Negation, Absurdity and Contrariety

by

Neil Tennant*

Department of Philosophy
230 North Oval Mall
The Ohio State University
Columbus Ohio 43210
Tel. 614-2921591
email tennant.9@osu.edu

September 20, 2004

Abstract

I argue for a rule-based account of negation answering to both constructivist and relevantist demands. We can give such an account in terms of basic contrarieties, and by co-inductively defining proofs and disproofs, without having to make explicit appeal to the absurdity constant $\perp$. If we do make such an appeal, it is to $\perp$ only as a structural punctuation marker within deductions, a device that allows us to assimilate disproofs to the general class of proofs. $\perp$ does not, in this rôle, need to be governed by any 'introduction' or 'elimination' rules of its own. Nor does $\perp$ need to be treated as a propositional constant eligible for embedding within other sentences. But even if we do treat $\perp$ as an embeddable propositional constant, it does not follow that negation can, let alone should, be defined in terms of it. Negation should be taken as primitive, and one should explain how a grasp of its sense arises from one's prior grasp of primitive metaphysical contrarieties within an interpreted language.

*To appear in D. Gabbay and H. Wansing, eds., *Negation*, Kluwer Academic Press. Much of the stimulus for this essay derives from exchanges with Michael Hand.

1 Introduction

There is a widespread view among logicians that can be summarized as follows. Absurdity (or falsity) — commonly designated as $\perp$ — is a propositional constant. It may be used, with the conditional connective $\supset$, to define negation, so that $\neg A =_{df} A \supset \perp$. Therefore, whether or not negation is correctly to be understood as classical in nature will depend, ultimately, on the logical properties correctly ascribable to ($\supset$ and to) $\perp$. Moreover, these properties can be correctly ascribed within systems of propositional logic that are standard in at least two regards:

1. their deducibility relations are unrestrictedly transitive[1] and they admit of a deduction theorem; and

2. they permit at least some non-trivial 'dilutions on the right': that is, they allow that when $\perp$ follows from a set Δ of premisses, then *by virtue of that fact alone*, some sentence A other than $\perp$ follows from Δ also.[2]

One variant of this view goes a little further, maintaining that the meaning — and therefore also the logical properties — of $\perp$ are to be captured by appropriate rules of natural deduction. This is thought to be analogous to the way that the meanings — and therefore also the logical properties — of the connectives and quantifiers are characterized by the rules of natural deduction that govern them.

Michael Hand holds the view just characterized[3]. He has interesting and apposite points to make about how $\perp$ fares in the respective propositional systems of minimal, intuitionistic and classical logic. He canvasses the possibility of an introduction rule for $\perp$ to balance the absurdity rule

[1] That this is an underlying systematic assumption worth challenging was the topic of my essay 'The Transmission of Truth and the Transitivity of Deduction', in D.Gabbay (ed.), *What is a Logical System?*, Oxford University Press, 1994, pp.161-177. Note that in sequent calculus, we do not mark inconsistencies explicitly with $\perp$ on the right, but use the empty succedent instead.

[2] This is immediate when the natural deduction system contains the 'absurdity rule' (also known as *ex falso quodlibet*) according to which any sentence may be inferred from $\perp$. Intuitionistic and classical logic both contain the absurdity rule. But minimal logic also permits non-trivial dilutions on the right. For it has the rule of negation introduction without any restriction to the effect that $\neg A$ may be inferred from $\perp$ only if A has actually been *used* as an assumption in the derivation of $\perp$. Thus in minimal logic we have $A, \neg A \vdash \neg B$, even though we do not have $A, \neg A \vdash B$.

[3] 'Antirealism and Falsity', this volume.

$$\frac{\perp}{A}$$

conceived of as the elimination rule for $\perp$.[4] Much of his essay is devoted to exhibiting alleged difficulties standing in the way of construing $\neg A$, when it is defined as $A \supset \perp$, as being the genuine *negation* of A. His conclusion seems to be one of pessimism about the prospects for a rule-based, anti-realist or constructivist account of negation via the absurdity constant $\perp$.

In this essay I intend to challenge the orthodox assumptions laid out above in order to undermine this pessimistic conclusion. I shall be considering the following questions:

1. Is $\perp$ a genuine propositional constant, eligible to be a constituent of other sentences?

2. Is $\neg A$ is equivalent to $A \supset \perp$? and should $\neg$ be taken as primitive, rather than defined in terms of $\perp$ in the usual way?

3. Is an antirealist who accounts for the meanings of the connectives and quantifiers by appealing to the introduction and elimination rules governing them committed to giving a slavishly similar account of the meaning of $\perp$?

4. What is the source of the semantic or pragmatic 'badness' that $\perp$ seeks to register? — and how does the denial $\neg A$ of A succeed in laying blame on A for the 'badness' in question?

Concerns (1)-(4) are addressed, respectively, in sections 2-5 below. There will, however, perforce be some degree of overlap among these sections.

[4] Here one can, without loss of generality, restrict A to be atomic. Then Hand suggests that the corresponding introduction rule for $\perp$ should be Dummett's rule of 'complete atomic entailment':

$$\frac{A \quad B \quad C \ \ldots}{\perp}$$

where A, B, C, ... is a complete list of the atomic sentences of the language (a list which may be infinite). This rule was stated by Dummett, in *The Logical Basis of Metaphysics*, Harvard University Press, 1991, at p.295.

2 Is $\perp$ a genuine propositional constant, eligible to be a constituent of other sentences?

On the face of it, no; unless $\perp$ is conventionally identified with a particular 'absurd' sentence of an already interpreted language. For on this view, of course, we would permit the embedding of that absurd sentence within other sentences. Such would be the case with the intuitionistic mathematicians' identification of $\perp$ with the atomic arithmetical sentence $0 = 1$. The latter is *arithmetically* absurd. If one could prove that $0 = 1$, then one could prove *any* sentence of arithmetic in the language based on the connectives $\wedge$, $\vee$ and $\supset$, and the quantifiers $\exists$ and $\forall$. (There is an easy inductive proof of this metatheorem. The main part of the proof involves establishing that if one could prove $0 = 1$ then one would be able to prove $m = n$ for every numeral m, n in the language of arithmetic. One would also be able to prove any 'negation' $\neg A$, that is, any sentence of the form $A \supset 0 = 1$.) Indeed, it is this consideration that leads intuitionistic mathematicians, without much further ado, to endorse the absurdity rule, *given* their identification of $\perp$ with the sentence $0 = 1$.

In my view, however, this is a strategic mistake. Instead of making this identification, one should rather admit only the atomic rule of inference

$$\frac{0 = 1}{\perp}$$

Here the occurrence of $\perp$ is *distinct* (both as token and as type) from the occurrence of $0 = 1$. Inferring *to* $\perp$ is the paradigmatic way of showing that $0 = 1$ is *itself* absurd. But this is not to make the sentence $0 = 1$ *type-identical to* the absurdity constant $\perp$. That $0 = 1$ is absurd is a meaning-fact arising out of the meanings of 0, $=$ and the successor sign $s(\ )$. (For 1 is simply $s(0)$.) It is the *internal composition* of the sentence $0 = 1$ that *both* makes it absurd *and* prevents it from being a primitive sign for absurdity! It is absurd to claim that 0 is identical to 1, in just the same way that is is absurd to say, of a physical object, that it is in two different spatial locations at the same time; or of two events, that one was both earlier and later than the other; or of a visible object, that it is both red all over and green all over; or to say, when speaking to another person, 'You and I are numerically identical, that is, the same person'. Each of these absurd claims can feature as the premiss A in an inference

$$\frac{A}{\perp}$$

Or, if one wishes to construe the absurd claims in question as consisting really of a pair of claims (such as 'this visible object is red all over' and 'this visible object is green all over'), then we have an instance of the general form

$$\frac{A \quad B}{\perp}$$

where it is now a particular *pair* of premisses A, B that gives rise to absurdity, by virtue of the particular expressions that they involve. But this basic metaphysico-semantical fact of absurdity could just as well be registered by the horizontal line:

$$\underline{A \quad B}$$

with nothing below it. More will be made of this simple point in due course.

One should not tie one's conception of absurdity too closely to any particular discourse, such as that of arithmetic. If absurdity is a properly logical notion, it should be discourse-independent. For even on the view being challenged here, negation is surely discourse-independent, and yet is to be defined in terms of absurdity. That is why one should refrain from identifying the absurdity constant $\perp$ with any particular sentence within a discourse. For, if one does so, one makes absurdity discourse-specific. Two unpalatable options would then be the only ones available, which I shall call the *unitary option* and the *fragmented option* respectively:

1. take the arithmetical absurdity $0 = 1$ as the only absurdity, and try to show that *it* would follow from sentences rightly regarded as absurd in *other* discourses (such as discourse about bodies in space and time, or colour and shape discourse); or

2. designate similar 'basic absurdities' in these other discourses, intended to play the rôle, within those discourses, that $0 = 1$ plays within arithmetical discourse.

The unitary option (1) is extremely difficult to justify, *even when* one has the absurdity rule, which on the view in question would be

$$\frac{0 = 1}{A}$$

For the problem is how one would justify inferential passage *from* patent absurdities in a non-arithmetical discourse *to* the supposedly primal absurdity

$0 = 1$! The rule just stated only deals with the converse direction. There has to be a violation of relevance here.

The fragmented option (2) is likewise extremely difficult to justify since there is, in general, no specific canonically absurd sentence in a given non-arithmetical discourse that would be suitably analogous to $0 = 1$ in the arithmetical case. In particular, it is difficult to see how one could meet, in any other than arithmetical discourse, the standard theorist's insistence that the absurdity rule itself should somehow be 'derived' once the canonically absurd sentence, such as $0 = 1$, has been identified. It is just too far-fetched to conceive of ordinary linguistic negations $\neg A$ in a discourse D as 'really' conditional-absurdity claims of the form $A \supset \perp$, where the constant $\perp$ in question is supposed to be some specific sentence, chosen from the discourse D, such as (say) 'I am you' or 'The Taj Mahal is red all over and the Taj Mahal is green all over'.

Nor does this point depend on the canonically absurd sentence B chosen. It has the same force regardless of what actual sentence B might be substituted for 'I am you' or 'The Taj Mahal is red all over and the Taj Mahal is green all over'. It is a clearly counterintuitive grammatical suggestion that explicit negation operators or prefixes (such as the English particles 'not', 'un-', 'im-' etc.) are the surface indicators of some underlying *conditional absurdity* (of the form '$\ldots \supset B$') at a 'deeper', more 'logical' level — a conditional absurdity, moreover, whose *consequent* B is some other grammatical sentence of the language whose identity is completely obscured in the 'surface' form using the negative particle in question!

But even if these linguistic intuitions were to be dismissed in favour of grander theory, the problem remains, with the fragmented option, that in each discourse there would be a canonically absurd sentence provincial to that discourse; and we would still need to show that these respective canonically absurd sentences were somehow inferentially equivalent, on pain of not being able to confer a uniform sense on the negation operator across all discourses. Absurdity is much more cosmopolitan a notion than the discourse-specific model would make it. And the cosmopolitanism required could be purchased, it would seem, only at the cost of irrelevance, by simply stipulating that all the 'absurdities' are interdeducible.

I conclude, then, that $\perp$ is not to be regarded as a propositional constant that may occur as a constituent in other sentences. This alone is enough to prevent us from defining $\neg A$ as $A \supset \perp$. We shall presently see that there is another cogent reason for refraining from such definition.

It remains, in this section, to consider whether, indeed, $\perp$ should be adopted as a *propositional constant*, albeit a non-embeddable one. Suppose we accept that we are prevented from identifying $\perp$ with any particular sentence within any given discourse. With what right, then, could we persist in claiming that $\perp$ itself may nevertheless still be regarded as (stating) a *proposition*? No-one ever utters a non-embeddable sentence that could be regimented as $\perp$. Even a cry of 'Contradiction!' or 'That's absurd!' is ineligible for such regimentation, for such a cry is properly a *metalinguistic commentary* on some argumentative predicament that one's dialectical opponent (real or imagined) should realise she is in. Accordingly, an occurrence of $\perp$ is appropriate only within a proof (or a disproof), as a kind of structural punctuation mark. It tells us where a story being spun out gets tied up in a particular kind of knot — the knot of patent absurdity, or of self-contradiction. This much is evident from the usual way that $\perp$ features in the natural deduction rules of $\neg$-Introduction and $\neg$-Elimination:

$$\dfrac{\begin{array}{c}\overline{A}^{(i)}\\ \vdots \\ \perp\end{array}}{\neg A}{\scriptstyle (i)}\ \neg I \qquad\qquad \dfrac{\neg A \qquad A}{\perp}\ \neg E$$

I turn now to some reflections on the rôle played by the absurdity sign $\perp$ as a piece of notation within proofs and disproofs involving applications of these rules.

Suppose one has a logical system for which the existence of proofs is indicated by the usual turnstile $\vdash$, a relation of exact deducibility holding between a set of premisses on the left and a conclusion on the right. The intuitive meaning of '$X \vdash A$' is that there is a proof whose conclusion is A and whose premisses (undischarged assumptions) form the set X. I reckon any premiss P to X 'just once', no matter how often P may have been 'used' as an assumption in the proof.

There are two extreme cases:

1. X is empty. Then '$\vdash A$' means that A is a theorem. That is, there is a proof of A 'from no assumptions'. Any assumptions used for the sake of argument within the proof will have been discharged by the stage at which we reach A as the conclusion. Example: The one-step proof

$$\dfrac{\overline{B}^{(1)}}{B \supset B}{(1)}$$

justifies the claim $\vdash B \supset B$.

2. A is 'empty'. Then '$X \vdash$' means that there is a disproof of X, that is, a deduction showing that X is inconsistent. Example: The one-step disproof, consisting of just one application of $\neg E$:

$$\dfrac{\neg B \quad B}{\bot}$$

justifies the claim $B, \neg B \vdash$.

In a single-conclusion natural deduction calculus[5] one usually uses the absurdity symbol $\bot$ as an explicit conclusion to mark this, and one writes '$X \vdash \bot$' instead of '$X \vdash$ '. Thus: $B, \neg B \vdash \bot$. It is the sequent calculus which prompts the use of the turnstile with nothing to the right, because in the sequent calculus we prove the inconsistency of X by deriving the sequent X : , with the empty succedent. Thus the last disproof, re-cast in the sequent calculus, would be

$$\dfrac{B : B}{B, \neg B \ :}$$

$\bot$ was introduced by modern logicians in the natural deduction context in very much the way that the ancient Hindus introduced the symbol 0 into arithmetic. Rather than writing nothing, we indicate that it's nothing that we intend, by writing something in particular, which is to stand for the nothing that we intend. Thus the rule $\neg E$ above of negation elimination in a system of natural deduction is written with $\bot$ as its explicit conclusion, instead of being cast in the form

$$\dfrac{\neg A \quad A}{}$$

where the absence of anything below the inference stroke would explicitly represent a logical dead-end.[6] Likewise, the rule $\neg I$ of negation introduction

[5] See, for example, the treatment in my *Natural Logic*, Edinburgh University Press, 2nd edn., 1990.

[6] As the reader will see, I prefer this way of construing negation elimination and all other rules that produce disproofs. The use of the horizontal line as a logical dead-end was foreshadowed above, in our discussion of primitive contraries in a language.

above has a schematic subproof of the explicit conclusion $\perp$, rather than being cast in the form

$$\begin{array}{l} \overline{}^{(i)} \\ A \\ \vdots \\ \overline{} \\ \overline{}^{(i)} \quad \neg I \\ \neg A \end{array}$$

where the absence of anything between the lower two inference strokes would once again represent a logical dead-end.

It is a pity that we have no convention of using empty spaces in natural deductions to represent emptiness. For I would like to be able to say that a natural deduction 'of $\perp$', that is, ending with an explicit occurrence of $\perp$, is just a proof of the empty conclusion; and I would like to say that, for example, the sequent $X : \perp$ is a sub-sequent of $X : A$.

I shall pretend that this notational quirk of natural deduction does not prevent us from saying such things. Thus we shall be able to treat occurrences of $\perp$ as if they 'really were not there'. After a logical dead-end there is to be no afterlife, no sentential resurrection in the form of a would-be 'propositional constant' $\perp$. Accordingly, there would be no need to give rules 'for' such a propositional constant; for it would be absent from the language. One would still be left with the problem of explaining the genesis of our grasp of negation; and this is a problem to which I shall be proposing a solution in due course.

One way of appreciating the upshot of this suggestion is to consider how the usual inductive definition of proof[7] might be changed so as to become a simultaneous inductive definition of both proof and disproof. Before, when one could make explicit appeal to $\perp$ (whether as propositional constant or as 'conclusion marker' of a reductio proof) one could treat disproofs as special kinds of proof, namely proofs of the 'conclusion' $\perp$. Now, however, given that we are eschewing $\perp$ altogether, it behooves us to take special care over the definition of proof and of disproof. To this end, I would propose a simultaneous inductive definition of the two notions

'Π is a proof of A from the set Δ of undischarged assumptions'

and

[7]See, for example, the conventional treatment set out in *Natural Logic, op.cit.*

'Π is a disproof of the set Δ of undischarged assumptions'.

Proofs and disproofs are both *deductions.* We shall treat proofs and disproofs as types. The ground type is that of single sentence occurrences. The basis clause would be:

Any occurrence of a sentence A is a proof of A from $\{A\}$.

(There is no basis clause specifically for disproofs.)

The higher types of deductions are obtained by applying rules of inference. Higher-type *proofs* are either of the form $\rho(\Pi_1, \ldots, \Pi_n, C)$, where ρ is the *introduction* rule applied at the last step, $\Pi_1, \ldots, \Pi_n$ are the immediate sub-proofs or sub-disproofs, and C is the conclusion of that terminal introduction; or of the form $\rho(M, \Pi_1, \ldots, \Pi_n, C)$, where ρ is the *elimination* rule applied at the last step, C is its conclusion, $\Pi_1, \ldots, \Pi_n$ are the immediate sub-proofs or sub-disproofs, and M is the major premiss of the terminal elimination. Higher-type *disproofs* are of the form $\rho(M, \Pi_1, \ldots, \Pi_n)$, where ρ is the *elimination* rule applied at the last step and M is the major premiss of that terminal elimination. Note that disproofs have no 'conclusion'.

Disproofs arise only through the terminal application of elimination rules. They cannot arise from the terminal application of introduction rules. Terminal application of introduction rules produces only proofs, not disproofs. But terminal application of elimination rules (other than $\neg E$) can produce either a proof or a disproof, depending on the nature of the immediate sub-deductions.

I shall use the convention that 'Δ, A' stands for $\Delta \cup \{A\}$ where A is understood not to be a member of Δ.

Building on the basis clause are the following inductive clauses. The introduction rules generate only proofs; the elimination rule for negation generates only disproofs; while the remaining elimination rules, for $\wedge$, $\vee$ and $\supset$, generate both proofs and disproofs.

¬-Introduction

If Π is a disproof of Δ, A then $\neg I(\Pi, \neg A)$ is a proof of $\neg A$ from Δ.

Graphically, ¬-Introduction could be rendered thus:

$$\dfrac{\underbrace{\Delta\ ,\ \overset{\square\,\overline{}\,(i)}{A}}_{\Pi}}{\neg A}\,(i)\quad \neg I$$

where the box appended to the discharge stroke over the assumption A indicates that A is required to have been used as an assumption in the disproof Π. Note that there is no explicit use of the absurdity constant $\perp$ as the 'last line' of Π immediately above the conclusion $\neg A$. Instead, the disproof Π will itself have 'ended' with just a horizontal line above the one now marked with '(i)' that represents the application of $\neg I$. All disproofs, when displayed graphically, just end with horizontal lines, rather than with a terminal occurrence of $\perp$.

$\neg$-Elimination generates only disproofs:
If Π is a proof of A from Δ then $\neg E(\neg A, \Pi)$ is a disproof of $\Delta \cup \neg A$.

Graphically, $\neg$-Elimination could be rendered thus:

$$\begin{array}{cc} & \Delta \\ & \Pi \\ \hline \neg A & A \end{array}$$

where the horizontal line has nothing below it.

Note also that our rule of $\neg$-Elimination is formulated in such a way that the major premiss $\neg A$ *stands proud*, that is, has no proof-work above it. This is because we are building normality into our definition of proof and disproof, not allowing any major premiss of an elimination to stand as the conclusion of *any* rule application.

This latter feature is very important for a proper appreciation of what is to follow. I shall be prohibiting major premisses of eliminations from standing as the conclusions of *any* rule applications. All major premisses for eliminations will stand proud. To secure this feature of proofs, we need to formulate the rules of $\wedge$-Elimination and of $\supset$-Elimination in their *parallel* forms rather than their usual *serial* forms (for which, see below).

These parallel forms for the elimination rules yield what I have called the *hybrid* system of proof, a blend of sequent calculus and of natural deduction.[8] A hybrid proof has the economy of form that one finds in a sequent proof, but the economy of node-labelling that one finds in a natural deduction. It is the perfect compromise between a sequent proof and a natural deduction, yielding great advantages for computational logic.

[8] See my *Autologic*, Edinburgh University Press, 1992.

The rules for negation have already displayed how the recursive definitions of proof and of disproof interact. This interaction is displayed by other rules as well, as we shall presently see.

$\wedge$-Introduction
If Π_1 is a proof of A from Δ_1 and Π_2 is a proof of B from Δ_2 then $\wedge I(\Pi_1, \Pi_2, A \wedge B)$ is a proof of $A \wedge B$ from $\Delta_1 \cup \Delta_2$.

Graphically:

$$\dfrac{\begin{array}{c}\Delta_1\\ \Pi_1\\ A\end{array} \qquad \begin{array}{c}\Delta_2\\ \Pi_2\\ B\end{array}}{A \wedge B}$$

$\wedge$-Elimination has two halves, one for *proofs* and one for *disproofs*. (Remember that introduction rules can create only proofs; whereas elimination rules can create both proofs and disproofs, except for the elimination rule for $\neg$, which can create only disproofs.) For *proofs* the rule of $\wedge$-Elimination is as follows:
If $\Delta \cap \{A, B\} \neq \emptyset$ and Π is a proof of C from Δ then $\wedge E(A \wedge B, \Pi, C)$ is a proof of C from $(\Delta \setminus \{A, B\}) \cup \{A \wedge B\}$.

Graphically:

$$\dfrac{A \wedge B \qquad \begin{array}{c}\underbrace{\Delta\ ,\ \overset{(i)\text{—}\square\text{—}(i)}{A\ ,\ B}}\\ \Pi\\ C\end{array}}{C}{\scriptstyle (i)}$$

where the box between the discharge strokes over the assumptions A and B indicates that at least one of them must have been used in the proof Π of C.

For *disproofs*, the **$\wedge$-Elimination** rule is:
If $\Delta \cap \{A, B\} \neq \emptyset$ and Π is a disproof of Δ then $\wedge E(A \wedge B, \Pi)$ is a disproof of $(\Delta \setminus \{A, B\}) \cup \{A \wedge B\}$.

Graphically:

$$\dfrac{A \wedge B \qquad \underbrace{\Delta\ ,\ \overset{(i)}{A}\ ,\ \overset{(i)}{B}}_{\Pi}}{}$$

where the horizontal line has nothing below it. Note also that the major premiss $A \wedge B$ stands proud, with no proof-work above it.

$\vee$**-Introduction**:
If Π is a proof of A from Δ then $\vee I(\Pi, A \vee B)$ is a proof of $A \vee B$ from Δ; and if Π is a proof of B from Δ then $\vee I(\Pi, A \vee B)$ is a proof of $A \vee B$ from Δ.

Graphically:

$$\dfrac{\begin{array}{c}\Delta\\ \Pi\\ A\end{array}}{A \vee B} \qquad\qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ B\end{array}}{A \vee B}$$

$\vee$**-Elimination** has four forms, depending on whether each of the cases yields a proof or a disproof:
(1) If Π_1 is a proof of C from Δ_1, A and Π_2 is a proof of C from Δ_2, B then $\vee E(A \vee B, \Pi_1, \Pi_2, C)$ is a proof of C from $\Delta_1 \cup \Delta_2 \cup \{A \vee B\}$.

Graphically:

$$\dfrac{A \vee B \qquad \begin{array}{c}\Delta_1\ ,\ \overset{\square(i)}{A}\\ \Pi_1\\ C\end{array} \qquad \begin{array}{c}\Delta_2\ ,\ \overset{\square(i)}{B}\\ \Pi_2\\ C\end{array}}{C}{\scriptstyle(i)}$$

(2) If Π_1 is a proof of C from Δ_1, A and Π_2 is a disproof of Δ_2, B then $\vee E(A \vee B, \Pi_1, \Pi_2, C)$ is a proof of C from $\Delta_1 \cup \Delta_2 \cup \{A \vee B\}$.

Graphically:

$$\dfrac{A \vee B \qquad \begin{array}{c}\Delta_1\ ,\ \overset{\square(i)}{A}\\ \Pi_1\\ C\end{array} \qquad \begin{array}{c}\Delta_2\ ,\ \overset{\square(i)}{B}\\ \Pi_2\end{array}}{C}{\scriptstyle(i)}$$

(3) If Π_1 is a disproof of Δ_1, A and Π_2 is a proof of C from Δ_2, B then $\vee E(A \vee B, \Pi_1, \Pi_2, C)$ is a proof of C from $\Delta_1 \cup \Delta_2 \cup \{A \vee B\}$.

Graphically:

$$\dfrac{A \vee B \qquad \begin{array}{c} \Delta_1 \,,\, \overset{\Box\text{---}(i)}{A} \\ \Pi_1 \end{array} \qquad \begin{array}{c} \Delta_2 \,,\, \overset{\Box\text{---}(i)}{B} \\ \Pi_2 \\ C \end{array}}{C}\,(i)$$

(4) If Π_1 is a disproof of Δ_1, A and Π_2 is a disproof of Δ_2, B then $\vee E(A \vee B, \Pi_1, \Pi_2)$ is a *disproof* of $\Delta_1 \cup \Delta_2 \cup \{A \vee B\}$.

Graphically:

$$\dfrac{A \vee B \qquad \begin{array}{c} \Delta_1 \,,\, \overset{\Box\text{---}(i)}{A} \\ \Pi_1 \end{array} \qquad \begin{array}{c} \Delta_2 \,,\, \overset{\Box\text{---}(i)}{B} \\ \Pi_2 \end{array}}{}\,(i)$$

where the horizontal line has nothing below it.

Note also that in all four cases of $\vee E$, the major premiss $A \vee B$ stands proud, with no proof-work above it.

The last introduction and elimination rules to deal with are for $\supset$. The rule of **$\supset$-Introduction** splits into two halves:

If Π is a disproof of Δ, A then $\supset I(\Pi, A \supset B)$ is a proof of $A \supset B$ from Δ.

Graphically:

$$\dfrac{\begin{array}{c} \Delta \,,\, \overset{\Box\text{---}(i)}{A} \\ \Pi_1 \end{array}}{A \supset B}\,(i)$$

where the box appended to the discharge stroke above A indicates that A must have been used in the disproof Π; and

If Π is a proof of B from Δ then $\supset I(\Pi, A \supset B)$ is a proof of $A \supset B$ from $\Delta \setminus \{A\}$.

Graphically:

$$
\begin{array}{c}
\Delta\,,\ \overset{\diamond\overline{}(i)}{A} \\
\Pi \\
B \\
\hline
A \supset B
\end{array}\ (i)
$$

where the diamond appended to the discharge stroke above A indicates that A need not have been used in the proof Π. If A was used, however, it is to be discharged.

The rule of $\supset$**-Elimination**, like the elimination rules for $\wedge$ and for $\vee$, produces both proofs and disproofs. The rule of $\supset$-Elimination for proofs is:

If Π_1 is a proof of A from Δ_1 and Π_2 is a proof of C from Δ_2, B then $\supset E(A \supset B, \Pi_1, \Pi_2, C)$ is a proof of C from $\Delta_1 \cup \Delta_2 \cup \{A \supset B\}$.

Graphically:

$$
\begin{array}{ccc}
 & \Delta_1 & \Delta_2\,,\ \overset{\square\overline{}(i)}{B} \\
 & \Pi_1 & \Pi_2 \\
A \supset B & A & C \\
\hline
 & C &
\end{array}\ (i)
$$

The rule of $\supset$-Elimination for disproofs is:

If Π_1 is a proof of A from Δ_1 and Π_2 is a disproof of Δ_2, B then $\supset E(A \supset B, \Pi_1, \Pi_2)$ is a disproof of $\Delta_1 \cup \Delta_2 \cup \{A \supset B\}$.

Graphically:

$$
\begin{array}{ccc}
 & \Delta_1 & \\
 & \Pi_1 & \Delta_2\,,\ \overset{\square\overline{}(i)}{B} \\
A \supset B & A & \Pi_2 \\
\hline
\end{array}\ (i)
$$

where the horizontal line has nothing below it.

Note also that in both cases of $\supset E$, the major premiss $A \supset B$ stands proud, with no proof-work above it.

The rules above form the system of *intuitionistic relevant logic.*

3 Is $\neg A$ equivalent to $A \supset \perp$? Should negation be taken as primitive, rather than defined in terms of $\perp$?

Twice again, no. $\neg A$ is not equivalent to $A \supset \perp$. Moreover, negation should be taken as primitive, and not defined in terms of $\perp$.

Why this heretical stance? The reason has to do with my opposition to systems displaying the properties already mentioned above:

1. their deducibility relations are unrestrictedly transitive[9] and they admit of a deduction theorem; and

2. they permit at least some non-trivial 'dilutions on the right'.

These systems admit of *fallacies of relevance* in their deducibility relations. It is in order to avoid the most notorious fallacy of relevance, the so-called first Lewis paradox

$$\neg A, A : B$$

that the relevant logician rejects the absurdity rule. It is, likewise, in order to avoid the related paradox

$$\neg A, A : \neg B$$

that we should go further and insist that $\neg I$ never be applied with 'vacuous discharge' of the absent 'assumption' B.

This is not the place to state the whole case for relevantist logical reform, and to chart the exact route that leads one to the system *IR* of intuitionistic relevant logic and the system *CR* of classical relevant logic (depending on one's attitude to classicism).[10] The rules of *IR* have been stated in the previous section. *IR* can lay claim to being an extremely faithful codification of intuitive constructive reasoning. It admits of the following relevantizability theorem:

[9] See footnote 1.

[10] The reader can find a detailed case for relevantizing in this recommended way in the following: 'Perfect validity, entailment and paraconsistency', *Studia Logica* XLIII, 1984, pp.179-198; 'Intuitionistic Mathematics Does Not Need Ex Falso Quodlibet', *Topoi* 1994, pp.127-133; 'Delicate Proof Theory', in J. Copeland, ed., *Logic and Reality: Essays on the Legacy of Arthur Prior*, Oxford University Press, 1966, pp.351-385.; and *Autologic*, Edinburgh University Press, 1992.

> If there is an intuitionistic proof of A from Δ, then in *IR* there is either a proof of A from (some subset of) Δ, or a disproof of (some subset of) Δ.

This means that one can welcome *IR*'s apparent loss of transitivity of deduction. Note that in *IR*

> there is a proof of $A \vee B$ from A; and
> there is a proof of B from $A \vee B$, $\neg A$; *but*
> there is no proof of B from A, $\neg A$.

Blocking the Lewis paradox while retaining disjunctive syllogism means restricting transitivity of deduction. But our relevantizability result means that these restrictions of transitivity are only ever made in the interests of *epistemic gain.*

Another casualty of restricted transitivity is the second half of the following conventional deduction theorem:

> If there is a proof of B from Δ, A then there is a proof of $A \supset B$ from Δ;
>
> and
>
> if there is a proof of $A \supset B$ from Δ then there is a proof of B from Δ, A.

In *IR* the first half of this deduction theorem is secured by the form of our $\supset$-introduction rule. But the second half fails. For in *IR* there is a proof of $A \supset B$ from $\neg A$; but there is no proof of B from A, $\neg A$.

Another interesting feature of *IR* is that even if we *were* to have an explicit absurdity constant $\perp$, which were allowed to feature as a subsentence, then $A \supset \perp$ would *not* be intersubstitutable *salva veritate* with $\neg A$ in all contexts of deducibility.[11] If we allowed $\perp$ to feature as a subsentence, then our rule of $\supset$-introduction would allow the proof

$$\dfrac{\dfrac{\neg A \quad A}{\perp}}{B \supset \perp} \qquad \text{('vacuous' discharge of } B\text{)}$$

But our rule of $\neg$-introduction does *not* allow the construction of the analogous

[11] This criterion of synonymy is from T.J.Smiley, 'The Independence of Connectives', *Journal of Symbolic Logic*, 27, 1962, pp.426-436.

$$\frac{\dfrac{\neg A \quad A}{\perp}}{\neg B}$$

because of the requirement that for such an application of $\neg I$ the sentence B should have been *used* as an assumption in the derivation of $\perp$. All the more reason, therefore, to treat of negation directly, as a primitive logical operator, and not seek to define it in terms of $\supset$ and $\perp$. *Constructivizing* had already taught logicians not to seek to use $\neg$ with one of $\wedge$, $\vee$ or $\supset$ to define the rest. Now we appreciate that *relevantizing* means that we should not seek to use $\supset$ and $\perp$ to define $\neg$.

4 Should $\perp$ be subject to introduction and elimination rules?

This question presupposes that the absurdity constant $\perp$ has a rôle to play in our deductive system. We have seen above, however, how one can eschew $\perp$ completely, avoiding employing it even as a structural punctuation marker in deductions. One simply adopts the co-inductive definition above of proof and of disproof, neither of which kinds of construction need ever contain (occurrences of) $\perp$. For one who follows this line, the question of introduction and elimination rules for $\perp$ would therefore not arise. And that is how I would prefer it. But suppose one relented anyway, and allowed for occurrences of $\perp$ within deductions. Thus $\perp$ would be the conclusion of any disproof. Disproofs could then simply be treated as proofs of $\perp$, and the definitions above (of proof and disproof) could accordingly be simplified. This indeed is the strongest — if not the only — reason for using $\perp$ in the first place.

Having thus admitted $\perp$, the question arises whether it should be subject to any rules governing *it*, in addition to those rules, such as the rules for negation, in whose statement it can appear, albeit not as the conclusion of an introduction or as the major premiss of an elimination.

The orthodox (that is, non-relevantist) intuitionist does admit at least one special rule governing $\perp$ — the aforementioned absurdity rule

$$\frac{\perp}{A}$$

Here it looks as though $\perp$ is featuring as the major premiss of an elimination. Indeed, that is how some intuitionist proof-theorists — most notably, Prawitz — have chosen to regard the absurdity rule. They do so even though $\perp$ differs from the connectives and quantifiers in not being a sentence-forming operator.

The question that naturally arises, then, is: what is the correct form of the introduction rule for $\perp$, to match the absurdity rule if and when the latter is taken as the elimination rule?

It was to this question that Dummett (as noted above) proposed the rule of 'complete atomic entailment':

$$\frac{A_1 \quad A_2 \quad A_3 \quad \ldots}{\perp}$$

where $A_1,\ A_2,\ A_3, \ldots$ are all the atomic sentences of the language. This proposal at least has the merit of making the elimination rule appear to balance the introduction rule *via* the following procedure:

$$\dfrac{\dfrac{\begin{array}{cccc}\Pi_1 & \Pi_2 & \Pi_3 & \ldots \\ A_1 & A_2 & A_3 & \ldots\end{array}}{\perp}}{A_i} \quad \rightarrow \quad \begin{array}{c}\Pi_i \\ A_i\end{array}$$

But the main drawback, of course, is that the introduction rule will in general be infinitary. This will mean that deductions involving it become unsurveyable.

Hand's further objection to Dummett's proposed introduction rule for $\perp$ was that there is no guarantee that the atomic sentences of the language form an inconsistent set. Thus $\perp$, as the conclusion of the introduction rule, need not register any inherent 'badness'. I think, however, that this particular objection is not well-taken. For in laying down rules for logical operators and for logical constants like $\perp$, one has to bear in mind the possibility of arbitrary linguistic extensions and innovations. Dummett is aware of this when he writes[12]

> We may know our language to be such that not every atomic statement can be true; but logic does not know that. As far as it is concerned, they might form a consistent set, as they are assumed to do in Wittgenstein's *Tractatus*. The principle of consistency is not a logical principle: logic does not require it, and no logical laws could be framed that would entail it.

[12] *The Logical Basis of Metaphysics*, p.295.

But even if, as it happened, one's present language had only a stock of mutually consistent atomic sentences, one could not rule out the possibility of that language being extended by the addition of new atomic sentences which *would* be inconsistent either among themselves or with the old atomic sentences already present in the existing language. One can agree with Dummett that logic 'does not know' (of our present language, say) 'that not every atomic statement can be true.' But logic does not know either that every atomic statement can be true! Logic has to allow for languages whose sets of atomic sentences may or may not be jointly consistent. The rule of $\perp$-introduction stated above has to be understood as potentially open-ended in this way: namely, that it should hold whatever extension of the language might be undertaken. And we must allow that some of those extensions could involve the inconsistency of the set of all atoms.

Now this does not just mean that, in order to derive $\perp$ in the existing language, it suffices to derive each atomic sentence of the latter. Rather, it means that in order to derive $\perp$ one has to be in a position to derive *any* atomic sentence of *any extension* of the language. Once one realizes that extensions can result in inconsistent sets of atomic sentences, one becomes aware of just how exigent such a requirement really is. *Indeed, it should be absurd that it should ever be met.* Yet *that*, and no less, is what is required in order to get around Hand's otherwise very licit criticism of Dummett's proposal.

Dummett tries to justify the absurdity rule *qua* elimination rule as being harmoniously balanced with the introduction rule suggested above. This is in the spirit of the intuitionistic meaning theory according to which it is a logical operator's introduction rule that fixes its meaning, and on the basis of which its elimination rule is to be justified by appeal to harmony.

Pace Dummett, other intuitionistic proof-theorists (such as Prawitz) have tried to justify the absurdity rule *qua* elimination rule another way. This is to claim that *there is no introduction rule for* $\perp$. For an introduction rule for a logical operator is supposed to be the form of the last step of any canonical proof (or warrant) of a conclusion with that operator dominant. When it is a constant and not a logical operator that is in question, we can say, analogously, that its introduction rule is supposed to be the form of the last step of any canonical proof (or warrant) of that constant as its conclusion. Canonical proofs, or warrants, however, are constructed with respect to *consistent atomic bases.* Thus there can be no warrant for $\perp$. Thus there is no general form of the putative last step of any warrant for $\perp$. Hence there is no introduction rule for $\perp$.

How, then, would one justify the absurdity rule for $\perp$ as the elimination rule harmoniously balancing the non-existent (or null) rule of $\perp$-introduction? The absurdity rule would be justified if one could exhibit an effective method m such that, from the claim that Π is a canonical proof of $\perp$ one could infer that $m(\Pi)$ is a canonical proof of A. *But there is no canonical proof of* $\perp$. Hence the appending of A to Π would be just such an effective method. That is, the metalinguistic argument

> Π is a canonical proof of $\perp$;
> *therefore*, (Π/A) is a canonical proof of A

is valid.

This justification, however, is too swift. It has an important defect that distinguishes it from all the other (successful) justifications of elimination rules for logical operators in terms of their introduction rules. The defect is that in any detailed metaproof codifying this justification of the absurdity rule as an 'elimination' rule for $\perp$, *one will be using that very elimination rule in the metalanguage*. But this is not the case with the usual justifications of the elimination rules for $\neg$, $\wedge$, $\vee$ and $\supset$.

There is something unnatural in speaking of introduction or elimination rules for a propositional constant rather than for a sentence-forming logical operator. In the case of a sentence-forming logical operator, the sub-sentences provide a focus for the specification of appropriate forms of sub-deductions. But there are no sub-sentences within a propositional constant such as $\perp$. It is doubly unnatural for it to have only an 'elimination' rule, with no introduction rule to which it is genuinely answerable.

The solution to this problem that is afforded by intuitionistic relevant logic is to say that, insofar as $\perp$ might appear as a structural punctuation marker within deductions, *it has no introduction rule and no elimination rule either*. The absurdity rule is abandoned because it is a source of irrelevance. This is the solution afforded by the system *IR*.

This much, however, is not yet a fully satisfactory account of the matter. For, insofar as $\perp$ might appear as a punctuation marker in deductions, we need an account of it *from which we might come to grasp the badness that negation tries to register*. Otherwise, we shall be at a loss to understand how the rule of $\neg$-introduction fixes the meaning of $\neg$ as that of *negation*. That account, however, is philosophically extra-systematic, and does not call for any logical extension of the system *IR* itself.

5 What is the source of the semantic or pragmatic 'badness' that $\perp$ seeks to register? How does the denial $\neg A$ of A succeed in blaming A as 'bad'?

The source of the 'badness' that $\perp$ seeks to register is *contrariety.* In interpreted languages there are atomic sentences that conflict by way of being contraries even if not contradictories. We gave various examples earlier, from colour discourse, discourse about place and time, and meta-discourse about the identity of speaker and hearer. Whereas Dummett seeks a logical basis for metaphysics, I think we need, at this point, to put it the other way round. One needs a metaphysical basis for logic, insofar as we seek an origin for our grasp of the meaning of negation. I believe this is to be found in our sense of contrariety, a sense that follows inexorably from our deploying perceptual concepts and objectual categories, and from our understanding of the fundamental features of bodies and events occupying space and time.

One could indeed have raised the objection earlier, to Hand's imagined language in which all atomic sentences were mutually consistent, that such a language would not be learnable. For in learning the meaning of an atomic sentence, one must be learning its truth-conditions; and in order to learn these, one must appreciate conditions under which the sentence in question fails to be true. Otherwise, there will be no telling apart the meaning of any one atomic sentence from that of any other. Hand's imagined case might well be impossible. In any event, with any natural language mastered by human beings, Hand's worries do not arise; for these languages are replete with collections of contrary atomic sentences. In learning them we acquire a primitive grasp of our not-being-able-to-hold-these-together: incompatible colour ascriptions, mutually exclusive (simultaneous) spatial locations for one and the same object,[13] etc.

The need for contrariety among at least some atomic sentences before any of them is learnable is particularly evident when we consider how we grasp the distinction between reference and predication. In order to grasp different predicates as expressing different concepts, or possessing different senses, we need to appreciate them as having different patterns of instantiation, that is, different extensions. When the extensions of two predicates with different senses happen to coincide, we can imagine circumstances, counterfactual if need be, in which they do not so coincide. Likewise, in order to grasp different referring terms as referring to distinct individuals, we

[13] This holds even in the relativistic case, within any given frame of reference.

need to appreciate that the individuals in question differ in the properties they have, that is, in the range of (primitive) predicates that they satisfy. These ranges can differ without there being any explicit use of negation. It is enough to have a few pairs of antonyms (such as short/tall; thin/fat; light/heavy; soft/hard; bright/dull; big/small; hot/cold), or contraries of a more general kind (such as red/green; round/square).

Within any such language, regimentation supplies various rules of the form

$$\frac{A_1 \ \ldots \ A_n}{}$$

where by this we are to understand that $A_1, \ldots, A_n$ are not jointly assertible, that is, that they are mutually inconsistent. It is important to realise that this mutual inconsistency can arise without any of these sentences containing an embedded negation. It arises, rather, by virtue of what the sentences mean and *various ways that we undertand the world simply cannot be.*

This is the primal 'badness' we are after. It long preceded the invention of arithmetic, with its provincial primal badness of '$0 = 1$'. If we use the absurdity constant $\perp$ to register the *metaphysical* primal badness of simultaneous predication of antonyms, or of conflicting colour or shape attributions, or conflicting spatial locations of one and the same body, or conflicting temporal orderings of distinct events etc., then that constant features thus:

$$\frac{A_1 \ \ldots \ A_n}{\perp}$$

Now we would be in a position to confer a sense upon $\neg$ by laying down the rule of $\neg$-Introduction. We are antecedently apprised of the 'badness' registered by $\perp$, and this now allows us to single out a particular sentence for blame within any bad collection of sentences. If the set Δ of sentences, along with some other sentence A leads to absurdity, then, if one holds to Δ, one must deny A.

But in exactly the same way we can confer a sense upon the negation sign $\neg$ by specifying its rule of introduction in the way we did above — where the terminal horizontal line tells all, and no use or mention is made of absurdity.

6 On classical negation.

In the context of the standard view described in my opening section, Hand went on to consider what the systems of minimal, intuitionistic and classical logic 'said about' the absurdity constant $\perp$. Minimal logic treats $\perp$ like any other constant. Intuitionistic logic distinguishes it by adding the absurdity rule. Finally, according to Hand, classical logic says in addition the following about $\perp$:

$$\frac{}{A \vee (A \supset \perp)} \qquad \text{Law of Excluded Middle}$$

$$\frac{\begin{array}{cc} \overline{A}^{(i)} & \overline{A \supset \perp}^{(i)} \\ \vdots & \vdots \\ B & B \end{array}}{B}{\scriptstyle (i)} \qquad \text{Dilemma}$$

$$\frac{(A \supset \perp) \supset \perp}{A} \qquad \text{Double 'Negation' Elimination}$$

$$\frac{\begin{array}{c} \overline{A \supset \perp}^{(i)} \\ \vdots \\ \perp \end{array}}{A}{\scriptstyle (i)} \qquad \text{Classical Reductio}$$

To regard the last four rules as *about* $\perp$, however, is to rest too much on the standard view's identification of $\neg A$ with $A \supset \perp$. I do not think it is at all helpful to try to understand the peculiar 'extra' of classical logic as consisting in the specification of further inferential powers for $\perp$ — especially when, as we have seen, there is no need to regard $\perp$ as a structural punctuation marker in deductions, let alone as a propositional constant that can be embedded as a subsentence of other sentences. Rather, the classical rules should be regarded as rules about *negation*, pure and simple — although the classical rules thereby make negation philosophically impure and logically simplistic:

$$\frac{}{A \vee \neg A} \qquad \text{Law of Excluded Middle}$$

$$\dfrac{\begin{matrix} \overline{A}^{(i)} & \overline{\neg A}^{(i)} \\ \vdots & \vdots \\ B & B \end{matrix}}{B}{\scriptstyle (i)}$$

Dilemma

$$\frac{\neg\neg A}{A}$$

Double Negation Elimination

$$\dfrac{\begin{matrix} \overline{\neg A}^{(i)} \\ \vdots \\ \bot \end{matrix}}{A}{\scriptstyle (i)}$$

Classical Reductio

Each of these rules suffices for the derivation of any of the other, *modulo* intuitionistic logic (which contains the absurdity rule). Excluded Middle and Dilemma can be derived, in minimal logic, from either Double Negation Elimination or Classical Reductio. In order to derive Double Negation Elimination or Classical Reductio from Excluded Middle or Dilemma, however, one needs to use the absurdity rule. Thus for any logical system lacking the absurdity rule it may seem as though there are two ways of 'classicizing' the system. The weaker way would be to add one of Excluded Middle or Dilemma; the stronger way would be to add one of Double Negation Elimination or Classical Reductio.

This is a correct way of viewing the matter, however, only for a system such as minimal logic. We must remember that in the system *IR* we have, by contrast, a 'liberalized' rule of $\vee$-Elimination. The four parts of it set out above can be captured in one graphic scheme as follows:

$$\dfrac{A \vee B \qquad \begin{matrix} \Delta_1 \,,\, \overset{\square}{\overline{A}}^{(i)} \\ \Pi_1 \\ \bot / C \end{matrix} \qquad \begin{matrix} \Delta_2 \,,\, \overset{\square}{\overline{B}}^{(i)} \\ \Pi_2 \\ \bot / C \end{matrix}}{\bot / C}{\scriptstyle (i)}$$

One may read this as saying that if either one of the case proofs ends with $\bot$ then one may bring the conclusion of the other case proof down as the main conclusion of the proof by cases.

In considering classical extensions of *IR* it is therefore natural to formulate the rule of Dilemma in a similarly liberalized form:

$$\dfrac{\begin{array}{cc} \overline{A}^{\,(i)} & \overline{\neg A}^{\,(i)} \\ \vdots & \vdots \\ B/\bot & B/\bot \end{array}}{B/\bot}\,(i) \qquad \text{Dilemma}$$

Now we can claim that each classical rule suffices for the derivation of any of the others, against the background of *IR*.

Let us focus now on just Dilemma and Classical Reductio in order to see how they extend *IR*.

Let us define the truth set $\underline{\tau}$ of a truth value assignment τ as the set formed by choosing the atom A if $\tau(A) = T$ and by choosing $\neg A$ if $\tau(A) = F$. When dealing with a sentence we shall think of assignments as defined only on the atoms occurring therein.

It is easy to prove the following result about *IR*, by induction on the complexity of sentences:

> If $\tau(\phi) = T$ then there is a proof of ϕ from $\underline{\tau}$;
> if $\tau(\phi) = F$ then there is a disproof of $\underline{\tau}, \phi$

Suppose ϕ is logically true. Then for every τ, $\tau(\phi) = T$, whence there is a proof of ϕ from $\underline{\tau}$. Multiple applications of Dilemma *on atoms* now suffice to ensure that there is a classical proof of ϕ.

Suppose ϕ is logically false. Then for every τ, $\tau(\phi) = F$, whence there is a disproof of $\underline{\tau}, \phi$. Multiple applications of $\neg$-Introduction now yield a disproof of ϕ (within *IR*). Hence if ψ is logically true (whence $\neg\psi$ is logically false) there is a disproof of $\neg\psi$ within *IR*. A single terminal step of Classical Reductio now yields a classical proof of ψ.

The immediate apparent difference between these two proofs of theorem-completeness of classical propositional logic is that when Dilemma is the classical rule, it may be restricted so as to apply only to atoms; whereas when Classical Reductio is the classical rule, it appears that it needs to be applied to complex sentences, albeit only once. Is this difference only apparent? It is not. To be sure,

1. applications of Classical Reductio to obtain conclusions of the form $\neg A$ are derivable in *IR*;

2. proofs containing applications of Classical Reductio to obtain conclusions of the form $A \wedge B$ can be transformed into ones in which applications of Classical Reductio are made only to A and to B; and

3. proofs containing applications of Classical Reductio to obtain conclusions of the form $A \supset B$ can be transformed into ones in which applications of Classical Reductio are made only to B.

But — it is *not* the case that proofs containing applications of Classical Reductio to obtain conclusions of the form $A \vee B$ can be transformed into ones in which applications of Classical Reductio are made only to A or to B. Thus the presence of *disjunction* prevents us from carrying through an inductive proof of the claim that one can classicize *IR* by adopting Classical Reductio for atomic conclusions. We have to be able to apply Classical Reductio to at least some complex sentences — notably, disjunctions — in order to achieve classical logic.

This is not the case, however, with Dilemma. And this at once makes Dilemma the more attractive route to full classical logic. The absurdity-free proof system set out above (for *IR*) can now be extended to one for Classical Relevant logic (*CR*) by adopting the following two-part rule of **Dilemma on atoms** A:

(1) If Π_1 is a proof of B from Δ_1, A and Π_2 is a proof of B from $\Delta_2, \neg A$ then $Dil(\Pi_1, \Pi_2, B)$ is a proof of B from $\Delta_1 \cup \Delta_2$.

Graphically:

$$\dfrac{\begin{array}{c}\Delta_1\,,\ \overset{\square\,\overline{\quad}\,(i)}{A} \\ \Pi_1 \\ B\end{array} \qquad \begin{array}{c}\Delta_2\,,\ \overset{\square\,\overline{\quad\quad}\,(i)}{\neg A} \\ \Pi_2 \\ B\end{array}}{B}\,(i)$$

(2) If Π_1 is a proof of B from Δ_1, A and Π_2 is a disproof of $\Delta_2, \neg A$ then $Dil(\Pi_1, \Pi_2, B)$ is a proof of B from $\Delta_1 \cup \Delta_2$.

Graphically:

$$\dfrac{\begin{array}{c}\Delta_1\,,\ \overset{\square\,\overline{}\,(i)}{A}\\ \Pi_1\\ B\end{array}\qquad\qquad \begin{array}{c}\Delta_2\,,\ \overset{\square\,\overline{}\,(i)}{\neg A}\\ \Pi_2\end{array}}{B}\,(i)$$

In the case where Π_1 is a disproof of Δ_1, A and Π_2 is a proof of B from $\Delta_2, \neg A$ there is no need for an application of Dilemma, since the net effect is derivable without it. One applies $\neg$-Introduction to Π_1 to obtain $\neg A$ as a conclusion, and proceeds with Π_2 . Likewise when Π_1 is a disproof of Δ_1, A and Π_2 is a disproof of $\Delta_2, \neg A$. Of course, the results of doing so will not in general be in normal form; but they can be normalized, so as to obtain a proof in which all major premisses of eliminations 'stand proud'.

7 Conclusion.

I hope to have dispelled undue pessimism about the prospects for a rule-based, anti-realist or constructivist account of negation. It is also relevantist to boot. We can give such an account in terms of basic contrarieties, and by co-inductively defining proofs and disproofs, without having to make explicit appeal to the absurdity constant $\perp$. If we do make such an appeal, it is to $\perp$ only as a structural punctuation marker within deductions, a device that allows us to assimilate disproofs to the general class of proofs. $\perp$ does not, in this rôle, need to be governed by any 'introduction' or 'elimination' rules of its own. Nor does $\perp$ need to be treated as a propositional constant eligible for embedding within other sentences. But even if we do treat $\perp$ as an embeddable propositional constant, it does not follow that negation can, let alone should, be defined in terms of it. Negation should be taken as a primitive, and one should explain how a grasp of its sense arises from one's prior grasp of primitive metaphysical contrarieties with an interpreted language.

$\perp$ is a sham logical operator. Some logicians like to think of it as a zero-place connective. I like to think of that as an admission that it has no place in logic.