CHAPTER VII

AXIOMS FOR A MATHEMATICAL MODEL OF EXPERIMENTAL SCIENCE

This chapter is a diversion from the main subject of this book, and it can be skipped without affecting the material that follows. However, we believe that the naive approach taken in this chapter toward the axiomatizing of experimental science serves as a good motivation for the mathematical theory developed in the following four chapters.

We describe here a set of axioms, first introduced by G.W. Mackey, to model experimental investigation of a system in nature. We suppose that we are studying a phenomenon in terms of various observations of it that we might make. We postulate that there exists a nonempty set S of what we shall call the possible *states* of the system, and we postulate that there is a nonempty set O of what we shall call the possible *observables* of the system. We give two examples.

(1) Suppose we are investigating a system that consists of a single physical particle in motion on an infinite straight line. Newtonian mechanics ($f = ma$) tells us that the system is completely determined for all future time by the current position and velocity, i.e., by two real numbers. Hence, the states of this system might well be identified with points in the plane. Two of the (many) possible observables of this system can be described as position and velocity observables. We imagine that there is a device which indicates where the particle is and another device that indicates its velocity. More realistically, we might have many yes/no devices that answer the observational questions: "Is the particle

between a and b?" "Is the velocity of the particle between c and d?"

Quantum mechanical models of this single particle are different from the Newtonian one. They begin by assuming that the (pure) states of this one-particle system are identifiable with certain square-integrable functions and the observables are identified with certain linear transformations. This model seems quite mysterious to most mathematicians, and Mackey's axioms form one attempt at justifying it.

(2) Next, let us imagine that we are investigating a system in which three electrical circuits are in a black box and are open or closed according to some process of which we are not certain. The states of this system might well be described as all triples of 0's and 1's (0 for open and 1 for closed). Suppose that we have only the following four devices for observing this system. First, we can press a button b_0 and determine how many of the three circuits are closed. However, when we press this button, it has the effect of opening all three circuits, so that we have no hope of learning exactly which of the three were closed. (Making the observation actually affects the system.) In addition, we have three other buttons b_1, b_2, b_3, b_i telling whether circuit i is open or closed. Again, when we press button b_i, all three circuits are opened, so that we have no way of determining if any of the circuits other than the ith was closed. This is a simple example in which certain simultaneous observations appear to be impossible, e.g., determining whether circuits 1 and 2 are both closed.

The axioms we introduce are concerned with the concept of interpreting what it means to make a certain observation of the system when the system is in a given state. The result of such an observation should be a real number, with some probability, depending on the state and on the observable.

AXIOM 1. To each state $\alpha \in S$ and observable $A \in O$ there corresponds a Borel probability measure $\mu_{\alpha,A}$ on $\mathbb{R}$.

REMARK. The probability measure $\mu_{\alpha,A}$ contains the information about the probability that the observation A will result in a certain value, when the system is in the state α.

EXERCISE 7.1. Write out in words, from probability theory, what the following symbols mean.

(a) $\mu_{\alpha,A}([3,5]) = 0.9$.

(b) $\mu_{\alpha,A}(\{0\}) = 1$.

AXIOM 2. (a) If A, B are observables for which $\mu_{\alpha,A} = \mu_{\alpha,B}$ for every state $\alpha \in S$, then $A = B$.

(b) If α, β are states for which $\mu_{\alpha,A} = \mu_{\beta,A}$ for every observable $A \in O$, then $\alpha = \beta$.

EXERCISE 7.2. Discuss the intuitive legitimacy of Axiom 2.

AXIOM 3. If $\alpha_1, \ldots \alpha_n$ are states, and $t_1, \ldots t_n$ are nonnegative real numbers for which $\sum_{i=1}^n t_i = 1$, then there exists a state α for which

$$\mu_{\alpha,A} = \sum_{i=1}^n t_i \mu_{\alpha_i,A}$$

for every observable A. This axiom can be interpreted as asserting that the set S of states is closed under convex combinations. If the α_i's are not all identical, we call this state α a *mixed state* and we write $\alpha = \sum_{i=1}^n t_i \alpha_i$.

We say that a state $\alpha \in S$ is a *pure state* if it is not a mixture of other states. That is, if $\alpha = \sum_{i=1}^n t_i \alpha_i$, with each $t_i > 0$ and $\sum_{i=1}^n t_i = 1$, then $\alpha_i = \alpha$ for all i.

EXERCISE 7.3. Discuss the intuitive legitimacy of Axiom 3. Think of a physical system, like a beaker of water, for which there are what we can interpret as pure states and mixed states.

AXIOM 4. If A is an observable, and $f : \mathbb{R} \to \mathbb{R}$ is a Borel function, then there exists an observable B such that

$$\mu_{\alpha,B}(E) = \mu_{\alpha,A}(f^{-1}(E))$$

for every state α and every Borel set $E \subseteq \mathbb{R}$. We denote this observable B by $f(A)$.

EXERCISE 7.4. Discuss the intuitive legitimacy of Axiom 4. Show that, if f is 1-1, the system is in the state α and the observable A results in a value t with probability p, the observable $B = f(A)$ results in the value $f(t)$ with the same probability p.

EXERCISE 7.5. (a) Prove that there exists an observable A such that $\mu_{\alpha,A}(-\infty, 0) = 0$ for every state α. That is, A is an observable that is nonnegative with probability 1 independent of the state of the system. HINT: Use $f(t) = t^2$ for example.

(b) Given a real number t, show that there exists an observable A such that $\mu_{\alpha,A} = \delta_t$ for every state α. That is, A is an observable that equals t with probability 1, independent of the state of the system.

(c) Show that the set of observables is closed under scalar multiplication. That is, if A is an observable and c is a nonzero real number, then there exists an observable B such that

$$\mu_{\alpha,B}(E) = \mu_{\alpha,A}((1/c)E).$$

We may then write $B = cA$.

(d) If A and B are observables, does there have to be an observable C that we could think of as the sum $A + B$?

(e) In what way must we alter the descriptions of the systems in Example 1 and Example 2 in order to incorporate these first four axioms (particularly Axioms 3 and 4)?

DEFINITION. We say that two observables A and B are *compatible, pairwise compatible*, or *simultaneously observable* if there exists an observable C and Borel functions f and g such that $A = f(C)$ and $B = g(C)$. A sequence $\{A_i\}$ is called *mutually compatible* if there exists an observable C and Borel functions $\{f_i\}$ such that $A_i = f_i(C)$ for all i.

EXERCISE 7.6. Is there a difference between a sequence $\{A_i\}$ of observables being pairwise compatible and being mutually compatible? In particular, is it possible that there could exist observables A, B, C, such that A and B are compatible, B and C are compatible, A and C are compatible, and yet A, B, C are not mutually compatible? HINT: Try to modify Example 2.

EXERCISE 7.7. (a) If A, B are observables, what should it mean to say that an observable C is the sum $A+B$ of A and B? Discuss why we do not hypothesize that there always exists such an observable C.

(b) If A and B are compatible, can we prove that there exists an observable C that can be regarded as $A + B$?

DEFINITION. An observable q is called a *question* or a *yes/no observable* if, for each state α, the measure $\mu_{\alpha,q}$ is supported on the two numbers 0 and 1. We say that the result of observing q, when the system is in the state α, is "yes" with probability $\mu_{\alpha,q}(\{1\})$, and it is "no" with probability $\mu_{\alpha,q}(\{0\})$.

THEOREM 7.1. *Let A be an observable.*

(1) *For each Borel subset E in $\mathbb{R}$, the observable $\chi_E(A)$ is a question.*
(2) *If g is a real-valued Borel function on $\mathbb{R}$, for which $g(A)$ is a question, then there exists a Borel set E such that $g(A) = \chi_E(A)$.*

(Note that condition 2 does not assert that g necessarily equals χ_E.)

PROOF. For each Borel set E, we have

$$\begin{aligned}\mu_{\alpha,\chi_E(A)}(\{1\}) &= \mu_{\alpha,A}(\chi_E^{-1}(\{1\})) \\ &= \mu_{\alpha,A}(E),\end{aligned}$$

and

$$\begin{aligned}\mu_{\alpha,\chi_E(A)}(\{0\}) &= \mu_{\alpha,A}(\chi_E^{-1}(\{0\})) \\ &= \mu_{\alpha,A}(\tilde{E}) \\ &= 1 - \mu_{\alpha,A}(E),\end{aligned}$$

which proves that $\mu_{\alpha,\chi_E(A)}$ is supported on the two points 0 and 1 for every α, whence $\chi_E(A)$ is a question and so part 1 is proved.

Given a g for which $q = g(A)$ is a question, set $E = g^{-1}(\{1\})$, and observe that for any $\alpha \in S$ we have

$$\begin{aligned}\mu_{\alpha,q}(\{1\}) &= \mu_{\alpha,g(A)}(\{1\}) \\ &= \mu_{\alpha,A}(E) \\ &= \mu_{\alpha,\chi_E(A)}(\{1\}).\end{aligned}$$

Since both q and $\chi_E(A)$ are questions, it follows from the preceding paragraph that

$$\mu_{\alpha,q}(\{0\}) = \mu_{\alpha,\chi_E(A)}(\{0\}),$$

showing that

$$\mu_{\alpha,q} = \mu_{\alpha,\chi_E(A)}$$

for every state α. Then, by Axiom 2, we have that

$$g(A) = q = \chi_E(A).$$

We now define some mathematical structure on the set Q of all questions. This set will form the fundamental ingredient of our model.

DEFINITION. Let Q denote the set of all questions. For each question $q \in Q$, define a real-valued function m_q on the set S of states by

$$m_q(\alpha) = \mu_{\alpha,q}(\{1\}).$$

If p and q are questions, we say that $p \leq q$ if $m_p(\alpha) \leq m_q(\alpha)$ for all $\alpha \in S$.

If p, q and r are questions, for which $m_r = m_p + m_q$, we say that p and q are *summable* and that r is the sum of p and q. We then write $r = p + q$. More generally, if $\{q_i\}$ is a countable (finite or infinite) set of questions, we say that the q_i's are *summable* if there exists a question q such that

$$m_q(\alpha) = \sum_i m_{q_i}(\alpha)$$

for every $\alpha \in S$. In such a case, we write $q = \sum_i q_i$.

Finally, a countable set $\{q_i\}$ is called *mutually summable* if every subset of the q_i's is summable.

REMARK. As mentioned above, the set Q will turn out to be the fundamental ingredient of our model, in the sense that everything else will be described in terms of Q.

THEOREM 7.2.

(1) *The set Q is a partially ordered set with respect to the ordering $\leq$ defined above.*
(2) *There exists a question $q_1 \in Q$, which we shall often simply call 1, for which $q \leq q_1$ for every $q \in Q$. That is, Q has a maximum element q_1.*
(3) *There exists a question $q_0 \in Q$, which we shall often simply call 0, for which $q_0 \leq q$ for every $q \in Q$. That is, Q has a minimum element q_0.*
(4) *For each question q, there exists a question $\tilde{q}$ such that*

$$m_q + m_{\tilde{q}} = q_1 = 1.$$

That is, every question has a complementary question.

PROOF. That Q is a partially ordered set is clear.

If A is any observable, and f is the identically 1 function, then the question $q_1 = f(A)$ satisfies

$$\begin{aligned} m_{q_1}(\alpha) &= \mu_{\alpha,q_1}(\{1\}) \\ &= \mu_{\alpha,f(A)}(\{1\}) \\ &= \mu_{\alpha,A}(f^{-1}(\{1\})) \\ &= \mu_{\alpha,A}(\mathbb{R}) \\ &= 1 \end{aligned}$$

for all α, and clearly then $q \leq q_1$ for every $q \in Q$.

Taking f to be the identically 0 function, we may define the question q_0 to be $f(A)$.

Finally, if f is the function defined by $f(t) = 1 - t$, and if $q \in Q$, then $f(q)$ is the desired question $\tilde{q}$. Indeed,

$$\begin{aligned}\mu_{\alpha,f(q)}(\{1\}) &= \mu_{\alpha,q}(f^{-1}(\{1\})) \\ &= \mu_{\alpha,q}(\{0\}),\end{aligned}$$

and

$$\begin{aligned}\mu_{\alpha,f(q)}(\{0\}) &= \mu_{\alpha,q}(f^{-1}(\{0\})) \\ &= \mu_{\alpha,q}(\{1\}),\end{aligned}$$

proving that $f(q)$ is a question and showing also that

$$m_{f(q)}(\alpha) = 1 - m_q(\alpha)$$

for every α, as desired.

DEFINITION. Two questions p and q are called *orthogonal* if $p \leq \tilde{q}$ or (equivalently) $q \leq \tilde{p}$. That is, p and q are orthogonal if $m_p + m_q \leq 1$.

REMARK. Clearly, if p and q are summable, then they are orthogonal, but the converse need not hold. Even if $m_p + m_q \leq 1$, there may not be a question r such that $m_r = m_p + m_q$. We have no axiom that ensures this.

Our next goal is to describe the observables in terms of the set Q.

THEOREM 7.3. *Let A be an observable. For each Borel set $E \subseteq \mathbb{R}$, put*

$$q_E^A = \chi_E(A).$$

Then, the mapping $E \to q_E^A$ satisfies:

(1) $q_{\mathbb{R}}^A = 1$ *and* $q_{\emptyset}^A = 0$.

(2) *If $\{E_i\}$ is a sequence of pairwise disjoint Borel sets, then $\{q_{E_i}^A\}$ is a sequence of mutually compatible, mutually summable, (pairwise orthogonal) questions, and*

$$q_{\cup_i E_i}^A = \sum_i q_{E_i}^A.$$

(3) *If A and B are observables, for which $q_E^A = q_E^B$ for every Borel set E, then $A = B$.*

PROOF. Since $\chi_{\mathbb{R}}$ is the identically 1 function, it follows that $q_{\mathbb{R}}^A = 1$. Similarly, $q_{\emptyset}^A = 0$.

If $\{F_i\}$ is any (finite or infinite) sequence of pairwise disjoint Borel sets, set $F = \cup F_i$. Then, clearly the questions $\{q_{F_i}^A\}$ are mutually compatible, since they are all functions of the observable A. Also, for any state α we have

$$\begin{aligned} m_{q_F^A}(\alpha) &= \mu_{\alpha,q_F^A}(\{1\}) \\ &= \mu_{\alpha,\chi_F(A)}(\{1\}) \\ &= \mu_{\alpha,A}(F) \\ &= \mu_{\alpha,A}(\cup F_i) \\ &= \sum_i \mu_{\alpha,A}(F_i) \\ &= \sum_i \mu_{\alpha,\chi_{F_i}(A)}(\{1\}) \\ &= \sum_i m_{q_{F_i}^A}(\alpha). \end{aligned}$$

Now let $\{E_i\}$ be a sequence of pairwise disjoint Borel sets. The preceding calculation, as applied to every subset of the E_i's, shows that the questions $\{q_{E_i}^A\}$ are mutually summable and that

$$q_E^A = \sum q_{E_i}^A.$$

And, in particular, since the $q_{E_i}^A$'s are pairwise summable, they are pairwise orthogonal.

Finally, if A and B are distinct observables, then, by Axiom 2, there exists a state α such that $\mu_{\alpha,A} \neq \mu_{\alpha,B}$. Hence, there is a Borel set E such that

$$\mu_{\alpha,A}(E) \neq \mu_{\alpha,B}(E),$$

or

$$\mu_{\alpha,\chi_E(A)}(\{1\}) \neq \mu_{\alpha,\chi_E(B)}(\{1\}),$$

or $q_E^A \neq q_E^B$, as desired.

DEFINITION. A mapping $E \to q_E$, from the σ-algebra $\mathcal{B}$ of Borel sets into Q, which satisfies the two properties below, is called a *question-valued measure.*

(1) $q_{\mathbb{R}} = 1$ and $q_{\emptyset} = 0$.

(2) If $\{E_i\}$ is a sequence of pairwise disjoint Borel sets, then $\{q_{E_i}\}$ is a sequence of mutually compatible, mutually summable, (pairwise orthogonal) questions, and

$$q_{\cup E_i} = \sum_i q_{E_i}.$$

REMARK. Theorem 7.3 asserts that each observable A determines a question-valued measure q^A and that the assignment $A \to q^A$ is 1-1.

EXERCISE 7.8. Let $E \to q_E$ be a question-valued measure.
(a) Prove that if $E \subseteq F$, then $q_E \leq q_F$; i.e., $E \to q_E$ is order-preserving.
(b) Show that $q_{\tilde{E}} = \tilde{q_E}$; i.e., $E \to q_E$ is complement-preserving.

AXIOM 5. If $E \to q_E$ is a question-valued measure, then there exists an observable A such that $q_E = q_E^A$ for all Borel sets E.

EXERCISE 7.9. Discuss the intuitive legitimacy of Axiom 5.

EXERCISE 7.10. Let $\{q_1, q_2, \dots\}$ be a mutually summable set of questions for which $\sum_i q_i = 1$. Prove that the q_i's are mutually compatible. HINT: Define a question-valued measure $E \to q_E$ by setting $q_{\{i\}} = q_i$ for each $i = 1, 2, \dots$, and define

$$q_E = \sum_{i \in E} q_{\{i\}}.$$

then use Axiom 5.

THEOREM 7.4. *Let p and q be questions. Then p and q are compatible if and only if there exist mutually summable questions r_1, r_2, r_3 and r_4 satisfying:*

(1) $p = r_1 + r_2$.
(2) $q = r_2 + r_3$.
(3) $r_1 + r_2 + r_3 + r_4 = 1$.

PROOF. If p and q are compatible, let A be an observable and let f and g be Borel functions such that $p = f(A)$ and $q = g(A)$. By Theorem 7.1, we may assume that $f = \chi_E$ and $g = \chi_F$, where E and F are Borel sets in $\mathbb{R}$. Define four pairwise disjoint Borel sets as follows:

$$E_1 = E - F,\ E_2 = E \cap F,\ E_3 = F - E,\ E_4 = \mathbb{R} - (E \cup F).$$

Now, define $r_i = \chi_{E_i}(A)$. The desired properties of the r_i's follow directly. For example,

$$\begin{aligned} m_p(\alpha) &= \mu_{\alpha,\chi_E(A)}(\{1\}) \\ &= \mu_{\alpha,A}(E) \\ &= \mu_{\alpha,A}(E_1 \cup E_2) \\ &= \mu_{\alpha,A}(E_1) + \mu_{\alpha,A}(E_2) \\ &= \mu_{\alpha,\chi_{E_1}(A)}(\{1\}) + \mu_{\alpha,\chi_{E_2}(A)}(\{1\}) \\ &= m_{r_1}(\alpha) + m_{r_2}(\alpha), \end{aligned}$$

showing that $p = r_1 + r_2$ as desired. We leave the other verifications to the exercise that follows.

Conversely, given $r_1, \ldots, r_4$ satisfying the conditions in the statement of the theorem, define a mapping $E \to q_E$ of the σ-algebra $\mathcal{B}$ of Borel sets into Q as follows:

$$q_E = \sum_{i \in E} r_i,$$

with the convention that $q_E = 0$ if E does not contain any of the numbers 1,2,3,4. Then $E \to q_E$ is a question-valued measure. (See the preceding exercise.) By Axiom 5, there exists an observable A such that $q_E = q_E^A$ for all E, and clearly $p = \chi_{[1,2]}(A)$ and $q = \chi_{[2,3]}(A)$ are both functions of A, as desired.

EXERCISE 7.11. Verify that $q = r_1 + r_3$ and that $r_1 + r_2 + r_3 + r_4 = 1$ in the first part of the preceding proof.

EXERCISE 7.12. (a) Prove that the map $q \to m_q$ is 1-1.

(b) Show, by identifying q with m_q, that the set Q can be given a natural Hausdorff topology.

(c) Let q be a question. Show that the set of all questions p, for which $p \leq q$, and the set of all questions p such that p is orthogonal to q are closed subsets of Q in the topology from part b.

(d) Prove that the map $q \to \tilde{q}$ is continuous with respect to the topology on Q from part b.

REMARK. We equip the set Q of all questions with the topology from the preceding exercise. That is, we identify each question q with the corresponding function m_q and use the topology of pointwise convergence of these functions. In this way, the set Q is a partially-ordered Hausdorff topological space having a maximum element and a minimum element. In addition to these topological and order structures on Q, there are notions of complement, of orthogonality, of summability, and of compatibility. We shall be interested in finding a mathematical object having these attributes.

EXERCISE 7.13. (a) Show that the closed interval $[0,1]$ has all the properties of Q. That is, show that $[0,1]$ is a partially-ordered topological space having a maximum and a minimum, and show that there is a notion of summability (not the usual one) on $[0,1]$ such that each element has a complement. Finally, prove that any two elements of $[0,1]$ that are summable are compatible. In a way, $[0,1]$ is the simplest model for Q. HINT: Use the characterization of compatibility in Theorem 7.4.

(b) Is the unit circle a possible model for Q?

Having described the set O of observables as question-valued measures, we turn next to the set S of states. We want to describe the states also in terms of the set Q.

DEFINITION. By an *automorphism* of Q we mean a 1-1 map ϕ of Q onto itself that satisfies:

(1) If $p \leq q$, then $\phi(p) \leq \phi(q)$; i.e., ϕ is order-preserving.
(2) $\phi(\tilde{q}) = \widetilde{\phi(q)}$ for all $q \in Q$; i.e., ϕ is complement-preserving.
(3) If $\{q_i\}$ is a summable set of questions, then $\{\phi(q_i)\}$ is a summable set of questions, and

$$\phi(\sum_i q_i) = \sum_i \phi(q_i).$$

If ϕ and ϕ^{-1} are Borel maps of the topological space Q, then ϕ is called a *Borel automorphism*.

By a *character* of the set Q of questions, we mean a continuous function $\mu : Q \to [0,1]$ that satisfies:

(1) If $p \leq q$, then $\mu(p) \leq \mu(q)$; i.e., μ is order-preserving.
(2) $\mu(\tilde{q}) = 1 - \mu(q)$; i.e., μ is complement-preserving.
(3) If $\{q_i\}$ is a summable sequence of questions, then $\mu(\sum q_i) = \sum \mu(q_i)$; i.e., μ is additive when possible.

DEFINITION. For each state α, define a function μ_α on Q by

$$\mu_\alpha(q) = m_q(\alpha) = \mu_{\alpha,q}(\{1\}).$$

EXERCISE 7.14. (a) Show that each function μ_α is a continuous order-preserving map of Q into [0,1].

(b) Show that $\mu_\alpha(\tilde{q}) = 1 - \mu_\alpha(q)$ for all $q \in Q$ and all $\alpha \in S$.

(c) If $\{q_i\}$ is a summable sequence of questions with $q = \sum q_i$, show that

$$\mu_\alpha(q) = \sum \mu_\alpha(q_i).$$

(d) Conclude that each function μ_α is a continuous character of Q.

(e) Show that the composition of a character of Q (e.g., μ_α) and a question-valued measure $E \to q_E$ defines a probability measure on the Borel subsets of $\mathbb{R}$.

(f) Show that the map $\alpha \to \mu_\alpha$ is 1-1 on S. Show further that if α is a mixed state, say $\alpha = \sum_{i=1}^n t_i\alpha_i$, then

$$\mu_\alpha = \sum_{i=1}^{n} t_i \mu_{\alpha_i};$$

i.e., $\alpha \to \mu_\alpha$ is an affine map on S.

REMARK. We give to S the Hausdorff topology obtained by identifying α with the continuous function μ_α on Q and considering this space of functions as topologized by the topology of pointwise convergence. Thus, we identify the set S of states of our system with certain continuous functions (characters) from the set Q of questions into $[0,1]$. Of course, not every continuous function $f : Q \to [0,1]$ need correspond to a state. Indeed, the functions corresponding to states must be characters.

We turn now to the evolution of the system in time. The axiom we take assumes that the system has always existed and will always exist. That is, the system can be thought of as evolving backward in time as well as forward. See part d of Exercise 7.15.

AXIOM 6. (Time Evolution of the System) For each nonnegative real number t, there exists a 1-1 transformation ϕ_t of S onto itself that describes the evolution of the system in time. In addition, for each nonnegative real number t, there exists a corresponding 1-1 transformation ϕ'_t, of the set Q onto itself, so that

(1) $\phi_{t+s} = \phi_t \circ \phi_s$ for all nonnegative s, t.

(2) For all $\alpha \in S,\ q \in Q$, and $t \geq 0$, we have

$$\mu_{\phi_t(\alpha),q} = \mu_{\alpha,\phi'_t(q)}.$$

(3) The map $(t,\alpha) \to \phi_t(\alpha)$ is a Borel map of $[0,\infty) \times S$ into S.

(4) The map $(t,q) \to \phi'_t(q)$ is a Borel map of $[0,\infty) \times Q$ into Q.

EXERCISE 7.15. (a) Discuss the intuitive legitimacy of Axiom 6. In particular, what is the interpretation of the transformation ϕ'_t?

(b) Show that $\phi'_{t+s} = \phi'_t \circ \phi'_s$ for all nonnegative t and s.

(c) Show that ϕ'_t is uniquely determined by ϕ_t and that ϕ_t is uniquely determined by ϕ'_t.

(d) Suppose α is a state. Given $t > 0$, show that there exists a unique state β such that if the system is in the state α now, then it was in the state β t units of time ago. (In other words, the evolution of the system can be reversed in time.)

REMARK. Of course, the primary goal of experimental investigation is to discover how to predict what will happen to a system as time goes by. In our development, then, we would want to discover the evolution transformations ϕ_t of S into itself.

Next, we turn to the notion of a symmetry of the system.

DEFINITION. If g denotes a (possibly hypothetical) 1-1 transformation of space, of the observer, of the system, etc., and if $\alpha \in S$ and $A \in O$ are given, we write $\mu^g_{\alpha,A}$ for the probability measure obtained by assuming that this transformation g has been performed, supposing that the system is in the state α, and by making the observation A. The transformation g is called a *symmetry* of the system if each $\mu^g_{\alpha,A} = \mu_{\alpha,A}$, i.e., if the "measurements" of the system are unchanged by performing the transformation g.

REMARK. We assume that the set G of all symmetries forms a group of transformations.

AXIOM 7. To each symmetry g of the system there corresponds a 1-1 transformation π_g of S onto itself and a 1-1 transformation π'_g of Q onto itself such that

(1) $\pi_{g_1 g_2} = \pi_{g_1} \circ \pi_{g_2}$ for all $g_1, g_2 \in G$.

(2) For all $\alpha \in S$ and all $q \in Q$, we have

$$\mu_{\pi_g(\alpha),q} = \mu_{\alpha,\pi'_g(q)}.$$

(3) If a subgroup H of the group of all symmetries has some "natural" topological structure, then the maps $(h, \alpha) \to \pi_h(\alpha)$ and $(h, q) \to \pi'_h(q)$ are Borel maps from $H \times S$ into S and $H \times Q$ into Q respectively.

(4) π_g commutes with each evolution transformation ϕ_t; i.e., $\pi_g \circ \phi_t = \phi_t \circ \pi_g$ for all $t \geq 0$ and all $g \in G$.

EXERCISE 7.16. (a) Discuss the intuitive legitimacy of Axiom 7. In particular, what is the interpretation of the assumption that each π_g commutes with each evolution transformation ϕ_t?

(b) Show that each π'_g is uniquely determined by π_g, and that $\pi'_{g_1 g_2} = \pi'_{g_1} \circ \pi'_{g_2}$ for all $g_1, g_2 \in G$.

(c) Prove that each transformation π'_g commutes with each evolution transformation ϕ'_t.

THEOREM 7.5. *Each of the time evolution transformations ϕ'_t and each of the symmetry transformations π'_g are Borel automorphisms of the set Q. That is,*

(1) *ϕ'_t, π'_g, and their inverses are Borel maps of Q onto itself.*
(2) *if $p \leq q$, then $\phi'_t(p) \leq \phi'_t(q)$ and $\pi'_g(p) \leq \pi'_g(q)$.*
(3) *$\phi'_t(\tilde{q}) = \widetilde{\phi'_t(q)}$ and $\pi'_g(\tilde{q}) = \widetilde{\pi'_g(q)}$.*
(4) *If $\{q_i\}$ is a summable sequence of questions, then $\{\phi'_t(q_i)\}$ and $\{\pi'_g(q_i)\}$ are summable sequences of questions, and*

$$\phi'_t(\sum q_i) = \sum \phi'_t(q_i)$$

and

$$\pi'_g(\sum q_i) = \sum \pi'_g(q_i).$$

PROOF. Suppose $p \leq q$ are questions. We have then for any α that

$$\begin{aligned} m_{\phi'_t(q)}(\alpha) &= \mu_{\alpha,\phi'_t(q)}(\{1\}) \\ &= \mu_{\phi_t(\alpha),q}(\{1\}) \\ &= m_q(\phi_t(\alpha)) \\ &\geq m_p(\phi_t(\alpha)) \\ &= m_{\phi'_t(p)}(\alpha), \end{aligned}$$

showing that $\phi'_t(q) \geq \phi'_t(p)$. An analogous computation shows that $\pi'_g(q) \geq \pi'_g(p)$.

We leave the rest of the proof to the exercise that follows.

EXERCISE 7.17. Complete the proof to the preceding theorem.

We summarize the ingredients in our model as follows:

(1) There exists a Hausdorff space Q that is a partially ordered set, having a maximum element 1 and a minimum element 0. There are

notions of compatibility, orthogonality, and summability for certain of the elements of Q. Compatibility is characterized in Theorem 7.4.

(2) Each $q \in Q$ has a complementary element $\tilde{q}$ satisfying $q + \tilde{q} = 1$.

(3) The set S of states is represented as a set of continuous homomorphisms (characters) μ of Q into [0,1]. Each of these homomorphisms is continuous, order-preserving, additive when possible, and complement-preserving. This set S of states is a topological space and is closed under convex combinations.

(4) The set O of observables is identified with the set of Q-valued measures.

(5) The time evolution of the system is described by a one-parameter semigroup ϕ'_t of Borel transformations (automorphisms) of Q. These transformations are additive when possible, complement-preserving, and order-preserving.

(6) To each symmetry g of the system there corresponds a 1-1 transformation (automorphism) π'_g of Q onto itself. The transformation π'_g is Borel, preserves order, addition when possible, and complements. Each symmetry transformation π'_g commutes with each evolution transformation ϕ'_t.

The goal is to find concrete mathematical examples of the objects Q, S, ϕ'_t and π'_g. Initially, we will select a model for Q, and this selection will depend very much on which particular system we are studying. The set S is then a subset of the characters of Q, which, in any particular case, we can hope to describe concretely. Of course, the ultimate aim is to determine the evolution transformations ϕ_t of S into itself. Sometimes it is possible to describe the symmetry transformations π'_g by using group theory. If so, we may be able to describe the evolution transformations ϕ'_t by examining what transformations commute with the concrete π'_g's we have. However, our first task is to find an appropriate model for Q, and this we do in the next chapter.

We mention next some possibly less intuitively acceptable axioms. From a mathematical point of view, however, they are technically simplifying.

AXIOM 8. If $\{\alpha_i\}$ is a sequence of states, and if $\{t_i\}$ is a sequence of positive real numbers for which $\sum t_i = 1$, then there exists a state α, which we denote by $\sum t_i \alpha_i$, such that

$$\mu_{\alpha,A} = \sum t_i \mu_{\alpha_i,A}$$

for every observable A.

AXIOM 9. If $\{q_i\}$ is a net of questions, such that the net $\{m_{q_i}\}$ of functions converges pointwise to a function m on S, then there exists a question q such that $m_q = m$.

AXIOM 10. If $\{\alpha_i\}$ is a net of states, for which the net $\{\mu_{\alpha_i}\}$ of characters on Q converges pointwise to a character μ, then there exists a state α such that $\mu_\alpha = \mu$.

AXIOM 11. If μ is a character of Q, then there exists a state α for which $\mu_\alpha = \mu$.

EXERCISE 7.18. Discuss the intuitive legitimacy of Axioms 8, 9, 10, and 11.

AXIOM 12. If p and q are (compatible) questions, such that $p \leq q$ and $p \leq \tilde{q}$, then $p = 0$.

EXERCISE 7.19. (a) Discuss the intuitive legitimacy of Axiom 12.

(b) Suppose that for each nonzero question q there exists a state α such that $m_q(\alpha) > 1/2$. Show that Axiom 12 must then be valid.