

## INDEX

Almost everywhere  $p$   
(a.e. $p$ ), 167  
Absolute value of an  
operator, 216  
Adjoining an identity, 191  
Adjoint, 93  
of an operator, 157  
Alaoglu's Theorem, 94  
Algebraic direct sum, 2  
Approximate identities, 104  
  
Baire measures, 26  
Baire sets, 26  
Banach algebra, 75, 166, 175  
adjoining an identity, 191  
adjoint, 189  
Gelfand transform, 195  
 $L^1(\mathbb{R})$ , 190  
normal element, 189  
projection element, 189  
selfadjoint subset, 189  
structure space, 194  
structure space  
of  $L^1(\mathbb{R})$ , 196  
unitary element, 189

Banach  $*$ -algebra, 189  
Banach limits, 38  
Banach means, 38  
Banach space, 65  
characterization, 66  
complex, 65  
reflexive, 93  
reflexive criterion, 96  
Bessel's Inequality, 147  
Basis:  
for a vector space, 2  
for a topology, 5  
orthonormal, 150  
Bilateral shift, 159  
Borel function, 19, 165  
Borel measure, 19  
Borel sets, 19, 165  
Borel space, 167  
  
 $C^*$ -algebra, 190  
sub  $C^*$ -algebra, 190  
Cartesian products, 1  
Cauchy-Schwarz  
Inequality, 142

Cayley transform, 159  
     of an unbounded selfadjoint operator, 234  
 Chain rule, 250  
 Character, 137, 240  
 Choquet's Theorem, 61  
     and the Riesz Representation Theorem, 97  
 Closed Graph Theorem, 71  
 Closure of an operator, 238  
 Cluster point, 35  
 Cofinal, 35  
 Compact operator, 221  
     characterization, 223  
     properties of the set, 223  
 Compact space, 5  
     locally compact, 5  
      $\sigma$ -compact, 5, 20  
 Compact support, 19  
 Completion of a normed linear space, 75  
 Complex Banach space, 65  
 Complex measure, 24, 91  
     norm, 92  
 Cone, 28  
 Conjugate linear, 143  
 Conjugate space, 90  
 Continuity (characterized), 49  
 Continuity and the graph, 50  
 Continuous spectrum, 217  
 Continuously differentiable, 253  
 Contraction map, 255  
 Contraction Mapping Theorem, 256  
 Convex hull, 54  
 Convex set, 54  
 Convolution kernels, 103, 104  
     on the circle, 103  
     operator, 103, 160  
 Convolution Theorem, 113  
 Countably normed space, 52  
 Cyclic vector, 171  
 Derivative, 245  
 Diagonalizable, 206  
 Diagonalizing a Hermitian matrix, 207  
 Diffeomorphism, 246  
 Differentiability, 245  
 Differentiable manifold, 261  
 Differential, 247  
 Dimension (vector space), 2  
 Direct product, 3  
 Direct sum, 3  
     Hilbert spaces, 153  
     normed linear spaces, 65  
     projection-valued measures, 169  
 Directed set, 35  
 Discontinuous linear functionals, 50  
 Discrete spectrum, 217  
 Distribution, 86  
     properties, 87  
     tempered function, 86  
     tempered measure, 87  
 Distributional derivative, 87  
 Distributions as derivatives of functions, 88  
 Domain of an unbounded selfadjoint operator, 233  
 Dual space, 81  
 Duality Theorem, 83  
 Eigenvalue, 158  
 Eigenvector, 158

Essential spectrum, 217  
 Essentially selfadjoint, 238  
 Extreme point, 59  
  
 Face, 59  
 Finite intersection  
     property, 8  
 Finite rank operator, 221  
 Finitely additive measure, 40  
     translation-invariant, 41  
 First derivative test, 251  
 Fourier transform, 112  
     inversion Theorem, 114  
     Gauss kernel, 113  
     on  $\mathbb{R}^n$ , 118  
      $L^2$  transform, 116  
     Plancherel Theorem, 114  
     tempered distribution, 117  
 Frechet derivative, 247  
 Frechet space, 44, 86  
  
 Gauss kernel, 108  
     Fourier transform, 113  
 Gelfand transform, 195  
 General spectral  
     Theorem, 202  
 Generator of a one-parameter  
     group, 239  
 Gleason's Theorem, 240  
 Gradient, 262  
 Gram-Schmidt process, 147  
 Green's functions, 108  
     for the Laplacian, 119?

Hahn-Banach Theorem:  
     complex version, 41  
     extreme point version, 59  
     locally convex version, 56  
     norm version, 33, 75  
     positive cone version, 28  
     semigroup-invariant  
         version, 39  
     seminorm version, 32  
 Hausdorff space, 5  
 Hausdorff-Young  
     Inequality, 117  
 Hausdorff Maximality  
     Principle, 1  
 Hermitian form, 143, 156  
 Hilbert space direct sum, 153  
 Hilbert transform, 120  
 Hilbert-Schmidt operator, 222  
     characterization, 225  
 Hilbert-Schmidt inner  
     product, 228  
 Hilbert-Schmidt norm, 228  
  
 Idempotent operator, 158  
 Implicit Function  
     Theorem, 256  
     in Hilbert space, 261  
     foliated version, 260  
 Independence of spectrum, 201  
 Infinite Cartesian  
     products, 1  
 Inner product, 143  
 Inner product space, 143  
 Integral kernel, 99  
 Integral operator, 99  
     bounded, 99  
 Invariance of the essential  
     spectrum, 236  
 Inverse Function

Theorem, 259  
 Involution, 189  
 Isometric isomorphism, 65  
 Isomorphism Theorem, 70  
  
 Jacobian, 249  
  
 Kernel and range, 3  
 Kernel, 99  
   convolution, 103  
   Gauss, 108  
   Poisson in  $\mathbb{R}^n$ , 107  
   Poisson on the circle, 107  
   Poisson on the line, 107  
   reproducing, 104  
   singular, 104  
 Krein-Milman Theorem, 60  
  
 Lagrange multipliers, 262  
 Linear functional, 4  
   bounded 21  
   conjugate linear, 143  
   norm, 90  
   positive, 12, 28  
 Linear isomorphism, 3  
 Linear transformation, 3  
 Locally compact, 5  
 Locally convex, 54  
  
 Matrix coefficient, 155  
 Max norm, 65  
 Mazur's Theorem, 193  
 Mean Value Theorem, 252  
 Mercer's Theorem, 233  
 Metric, 6  
 Metrizable space, 6  
 Minkowski functional, 56  
 Mixed partials, 254  
 Mixed state, 129  
 Multiplication operator, 162  
  
 Multiplicity, 217  
 Multipliers on the line, 120  
  
 Natural map, 4  
 Neighborhood, 5  
 Net, 35  
 Non-locally convex space, 58  
  
 Norm:  
   of a linear functional, 90  
   of an operator, 72  
 Normable spaces, 66  
 Normal operator, 158  
 Normal topological space, 5  
  
 Observable, 127  
   compatible, 130  
   simultaneously, 130  
 Open Mapping Theorem, 71  
 Orthogonal complement, 146  
 Orthogonal projection, 158  
 Orthogonality, 146  
 Orthonormal basis, 150  
 Orthonormal set, 146  
  
 Parallelogram Law, 146  
 Parseval's Equality, 148  
 Partial isometry, 215  
 Plancherel Theorem, 114  
 Point spectrum, 217  
  
 Poisson kernel:  
   on the circle, 107  
   on the line, 107  
   in  $\mathbb{R}^n$ , 107  
 Polar decomposition, 215  
  
 Polarization Identity, 116, 145  
   for an operator, 155  
 Positive part of a self-adjoint operator, 214  
 Positive cone, 28

Positive functional, 12, 28  
 Positive operator, 158  
 Positive square root of a  
     positive operator, 213  
 Product topology, 7  
 Projection onto a  
     subspace, 152  
 Projection Theorem, 151  
 Projection-valued  
     measure, 167  
     canonical, 170  
     change of variables, 178  
     direct sum, 169  
     integral of a bounded  
         function, 177  
     integral of an unbounded  
         function, 186  
     invariant subspace, 167  
      $L^\infty(p)$ , 168  
     Riesz representation  
         Theorem, 179  
     unitarily equivalent, 168  
 Pure state, 129  
 Purely atomic spectrum, 217  
  
 Questions, 128  
     complementary, 130  
     ordering, 129  
     orthogonality, 131  
     sum, 130  
 Question-valued measure, 132  
 Quotient space, 4  
 Quotient topological vector  
     space, 48  
 Quotient topology, 8  
  
 Reflexive Banach space, 93  
     criterion, 96  
 Regular topological space, 5  
  
 Reproducing kernels, 104  
 Resolution of the  
     identity, 205  
 Resolvent, 197, 235  
 Restriction of a projection-  
     valued measure, 167  
 Riemann-Lebesgue  
     Theorem, 113  
 Riesz Representation  
     Theorem:  
         for  $C_0(\Delta)$ , 21  
         complex version, 25  
         in Hilbert space, 153  
         for projection-  
             valued measures, 179  
 Riesz Interpolation  
     Theorem, 76  
 Ring of sets, 40  
  
 Schwartz space  $\mathcal{S}$ , 53  
 Second countable, 5  
 Second derivative test, 255  
 Second dual, 93  
 Self dual, 154  
 Selfadjoint extension, 239  
 Selfadjoint operator, 158  
     unbounded, 233  
 Selfadjoint subset, 189  
 Seminorm, 31  
     on a complex space, 41  
 Separating vector, 171  
 Separation Theorem, 58  
 Sgn (signum function), 78, 117  
 Simultaneously diagona-  
     lizable, 207  
 Singular kernels, 104  
 Spectral mapping  
     Theorem, 197, 205

- Spectral measure, 204
  - unbounded selfadjoint operator, 234
- Spectral radius formula, 199
- Spectral radius, 197
- Spectral Theorem:
  - general, 202
  - normal operator, 202
  - selfadjoint operator, 205
  - unbounded selfadjoint operator, 233
- Spectrum, 197, 235
  - characterization of point spectrum, 217
  - continuous, 217
  - discrete, 217
  - essential, 217
  - independence, 201
  - invariance of essential spectrum, 236
  - multiplicity, 217
  - point, 217
  - properties, 220
  - purely atomic, 217
- States, 127
  - mixed, 129
  - pure, 129
- Stone's Axiom, 11
- Stone's Theorem, 207, 239
- Strong topology, 81
- Structure space, 194
  - of  $L^1(\mathbb{R})$ , 196
- Subadditive functional, 31, 55
- Sub  $C^*$ -algebra, 190
- Subnet, 35
- Supporting vector, 171
- Supremum norm, 21
- Symmetry, 139
- Tangent vector, 261
- Tempered distribution, 86
- Tempered function, 86
- Tempered measure, 87
- Time evolution, 138
- Topologically isomorphic, 44
- Topology on a set, 5
  - basis for, 5
  - product, 7
  - relation among the weak, weak\*, and norm, 93
  - $\sigma$ -compact, 5, 20
  - weak and strong, 81
  - relative, 5
  - strong, 81
  - weak topologies and metrizable, 93
  - weak, 7, 81
  - weak\*, 83
- Total variation norm, 92
- Trace class operator, 224
  - the set, 225, 228
- Trace class norm, 231
- Trace of an operator, 231
- Transpose, 93
- Twice differentiable, 253
- Tychonoff Theorem, 36
- Urysohn's Lemma, 8
- Unbounded selfadjoint operator, 233
  - Cayley transform, 234
  - domain, 233
  - positive, 233
  - resolvent, 235
  - spectral measure, 235
  - Spectral Theorem, 233
  - spectrum, 235
- Uniform Boundedness

Principle, 73  
Unilateral shift, 159  
Unitary operator, 158  
    characterization, 158  
  
Vanish at infinity, 19  
Vector lattice, 11  
Vector space direct  
    product, 3  
  
Weak topology, 7, 81  
Weak\* topology, 93  
Wigner's Theorem, 241  
  
Young's Inequality, 102