65540

RG(19, 6) = 12218 90111 37527 12984 45845 84754 80428

RG(19, 8) = 260 51341 89838 73007 07368 58744 50318 27810

RG(20, 1) = 6547 29075

RG(20, 2) = 140 46235 58216 28771

RG(20, 3) = 9762 73961 16036 31721 31825

RG(20, 4) = 35792 05185 12934 32427 84678 20756

RG(20, 5) = 10148 61308 10401 17624 31953 69019 32188

RG(20, 6) = 289 16028 25271 77828 14901 37004 21235 35900

RG(20, 7) = 98830 18890 80323 33162 33360 72448 97992 27748

RG(20, 8) = 45 54732 95761 28860 30974 46621 93995 60781 33650

RG(20, 9) = 304 00592 81615 70414 70070 85764 67967 97406 91838

RG(21, 2) = 2883 01399 43484 84940

RG(21, 4) = 101 64451 01548 76155 55025 02930 31135
RG(21, 6) = 7 89642 45348 87112 95542 07357 67585 13220 40800
RG(21, 8) = 9 52659 09036 62623 49618 60731 83718 44302 86563 11550
RG(21, 10) = 431 99239 38801 24746 08477 96806 54972 03596 19669 98592
RG(22, 10) = 745 97015 24698 60833 84362 42835 75087 30776 06371 61906 67288

(e) labelled regular bipartite graphs

RBG(2, 1) = 2
RBG(3, 1) = 6
RBG(4, 1) = 24
RBG(4, 2) = 90
RBG(5, 1) = 120
RBG(5, 2) = 2040
RBG(6, 1) = 720
RBG(6, 2) = 67950
RBG(6, 3) = 2 97200
RBG(7, 1) = 5040
RBG(7, 2) = 31 10940
RBG(7, 3) = 689 38800
RBG(8, 1) = 40320
RBG(8, 2) = 1875 30840
RBG(8, 3) = 2 40461 89440
RBG(8, 4) = 11 69637 96250
RBG(9, 1) = 3 62880
RBG(9, 2) = 1 43981 71200
RBG(9, 3) = 1202 57808 92160
RBG(9, 4) = 31503 14008 02720
RBG(10, 1) = 36 28800
RBG(10, 2) = 137 17853 98200
RBG(10, 3) = 8 30281 64994 43200
RBG(10, 4) = 1289 14458 41435 23800
RBG(10, 5) = 6736 21828 74304 60752
RBG(11, 1) = 399 16800
RBG(11, 2) = 15881 53879 62000
RBG(11, 3) = 7673 68877 74636 32000
RBG(11, 4) = 77 22015 01701 39844 56000
RBG(11, 5) = 2268 85231 70021 57135 35680
RBG(12, 1) = 4790 01600
RBG(12, 2) = 21 95954 74100 77200
RBG(12, 3) = 92 54768 77016 01242 88000
RBG(12, 4) = 6 55998 39591 25190 89827 12750
RBG(12, 5) = 1164 93371 08041 07898 07329 43360

RBG(12,6) = 6405 13758 89927 38003 55498 04336

RBG(13,1) = 62270 20800

RBG(13,2) = 3574 34059 91044 75200

RBG(13,3) = 1 42556 16537 57873 59868 67200

RBG(13,4) = 76923 70719 09157 57910 85711 90000

RBG(13,5) = 885 28277 62101 20715 08671 56197 24160

RBG(13,6) = 28278 44745 41650 11203 55173 45844 21120

RBG(14,1) = 8 71782 91200

RBG(14,2) = 6 76508 13362 31358 14000

RBG(14,3) = 2753 71524 49960 68059 77394 68800

RBG(14,4) = 12163 52574 13474 97524 17830 77409 04300

RBG(14,5) = 969 86285 29415 10660 94112 97026 27979 53280

RBG(14,6) = 1 90404 19266 27879 97666 31032 46184 91390 13040

RBG(14,7) = 10 87381 82111 44649 86147 05217 75461 49763 71200

RBG(15,1) = 130 76743 68000

RBG(15,2) = 1473 20988 74154 20994 84000

RBG(15,3) = 65 66204 06980 02721 81065 90051 84000

RBG(15,4) = 2541 43667 82268 66358 50590 66155 50954 68000

RBG(15,5) = 1496 26288 16774 97094 07727 77740 08499 85217 38256

RBG(15,6) = 19 09217 49838 17380 04722 91626 51397 27069 77650 56000

RBG(15,7) = 649 92060 69716 25785 99383 34721 82540 93312 18314 48000

RBG(16,1) = 2092 27898 88000

RBG(16,2) = 3 65747 51938 49174 83413 60000

RBG(16,3) = 1 90637 22850 65358 83540 30203 83641 60000

RBG(16,4) = 689 26927 35539 27875 30584 56514 22173 77622 15000

RBG(16,5) = 3183 52962 46458 47695 37507 81437 69686 74106 56203 16160

RBG(16,6) = 278 87419 38278 12806 74144 42355 62382 83796 79491 75021 53600

RBG(16,7) = 59195 05849 16356 81493 50844 92609 53867 86290 52742 02726 40000

RBG(16,8) = 3 48122 90428 17629 82853 94893 93677 37079 51192 22412 42397 96250

RBG(17,1) = 35568 74280 96000

RBG(17,2) = 1026 89029 98771 35115 73271 04000

RBG(17,3) = 6658 25560 53277 22511 75492 20297 29382 40000

RBG(17,4) = 238 87159 61292 85108 31568 47890 88803 52576 29425 60000

RBG(17,5) = 9168 86244 09579 91340 21184 73210 01642 08874 60056 79011 12320

RBG(17,6) = 5814 83639 87656 80061 78350 59113 77029 77810 73474 79697 64240 79360

RBG(17,7) = 80 57126 57304 85075 54350 27526 80373 84983 17410 00393 05713 37498 62400

RBG(17,8) = 2883 59745 10435 13013 53483 15975 14855 70644 70507 35254 64474 82832 00000

RBG(18,1) = 6 40237 37057 28000

RBG(18,2) = 3 23741 52474 16050 49157 79711 84000

RBG(18,3) = 276 78064 80542 21157 16515 50187 27922 24727 04000

RBG(18,4) = 104 31401 63479 38179 06193 87316 37674 79249 71079 75682 00000

RBG(18,5) = 35153 42842 78878 21891 66618 02093 88087 91403 55655 30999 24443 95520

RBG(18,6) = 1 69902 96408 43749 25521 22113 93198 75383 12068 34714 11242 51843 10626 98496

RBG(18,7) = 16085 61169 52745 41217 70507 56300 36200 29191 00798 86551 64134 37889 41131 77600

RBG(18,8) = 36 35049 05535 87197 69285 33373 95779 86374 44106 30573 95414 70067 07586 25750 50400

RBG(18,9) = 218 82630 32066 76892 25357 10968 72403 64487 59525 15497 73489 44382 85330 14608 50000

RBG(19,1) = 121 64510 04088 32000

RBG(19,2) = 1138 80369 80465 07486 98191 89710 40000

RBG(19,3) = 13 56440 63609 15457 77172 03991 43711 43095 22677 76000

RBG(19,4) = 56 68952 42033 46063 74610 17821 03742 97911 82821 01595 34954 88000

RBG(19,5) = 1 76738 29490 59962 13547 52959 24031 59232 11049 90709 04289 49362 97904 02560

RBG(19,6) = 68 39994 13471 98950 14161 42783 54058 12008 45617 29294 45342 88003 77070 00536 47360

RBG(19,7) = 46 28330 74905 16592 14753 95804 92567 77549 15117 64842 78047 72565 64810 46207 79892 73600

RBG(19,8) = 68612 67651 67803 58177 39055 83926 29673 61238 64947 79212 03431 41661 26668 95590 47174 40000

RBG(19,9) = 25 56950 74974 13510 95377 56379 30189 11334 91302 77163 95253 81023 18214 58496 11646 29973 05600

RBG(20,1) = 2432 90200 81766 40000

RBG(20,2) = 4 44432 47430 08447 87327 72568 49694 40000

RBG(20,3) = 77705 10468 93402 39554 38806 16451 33412 62150 71334 40000

RBG(20,4) = 37 91158 96134 25952 73339 37182 64069 14767 88778 77626 16902 20245 15000

RBG(20,5) = 11 49470 69528 37749 55905 66828 13209 87092 83275 36743 80465 80548 62081 24926 73024

RBG(20,6) = 3735 88018 43970 43563 78290 38407 62733 51873 54506 30512 97912 08448 13042 47703 03792 00000

RBG(20,7) = 18880 43648 94220 46737 94518 29831 85350 75165 35678 19041 66261 14575 17750 16381 89936 23449 60000

RBG(20,8) = 1908 47751 25745 21314 93459 20586 81239 88140 51323 24595 15599 80401 96602 54318 22577 35472 85125 35000

RBG(20,9) = 4 54023 31876 50987 86174 37844 66977 64598 85762 04689 01278 65489 65759 92995 91260 69068 17792 03609 60000

RBG(20,10) = 27 85672 66191 48825 49478 40785 80561 90269 80295 61072 47731 39219 03857 59473 98137 13335 06957 28357 19440

RBG(21,3) = 5163 44916 59246 59859 16193 13295 56346 68149 85331 42528 00000

RBG(21,4) = 30 88496 93867 80026 62586 22352 50047 09655 84171 50767 55052 99965 88866 00000

RBG(21,5) = 95 52333 76645 53264 78146 00896 22799 78720 90794 24811 30861 19762 31773 74095 20222 38720

RBG(21,6) = 2 72943 86329 93987 39582 40923 08055 84611 70115 40182 32073 61888 63349 12640 80340 76447 66351 36000

RBG(22,3) = 395 22359 63968 63623 53739 20011 83776 40822 75323 31216 56832 00000

RBG(22,4) = 30 36905 52732 35097 17071 02530 20386 08437 20388 55938 31697 81279 41004 52216 50000

RBG(22,5) = 1002 95780 68852 10288 65434 81690 81933 91633 29022 63902 49845 04722 50331 71927 62732 14244 45440

RBG(22,6) = 263 30984 84807 14038 14616 04127 79422 95198 84701 14580 77593 27493 19435 77046 64198 62807 53259 80717 85600

RBG(23,3) = 34 62586 57510 56601 78604 09390 91232 06792 50540 65762 03499 20870 40000

RBG(23,4) = 35 74210 97188 85426 83805 46529 17177 75117 19323 00422 06859 35977 99569 49256 51220 00000

RBG(23,5) = 13169 00713 84232 47912 69779 02614 82287 52605 98137 85932 69903 81173 78417 68750 80907 31520 42843 75040

RBG(24,3) = 3 45216 77003 53276 87605 58513 86158 97466 61018 46820 80602 69335 41273 60000

RBG(24,4) = 49 96413 67549 25549 94842 75577 04072 25675 62912 75102 47792 71005 27069 84849 71227 46036 90000

(f) eulerian circuits in the complete graph

EK(3) = 2

EK(5) = 264

EK(7) = 10 15440

EK(9) = 9 04492 51200

EK(11) = 169 10704 34783 65440

EK(13) = 62674 16821 16507 92035 99360

EK(15) = 4435 71127 63059 05572 69512 76764 67200

EK(17) = 5839 30527 51308 54565 39291 38771 58038 68245 19680