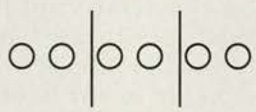
MAGICAL HOPES

Manipulatives and the Reform of Math Education

BY DEBORAH LOEWENBERG BALL

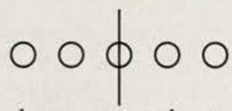
THIS ARTICLE begins with a story from my own teaching of third-grade mathematics.[1] It centers on an unusual idea about odd and even numbers that one of my students proposed.[2] The crux of the story, however, is the response I've received whenever I've shown a segment of videotape from that particular lesson to groups of educators.

First, what happened in the class: One day, as we began class, Sean announced, seemingly out of the blue, that he had been thinking that six could be both odd *and* even because it was made of "three twos." He drew the following on the board to demonstrate his point:



He explained that since three was an odd number, and there were three *groups*, this showed that six could be both even and odd. We had been working with even and odd numbers and exploring patterns that the children had noticed such as, "An even number plus an even number will always equal an even number." At this point, the definition of even numbers that we shared was that a number was even "if you can split it in half without having to use halves":

Six is even because you can split it in half without having to use halves.



Five is not even because you have to split one in half. Five is odd.

Sean was apparently dividing six into *groups of two* rather than into *two groups*. Although the other children were pretty sure that six could not be considered odd, they were intrigued. Mei thought she could explain what he was thinking. She tried:

> I think I know what he is saying . . . is that it's, see. I think what he's saying is that you have three groups of two. And three is an odd number so six can be an odd number *and* an even number.

Sean nodded in assent. Then Mei said she disagreed with him. "Can I show it on the board?" she asked. Paus-

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ILLUSTRATED BY DAN SHERBO


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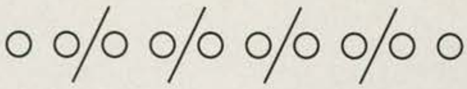
ing for a moment to decide what number to use, she drew ten circles and divided them into five groups of two:



Mei: Then why don't you call *other* numbers an odd number and an even number? What about ten? Why don't you call ten an even and an odd number?

Sean: *(paused, studying her drawing calmly and carefully)* I didn't think of it that way. Thank you for bringing it up, and I agree. I say ten *can* be odd or even.

Mei: *(with some agitation)* What about *other* numbers? Like, if you keep on going on like that and you say that *other* numbers are odd and even, maybe we'll end up with *all* numbers are odd and even! Then it won't make sense that all numbers should be odd and even, because if all numbers were odd and even, we wouldn't be even *having* this discussion!

I think this episode illustrates the dilemma faced by teachers who are committed to respecting students' ideas and yet also feel responsible for covering the curriculum. On the one hand, numbers are *not* conventionally considered both odd and even. Why not just tell Sean this and clarify for all the students that the definition of an even number does not depend on how many groups of two one can make? On the other hand, Sean *was* beginning to engage in a kind of activity that is essential to number theory: namely, noticing and exploring patterns with numbers, and, as such, his idea was worth encouraging. As the conversation unfolded in the class, Sean sparked the other children to discover that alternating even numbers (i.e., 2, 6, 10, 14, 18, etc.) had the same property he had first observed with six. Fourteen is seven groups of two, eighteen is nine groups of two, and so on. Each of these numbers is composed of an odd number of groups of two, and could be considered, according to Sean, both odd *and* even.

I have shown a small portion of the videotape from this class to other educators on several occasions. My intention has been to provoke some discussion about how to handle this situation: Should I seek other students' opinions? Clarify the definition of even numbers? Agree with Mei and move on to the plan for the day? Is this an opportunity or a problem to solve? Every time I show this tape, however, several teachers immediately inquire whether we used manipulatives for our work with even and odd numbers. When I say that we made drawings but did not use any concrete materials, these teachers have argued fiercely that that was "the problem" in this episode: Had I given the children counters as the medium for talking about even and odd numbers, then Sean would not have had this "confusion" about what makes a number even.

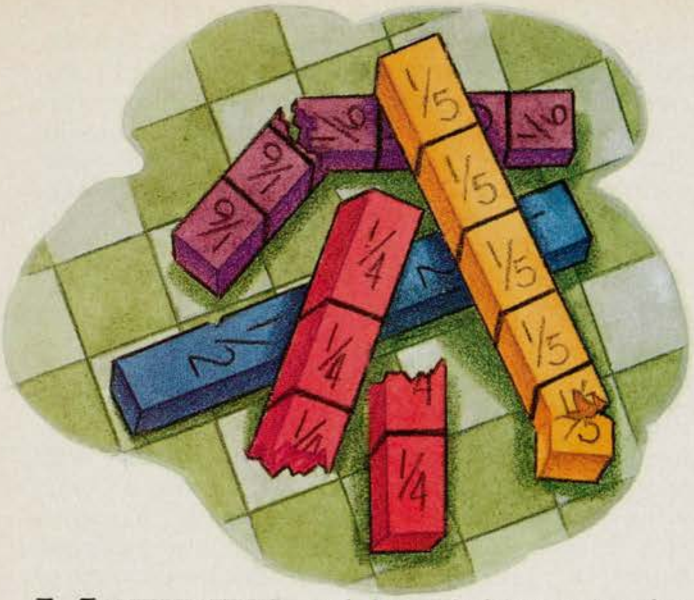
This response has baffled me. I am unable to discern how using counters and separating them into groups would have forestalled Sean's discovery that, if you group by twos, some numbers will yield an odd number of groups of two. Couldn't he have just moved six

counters on his desk into three piles of two and made the same observation?

I am not convinced that manipulatives were the key to dealing with Sean's observation. Now, of course, I could have used manipulatives and *told* the children to divide the counters into two equal piles and if one were left over, then the number was odd. In other words, I could have guided their work more firmly, toward the desired conclusions. But I could have done this in guiding their use of drawings as well. However, as a teacher, I am not necessarily interested in preventing the sorts of discoveries that Sean made. Moreover, I do not think that the point being made here had anything to do with whether the students were using manipulatives.

Some teachers are convinced that manipulatives would have been the way to prevent the students' "confusion" about odd and even numbers. This reaction makes sense in the current context of educational reform. In much of the talk about improving mathematics education, manipulatives have occupied a central place. Mathematics curricula are assessed by the extent to which manipulatives are used and how many "things" are provided to teachers who purchase the curriculum. Inservice workshops on manipulatives are offered, are usually popular, and well attended. Parents and teachers alike laud classrooms in which children use manipulatives, and Piaget is widely cited as having "shown" that young children need concrete experiences in order to learn. Some argue that all learning must proceed from the concrete to the abstract. "Concrete" is inherently good; "abstract" inherently not appropriate—at least at the beginning, at least for young learners. Whether termed "manipulatives," "concrete materials," or "concrete objects," physical materials are widely touted as crucial to the improvement of mathematics learning. From Unifix cubes, counters, and fraction pieces to base-ten blocks, Cuisenaire rods, and dice, mathematics educators emphasize the role of manipulatives in promoting student learning.

One notable exception to this emphasis on manipulatives can be found in the *Professional Standards for Teaching Mathematics* (1991) published by the National Council of Teachers of Mathematics (NCTM). The use of manipulatives is not the centerpiece of this document's vision of mathematics teaching. Instead, the *Standards* hold that teachers should encourage the use of a wide range of "tools" for exploring, representing, and communicating mathematical ideas. "Tools" include concrete models and materials, graphs and pictures, calculators and computers, and nonstandard and conventional notation. Manipulatives—or concrete objects—are important but no more so than other vehicles in NCTM's vision of mathematics teaching and learning. Still, because the passion for manipulatives runs so deep in the current discourse, many people read the *Standards* as a treatise that puts manipulatives at the center of mathematics teaching.

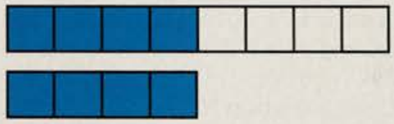


MANIPULATIVES—and the underlying notion that understanding comes through the fingertips—have become part of educational dogma: Using them helps students; not using them hinders students. There is little open, principled debate about the purposes of using manipulatives and their appropriate role in helping students learn. Little discussion occurs about possible uses of different kinds of concrete materials with different students investigating a variety of mathematical content. Likewise, how to sort among alternatives, distinguishing the fruitful from the flat, receives little attention. Articles in teacher journals, workshops, and new curricula all illustrate how to use particular concrete materials—how to use fraction bars to help students find equivalent fractions, or beansticks to understand computation with regrouping. But rarely are alternative manipulatives compared side by side. For example, in teaching place value, what are the relative merits of base-ten blocks and beansticks? Is money an equivalently workable model? How do bundled Popsicle sticks fit with the other options available? Rarely is the relative merit—in a specific context—of symbolic, pictorial, and concrete approaches explored. In teaching fractions, for example, what is gained from using fraction bars? Might drawing one's own pictures offer other opportunities? And rarely is the difficult problem of helping students make connections among these materials examined. Many teachers have seen students operate competently with base-ten blocks in modeling and computing subtraction problems, only to fall back to the familiar "subtract-up" strategy when they move into the symbolic realm.[3] This lack of specific talk leaves teachers in the position of hearing that manipulatives are good, maybe even believing that manipulatives can be very helpful, but without adequate opportunities for developing their thinking about them as one of several useful pedagogical alternatives.

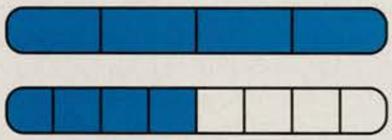
A close examination of some widely used instructional materials reveals an assumption that mathematical truths can be directly "seen" through the use of concrete objects: "Because the materials are real, and physically present before the child, they engage the child's senses . . . . Real materials . . . can be manipulated to illustrate the concept concretely, and can be experienced visually by the child" (p. xiv).[4] Teachers' guides also often convey the impression that, when students use manipulatives, they will most likely draw correct conclusions. This approach suggests that the desired conclusions reside palpably within the materials themselves.

One of the reasons that we as adults may overstate the power of concrete representations to deliver accurate mathematical messages is that we are "seeing" concepts that we already understand. That is, we who already have the conventional mathematical understandings *can* "see" correct ideas in the material representations. But for children who do not have the same mathematical understandings that we have, other things can reasonably be "seen":

"Can I have a few of the blue fraction bars—the thirds ones?" asks Jerome. Dina passes him two and he piles them with his other fraction bars. "Is four eighths greater than or less than four fourths?" asks Ms. Jackson. Jerome thinks this is a silly question. "Four eighths has to be more," he says to himself, "because eight is more than four." Lennie, sitting next to him, makes a picture:



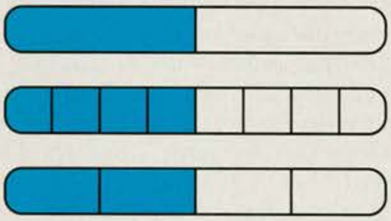
"Yup," says Jerome, looking at Lennie's drawing. "That's what I was thinking." But because he knows that he is supposed to show his answer in terms of fraction bars, Jerome lines up two fraction bars and is surprised by the result:



"Four fourths is more?" he wonders. He hears Ms. Jackson saying something about that four fourths means that the whole thing is shaded in, which is the same as what he has in front of him. It doesn't quite make sense, because the pieces in one bar are much bigger than the pieces in the other one. He does not quite understand what's wrong with Lennie's drawing, either. He moves some of the fraction bars around on his desk and waits for Ms. Jackson's next question. She asks, "Which is more—three thirds or five fifths?" Jerome moves two fraction bars in front of him and sees that both have all the pieces shaded. "Five fifths is more, though," he decides, "because there are more pieces."

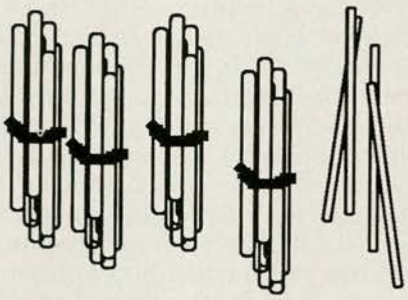
Jerome is struggling to figure out what he should pay attention to about the fraction models—is it the number of pieces that are shaded? The size of the pieces that are shaded? How much of the bar is shaded? The length of the bar itself?

This vignette illustrates the fallacy of assuming that students will automatically draw the conclusions their teachers want simply by interacting with particular manipulatives. Because students may well see and do other things with the materials, some teachers strive to tightly structure students' use of manipulatives. This is usually done in one of two ways. One way is to use materials that are relatively rigid. For example, if you use fraction bars to find equivalent fractions, it is difficult to come up with anything other than appropriate matches. The materials force you to get the right answers:



Find fractions that are equivalent to $\frac{1}{2}$

It is very hard to go wrong with these materials. Students' answers will likely be what we want: e.g., $\frac{4}{8}$, $\frac{2}{4}$, and so on. Another strategy often used to control students' thinking with manipulatives is to make rules about how to operate with the manipulatives so that students are less likely to wander into other conclusions or ideas. Fuson and Briars, for example, argue that any fruitful approach must lead the child to "construct the necessary meanings by using . . . a physical embodiment that can direct their attention to crucial meanings and help constrain their actions with the embodiments to those consistent with the mathematical features of the systems."[5] Nesher also emphasizes that any learning system must be built in with clear rules about how to use it.[6] For example, bundles of Popsicle sticks are often used to teach addition and subtraction with regrouping. Although the manipulatives in this case are relatively flexible, teachers will usually tell students that they must always group by tens and that when they need to subtract, they cannot do it unless they unbundle an entire group of ten. Without such instructions, many second graders I know would simply remove a few sticks from a bundle—just enough sticks to make the subtraction possible. But instead they follow the rules:

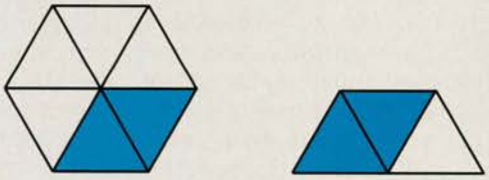


$$\begin{array}{r} 44 \\ \underline{-27} \end{array}$$

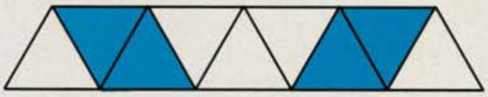
This works very well: Students unbundle a group of ten and count that they have fourteen sticks. Next they take away seven sticks. They then take two bundles of ten sticks away from the remaining three bundles, and they happily write down 17. Their answer is right. Following the rules, they readily arrive at the correct answers. In a sense, the manipulatives are employed as "training wheels" for students' mathematical thinking. However, most teachers have encountered directly the frustration when the training wheels are removed. Students, rather than riding their mathematical "bicycles" smoothly, fall off, reverting to "subtracting up" and other symbol-associated methods for subtraction. Even with close controls over how students work in the concrete domain, there are no assurances about the robustness of what they are learning. These training wheels do not work magic. Seeing students work well within the manipulative context can mislead—and later disappoint—teachers about what their students know.

MY MAIN concern about the enormous faith in the power of manipulatives, in their almost magical ability to enlighten, is that we will be misled into thinking that mathematical knowledge will automatically arise from their use. Would that it were so! Unfortunately, creating effective vehicles for learning mathematics requires more than just a catalog of promising manipulatives. The context in which any vehicle—concrete or pictorial—is used is as important as the material itself. By context, I mean the ways in which students work with the material, toward what purposes, with what kinds of talk and interaction. The creation of a shared learning context is a joint enterprise between teacher and students and evolves during the course of instruction. Developing this broader context is a crucial part of working with any manipulative. The manipulative itself cannot on its own carry the intended meanings and uses.

The need to develop these shared contexts was underscored for me when, in my class, we were using pattern blocks to develop some ideas about fractions. The children were able to build such patterns as:



and to label them as, respectively, two sixths and two thirds. They were able to interpret the two triangles as sixths in the first arrangement and the very same triangular pieces as thirds in the second. This attention to the *unit* is crucial both to understanding fractions in general as well as to using these blocks to develop such understandings. The students were also able to build arrangements that modeled other fractions, such as:



$$\frac{4}{9}$$

One day they were trying to figure out what one sixth plus one sixth would be. A disagreement developed between those who thought the answer was two sixths and those who thought it was two twelfths. Charlie argued that the answer had to be two twelfths, "because one plus one equals two, and six plus six is twelve."

$$\frac{1}{6} + \frac{1}{6} = \frac{2}{12}$$

(Continued on page 46)

formed a substantial majority of the [direct] caregiving tasks for the child."

THESE AND similar proposals will help custodial mothers and their children pick up the pieces after divorce, but they will do little to reduce the incidence of divorce. For Furstenberg and Cherlin, this is all that can be done: "We are inclined to accept the irreversibility of high levels of divorce as our starting point for thinking about changes in public policy." Hewlett is more disposed to grasp the nettle. While rejecting a return to the fault-based system of the past, she believes that the current system makes divorce too easy and too automatic. Government should send a clearer moral signal that families with children are worth preserving. In this spirit, she suggests that parents of minor children seeking divorce undergo an eighteen-month waiting period, during which they would be obliged to seek counseling and to reach a binding agreement that truly safeguards their children's future.

The generation that installed the extremes of self-expression and self-indulgence at the heart of American culture must now learn some hard old lessons about commitment, self-sacrifice, the deferral of gratification, and simple endurance. It will not be easy. But other sorts of gratifications may be their reward. Perhaps the old morality was not wrong to suggest that a deeper kind of satisfaction awaits those who accept and fulfill their essential human responsibilities. □

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[1] Mare, Robert D. and Winship, Christopher, "Socio-economic Change and the Decline of Marriage for Blacks and Whites." In *The Urban Underclass,* edited by Christopher Jencks and Paul Peterson. Washington, D.C.: The Brookings Institute, 1991.

[2] Smith, Douglas A., Jarjoura, G. Roger, "Social Structure and Criminal Victimization." In *Journal of Research in Crime and Delinquency,* Vol. 25, No. 1, February 1988.

[3] Guidubaldi, J., Cleminshaw, H.K., Perry, J.D., Nastasi, B.K., and Lightel, J., "The Role of Selected Family Environment Factors in Children's Post-Divorce Adjustment." In *Family Relations,* Vol. 35, 1986.

[4] Wallerstein, Judith S., and Blakeslee, Sandra, *Second Chances: Men, Women, and Children a Decade after Divorce.* New York: Ticknor and Fields, 1989.

[5] Sally Banks Zakariya, "Another Look at the Children of Divorce," *Principal Magazine,* September 1982, p. 35. See also, R.B. Zajonc, "Family Configuration and Intelligence," *Science,* Vol. 192, April 16, 1976, pp. 227-236. In a later and more methodologically sophisticated study, the authors try to define more completely what it is about two-parent families that make them better at preparing students for educational success. Income clearly stands out as the most important variable; but the close relationship between one-parent status, lower income, and lack of time for things like homework help and attendance at parent teacher conferences—to name a few of the variables considered—led the authors to say that "the negative effects of living in a one-parent family work primarily through other variables in our model." Ann M. Milne, David E. Myers, Alvin S. Rosenthal, and Alan Ginsburg, "Single Parents, Working Mothers, and the Educational Achievement of School Children," *Sociology of Education,* 1986, Vol. 59 (July), p. 132.

[6] Glendon, Mary Ann, *Abortion and Divorce in Western Law.* Cambridge, MA: Harvard University Press, 1987 (pp. 93-95).

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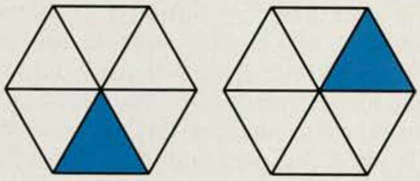
(Continued from page 18)

Most of the children thought that made sense. Dalia disagreed and showed on the overhead with the transparent pattern blocks that the answer had to be two *sixths*:



The other children easily agreed with Dalia. Following this, I thought the manipulative had convincingly helped students move toward the appropriate understanding until I heard Robbie explain, *"Both.* Both are right, because the answer is two twelfths with numbers, but two sixths with the blocks." Several others murmured assent. Juliette explained, "With numbers you add the one and the one and then you add the six and the six, and so you get two twelfths, but with the blocks, you have two of the one sixths, so you have two sixths." No one seemed at all disturbed that these answers did not correspond, and I realized that to know that these things were supposed to be congruent is something that has to be learned. The students had had plenty of experience with how context can affect both one's perspectives and one's answers. It made sense to them that the answers would vary in this case. They also had experience with mathematics problems having multiple solutions and, to them, this seemed like an example of such a problem. When Soo-Yung noted that Dalia's arrangement was also a picture of two twelfths (two pieces out of twelve), I knew we had a considerable way to go to use these materials toward some common understanding. Of course Soo-Yung was right. As was Dalia. I was beginning to understand how much work we needed to do in considering the question of *unit* in fractions.

The story of Soo-Yung and Dalia highlights the importance of the language we use around manipulatives. And how, even though they are more concrete than numbers floating on a page, there is much room for multiple interpretation and confusion. We need a lot more opportunity to discuss and develop ways to guide students' use of concrete materials in helping students learn mathematics. We need to listen more to what our students say and watch what they do. We cannot assume that apparently correct—or incorrect—answers, operations, or displays reflect the understandings that they appear to. Most of all, we need to put aside magical hopes for what manipulatives can do as we strive to improve mathematics teaching and learning.

IF WE PIN our hopes for the improvement of mathematics education on manipulatives, I predict that we will be sadly let down. Manipulatives alone cannot—and should not—be expected to carry the burden of the many problems we face in improving mathematics education in this country. The vision of reform in mathematics teaching and learning encompasses not just questions of the materials we use but of the very curriculum we choose to teach, in what ways, to whom, and in what kinds of classroom environments and discourse. It centers on new notions about what counts as worthwhile mathematical knowledge. These issues are numerous and complex. For instance, we need to shift from an emphasis on computational proficiency to an emphasis on meaning and estimation, from an emphasis on individual practice to an emphasis on discussion and on ideas, reasoning, and solution strategies. We need to alter the balance of the elementary curriculum from a dominant focus on numbers and operations to a broader range of mathematical topics, such as probability and geometry. We need to shift from a cut-and-dried, right-answer orientation to one that supports and encourages multiple modes of representation, exploration, and expression. We need to increase the participation, enthusiasm, and success of a much wider range of students. Manipulatives undoubtedly have a role to play in these aims, by enhancing the modes of learning and communication available to our students. But simply getting manipulatives into every elementary classroom cannot possibly suffice to fulfill these aims.

Why not? First of all, much more support is needed to make possible the wise use of manipulatives. Many teachers, who themselves did not learn mathematics represented in a wide range of ways, do not find it easy to distinguish among a variety of models for mathematical ideas, nor to invent them for some ideas. Teaching with manipulatives is not just a matter of pedagogical strategy and technique. Few well-educated adults—not just teachers—can devise or use legitimate representations for many elementary mathematical concepts and procedures—from fractions to multiplication to chance.[7] It should not be surprising to discover this. Consider merely the kinds of opportunities to explore and understand mathematics that most adults have had. Although a number are competent with procedures, many have not had the opportunity to develop the accompanying conceptual understandings that are necessary to manage the development of appropriate concrete contexts for learning mathematics and to respond to students' discoveries (e.g., Soo-Yung's observation that the arrangement of triangles on top of hexagons showed that $\frac{1}{6} + \frac{1}{6} = \frac{2}{12}$). Most adults simply remember learning that, with fractions, you do not add the bottom numbers. Why not? Few can explain or model it. And still fewer can explain what is going on with Soo-Yung's observation. Modeling addition and subtraction is one thing; modeling probability, factoring, or operations with fractions is another.

We also need to question and talk more openly about what we know about learning and about knowledge. Although kinesthetic experience can enhance perception and thinking, understanding does not travel through the fingertips and up the arm. And children also clearly learn from many other sources—even from highly verbal and abstract, imaginary contexts. Although concrete materials can offer students contexts and tools for making sense of the content, mathematical ideas really do not reside in cardboard and plastic materials.

More opportunities for talk and exchange—not just of techniques, but of students' thinking, of the pitfalls and advantages of alternative models, and of ways of assessing what students are learning—are needed. If manipulatives are to find their appropriate and fruitful place among the many possible improvements to mathematics education, there will have to be more opportunities for individual reflection and professional discourse. Like so many other reforms, these sorts of support imply the need for restructuring. Delivering boxes of plastic links, wooden cubes, and pattern blocks is insufficient to affect the practice of mathematics teaching and learning. At best, such deliveries can alter the surfaces of mathematics classrooms. They do not necessarily change the basic orientation to mathematical knowledge and to what counts as worth knowing. They do not necessarily provide students with conceptual understandings. They are not necessarily engaging for all students. In a few years, the boxes of manipulatives will sadly be collecting dust in the corners of our classrooms, next to the artifacts of our past magical hopes. Manipulatives will continue to play a very important role—both as an appealing lever to motivate and inspire change and as an important tool in teaching and learning. But it is time to stop pretending that they are magic and turn to more serious and sustained talk and work. Then we will begin to move beyond quick fixes and panaceas and face off with the difficult challenge of improving students' learning. □

Acknowledgement

I would like to acknowledge several colleagues whose ideas about this article and about the uses of manipulatives have influenced my own thinking: Daniel Chazan, David Cohen, Magdalene Lampert, Dirck Roosevelt, Kara Suzuka, and Suzanne Wilson.

References

[1] I teach mathematics daily in a local elementary school, in Sylvia Rundquist's third-grade class in East Lansing, Michigan. She and I have been collaborating since 1988; I teach mathematics and she teaches all the other subjects. In our regular meetings (and the conversations in between) we talk about the children, the culture of the classroom we are sharing, and about our role in helping students learn. My aim in this work is to investigate some of the issues that arise in trying to teach mathematics in the spirit of the current reforms (e.g., the NCTM *Standards* [1989, 1991]). It is a kind of research *into* teaching that I see as complementary to other research on teaching.

[2] I have written about this story more extensively in "With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics," which will appear in the *Elementary School Journal*.

[3] The "subtract-up" strategy, familiar to all elementary teachers, consists of looking at a problem like:

$$\begin{array}{r} 57 \\ \underline{-39} \end{array}$$

and computing 9-7 instead of regrouping to subtract 9 from 17. This is one of the most persistent computational procedures that young children use.

[4] Baratta-Lorton, M. (1976). *Mathematics Their Way*. Menlo Park: Addison-Wesley.

[5] Fuson, K., & Briars, D. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place value and multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 21, 180-206.

[6] Nesher, P. (1989). Microworlds in mathematical education: A pedagogical realism. In L. B. Resnick (Ed.), *Knowing, learning and instruction: Essays in honor of Robert Glaser* (pp. 187-215). Hillsdale, NJ: Erlbaum, p. 188.

[7] In our research (e.g., Ball, 1990), we asked college students and other adults to make up a story, draw a picture, or use concrete objects to model division of fractions: $1\frac{3}{4} \div \frac{1}{2}$. Only a very small percentage of adults in any category were able to correctly represent this statement. Most modelled $1\frac{3}{4} \div 2$ instead of dividing by $\frac{1}{2}$. A sizable proportion said that this statement was not possible to model in any meaningful way.