CHAPTER VIII

HILBERT SPACES

DEFINITION Let X and Y be two complex vector spaces. A map $T : X \to Y$ is called a *conjugate-linear* transformation if it is a real-linear transformation from X into Y, and if

$$T(\lambda x) = \overline{\lambda} T(x)$$

for all $x \in X$ and $\lambda \in \mathbb{C}$.

Let X be a complex vector space. An *inner product* or *Hermitian form* on X is a mapping from $X \times X$ into $\mathbb{C}$ (usually denoted by (x, y)) which satisfies the following conditions:

(1) $(x, y) = \overline{(y, x)}$ for all $x, y \in X$.
(2) For each fixed $y \in X$, the map $x \to (x, y)$ is a linear functional on X.
(3) $(x, x) > 0$ for all nonzero $x \in X$.

Note that conditions 1 and 2 imply that for each fixed vector x the map $y \to (x, y)$ is conjugate-linear. It also follows from condition 2 that $(0, x) = 0$ for all $x \in X$.

The complex vector space X, together with an inner product $(\, , \,)$, is called an *inner product space*.

REMARK. We treat here primarily complex inner product spaces and complex Hilbert spaces. Corresponding definitions can be given for real inner product spaces and real Hilbert spaces, and the results about these spaces are occasionally different from the complex cases.

EXERCISE 8.1. (a) Let X be the complex vector space of all continuous complex-valued functions on [0,1], and define

$$(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx.$$

Show that X, with this definition of (,), is an inner product space.

(b) Let $X = \mathbb{C}^n$, and define

$$(x,y) = \sum_{j=1}^{n} x_j\overline{y_j},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Prove that X, with this definition of (,), is an inner product space.

(c) (General l^2) Let μ be counting measure on a countable set (sequence) S. Let $X = L^2(\mu)$, and for $f, g \in X$ define

$$(f,g) = \int_S f(s)\overline{g(s)}\,d\mu(s) = \sum_{s\in S} f(s)\overline{g(s)}.$$

Prove that X is an inner product space with respect to this definition.

(d) Specialize the inner product space defined in part c to the two cases first where S is the set of nonnegative integers and then second where S is the set $\mathbb{Z}$ of all integers.

THEOREM 8.1. *Let X be an inner product space.*

(1) *(Cauchy-Schwarz Inequality) For all $x, y \in X$,*

$$|(x,y)| \le \sqrt{(x,x)}\sqrt{(y,y)}.$$

(2) *The assignment $x \to \sqrt{(x,x)}$ is a norm on X, and X equipped with this norm is a normed linear space.*

PROOF. Fix x and y in X. If either x or y is 0, then part 1 is immediate. Otherwise, define a function f of a complex variable λ by

$$f(\lambda) = (x + \lambda y\ ,\ x + \lambda y),$$

and note that $f(\lambda) \ge 0$ for all λ. We have that

$$f(\lambda) = (x,x) + \lambda(y,x) + \overline{\lambda}(x,y) + (y,y)|\lambda|^2.$$

Substituting $\lambda = -(x,y)/(y,y)$, and using the fact that $f(\lambda) \geq 0$ for all λ, the general case of part 1 follows.

To see that $x \to \sqrt{(x,x)}$ defines a norm $\|x\|$ on X, we need only check that $\|x+y\| \leq \|x\| + \|y\|$. But

$$\begin{aligned}\|x+y\|^2 &= (x+y\ ,\ x+y) \\ &= (x,x) + 2\Re((x,y)) + (y,y) \\ &\leq \|x\|^2 + 2|(x,y)| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2,\end{aligned}$$

which completes the proof of part 2.

EXERCISE 8.2. (a) Show that equality holds in the Cauchy-Schwarz inequality, i.e.,

$$|(x,y)| = \|x\|\|y\|,$$

if and only if one of the vectors is a scalar multiple of the other. Conclude that equality holds in the triangle inequality for the norm if and only if one of the vectors is a nonnegative multiple of the other.

(b) Let y and z be elements of an inner product space X. Show that $y = z$ if and only if $(x,y) = (x,z)$ for all $x \in X$.

(c) Prove the *polarization identity* and the *parallelogram law* in an inner product space X; i.e., show that for $x, y \in X$, we have

$$(x,y) = (1/4)\sum_{j=0}^{3} i^j\,\|x + i^j y\|^2$$

and

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(d) Suppose X and Y are inner product spaces and that T is a linear isometry of X into Y. Prove that T preserves inner products. That is, if $x_1, x_2 \in X$, then

$$(T(x_1), T(x_2)) = (x_1, x_2).$$

(e) Suppose X is an inner product space, that Y is a normed linear space, and that T is a linear isometry of X onto Y. Show that there exists an inner product $(\ ,\ )$ on Y such that $\|y\| = \sqrt{(y,y)}$ for every $y \in Y$; i.e., Y is an inner product space and the norm on Y is determined by that inner product.

(f) Suppose Y is a normed linear space whose norm satisfies the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all $x, y \in Y$. Show that there exists an inner product $(\ ,\ )$ on Y such that $\|y\| = \sqrt{(y,y)}$ for every $y \in Y$; i.e., Y is an inner product space and the given norm on Y is determined by that inner product. HINT: Use the polarization identity to define (x, y). Show directly that $(y, x) = \overline{(x, y)}$ and that $(x, x) > 0$ if $x \neq 0$. For a fixed y, define $f(x) = (x, y)$. To see that f is linear, first use the parallelogram law to show that

$$f(x + x') + f(x - x') = 2f(x),$$

from which it follows that $f(\lambda x) = \lambda f(x)$ for all $x \in Y$ and $\lambda \in \mathbb{C}$. Then, for arbitrary elements $u, v \in Y$, write $u = x + x'$ and $v = x - x'$.

(g) Show that the inner product is a continuous function of $X \times X$ into $\mathbb{C}$. In particular, the map $x \to (x, y)$ is a continuous linear functional on X for every fixed $y \in X$.

DEFINITION. A (complex) *Hilbert space* is an inner product space that is complete in the metric defined by the norm that is determined by the inner product. An inner product space X is called *separable* if there exists a countable dense subset of the normed linear space X.

REMARK. Evidently, a Hilbert space is a special kind of complex Banach space. The inner product spaces and Hilbert spaces we consider will always be assumed to be separable.

EXERCISE 8.3. Let X be an inner product space. Show that any subspace $M \subseteq X$ is an inner product space, with respect to the restriction of the inner product on X, and show that a closed subspace of a Hilbert space is itself a Hilbert space. If M is a closed subspace of a Hilbert space H, is the quotient space H/M necessarily a Hilbert space?

DEFINITION. Let X be an inner product space. Two vectors x and y in X are called *orthogonal* or *perpendicular* if $(x, y) = 0$. Two subsets S and T are *orthogonal* if $(x, y) = 0$ for all $x \in S$ and $y \in T$. If S is a subset of X, then $S^\perp$ will denote what we call the *orthogonal complement* to S and consists of the elements $x \in X$ for which $(x, y) = 0$ for all $y \in S$. A collection of pairwise orthogonal unit vectors is called an *orthonormal* set.

EXERCISE 8.4. Let X be an inner product space.

(a) Show that a collection $x_1, \ldots, x_n$ of nonzero pairwise orthogonal vectors in X is a linearly independent set. Verify also that

$$\|\sum_{i=1}^{n} c_i x_i\|^2 = \sum_{i=1}^{n} |c_i|^2 \|x_i\|^2.$$

(b) (Gram-Schmidt Process) Let $x_1, \ldots$ be a (finite or infinite) sequence of linearly independent vectors in X. Show that there exists a sequence $w_1, \ldots$ of orthonormal vectors such that the linear span of $x_1, \ldots, x_i$ coincides with the linear span of $w_1, \ldots, w_i$ for all $i \geq 1$. HINT: Define the w_i's recursively by setting

$$w_i = \frac{x_i - \sum_{k=1}^{i-1} (x_i, w_k) w_k}{\|x_i - \sum_{k=1}^{i-1} (x_i, w_k) w_k\|}.$$

(c) Show that if X is a separable inner product space, then there exists an orthonormal sequence $\{x_i\}$ whose linear span is dense in X.

(d) If M is a subspace of X, show that the set $M^\perp$ is a closed subspace of X. Show further that $M \cap M^\perp = \{0\}$.

(e) Let $X = C([0,1])$ be the inner product space from part a of Exercise 8.1. For each $0 < t < 1$, let M_t be the set of all $f \in X$ for which $\int_0^t f(x)\,dx = 0$. Show that the collection $\{M_t\}$ forms a pairwise distinct family of closed subspaces of X. Show further that $M_t^\perp = \{0\}$ for all $0 < t < 1$. Conclude that, in general, the map $M \to M^\perp$ is not 1-1.

(f) Suppose X is a Hilbert Space. If M and N are orthogonal closed subspaces of X, show that the subspace $M+N$, consisting of the elements $x + y$ for $x \in M$ and $y \in N$, is a closed subspace.

THEOREM 8.2. *Let H be a separable infinite-dimensional (complex) Hilbert space. Then*

(1) *Every orthonormal set must be countable.*
(2) *Every orthonormal set in H is contained in a (countable) maximal orthonormal set. In particular, there exists a (countable) maximal orthonormal set.*
(3) *If $\{\phi_1, \phi_2, \ldots\}$ is an orthonormal sequence in H, and $\{c_1, c_2, \ldots\}$ is a square summable sequence of complex numbers, then the infinite series $\sum c_n \phi_n$ converges to an element in H.*
(4) (Bessel's Inequality) *If $\phi_1, \phi_2, \ldots$ is an orthonormal sequence in H, and if $x \in H$, then*

$$\sum_n |(x, \phi_n)|^2 \leq \|x\|^2,$$

implying that the sequence $\{(x, \phi_n)\}$ is square-summable.

(5) *If $\{\phi_n\}$ denotes a maximal orthonormal sequence (set) in H, then every element $x \in H$ is uniquely expressible as a (infinite) sum*

$$x = \sum_n c_n \phi_n,$$

where the sequence $\{c_n\}$ is a square summable sequence of complex numbers. Indeed, we have that $c_n = (x, \phi_n)$.

(6) *If $\{\phi_n\}$ is any maximal orthonormal sequence in H, and if $x, y \in H$, then*

$$(x, y) = \sum_n (x, \phi_n)\overline{(y, \phi_n)}.$$

(7) (Parseval's Equality) *For any $x \in H$ and any maximal orthonormal sequence $\{\phi_n\}$, we have*

$$\|x\|^2 = \sum_n |(x, \phi_n)|^2.$$

(8) *Let $\{\phi_1, \phi_2, \dots\}$ be a maximal orthonormal sequence in H, and define $T : l^2 \to H$ by*

$$T(\{c_n\}) = \sum_{n=1}^{\infty} c_n \phi_n.$$

Then T is an isometric isomorphism of l^2 onto H. Consequently, any two separable infinite-dimensional Hilbert spaces are isometrically isomorphic.

PROOF. Suppose an orthonormal set in H is uncountable. Then, since the distance between any two distinct elements of this set is $\sqrt{2}$, it follows that there exists an uncountable collection of pairwise disjoint open subsets of H, whence H is not separable. Hence, any orthonormal set must be countable, i.e., a sequence.

Let S be an orthonormal set in H. The existence of a maximal orthonormal set containing S now follows from the Hausdorff maximality principle, applied to the collection of all orthonormal sets in H that contain S.

Next, let $\{\phi_1, \phi_2, \dots\}$ be an orthonormal sequence, and let $x \in H$ be given. For each positive integer i, set $c_i = (x, \phi_i)$. Then, for each positive integer n We have

$$
\begin{aligned}
0 &\leq \|x - \sum_{i=1}^{n} c_i\phi_i\|^2 \\
&= ((x - \sum_{i=1}^{n} c_i\phi_i), (x - \sum_{j=1}^{n} c_j\phi_j)) \\
&= (x, x) - \sum_{j=1}^{n} \overline{c_j}(x, \phi_j) - \sum_{i=1}^{n} c_i(\phi_i, x) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}(\phi_i, \phi_j) \\
&= (x, x) - \sum_{j=1}^{n} \overline{c_j}c_j - \sum_{i=1}^{n} c_i\overline{c_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\delta_{ij} \\
&= \|x\|^2 - \sum_{i=1}^{n} |c_i|^2.
\end{aligned}
$$

Since this is true for an arbitrary n, Bessel's inequality follows.

If $\{\phi_1, \phi_2, \dots\}$ is an orthonormal sequence and $\{c_1, c_2, \dots\}$ is a square summable sequence of complex numbers, then the sequence of partial sums of the infinite series

$$\sum_{n=1}^{\infty} c_n\phi_n$$

is a Cauchy sequence. Indeed,

$$\|\sum_{n=1}^{j} c_n\phi_n - \sum_{n=1}^{k} c_n\phi_n\|^2 = \sum_{n=k+1}^{j} |c_n|^2.$$

See part a of Exercise 8.4. This proves part 4.

Now, if $\{\phi_1, \phi_2, \dots\}$ is a maximal orthonormal sequence in H, and x is an element of H, we have from Bessel's inequality that $\sum_n |(x, \phi_n)|^2$ is finite, and therefore $\sum(x, \phi_n)\phi_n$ converges in H by part 4. If we define

$$y = \sum_n (x, \phi_n)\phi_n.$$

Clearly $((x - y), \phi_n) = 0$ for all n, implying that, if $x - y \neq 0$, then $(x - y)/\|x - y\|$ is a unit vector that is orthogonal to the set $\{\phi_n\}$. But since this set is maximal, no such vector can exist, and we must have $x = y$ as desired. To see that this representation of x as an infinite series is unique, suppose $x = \sum c'_n\phi_n$, where $\{c'_n\}$ is a square-summable sequence of complex numbers. Then, for each k, we have

$$(x, \phi_k) = \sum_n c'_n(\phi_n, \phi_k) = c'_k,$$

showing the uniqueness of the coefficients.

Because the inner product is continuous in both variables, we have that

$$(x, y) = \sum_n \sum_k ((x, \phi_n)\phi_n, (y, \phi_k)\phi_k) = \sum_n (x, \phi_n)\overline{(y, \phi_n)},$$

proving part 6.

Parseval's equality follows from part 6 by setting $y = x$.

Part 7 is now immediate, and this completes the proof.

DEFINITION. We call a maximal orthonormal sequence in a separable Hilbert space H an *orthonormal basis* of H.

EXERCISE 8.5. (a) Prove that $L^2[0,1]$ is a Hilbert space with respect to the inner product defined by

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\,dx.$$

(b) For each integer n, define an element $\phi_n \in L^2[0,1]$ by $\phi_n(x) = e^{2\pi i n x}$. Show that the ϕ_n's form an orthonormal sequence in $L^2[0,1]$.

(c) For each $0 < r < 1$, define a function k_r on $[0,1]$ by

$$k_r(x) = \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi x)}.$$

(See part d of Exercise 6.7.) Show that

$$k_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|}\phi_n(x),$$

whence $\int_0^1 k_r(x)\,dx = 1$ for every $0 < r < 1$. Show further that if $f \in L^2[0,1]$, then

$$f = \lim_{r\to 1} k_r * f,$$

where the limit is taken in L^2, and where $*$ denotes convolution; i.e.,

$$(k_r * f)(x) = \int_0^1 k_r(x-y)f(y)\,dy.$$

(d) Suppose $f \in L^2[0,1]$ satisfies $(f, \phi_n) = 0$ for all n. Show that f is the 0 element of $L^2[0,1]$. HINT: $(k_r * f)(x) = 0$ for every $r < 1$.

(e) Conclude that the set $\{\phi_n\}$ forms an orthonormal basis for $L^2[0,1]$.

(f) Using $f(x) = x$, show that

$$\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6.$$

Then, using $f(x) = x^2 - x$, show that

$$\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90.$$

HINT: Parseval's equality.

(g) Let M be the set of all functions $f = \sum c_n\phi_n$ in $L^2[0,1]$ for which $c_{2n+1} = 0$ for all n, and let N be the set of all functions $g = \sum c_n\phi_n$ in $L^2[0,1]$ for which $c_{2n} = (1+|n|)c_{2n+1}$ for all n. Prove that both M and N are closed subspaces of $L^2[0,1]$.

(h) For M and N as in part g, show that $M+N$ contains each ϕ_n and so is a dense subspace of $L^2[0,1]$. Show further that if $h = \sum_n c_n\phi_n \in M+N$, then

$$\sum_n n^2|c_{2n+1}|^2 < \infty.$$

(i) Conclude that the sum of two arbitrary closed subspaces of a Hilbert space need not be closed. Compare this with part f of Exercise 8.4.

THEOREM 8.3. (Projection Theorem) *Let M be a closed subspace of a separable Hilbert space H. Then:*

1. *H is the direct sum $H = M \bigoplus M^\perp$ of the closed subspaces M and $M^\perp$; i.e., every element $x \in H$ can be written uniquely as $x = y + z$ for $y \in M$ and $z \in M^\perp$.*
2. *For each $x \in H$ there exists a unique element $y \in M$ for which $x - y \in M^\perp$. We denote this unique element y by $p_M(x)$.*
3. *The assignment $x \to p_M(x)$ of part 2 defines a continuous linear transformation p_M of H onto M that satisfies $p_M^2 = p_M$.*

PROOF. We prove part 1 and leave the rest of the proof to an exercise. Let $\{\phi_n\}$ be a maximal orthonormal sequence in the Hilbert space M, and extend this set, by Theorem 8.2, to a maximal orthonormal sequence $\{\phi_n\} \cup \{\psi_k\}$ in H. If $x \in H$, then, again according to Theorem 8.2, we have

$$x = \sum_n (x, \phi_n)\phi_n + \sum_k (x, \psi_k)\psi_k = y + z,$$

where $y = \sum_n (x, \phi_n)\phi_n$ and $z = \sum_k (x, \psi_k)\psi_k$. Clearly, $y \in M$ and $z \in M^\perp$. If $x = y' + z'$, for $y' \in M$ and $z' \in M^\perp$, then the element $y - y' = z' - z$ belongs to $M \cap M^\perp$, whence is 0. This shows the uniqueness of y and z and completes the proof of part 1.

EXERCISE 8.6. (a) Complete the proof of the preceding theorem.

(b) For p_M as in part 3 of the preceding theorem, show that

$$\|p_M(x)\| \leq \|x\|$$

for all $x \in H$; i.e., p_M is norm-decreasing.

(c) Again, for p_M as in part 3 of the preceding theorem, show that

$$(p_M(x), y) = (x, p_M(y))$$

for all $x, y \in H$.

(d) Let S be a subset of a Hilbert space H. Show that $(S^\perp)^\perp$ is the smallest closed subspace of H that contains S. Conclude that if M is a closed subspace of a Hilbert space H, then $M = (M^\perp)^\perp$.

(e) Let M be a subspace of a Hilbert space H. Show that M is dense in H if and only if $M^\perp = \{0\}$. Give an example of a proper closed subspace M of an inner product space X (necessarily not a Hilbert space) for which $M^\perp = \{0\}$.

(f) Let M be a closed subspace of a Hilbert space H. Define a map $T : M^\perp \to H/M$ by $T(x) = x + M$. Prove that T is an isometric isomorphism of $M^\perp$ onto H/M. Conclude then that the quotient space H/M, known to be a Banach space, is in fact a Hilbert space.

DEFINITION. If M is a closed subspace of a separable Hilbert space H, then the transformation p_M of the preceding theorem is called the *projection* of Honto M.

REMARK. The set $\mathcal{M}$ of all closed subspaces of a Hilbert space H is a candidate for the set Q of questions in our mathematical model of experimental science. (See Chapter VII.) Indeed, $\mathcal{M}$ is obviously a partially-ordered set by inclusion; it contains a maximum element H and a minimum element $\{0\}$; the sum of two orthogonal closed subspaces is a closed subspace, so that there is a notion of summability for certain pairs of elements of $\mathcal{M}$; each element $M \in \mathcal{M}$ has a complement $M^\perp$ satisfying $M + M^\perp = H$. Also, we may define M and N to be compatible if there exist four pairwise orthogonal closed subspaces $M_1, \dots, M_4$ satisfying

(1) $M = M_1 + M_2$.
(2) $N = M_2 + M_3$.
(3) $M_1 + M_2 + M_3 + M_4 = H$.

We study this candidate for Q in more detail later by putting it in 1-1 correspondence with the corresponding set of projections.

DEFINITION. Let $\{H_n\}$ be a sequence of Hilbert spaces. By the *Hilbert space direct sum* $\bigoplus H_n$ of the H_n's, we mean the subspace of the direct product $\prod_n H_n$ consisting of the sequences $\{x_n\}$, for which $x_n \in H_n$ for each n, and for which $\sum_n \|x_n\|^2 < \infty$.

EXERCISE 8.7. (a) Prove that the Hilbert space direct sum $\bigoplus H_n$ of Hilbert spaces $\{H_n\}$ is a Hilbert space, where the vector space operations are componentwise and the inner product is defined by

$$(\{x_n\}, \{y_n\}) = \sum_n (x_n, y_n).$$

Show that the ordinary (algebraic) direct sum of the vector spaces $\{H_n\}$ can be naturally identified with a dense subspace of the Hilbert space direct sum of the H_n's. Verify that if each H_n is separable then so is $\bigoplus_n H_n$.

(b) Suppose $\{M_n\}$ is a pairwise orthogonal sequence of closed subspaces of a Hilbert space H and that M is the smallest closed subspace of H that contains each M_n. Construct an isometric isomorphism between M and the Hilbert space direct sum $\bigoplus M_n$, where we regard each closed subspace M_n as a Hilbert space in its own right.

THEOREM 8.4. (Riesz Representation Theorem for Hilbert Space) *Let H be a separable Hilbert space, and let f be a continuous linear functional on H. Then there exists a unique element y_f of H for which $f(x) = (x, y_f)$ for all $x \in H$. That is, the linear functional f can be represented as an inner product. Moreover, the map $f \to y_f$ is a conjugate-linear isometric isomorphism of the conjugate space H^* onto H.*

PROOF. Let $\{\phi_1, \phi_2, \dots\}$ be a maximal orthonormal sequence in H, and for each n, define $c_n = f(\phi_n)$. Note that $|c_n| \leq \|f\|$ for all n; i.e., the sequence $\{c_n\}$ is bounded. For any positive integer n, write $w_n = \sum_{j=1}^n \overline{c_j}\phi_j$, and note that

$$\|w_n\|^2 = \sum_{j=1}^n |c_j|^2 = |f(w_n)| \leq \|f\|\|w_n\|,$$

whence

$$\sum_{j=1}^n |c_j|^2 \leq \|f\|^2,$$

showing that the sequence $\{\overline{c_n}\}$ belongs to l^2. Therefore, the series $\sum_{n=1}^\infty \overline{c_n}\phi_n$ converges in H to an element y_f. We see immediately that $(y_f, \phi_n) = \overline{c_n}$ for every n, and that $\|y_f\| \leq \|f\|$. Further, for each $x \in H$, we have by Theorem 8.2 that

$$\begin{aligned} f(x) &= f(\sum (x, \phi_n)\phi_n) \\ &= \sum (x, \phi_n)c_n \\ &= \sum (x, \phi_n)\overline{(y_f, \phi_n)} \\ &= (x, y_f), \end{aligned}$$

showing that $f(x) = (x, y_f)$ as desired.

From the Cauchy-Schwarz inequality, we then see that $\|f\| \leq \|y_f\|$, and we have already seen the reverse inequality above. Hence, $\|f\| = \|y_f\|$. We leave the rest of the proof to the exercise that follows.

EXERCISE 8.8. (a) Prove that the map $f \to y_f$ of the preceding theorem is conjugate linear, isometric, and onto H. Conclude that the map $f \to y_f$ is a conjugate-linear, isometric isomorphism of the conjugate space H^* of H onto H. Accordingly, we say that a Hilbert space is *self dual.*

(b) Let H be a Hilbert space. Show that a net $\{x_\alpha\}$ of vectors in H converges to an element x in the weak topology of H if and only if

$$(x, y) = \lim_\alpha (x_\alpha, y)$$

for every $y \in H$.

(c) Show that the map $f \to y_f$ of the preceding theorem is a homeomorphism of the topological vector space $(H^*, \mathcal{W}^*)$ onto the topological vector space $(H, \mathcal{W})$.

(d) Let H be a separable Hilbert space. Prove that the closed unit ball in H is compact and metrizable in the weak topology.

(e) Let H be a separable Hilbert space and let $\{x_n\}$ be a sequence of vectors in H. If $\{x_n\}$ converges weakly to an element $x \in H$, show that the sequence $\{x_n\}$ is uniformly bounded in norm. Conversely, if the sequence $\{x_n\}$ is uniformly bounded in norm, prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that is weakly convergent. HINT: Uniform Boundedness Principle and Alaoglu's Theorem.

DEFINITION. Let H be a Hilbert space, and let $B(H)$ denote the set $L(H, H)$ of all bounded linear transformations of H into itself. If $T \in B(H)$ and $x, y \in H$, we call the number $(T(x), y)$ a *matrix coefficient* for T.

Let T be an element of $B(H)$. Define, as in Chapter IV, $\|T\|$ by

$$\|T\| = \sup_{\substack{x \in H \\ \|x\| \leq 1}} \|T(x)\|.$$

EXERCISE 8.9. (a) For $T \in B(H)$, show that

$$\|T\| = \sup_{\substack{x, y \in H \\ \|x\| \leq 1, \|y\| \leq 1}} |(T(x), y)|.$$

(b) For $T \in B(H)$ and $x, y \in H$, prove the following polarization identity:

$$(T(x), y) = (1/4) \sum_{j=0}^{3} i^j (T(x + i^j y), (x + i^j y)).$$

(c) If $S, T \in B(H)$, show that $\|TS\| \leq \|T\| \|S\|$. Conclude that $B(H)$ is a *Banach algebra*; i.e., $B(H)$ is a Banach space on which there is also defined an associative multiplication $\times$, which is distributive over addition, commutes with scalar multiplication, and which satisfies $\|T \times S\| \leq \|T\| \|S\|$.

(d) If $S, T \in B(H)$ satisfy $(T(x), y) = (S(x), y)$ for all $x, y \in H$ (i.e., they have the same set of matrix coefficients), show that $S = T$. Show further that $T = S$ if and only if $(T(x), x) = (S(x), x)$ for all $x \in H$.

(This is a result that is valid in complex Hilbert spaces but is not valid in Hilbert spaces over the real field. Consider the linear transformation on $\mathbb{R}^2$ determined by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.)

(e) If F, G are continuous linear transformations of H into any topological vector space X, and if $F(\phi_n) = G(\phi_n)$ for all ϕ_n in a maximal orthonormal sequence, show that $F = G$.

THEOREM 8.5. *Let H be a complex Hilbert space, and let L be a mapping of $H \times H$ into $\mathbb{C}$ satisfying:*

(1) *For each fixed y, the map $x \to L(x, y)$ is a linear functional on H.*

(2) *For each fixed x, the map $y \to L(x, y)$ is a conjugate linear transformation of H into $\mathbb{C}$.*

(3) *There exists a positive constant M such that*

$$|L(x, y)| \leq M\|x\|\|y\|$$

for all $x, y \in H$.

(Such an L is called a bounded Hermitian form *on H.) Then there exists a unique element $S \in B(H)$ such that*

$$L(x, y) = (x, S(y))$$

for all $x, y \in H$.

PROOF. For each fixed $y \in H$, we have from assumptions (1) and (3) that the map $x \to L(x, y)$ is a continuous linear functional on H. Then, by the Riesz representation theorem (Theorem 8.4), there exists a unique element $z \in H$ for which $L(x, y) = (x, z)$ for all $x \in H$. We denote z by $S(y)$, and we need to show that S is a continuous linear transformation of H into itself.

Clearly,

$$\begin{aligned} (x, S(y_1 + y_2)) &= L(x, y_1 + y_2) \\ &= L(x, y_1) + L(x, y_2) \\ &= (x, S(y_1)) + (x, S(y_2)) \\ &= (x, S(y_1) + S(y_2)) \end{aligned}$$

for all x, showing that $S(y_1 + y_2) = S(y_1) + S(y_2)$. Also,

$$\begin{aligned} (x, S(\lambda y)) &= L(x, \lambda y) \\ &= \overline{\lambda} L(x, y) \\ &= \overline{\lambda}(x, S(y)) \\ &= (x, \lambda S(y)) \end{aligned}$$

for all x, showing that $S(\lambda y) = \lambda S(y)$, whence S is linear.

Now, since $|(x, S(y))| = |L(x, y)| \leq M\|x\|\|y\|$, it follows by setting $x = S(y)$ that S is a bounded operator of norm $\leq M$ on H, as desired.

Finally, the uniqueness of S is evident since any two such operators S_1 and S_2 would have identical matrix coefficients and so would be equal.

DEFINITION. Let T be a bounded operator on a (complex) Hilbert space H. Define a map L_T on $H \times H$ by

$$L_T(x, y) = (T(x), y).$$

By the *adjoint* of T, we mean the unique bounded operator $S = T^*$, whose existence is guaranteed by the previous theorem, that satisfies

$$(x, T^*(y)) = L_T(x, y) = (T(x), y)$$

for all $x, y \in H$.

THEOREM 8.6. *The adjoint mapping $T \to T^*$ on $B(H)$ satisfies the following for all $T, S \in B(H)$ and $\lambda \in \mathbb{C}$.*

(1) $(T + S)^* = T^* + S^*$.
(2) $(\lambda T)^* = \overline{\lambda} T^*$.
(3) $(TS)^* = S^* T^*$.
(4) *If T is invertible, then so is T^*, and $(T^*)^{-1} = (T^{-1})^*$.*
(5) *The kernel of T^* is the orthogonal complement of the range of T; i.e., $y \in M^\perp$ if and only if $T^*(y) = 0$.*
(6) $(T^*)^* = T$.
(7) $\|T^*\| = \|T\|$.
(8) $\|T^*T\| = \|TT^*\| = \|T\|^2$.

PROOF. We prove parts 3 and 8 and leave the remaining parts to an exercise.

We have

$$\begin{aligned}(x, (TS)^*(y)) &= (T(S(x)), y) \\ &= (S(x), T^*(y)) \\ &= (x, S^*(T^*(y))),\end{aligned}$$

showing part 3.

Next, we have that $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$ by part 7, so to obtain part 8 we need only show the reverse inequality. Thus,

$$\begin{aligned}
\|T\|^2 &= \sup_{\substack{x \in H \\ \|x\| \leq 1}} \|T(x)\|^2 \\
&= \sup_{\substack{x \in H \\ \|x\| \leq 1}} (T(x), T(x)) \\
&= \sup_{\substack{x \in H \\ \|x\| \leq 1}} (x, T^*(T(x))) \\
&\leq \|T^*T\|,
\end{aligned}$$

as desired.

EXERCISE 8.10. Prove the remaining parts of Theorem 8.6.

DEFINITION. Let H be a (complex) Hilbert space. An element $T \in B(H)$ is called *unitary* if it is an isometry of H onto H. A linear transformation U from one Hilbert space H_1 into another Hilbert space H_2 is called a *unitary* map if it is an isometry of H_1 onto H_2.

An element $T \in B(H)$ is called *selfadjoint* or *Hermitian* if $T^* = T$.

An element $T \in B(H)$ is called *normal* if T and T^* commute, i.e., if $TT^* = T^*T$.

An element T in $B(H)$ is called *positive* if $(T(x), x) \geq 0$ for all $x \in H$.

An element $T \in B(H)$ is called *idempotent* if $T^2 = T$.

If $p \in B(H)$ is selfadjoint and idempotent, we say that p is an *orthogonal projection* or (simply) a *projection*.

An *eigenvector* for an operator $T \in B(H)$ is a nonzero vector $x \in H$ for which there exists a scalar λ satisfying $T(x) = \lambda x$. The scalar λ is called an *eigenvalue* for T, and the eigenvector x is said to *belong* to the eigenvalue λ.

EXERCISE 8.11. (a) Prove that the L^2 Fourier transform U is a unitary operator on $L^2(\mathbb{R})$.

(b) Suppose μ and ν are σ-finite measures on a σ-algebra $\mathcal{B}$ of subsets of a set S, and assume that ν is absolutely continuous with respect to μ. Let f denote the Radon-Nikodym derivative of ν with respect to μ, and define $U : L^2(\nu) \to L^2(\mu)$ by

$$U(g) = \sqrt{f}g.$$

Prove that U is a norm-preserving linear transformation of $L^2(\nu)$ into $L^2(\mu)$, and that it is a unitary transformation between these two Hilbert spaces if and only if μ and ν are mutually absolutely continuous.

(c) (Characterization of unitary transformations) Let U be a linear transformation of a Hilbert space H_1 into a Hilbert space H_2. Prove that U is a unitary operator if and only if it is onto H_2 and is *inner-product preserving;* i.e.,

$$(U(x), U(y)) = (x, y)$$

for all $x, y \in H_1$.

(d) (Another characterization of unitary operators) Let U be an element of $B(H)$. Prove that U is a unitary operator if and only if

$$UU^* = U^*U = I.$$

(e) (The bilateral shift) Let $\mathbb{Z}$ denote the set of all integers, let μ be counting measure on $\mathbb{Z}$, and let H be $L^2(\mu)$. Define a transformation U on H by

$$[U(x)]_n = x_{n+1}.$$

Prove that U is a unitary operator on H. Compute its adjoint (inverse) U^*.

(f) (The unilateral shift) Let S be the set of all nonnegative integers, let μ be counting measure on S, and let $H = L^2(\mu)$. Define a transformation T on H by

$$[T(x)]_n = x_{n+1}.$$

Show that T is not a unitary operator. Compute its adjoint T^*.

THEOREM 8.7. *Let H be a (complex) Hilbert space.*

(1) *If $T \in B(H)$, then there exist unique selfadjoint operators T_1 and T_2 such that $T = T_1 + iT_2$. T_1 and T_2 are called respectively the real and imaginary parts of the operator T.*
(2) *The set of all selfadjoint operators forms a real Banach space with respect to the operator norm, and the set of all unitary operators forms a group under multiplication.*
(3) *An element $T \in B(H)$ is selfadjoint if and only if $(T(x), x) = (x, T(x))$ for all x in a dense subset of H.*
(4) *An element $T \in B(H)$ is selfadjoint if and only if $(T(x), x)$ is real for every $x \in H$. If λ is an eigenvalue for a selfadjoint operator T, then λ is real.*
(5) *Every positive operator is selfadjoint.*
(6) *Every orthogonal projection is positive.*
(7) *If T is selfadjoint, then $I \pm iT$ is 1-1, onto, and $\|(I \pm iT)(x)\| \geq \|x\|$ for all $x \in H$, whence $(I \pm iT)^{-1}$ is a bounded operator on H.*

(8) *If T is selfadjoint, then $U = (I - iT)(I + iT)^{-1}$ is a unitary operator, for which -1 is not an eigenvalue; i.e., $I + U$ is 1-1. Moreover,*

$$T = -i(I - U)(I + U)^{-1}.$$

This unitary operator U is called the Cayley transform of T.

(9) *A continuous linear transformation $U : H_1 \to H_2$ is unitary if and only if its range is a dense subspace of H_2, and*

$$(U(x), U(x)) = (x, x)$$

for all x in a dense subset of H_1.

PROOF. Defining $T_1 = (1/2)(T + T^*)$ and $T_2 = (1/2i)(T - T^*)$, we have that $T = T_1 + iT_2$, and both T_1 and T_2 are selfadjoint. Further, if $T = S_1 + iS_2$, where both S_1 and S_2 are selfadjoint, then $T^* = S_1 - iS_2$, whence $2S_1 = T + T^*$ and $2iS_2 = T - T^*$, from which part 1 follows.

Parts 2 through 6 are left to the exercises.

To see part 7, notice first that

$$\begin{aligned}
\|(I + iT)(x)\|^2 &= ((I + iT)(x), (I + iT)(x)) \\
&= (x, x) + i(T(x), x) - i(x, T(x)) + (T(x), T(x)) \\
&= \|x\|^2 + \|T(x)\|^2 \\
&\geq \|x\|^2,
\end{aligned}$$

which implies that $I + iT$ is 1-1 and norm-increasing. Moreover, it follows that the range of $I + iT$ is closed in H. For, if $y \in H$ and $y = \lim y_n = \lim(I + iT)(x_n)$, then the sequence $\{y_n\}$ is Cauchy, and hence the sequence $\{x_n\}$ must be Cauchy by the above inequality. Therefore $\{x_n\}$ converges to an $x \in H$. Then $y = (I + iT)(x)$, showing that the range of $I + iT$ is closed.

If $z \in H$ is orthogonal to the range of $I + iT$, then $((I + iT)(z), z) = 0$, which implies that $(z, z) = -i(T(z), z)$, which can only happen if $z = 0$, since $(T(z), z)$ is real if T is selfadjoint. Hence, the range of $I + iT$ only has 0 in its orthogonal complement; i.e., this range is dense. Since it is also closed, we have that the range of $I + iT = H$, and $I + iT$ is onto. Since $I + iT$ is norm-increasing, we see that $(I + iT)^{-1}$ exists and is norm-decreasing, hence is continuous.

Of course, an analogous argument proves that $(I - iT)^{-1}$ is continuous.

Starting with

$$(I + iT)(I + iT)^{-1} = I,$$

we see by taking the adjoint of both sides that

$$((I + iT)^{-1})^* = (I - iT)^{-1}.$$

It follows also then that $I - iT$ and $(I + iT)^{-1}$ commute. But now

$$\begin{aligned} I &= (I - iT)(I - iT)^{-1}(I + iT)^{-1}(I + iT) \\ &= (I - iT)(I + iT)^{-1}(I - iT)^{-1}(I + iT) \\ &= (I - iT)(I + iT)^{-1}[(I - iT)(I + iT)^{-1}]^*, \end{aligned}$$

showing that $U = (I - iT)(I + iT)^{-1}$ is unitary. See part d of Exercise 8.11. Also,

$$\begin{aligned} I + U &= (I + iT)(I + iT)^{-1} + (I - iT)(I + iT)^{-1} \\ &= 2(I + iT)^{-1}, \end{aligned}$$

showing that $I + U$ is 1-1 and onto. Finally,

$$I - U = (I + iT)(I + iT)^{-1} - (I - iT)(I + iT)^{-1} = 2iT(I + iT)^{-1},$$

whence

$$-i(I - U)(I + U)^{-1} = -i \times 2iT(I + iT)^{-1}(1/2)(I + iT) = T,$$

as desired.

Finally, if a continuous linear transformation $U : H_1 \to H_2$ is onto a dense subspace of H_2, and $(U(x), U(x)) = (x, x)$ for all x in a dense subset of H_1, we have that U is an isometry on this dense subset, whence is an isometry of all of H_1 into H_2. Since H_1 is a complete metric space, it follows that the range of U is complete, whence is a closed subset of H_2. Since this range is assumed to be dense, it follows that the range of $U = H_2$, and U is unitary.

EXERCISE 8.12. Prove parts 2 through 6 of the preceding theorem.

EXERCISE 8.13. Let H be a Hilbert space.

(a) If $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series function with radius of convergence r, and if T is an element of $B(H)$ for which $\|T\| < r$, show that the infinite series $\sum_{n=0}^{\infty} a_n T^n$ converges to an element of $B(H)$. (We may call this element $\phi(T)$.)

(b) Use part a to show that $I+T$ has an inverse in $B(H)$ if $\|T\| < 1$.

(c) For each $T \in B(H)$, define

$$e^T = \sum_{n=0}^{\infty} T^n/n!.$$

Prove that

$$e^{T+S} = e^T e^S$$

if T and S commute. HINT: Show that the double series $\sum T^n/n! \times \sum S^j/j!$ converges independent of the arrangement of the terms. Then, rearrange the terms into groups where $n + j = k$.

(d) Suppose T is selfadjoint. Show that

$$e^{iT} = \sum_{n=0}^{\infty} (iT)^n/n!$$

is unitary.

EXERCISE 8.14. (Multiplication Operators) Let (S, μ) be a σ-finite measure space. For each $f \in L^\infty(\mu)$, define the operator m_f on the Hilbert space $L^2(\mu)$ by

$$m_f(g) = fg.$$

These operators m_f are called *multiplication operators*.

(a) Show that each operator m_f is bounded and that

$$\|m_f\| = \|f\|_\infty.$$

(b) Show that $(m_f)^* = m_{\overline{f}}$. Conclude that each m_f is normal, and that m_f is selfadjoint if and only if f is real-valued a.e.μ.

(c) Show that m_f is unitary if and only if $|f| = 1$ a.e.μ.

(d) Show that m_f is a positive operator if and only if $f(x) \geq 0$ a.e.μ.

(e) Show that m_f is a projection if and only if $f^2 = f$ a.e.μ, i.e., if and only if f is the characteristic function of some set E.

(f) Show that λ is an eigenvalue for m_f if and only if $\mu(f^{-1}(\{\lambda\})) > 0$.

(g) Suppose $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series function with radius of convergence r, and suppose $f \in L^\infty(\mu)$ satisfies $\|f\|_\infty < r$. Show that $\phi(m_f) = m_{\phi \circ f}$. (See the previous exercise.)

EXERCISE 8.15. Let H be the complex Hilbert space $L^2(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, write T_f for the operator on H determined by convolution by f. That is, for $g \in L^2(\mathbb{R})$, we have $T_f(g) = f * g$. See Theorem 6.2.

(a) Prove that $T_f \in B(H)$ and that the map $f \to T_f$ is a norm-decreasing linear transformation of $L^1(\mathbb{R})$ into $B(H)$. See Theorem 6.2.

(b) For $g, h \in L^2(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, show that

$$(T_f(g), h) = (\hat{f}U(g), U(h)) = (m_{\hat{f}}(U(g)), U(h)),$$

where $\hat{f}$ denotes the Fourier transform of f and $U(g)$ and $U(h)$ denote the L^2 Fourier transforms of g and h. Conclude that the map $f \to T_f$ is 1-1.

(c) Show that $T_f^* = T_{f^*}$, where $f^*(x) = \overline{f(-x)}$.

(d) Show that

$$T_{f_1 * f_2} = T_{f_1} \circ T_{f_2}$$

for all $f_1, f_2 \in L^1(\mathbb{R})$. Conclude that T_f is always a normal operator, and that it is selfadjoint if and only if $f(-x) = \overline{f(x)}$ for almost all x. HINT: Fubini's Theorem.

(e) Prove that T_f is a positive operator if and only if $\hat{f}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$.

(f) Show that T_f is never a unitary operator and is never a nonzero projection. Can T_f have any eigenvectors?

We return now to our study of the set $\mathcal{M}$ of all closed subspaces of a Hilbert space H. The next theorem shows that $\mathcal{M}$ is in 1-1 correspondence with a different, and perhaps more tractable, set $\mathcal{P}$.

THEOREM 8.8. *Let p be an orthogonal projection on a Hilbert space H, let M_p denote the range of p and let K_p denote the kernel of p. Then:*

(1) *$x \in M_p$ if and only if $x = p(x)$.*

(2) *$M_p = K_p^\perp$, whence M_p is a closed subspace of H. Moreover, p is the projection of H onto M_p.*

(3) *The assignment $p \to M_p$ is a 1-1 correspondence between the set $\mathcal{P}$ of all orthogonal projections on H and the set $\mathcal{M}$ of all closed subspaces of H.*

(4) *M_p and M_q are orthogonal subspaces if and only if $pq = qp = 0$ which implies that $p+q$ is a projection. In fact, $M_{p+q} = M_p + M_q$.*

(5) *$M_p \subseteq M_q$ if and only if $pq = qp = p$, which implies that $r = q - p$ is a projection, and $q = p + r$.*

PROOF. We leave the proof of part 1 to the exercise that follows.

If $x \in M_p$, and $y \in K_p$, then, $x = p(x)$ by part 1. Therefore,

$$(x, y) = (p(x), y) = (x, p(y)) = 0,$$

showing that $M_p \subseteq K_p^\perp$. Conversely, if $x \in K_p^\perp$, then $x - p(x)$ is also in $K_p^\perp$. But

$$p(z - p(z)) = p(z) - p^2(z) = 0$$

for any $z \in H$, whence $x - p(x) \in K_p \cap K_p^\perp$, and this implies that $x = p(x)$, and $x \in M_p$. This proves the first part of 2. We see also that for any $z \in H$ we have that

$$z = p(z) + (z - p(z)),$$

and that $p(z) \in M_p$, and $z - p(z) \in K_p$. It follows then that p is the projection of H onto the closed subspace M_p. See the Projection Theorem (8.3).

Part 3 follows directly from Theorem 8.3.

Let M_p and M_q be orthogonal subspaces. If x is any element of H, then $q(x) \in M_q$ and $M_q \subseteq K_p$. Therefore $p(q(x)) = 0$ for every $x \in H$; i.e., $pq = 0$. A similar calculation shows that $qp = 0$. Then it follows directly that $p + q$ is selfadjoint and that $(p + q)^2 = p + q$. Conversely, if $pq = qp = 0$, then $M_p \subseteq K_q$, whence M_p is orthogonal to M_q.

We leave part 5 to the exercises.

EXERCISE 8.16. (a) Prove parts 1 and 5 of the preceding theorem.

(b) Let p be a projection with range M_p. Show that a vector x belongs to M_p if and only if $\|p(x)\| = \|x\|$.

REMARK. We now examine the set $\mathcal{P}$ of all projections on a separable complex Hilbert space H as a candidate for the set Q of all questions in our development of axiomatic experimental science. The preceding theorem shows that $\mathcal{P}$ is in 1-1 correspondence with the set $\mathcal{M}$ of all closed subspaces, and we saw earlier that $\mathcal{M}$ could serve as a model for Q. The following theorem spells out the properties of $\mathcal{P}$ that are relevant if we wish to use $\mathcal{P}$ as a model for Q.

THEOREM 8.9. *Consider the set $\mathcal{P}$ of all projections on a separable complex Hilbert space H as being in 1-1 correspondence with the set $\mathcal{M}$ of all closed subspaces of H, and equip $\mathcal{P}$ with the notions of partial order, complement, orthogonality, sum, and compatibility coming from this identification with $\mathcal{M}$. Then:*

(1) *$p \leq q$ if and only if $pq = qp = p$.*

(2) *p and q are orthogonal if and only if $pq = qp = 0$.*
(3) *p and q are summable if and only if they are orthogonal.*
(4) *p and q are compatible if and only if they commute, i.e., if and only if $pq = qp$.*
(5) *If $\{p_i\}$ is a sequence of pairwise orthogonal projections, then there exists a (unique) projection p such that $p(x) = \sum_i p_i(x)$ for all $x \in H$.*

PROOF. Parts 1 and 2 follow from the preceding theorem. It also follows from that theorem that if p and q are orthogonal then $p+q$ is a projection, implying that p and q are summable. Conversely, if p and q are summable, then $p+q$ is a projection, and

$$p + q = (p + q)^2 = p^2 + pq + qp + q^2 = p + q + pq + qp,$$

whence $pq = -qp$. But then

$$\begin{aligned} -pq &= -p^2q^2 \\ &= -ppqq \\ &= p(-pq)q \\ &= pqpq \\ &= (-qp)(-qp) \\ &= qpqp \\ &= q(-qp)p \\ &= -qp, \end{aligned}$$

implying that $pq = qp$. But then $pq = qp = 0$, whence p and q are orthogonal. This completes the proof of part 3.

Suppose now that p and q commute, and write $r_2 = pq$. Let $r_1 = p - r_2$, $r_3 = q - r_2$, and $r_4 = I - r_1 - r_2 - r_3$. It follows directly that the r_i's are pairwise orthogonal projections, that $p = r_1 + r_2$ and that $q = r_2 + r_3$. Hence p and q are compatible. Conversely, if p and q are compatible, and $p = r_1 + r_2$ and $q = r_2 + r_3$, where r_1, r_2, r_3 are pairwise orthogonal projections, then $pq = qp = r_2$ and p and q commute.

Finally, to see part 5, let M be the Hilbert space direct sum $\bigoplus M_{p_i}$ of the closed subspaces $\{M_{p_i}\}$, and let p be the projection of H onto M. Then, if $x \in M^\perp$, we have that $p(x) = p_i(x) = 0$ for all i, whence

$$p(x) = \sum p_i(x).$$

On the other hand, if $x' \in M$, then $x' = \sum x'_i$, where for each i, $x'_i \in M_{p_i}$. Obviously then $p(x') = \sum p_i(x'_i) = \sum p_i(x')$. Finally, if $z \in H$, then $z = x + x'$, where $x \in M^\perp$ and $x' \in M$. Clearly, we have $p(z) = \sum p_i(z)$, and the proof is complete.

DEFINITION. Let H be a separable Hilbert space. If $\{p_i\}$ is a sequence of pairwise orthogonal projections in $B(H)$, then the projection $p = \sum_i p_i$ from part 5 of the preceding theorem is called the *sum* of the p_i's.

EXERCISE 8.17. (a) Show that the set $\mathcal{P}$ satisfies all the requirements of the set Q of all questions. (See Chapter VII.)

(b) Show that in $\mathcal{P}$ a stronger property holds than is required for Q. That is, show that a sequence $\{p_i\}$ is mutually summable if and only if it is pairwise orthogonal.