# CHAPTER VI

# APPLICATIONS TO ANALYSIS

We include in this chapter several subjects from classical analysis to which the notions of functional analysis can be applied. Some of these subjects are essential to what follows in this text, e.g., convolution, approximate identities, and the Fourier transform. The remaining subjects of this chapter are highly recommended to the reader but will not specifically be referred to later.

## Integral Operators

Let $(S, \mu)$ and $(T, \nu)$ be $\sigma$-finite measure spaces, and let $k$ be a $\mu \times \nu$-measurable, complex-valued function on $S \times T$. We refer to the function $k$ as a *kernel*, and we are frequently interested in when the formula

$$[K(f)](s) = \int_T k(s,t) f(t) \, d\nu(t) \tag{6.1}$$

determines a bounded operator $K$ from $L^p(\nu)$ into $L^r(\mu)$, for some $1 \leq p \leq \infty$ and some $1 \leq r \leq \infty$. Ordinarily, formula (6.1) is only defined for certain functions $f$, the so-called *domain* $D(K)$ of $K$, i.e., the functions $f$ for which $s \to k(s,t)f(t)$ is $\nu$-integrable for $\mu$ almost all $s \in S$. In any event, $D(K)$ is a vector space, and on this domain, $K$ is clearly a linear transformation. More precisely, then, we are interested in when formula (6.1) determines a linear transformation $K$ that can be extended to a bounded operator on all of $L^p(\nu)$ into $L^r(\mu)$. Usually, the domain $D(K)$ is a priori dense in $L^p(\nu)$, and the question above then reduces to whether $K$ is a bounded operator from $D(K)$ into $L^r(\mu)$. That is, does there exist a constant $M$ such that

$$\|K(f)\|_r = (\int_S |\int_T k(s,t)f(t)\,d\nu(t)|^r\,d\mu(s))^{1/r} \leq M\|f\|_p$$

for all $f \in D(K)$. In such a case, we say that $K$ is a *bounded integral operator*. In general, we say that the linear transformation $K$ is an *integral operator determined by the kernel* $k(s,t)$.

The elementary result below is basically a consequence of Hoelder's inequality and the Fubini theorem.

THEOREM 6.1. *Suppose $p, r$ are real numbers strictly between 1 and $\infty$, and let $p'$ and $r'$ satisfy*

$$1/p + 1/p' = 1/r + 1/r' = 1.$$

*Suppose that $k(s,t)$ is a $\mu \times \nu$-measurable function on $S \times T$, and assume that the set $D(K)$ of all $f \in L^p(\nu)$ for which Equation (6.1) is defined is a dense subspace of $L^p(\nu)$. Then:*

(1) *If the function $b$ defined by $s \to \int_T |k(s,t)|^{p'}\,d\nu(t)$ is an element of $L^{r/p'}(\mu)$, then $K$ is a bounded integral operator from $L^p(\nu)$ into $L^r(\mu)$.*

(2) *Suppose $p = r$ and that there exists an $\alpha \in [0,1]$ for which the function $s \to \int_T |k(s,t)|^{\alpha p'}\,d\nu(t)$ is an element of $L^\infty(\mu)$ with $L^\infty$ norm $c_1$, and the function $t \to \int_S |k(s,t)|^{(1-\alpha)p}\,d\mu(s)$ is an element of $L^\infty(\nu)$ with $L^\infty$ norm $c_2$. Then $K$ is a bounded integral operator from $L^p(\nu)$ into $L^p(\mu)$. Moreover, we have that*

$$\|K(f)\|_p \leq c_1^{1/p'} c_2^{1/p} \|f\|_p$$

*for all $f \in L^p(\nu)$.*

PROOF. Let $f \in D(K)$ be fixed. We have that

$$\begin{aligned} |\int_T k(s,t)f(t)\,d\nu(t)| &\leq \int_T |k(s,t)f(t)|\,d\nu(t) \\ &\leq (\int_T |k(s,t)|^{p'}\,d\nu(t))^{1/p'} \times (\int_T |f(u)|^p\,d\nu(u))^{1/p} \\ &= b(s)^{1/p'}\|f\|_p, \end{aligned}$$

from which it follows that $D(K)$ is all of $L^p(\nu)$ for part 1, and

$$\|K(f)\|_r = (\int_S |\int_T k(s,t)f(t)\,d\nu(t)|^r\,d\mu(s))^{1/r} \leq \|b\|_{r/p'}^{1/p'}\|f\|_p.$$

This proves part 1.

Again, for $f \in D(K)$ we have that

$$\begin{aligned}
|\int_T k(s,t)f(t)\,d\nu(t)| &\leq \int_T |k(s,t)|^\alpha |k(s,t)|^{1-\alpha}|f(t)|\,d\nu(t) \\
&\leq (\int_T |k(s,t)|^{\alpha p'}\,d\nu(t))^{1/p'} \\
&\quad\times (\int_T |k(s,u)|^{(1-\alpha)p}|f(u)|^p\,d\nu(u))^{1/p} \\
&\leq c_1^{1/p'}(\int_T |k(s,u)|^{(1-\alpha)p}|f(u)|^p\,d\nu(u))^{1/p},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\|K(f)\|_p &= (\int_S |\int_T k(s,t)f(t)\,d\nu(t)|^p\,d\mu(s))^{1/p} \\
&\leq c_1^{1/p'}(\int_S\int_T |k(s,u)|^{(1-\alpha)p}|f(u)|^p\,d\nu(u)\,d\mu(s))^{1/p} \\
&= c_1^{1/p'}(\int_T\int_S |k(s,u)|^{(1-\alpha)p}|f(u)|^p\,d\mu(s)\,d\nu(u))^{1/p} \\
&\leq c_1^{1/p'}c_2^{1/p}\|f\|_p,
\end{aligned}$$

and this proves part 2.

EXERCISE 6.1. (a) Restate part 1 of the above theorem for $p = r$.

(b) Restate part 1 of the above theorem for $r = p'$. Restate both parts of the theorem for $p = r = 2$.

(c) As a special case of part 2 of the theorem above, reprove it for $p = r = 2$ and $\alpha = 1/2$.

(d) How can we extend the theorem above to the case where $p$ or $r$ is 1 or $\infty$?

EXERCISE 6.2. Suppose both $\mu$ and $\nu$ are finite measures.

(a) Show that if the kernel $k(s,t)$ is a bounded function on $S \times T$, then (6.1) determines a bounded integral operator $K$ for all $p$ and $r$.

(b) Suppose $S = T = [a,b]$ and that $\mu$ and $\nu$ are both Lebesgue measure. Define $k$ to be the characteristic function of the set of all pairs

$(s,t)$ for which $s \geq t$. Show that (6.1) determines a bounded integral operator $K$ from $L^1(\mu)$ into $L^1(\nu)$. Show further that $K(f)$ is always differentiable almost everywhere, and that $[K(f)]' = f$.

(c) Suppose $k$ is an element of $L^2(\mu \times \nu)$. Use Theorem 6.1 to show that (6.1) determines a bounded integral operator $K$ from $L^2(\nu)$ into $L^2(\mu)$.

(d) Is part c valid if $\mu$ and $\nu$ are only assumed to be $\sigma$-finite measures?

**Convolution Kernels**

THEOREM 6.2. (Young's Inequality) *Let $f$ be a complex-valued measurable function on $\mathbb{R}^n$, and define $k \equiv k_f$ on $\mathbb{R}^n \times \mathbb{R}^n$ by*

$$k(x,y) = f(x-y).$$

*If $f \in L^1(\mathbb{R}^n)$ and $1 \leq p \leq \infty$, then (6.1) determines a bounded integral operator $K \equiv K_f$ from $L^p(\mathbb{R}^n)$ into itself, where we equip each space $\mathbb{R}^n$ with Lebesgue measure. Moreover, $\|K_f(g)\|_p \leq \|f\|_1 \|g\|_p$ for every $g \in L^p(\mathbb{R}^n)$.*

PROOF. Suppose first that $p = \infty$. We have that

$$\begin{aligned}\|K_f(g)\|_\infty &= \sup_x \left|\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right| \\ &\leq \sup_x \int_{\mathbb{R}^n} |f(x-y)| \|g\|_\infty \,dy \\ &= \|f\|_1 \|g\|_\infty,\end{aligned}$$

as desired.

Now, suppose $1 \leq p < \infty$. Let $g$ be in $L^p(\mathbb{R}^n)$ and $h$ be in $L^{p'}(\mathbb{R}^n)$, $(1/p + 1/p' = 1)$. By Tonelli's Theorem, we have that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)g(y)h(x)|\,dydx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(-y)g(x+y)h(x)|\,dydx \\ &\leq \|f\|_1 \sup_y \int_{\mathbb{R}^n} |g(x+y)h(x)|\,dx \\ &\leq \|f\|_1 \sup_y \|g\|_p \|h\|_{p'},\end{aligned}$$

which shows that the function $f(x-y)g(y)h(x)$ is integrable on $\mathbb{R}^n \times \mathbb{R}^n$. Therefore, for almost all $x$, the function $y \to f(x-y)g(y)$ is integrable on $\mathbb{R}^n$. Because the inequality above holds for every $h \in L^{p'}(\mathbb{R}^n)$, the resulting function $K_f(g)$ of $x$ belongs to $L^p(\mathbb{R}^n)$. Moreover,

$$\|K_f(g)\|_p \leq \|f\|_1 \|g\|_p,$$

and the proof is complete.

By $\mathbb{T}$ we shall mean the half-open interval [0,1), and we shall refer to $\mathbb{T}$ as the *circle*. By $L^p(\mathbb{T})$ we shall mean the set of all Lebesgue measurable functions on $\mathbb{R}$, which are periodic with period 1, satisfying $\int_0^1 |f(x)|^p \, dx < \infty$.

EXERCISE 6.3. (a) Use part 2 of Theorem 6.1 to give an alternative proof of Theorem 6.2 in the case $1 < p < \infty$.

(b) (Convolution on the circle) If $f \in L^1(\mathbb{T})$, define $k = k_f$ on $\mathbb{T} \times \mathbb{T}$ by $k_f(x, y) = f(x - y)$. Prove that $K_f$ is a bounded integral operator from $L^p(\mathbb{T})$ into itself for all $1 \leq p \leq \infty$. In fact, prove this two ways: Use Theorem 6.1, and then mimic the proof of Theorem 6.2.

DEFINITION. If $f \in L^1(\mathbb{R}^n)$ ($L^1(\mathbb{T})$), then the bounded integral operator $K_f$ of the preceding theorem (exercise) is called the *convolution operator* by $f$, and we denote $K_f(g)$ by $f * g$. The kernel $k_f(x, y) = f(x - y)$ is called a *convolution kernel*.

EXERCISE 6.4. (a) Suppose $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$. Show that the function $f * g$, defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, dy,$$

is everywhere well-defined. Show further that $f * g$ is continuous and vanishes at infinity. Show finally that $f * g = g * f$.

(b) If $f, g, h \in L^1$, show that $f*g = g*f$ and that $(f*g)*h = f*(g*h)$.

The next result is a useful generalization of Theorem 6.2.

THEOREM 6.3. *Let $f$ be an element of $L^p(\mathbb{R}^n)$. Then, for any $1 \leq q \leq p'$, convolution by $f$ is a bounded operator from $L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, where $1/p + 1/q - 1/r = 1$.*

EXERCISE 6.5. Use the Riesz Interpolation Theorem, Theorem 6.2, and Exercise 6.4 to prove Theorem 6.3.

REMARK. Later, we will be interested in convolution kernels $k_f$ where the function $f$ does not belong to any $L^p$ space. Such kernels are called *singular kernels*. Though the arguments above cannot be used on such singular kernels, nevertheless these kernels often define bounded integral operators.

## Reproducing Kernels and Approximate Identities

DEFINITION. Let $(S, \mu)$ be a $\sigma$-finite measure space and let $k(x, y)$ be a $\mu \times \mu$-measurable kernel on $S \times S$. Suppose that the operator $K$, defined by (6.1), is a bounded integral operator from $L^p(\mu)$ into itself.

Then $K$ is called a *reproducing kernel* for a subspace $V$ of $L^p(\mu)$ if $K(g) = g$ for all $g \in V$. A parameterized family $\{k_t\}$ of kernels is called an *approximate identity* for a subspace $V$ of $L^p(\mu)$ if all the corresponding operators $K_t$ are bounded integral operators, and $\lim_{t\to 0} K_t(g) = g$ for every $g \in V$, where the limit is taken in $L^p(\mu)$.

THEOREM 6.4. *Let $S$ be the closed unit disk in $\mathbb{C}$. Using the Riesz Representation Theorem (Theorem 1.5), let $\mu$ be the measure on $S$ whose corresponding integral is defined on the space $C(S)$ of continuous functions on $S$ by*

$$\int_S f(z)\,d\mu(z) = \int_0^{2\pi} f(e^{i\theta})\,d\theta.$$

*Let $p = 1$, and let $H$ be the subspace of $L^1(\mu)$ consisting of the (complex-valued) functions that are continuous on $S$ and analytic on the interior of $S$. Let $k(z,\zeta)$ be the kernel on $S \times S$ defined by*

$$k(z,\zeta) = \frac{1}{2\pi}\frac{1}{1-(z/\zeta)},$$

*if $z \neq \zeta$, and*

$$k(z,z) = 0$$

*for all $z \in S$. Then $k$ is a reproducing kernel for $H$.*

EXERCISE 6.6. Prove Theorem 6.4. HINT: Cauchy's formula.

REMARK. Among the most interesting reproducing kernels and approximate identities are the ones that are convolution kernels.

THEOREM 6.5. *Let $k$ be a nonnegative Lebesgue-measurable function on $\mathbb{R}^n$ for which $\int k(x)\,dx = 1$. For each positive $t$, define*

$$k_t(x) = (1/t^n)k(x/t),$$

*and set*

$$K(x) = \int_{\|x\|\leq\|y\|} k(y)\,dy.$$

*Then:*

(1) *If $f$ is uniformly continuous and bounded on $\mathbb{R}^n$, then $k_t * f$ converges uniformly to $f$ on $\mathbb{R}^n$ as $t$ approaches 0.*
(2) *If $K \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$, then $k_t * f$ converges to $f$ in $L^p(\mathbb{R}^n)$ for every $f \in L^p(\mathbb{R}^n)$ as $t$ approaches 0.*
(3) *If $k \in L^{p'}(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty$ and $1/p + 1/p' = 1)$, and $f$ is continuous at a point $x$, then $(k_t * f)(x)$ converges to $f(x)$ as $t$ approaches 0.*

PROOF. To prove part 1, we must show that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $t < \delta$ then $|(k_t * f)(x) - f(x)| < \epsilon$ for all $x$. Note first that $\int k_t(x)\,dx = 1$ for all $t$. Write

$$(k_t * f)(x) = \int k_t(x-y)f(y)\,dy = \int k_t(y)f(x-y)\,dy.$$

So, we have that

$$\begin{aligned} |(k_t * f)(x) - f(x)| &= |\int k_t(y)f(x-y)\,dy - f(x)\int k_t(y)\,dy| \\ &\leq \int k_t(y)|f(x-y) - f(x)|\,dy \\ &= \int_{\|y\|\leq h} k_t(y)|f(x-y) - f(x)|\,dy \\ &\quad + \int_{\|y\|>h} k_t(y)|f(x-y) - f(x)|\,dy \end{aligned}$$

for any positive $h$. Therefore, given $\epsilon > 0$, choose $h$ so that $|f(w) - f(z)| < \epsilon/2$ if $\|w - z\| < h$, and set $M = \|f\|_\infty$. Then

$$\begin{aligned} |(k_t * f)(x) - f(x)| &\leq \int_{\|y\|\leq h} k_t(y)(\epsilon/2)\,dy + \int_{\|y\|>h} k_t(y)2M\,dy \\ &\leq (\epsilon/2)\int k_t(y)\,dy + 2M\int_{\|y\|>h} k_t(y)\,dy \\ &= (\epsilon/2) + 2M\int_{\|y\|>h/t} k(y)\,dy \end{aligned}$$

for all $x$. Finally, since $k \in L^1(\mathbb{R}^n)$, there exists a $\rho > 0$ such that

$$\int_{\|y\|>\rho} k(y)\,dy < \epsilon/(4M),$$

whence

$$|(k_t * f)(x) - f(x)| < \epsilon$$

for all $x$ if $t < \delta = h/\rho$. This proves part 1.

By Theorem 6.2 we have that $\|k_t * f\|_p \leq \|f\|_p$ for all $t$. Hence, if $f \in L^p(\mathbb{R}^n)$ and $\{f_j\}$ is a sequence of continuous functions with compact support that converges to $f$ in $L^p$ norm, then

$$\|k_t * f - f\|_p \leq \|k_t * (f - f_j)\|_p + \|k_t * f_j - f_j\|_p + \|f_j - f\|_p.$$

Given $\epsilon > 0$, choose $j$ so that the first and third terms are each bounded by $\epsilon/3$. Hence, we need only verify part 2 for an $f \in L^p(\mathbb{R}^n)$ that is continuous and has compact support. Suppose the support of such an $f$ is contained in the ball of radius $a$ around 0. From the proof above for part 1, we see that $|(k_t * f)(x) - f(x)| \leq 2M$ for all $x$. Moreover, if $\|x\| \geq 2a$ and $t < 1/2$, then

$$\begin{aligned}
|(k_t * f)(x) - f(x)| &= |\int k_t(y)(f(x-y) - f(x))\,dy| \\
&\leq \int k_t(y)|f(x-y)|\,dy \\
&= \int_{\|x\|-a\leq\|y\|\leq\|x\|+a} k_t(y)|f(x-y)|\,dy \\
&\leq M \int_{\|x\|-a\leq\|y\|} k_t(y)\,dy \\
&\leq M \int_{\|x/2\|\leq\|y\|} k_t(y)\,dy \\
&= M \int_{\|x/2t\|\leq\|y\|} k(y)\,dy \\
&\leq M \int_{\|x\|\leq\|y\|} k(y)\,dy \\
&= MK(x).
\end{aligned}$$

Hence, $|(k_t * f)(x) - f(x)|$ is bounded for all $t < 1/2$ by a fixed function in $L^p(\mathbb{R}^n)$, so that part 2 follows from part 1 and the dominated convergence theorem.

We leave part 3 to the exercises.

EXERCISE 6.7. (a) Prove part 3 of the preceding theorem.

(b) (Poisson Kernel on the Line) For each $t > 0$ define a kernel $k_t$ on $\mathbb{R} \times \mathbb{R}$ by

$$k_t(x, y) = (t/\pi)(1/(t^2 + (x-y)^2)).$$

Prove that $\{k_t\}$ is an approximate identity for $L^p(\mathbb{R})$ for $1 \leq p < \infty$. HINT: Note that the theorem above does not apply directly. Alter the proof.

(c) (Poisson Kernel in $\mathbb{R}^n$) Let $c = \int_{\mathbb{R}^n} 1/(1 + \|x\|^2)^{(n+1)/2}\,dx$. For each positive $t$, define a kernel $k_t$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$k_t(x, y) = \frac{t/c}{(t^2 + \|x-y\|^2)^{(n+1)/2}}.$$

Prove that $\{k_t\}$ is an approximate identity for $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

(d) (Poisson Kernel on the Circle) For each $0 < r < 1$ define a function $k_r$ on $\mathbb{T}$ by

$$k_r(x) = \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi x)}.$$

Show that $k_r(x) \geq 0$ for all $r$ and $x$, and that

$$k_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n x},$$

whence $\int_0^1 k_r(x)\,dx = 1$ for every $0 < r < 1$. Prove that $\{k_r\}$ is an approximate identity for $L^p(\mathbb{T})$ $(1 \leq p < \infty)$ in the sense that

$$f = \lim_{r \to 1} k_r * f,$$

where the limit is taken in $L^p(\mathbb{T})$.

EXERCISE 6.8. (Gauss Kernel) (a) Define $g$ on $\mathbb{R}$ by

$$g(x) = (1/\sqrt{2\pi})e^{-x^2/2},$$

and set

$$g_t(x) = (1/\sqrt{t})g(x/\sqrt{t}) = (1/\sqrt{2\pi t})e^{-x^2/2t}.$$

Prove that $\{g_t\}$ is an approximate identity for $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

(b) Define $g$ on $\mathbb{R}^n$ by

$$g(x) = (1/(2\pi)^{n/2})e^{-\|x\|^2/2},$$

and set

$$g_t(x) = (1/t^{n/2})g(x/\sqrt{t}) = (2\pi t)^{-n/2}e^{-\|x\|^2/2t}.$$

Prove that $\{g_t\}$ is an approximate identity for $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

**Green's Functions**

DEFINITION. Let $\mu$ be a $\sigma$-finite Borel measure on $\mathbb{R}^n$, let $D$ be a dense subspace of $L^p(\mu)$, and suppose $L$ is a (not necessarily continuous) linear transformation of $D$ into $L^p(\mu)$. By a *Green's function for* $L$ we shall mean a $\mu \times \mu$-measurable kernel $g(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n$, for which the corresponding (not necessarily bounded) integral operator $G$ satisfies the following: If $v$ belongs to the range of $L$, then $G(v)$, defined by

$$[G(v)](x) = \int_{\mathbb{R}^n} g(x,y)v(y)\,d\mu(y),$$

belongs to $D$, and $L(G(v)) = v$. That is, the integral operator $G$ is a right inverse for the transformation $L$.

Obviously, knowing a Green's function for an operator $L$ is of use in solving for $u$ in an equation like $L(u) = f$. Not every (even invertible) linear transformation $L$ has a Green's function, although many classical transformations do. There are various techniques for determining Green's functions for general kinds of transformations $L$, but the most important $L$'s are differential operators. The following exercise gives a classical example of the construction of a Green's function for such a transformation.

EXERCISE 6.9. Let $b$ be a positive real number, let $f$ be an element of $L^1([0,b])$, and consider the $n$th order ordinary differential equation:

$$u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1u' + a_0u = f, \qquad 6.2)$$

where the coefficients $a_0, \ldots, a_{n-1}$ are constants. Let $D$ denote the set of all $n$ times everywhere-differentiable functions $u$ on $[0,b]$ for which $u^{(n)} \in L^1([0,b])$, and let $L$ be the transformation of $D$ into $L^1([0,b]) \subset L^1(\mathbb{R})$ defined by

$$L(u) = u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1u' + a_0u.$$

Let $A$ denote the $n \times n$ matrix defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix},$$

let $\vec{F}(t)$ be the vector-valued function given by

$$\vec{F}(t) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f(t) \end{bmatrix},$$

and consider the vector-valued differential equation

$$\dot{\vec{U}} = A\vec{U} + \vec{F}. \tag{6.3}$$

(a) Show that if $\vec{U}$ is a solution of Equation (6.3), then $u_1$ is a solution of Equation (6.2), where $u_1$ is the first component of $\vec{U}$.

(b) If $B$ is an $n \times n$ matrix, write $e^B$ for the matrix defined by

$$e^B = \sum_{j=0}^{\infty} B^j/j!.$$

Define $\vec{U}$ on $[0, b]$ by

$$\vec{U}(t) = \int_0^t e^{(t-s)A} \times \vec{F}(s)\, ds.$$

Prove that $\vec{U}$ is a solution of Equation (6.3).

(c) For $1 \le i, j \le n$, write $c_{ij}(t)$ for the $ij$th component of the matrix $e^{tA}$. Define $g(t, s) = c_{1n}(t - s)$ if $s \le t$ and $g(t, s) = 0$ otherwise. Prove that $g$ is a Green's function for $L$.

We give next two general, but certainly not all-inclusive, results on the existence of Green's functions.

If $h(x, y)$ is a function of two variables, we denote by $h^y$ the function of $x$ defined by $h^y(x) = h(x, y)$.

THEOREM 6.6. *Let $\mu$ be a regular (finite on compact sets) $\sigma$-finite Borel measure on $\mathbb{R}^n$, let $D$ be a dense subspace of $L^p(\mu)$ ($1 \le p \le \infty$), and let $L$ be a (not necessarily continuous) linear transformation of $D$ into $L^1(\mu)$. Assume that:*

(1) *There exists a bounded integral operator $K$ from $L^1(\mu)$ into $L^1(\mu)$, determined by a kernel $k(x, y)$, for which $k$ is a reproducing kernel for the range $V$ of $L$, and such that the map $y \to k^y$ is uniformly continuous from $\mathbb{R}^n$ into $L^1(\mu)$.*

(2) *There exists a bounded integral operator $G$ from $L^1(\mu)$ into $L^p(\mu)$, determined by a kernel $g(x, y)$, such that the map $y \to g^y$ is uniformly continuous from $\mathbb{R}^n$ into $D$, and such that $L(g^y) = k^y$ for all $y$.*

(3) *The graph of $L$, thought of as a subset of $L^p(\mu) \times L^1(\mu)$, is closed. That is, if $\{u_j\}$ is a sequence of elements of $D$ that converges to an element $u \in L^p$, and if the sequence $\{L(u_j)\}$ converges in $L^1$ to a function $v$, then the pair $(u, v)$ belongs to the graph of $L$; i.e., $u \in D$ and $v = L(u)$.*

*Then $g$ is a Green's function for $L$.*

PROOF. Let $v$ be in the range of $L$, and let $\{\phi_j\}$ be a sequence of simple functions having compact support that converges to $v$ in $L^1(\mu)$. Because $\mu$ is regular and $\sigma$-finite, we may assume that

$$\phi_j = \sum_{i=1}^{n_j} a_{i,j}\chi_{E_{i,j}},$$

where

$$\lim_{j\to\infty} \max_i \operatorname{diam}(E_{i,j}) \equiv \lim \delta_j = 0.$$

For each $j = 1, 2, \ldots$ and each $1 \le i \le n_j$, let $y_{i,j}$ be an element of $E_{i,j}$, and define functions $v_j$ and $u_j$ by

$$v_j(x) = \sum_{i=1}^{n_j} a_{i,j}\mu(E_{i,j})k^{y_{i,j}}(x) = \sum_{i=1}^{n_j} a_{i,j}\mu(E_{i,j})k(x, y_{i,j})$$

and

$$u_j(x) = \sum_{i=1}^{n_j} a_{i,j}\mu(E_{i,j})g^{y_{i,j}}(x). = \sum_{i=1}^{n_j} a_{i,j}\mu(E_{i,j})g(x, y_{i,j}).$$

Notice that each $u_j \in D$ and that $v_j = L(u_j)$. Finally, for each positive $\delta$, define $\epsilon_1(\delta)$ and $\epsilon_2(\delta)$ by

$$\epsilon_1(\delta) = \sup_{\|y-y'\|<\delta} \|k^y - k^{y'}\|_1$$

and

$$\epsilon_2(\delta) = \sup_{\|y-y'\|<\delta} \|g^y - g^{y'}\|_p.$$

By the uniform continuity assumptions on the maps $y \to k^y$ and $y \to g^y$, we know that

$$0 = \lim_{\delta\to 0} \epsilon_i(\delta).$$

First, we have that $v = \lim_j v_j$. For

$$\begin{aligned}
\|v - v_j\|_1 &= \|K(v) - v_j\|_1 \\
&\leq \|K(v - \phi_j)\|_1 + \|K(\phi_j) - v_j\|_1 \\
&\leq \|K\|\|v - \phi_j\|_1 + \int |[K(\phi_j)](x) - v_j(x)|\, d\mu(x) \\
&= \|K\|\|v - \phi_j\|_1 + \int | \int k(x,y) \sum_{i=1}^{n_j} a_{i,j}\chi_{E_{i,j}}(y)\, dy \\
&\quad - \sum_{i=1}^{n_j} a_{i,j}\mu(E_{i,j})k(x, y_{i,j})|\, dx \\
&= \|K\|\|v - \phi_j\|_1 + \int | \sum_{i=1}^{n_j} [\int k(x,y)a_{i,j}\chi_{E_{i,j}}(y) \\
&\quad - k(x, y_{i,j})a_{i,j}\chi_{E_{i,j}}(y)\, dy]|\, dx \\
&\leq \|K\|\|v - \phi_j\|_1 \\
&\quad + \sum_{i=1}^{n_j} |a_{i,j}| \int \chi_{E_{i,j}}(y) \int |k(x,y) - k(x, y_{i,j})|\, dxdy \\
&= \|K\|\|v - \phi_j\|_1 + \sum_{i=1}^{n_j} |a_{i,j}| \int \|k^y - k^{y_{i,j}}\|_1 \chi_{E_{i,j}}(y)\, dy \\
&\leq \|K\|\|v - \phi_j\|_1 + \epsilon_1(\delta_j)\|\phi_j\|_1,
\end{aligned}$$

which tends to zero as $j$ tends to $\infty$.

Similarly, we have that $G(v) = \lim_j u_j$. (See the following exercise.) So, since the graph of $L$ is closed, and since $L(u_j) = v_j$ for all $j$, we see that $G(v) \in D$ and $L(G(v)) = v$, as desired.

EXERCISE 6.10. In the proof of Theorem 6.6, verify that $G(v)$ is the $L^p$ limit of the sequence $\{u_j\}$. HINT: Use the integral form of Minkowski's inequality. See Exercise 4.13.

THEOREM 6.7. *Let $\mu, D$, and $L$ be as in the preceding theorem. Suppose $\{g_t(x,y)\}$ is a parameterized family of kernels on $\mathbb{R}^n \times \mathbb{R}^n$ such that, for each $t$, the operator $G_t$ determined by the kernel $g_t$ is a bounded integral operator from $L^1(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, and the map $y \to g_t^y$ is uniformly continuous from $\mathbb{R}^n$ into $D$. Suppose that $\{k_t(x,y)\}$ is an approximate identity for the range of $L$, that for each $t$ the map $y \to k_t^y$ is uniformly continuous from $\mathbb{R}^n$ into $L^1(\mu)$, and that $L(g_t^y) = k_t^y$ for* *all $t$ and $y$. Suppose finally that* $\lim_{t\to 0} g_t(x,y) = g(x,y)$ *for almost all $x$ and $y$, and that* $\lim_{t\to 0} G_t(v) = G(v)$ *for each $v$ in the range of $L$, where $G$ is the integral operator determined by the kernel $g$. Then $g$ is a Green's function for $L$.*

EXERCISE 6.11. Prove Theorem 6.7. HINT: For $v$ in the range of $L$, show that $G_t(v) \in D$ and that $L(G_t(v)) = K_t(v)$. Then use again the fact that the graph of $L$ is closed.

EXERCISE 6.12. Let $\mu$ be Lebesgue measure on $\mathbb{R}^n$, and suppose $D$ and $L$ are as in the preceding two theorems. Assume that $L$ is homogeneous of degree $d$. That is, if $\delta_t$ is the map of $\mathbb{R}^n$ into itself defined by $\delta_t(x) = tx$, then

$$L(u \circ \delta_t) = t^d [L(u)] \circ \delta_t.$$

(Homogeneous differential operators fall into this class.) Suppose $p$ is a nonnegative function on $\mathbb{R}^n$ of integral 1, and that $u_0$ is an element of $D$ for which $L(u_0) = p$. Define $g_t(x) = t^{d-n} u_0(x/t)$, and assume that, for each $v$ in the range of $L$, $\lim_{t\to 0} g_t * v$ exists and that $g_t$ converges, as $t$ approaches 0, almost everywhere to a function $g$. Show that $g$ is a Green's function for $L$. HINT: Use Theorem 6.5 to construct an approximate identity from the function $p$. Then verify that the hypotheses of Theorem 6.7 hold.

## Fourier Transform

DEFINITION. If $f$ is a complex-valued function in $L^1(\mathbb{R})$, define a function $\hat{f}$ on $\mathbb{R}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx.$$

The function $\hat{f}$ is called the *Fourier transform* of $f$.

EXERCISE 6.13. (a) (Riemann-Lebesgue Theorem) For $f \in L^1$, show that the Fourier transform $\hat{f}$ of $f$ is continuous, vanishes at infinity, and $\|\hat{f}\|_\infty \leq \|f\|_1$. HINT: Do this first for $f$ the characteristic function of a finite interval $(a,b)$ and then approximate (in $L^1$ norm) an arbitrary $f$ by step functions.

(b) If $f$ and $g$ are elements of $L^1$, prove the *Convolution Theorem*

$$\widehat{f * g} = \hat{f}\hat{g}$$

and the *Exchange Theorem*

$$\int f\hat{g} = \int \hat{f}g.$$

(c) For $f \in L^1$, define $f^*$ by

$$f^*(x) = \overline{f(-x)}.$$

Show that $\widehat{f^*} = \overline{\hat{f}}$.

(d) If $|x||f(x)| \in L^1$, show that $\hat{f}$ is differentiable, and

$$\hat{f}'(\xi) = -2\pi i \int xf(x)e^{-2\pi ix\xi}\,dx.$$

(e) If $f$ is absolutely continuous ($f(x) = \int_{-\infty}^{x} f'$), and both $f$ and $f'$ are in $L^1(\mathbb{R})$, show that $\xi\hat{f}(\xi) \in C_0$.

(f) Show that the Fourier transform sends Schwartz space $\mathcal{S}$ into itself.

(g) If $f(x) = e^{-2\pi|x|}$, show that

$$\hat{f}(\xi) = (1/\pi)\frac{1}{1+\xi^2}.$$

(h) If $g(x) = e^{-\pi x^2}$, show that

$$\hat{g}(\xi) = e^{-\pi\xi^2} = g(\xi).$$

That is, $\hat{g} = g$. HINT: Show that $\hat{g}$ satisfies the differential equation

$$\hat{g}'(\xi) = -2\pi\xi\hat{g}(\xi),$$

and

$$\hat{g}(0) = 1.$$

Recall that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}.$$

EXERCISE 6.14. (Inversion Theorem)

(a) (Fourier transform of the Gauss kernel) If $g_t$ is the function defined by

$$g_t(x) = (1/\sqrt{2\pi t})e^{-x^2/2t},$$

use part h of the preceding exercise to show that

$$\widehat{g_t}(\xi) = e^{-2\pi^2 t\xi^2},$$

whence

$$g_t(x) = \int \hat{g}_t(\xi) e^{2\pi i x\xi}\, d\xi.$$

(b) Show that for any $f \in L^1$, for which $\hat{f}$ also is in $L^1$, we have that $f$ is continuous and

$$f(x) = \int \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi.$$

HINT: Make use of the fact that the $g_t$'s of part a form an approximate identity. Establish the equality

$$\int \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = \lim_{t\to 0} \int \hat{f}(\xi)\hat{g}_t(\xi) e^{2\pi i x\xi}\, d\xi,$$

and then use the convolution theorem.

(c) Conclude that the Fourier transform is 1-1 on $L^1$.

(d) Show that Schwartz space is mapped 1-1 and onto itself by the Fourier transform. Show further that the Fourier transform is a topological isomorphism of order 4 from the locally convex topological vector space $\mathcal{S}$ onto itself.

THEOREM 6.8. (Plancherel Theorem) *If* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then* $\hat{f} \in L^2(\mathbb{R})$ *and* $\|f\|_2 = \|\hat{f}\|_2$. *Consequently, if* $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then*

$$\int f(x)\overline{g(x)}\, dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi.$$

PROOF. Suppose first that $f$ is in Schwartz space $\mathcal{S}$, and write $f^*$ for the function defined by

$$f^*(x) = \overline{f(-x)}.$$

Then $f * f^* \in L^1$, and $\widehat{f * f^*} = |\hat{f}|^2 \in L^1$. So, by the Inversion Theorem,

$$\begin{aligned}
\|f\|_2^2 &= \int f(x)\overline{f(x)}\,dx \\
&= \int f(x) f^*(-x)\,dx \\
&= (f * f^*)(0) \\
&= \int \widehat{f * f^*}(\xi) e^{2\pi i 0\times\xi}\,d\xi \\
&= \int \widehat{f * f^*}(\xi)\,d\xi \\
&= \int |\hat{f}(\xi)|^2\,d\xi \\
&= \|\hat{f}\|_2^2.
\end{aligned}$$

Now, if $f$ is an arbitrary element of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and if $\{f_n\}$ is a sequence of elements of $\mathcal{S}$ that converges to $f$ in $L^2$ norm, then the sequence $\{\widehat{f_n}\}$ is a Cauchy sequence in $L^2(\mathbb{R})$, whence converges to an element $g \in L^2(\mathbb{R})$. We need only show that $g$ and $\hat{f}$ agree almost everywhere. If $h$ is any element of $\mathcal{S}$ we have, using part b of Exercise 6.13, that

$$\begin{aligned}
\int g(\xi)h(\xi)\,d\xi &= \lim \int \widehat{f_n}(\xi)h(\xi)\,d\xi \\
&= \lim \int f_n(\xi)\hat{h}(\xi)\,d\xi \\
&= \int f(\xi)\hat{h}(\xi)\,d\xi \\
&= \int \hat{f}(\xi)h(\xi)\,d\xi,
\end{aligned}$$

showing that $\hat{f}$ and $g$ agree as $L^2$ functions. (Why?) It follows then that $\hat{f} \in L^2$ and $\|\hat{f}\|_2 = \|f\|_2$.

The final equality of the theorem now follows from the polarization identity in $L^2(\mathbb{R})$. That is, for any $f, g \in L^2(\mathbb{R})$, we have

$$\int f\overline{g} = (1/4)\sum_{k=0}^{3} i^k \int (f + i^k g)\overline{(f + i^k g)},$$

which can be verified by expanding the right-hand side.

REMARK. The Plancherel theorem asserts that the Fourier transform is an isometry in the $L^2$ norm from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. Since Schwartz space is in the range of the Fourier transform on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform maps $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ onto a dense subspace of $L^2(\mathbb{R})$, whence there exists a unique extension $U$ of the Fourier transform from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to an isometry on all of $L^2(\mathbb{R})$. This $U$ is called the $L^2$ *Fourier transform.* It is an isometry of $L^2(\mathbb{R})$ onto itself.

EXERCISE 6.15. (a) Suppose $f(x)$ and $xf(x)$ are both elements of $L^2(\mathbb{R})$. Prove that $U(f)$ is differentiable almost everywhere and compute $[U(f)]'(\xi)$.

(b) If $f$ is absolutely continuous and both $f$ and $f'$ belong to $L^2(\mathbb{R})$, show that $f(x) = \int_{-\infty}^{x} f'(t)\,dt$ and that $[U(f')](\xi) = 2\pi i\xi[U(f)](\xi)$. State and prove results for the $L^2$ Fourier transform that are analogous to parts b and c of Exercise 6.13.

(c) Suppose $f$ is in $L^2(\mathbb{R})$ but not $L^1(\mathbb{R})$. Assume that for almost every $\xi$, the function $f(x)e^{-2\pi ix\xi}$ is improperly Riemann integrable. That is, assume that there exists a function $g$ such that

$$\lim_{B\to\infty} \int_{-B}^{B} f(x)e^{-2\pi ix\xi}\,dx$$

exists and equals $g(\xi)$ for almost all $\xi$. Prove that $g = U(f)$.

(d) Define the function $f$ by $f(x) = \sin(x)/x$. Prove that $f \in L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$. Show that $f$ is improperly Riemann integrable, and establish that

$$\lim_{B\to\infty} \int_{-B}^{B} f(x)\,dx = \pi.$$

HINT: Verify the following equalities:

$$\begin{aligned}
\lim_{B\to\infty}\int_{-B}^{B} f(x)\,dx &= 2\lim_n \int_0^{\pi(n+1/2)} f(x)\,dx \\
&= \lim_n \int_0^{\pi} \frac{\sin((n+1/2)x)}{x/2}\,dx \\
&= \lim_n \int_0^{\pi} \frac{\sin((n+1/2)x)}{\sin((1/2)x)}\,dx \\
&= \lim_n \int_0^{\pi} \frac{e^{-i(n+1/2)x} - e^{i(n+1/2)x}}{e^{-i(1/2)x} - e^{i(1/2)x}}\,dx \\
&= \lim_n \int_0^{\pi} \sum_{k=-n}^{n} e^{ikx}\,dx \\
&= \pi.
\end{aligned}$$

(e) Fix a $\delta > 0$, and let $f_\delta(x) = 1/x$ for $|x| \geq \delta$. Use part c to show that

$$[U(f_\delta)](\xi) = -i\mathrm{sgn}(\xi) \lim_{B\to\infty} \int_{2\pi\delta|\xi|\leq|x|\leq B} \sin(x)/x\,dx,$$

where sgn denotes the *signum* function defined on $\mathbb{R}$ by

$$\mathrm{sgn}(t) = 1, \ \text{for}\, t > 0$$

$$\mathrm{sgn}(0) = 0$$

$$\mathrm{sgn}(t) = -1, \ \text{for}\, t < 0.$$

Using part d, conclude that $[U(f_\delta)](\xi)$ is uniformly bounded in both the variables $\delta$ and $\xi$, and show that

$$\lim_{\delta\to 0}[U(f_\delta)](\xi) = -\pi i\mathrm{sgn}(\xi).$$

(We may say then that the Fourier transform of the non-integrable and non-square-integrable function $1/x$ is the function $-\pi i$sgn.)

EXERCISE 6.16. (Hausdorff-Young Inequality) Suppose $f \in L^1 \cap L^p$ for $1 \leq p \leq 2$. Prove that $\hat{f} \in L^{p'}$, for $1/p + 1/p' = 1$, and that $\|\hat{f}\|_{p'} \leq \|f\|_p$. HINT: Use the Riesz Interpolation Theorem.

DEFINITION. If $u$ is a tempered distribution, i.e., an element of $\mathcal{S}'$, define the *Fourier transform* $\hat{u}$ of $u$ to be the linear functional on $\mathcal{S}$ given by

$$\hat{u}(f) = u(\hat{f}).$$

EXERCISE 6.17. (a) Prove that the Fourier transform of a tempered distribution is again a tempered distribution.

(b) Suppose $h$ is a tempered function in $L^1(\mathbb{R})$ ($L^2(\mathbb{R})$), and suppose that $u$ is the tempered distribution $u_h$. Show that $\hat{u} = u_{\hat{h}}$ ($u_{U(h)}$).

(c) If $u$ is the tempered distribution defined by

$$u(f) = \lim_{\delta\to 0}\int_{|t|\geq\delta}[f(t)/t]\,dt,$$

show that $\hat{u} = u_{-\pi i\mathrm{sgn}}$. See part b of Exercise 5.8.

(d) If $u$ is a tempered distribution, show that the Fourier transform of the tempered distribution $u'$ is the tempered distribution $v = m\hat{u}$, where $m$ is the $C^\infty$ tempered function given by $m(\xi) = 2\pi i\xi$. That is,

$$\widehat{u'}(f) = v(f) = \hat{u}(mf).$$

(e) Suppose $u$ and its distributional derivative $u'$ are both tempered distributions corresponding to $L^2$ functions $f$ and $g$ respectively. Prove that $f$ is absolutely continuous and that $f'(x) = -g(x)$ a.e.

(f) Suppose both $u$ and its distributional derivative $u'$ are tempered distributions corresponding to $L^2$ functions $f$ and $g$ respectively. Assume that there exists an $\epsilon > 0$ such that $|\xi|^{(3/2)+\epsilon}\hat{u}(\xi)$ is in $L^2(\mathbb{R})$. Prove that $f$ is in fact a $C^1$ function.

DEFINITION. For vectors $x$ and $y$ in $\mathbb{R}^n$, write $(x, y)$ for the dot product of $x$ and $y$. If $f \in L^1(\mathbb{R}^n)$, define the *Fourier transform* $\hat{f}$ of $f$ on $\mathbb{R}^n$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i(x,\xi)}\,dx.$$

EXERCISE 6.18. (a) Prove the Riemann-Lebesgue theorem: If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

(b) Prove the convolution theorem: If $f, g \in L^1(\mathbb{R}^n)$, then

$$\widehat{f * g} = \hat{f}\hat{g}.$$

Show also that $\int \hat{f}g = \int f\hat{g}$.

(c) Let $1 \leq j \leq n$, and suppose that both $f$ and its partial derivative $\dfrac{\partial f}{\partial x_j}$ belong to $L^1(\mathbb{R}^n)$. Show that

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = 2\pi i\xi_j\hat{f}(\xi).$$

Generalize this equality to higher order and mixed partial derivatives.

(d) For $t > 0$ define $g_t$ on $\mathbb{R}^n$ by

$$g_t(x) = (2\pi t)^{-n/2} e^{-\|x\|^2/2t}.$$

Show that

$$\widehat{g_t}(\xi) = e^{-2\pi t\|\xi\|^2}.$$

(e) Prove the Inversion Theorem for the Fourier transform on $L^1(\mathbb{R}^n)$; i.e., if $f, \hat{f} \in L^1(\mathbb{R}^n)$, show that

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i(x,\xi)}\, d\xi$$

for almost all $x \in \mathbb{R}^n$.

(f) Prove the Plancherel Formula for the Fourier transform on $L^2(\mathbb{R}^n)$; i.e., for $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, show that

$$\int f\overline{g} = \int \hat{f}\overline{\hat{g}}.$$

Verify that the Fourier transform has a unique extension from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to an isometry from $L^2(\mathbb{R}^n)$ onto itself. We denote this isometry by $U$ and call it the $L^2$ *Fourier transform* on $\mathbb{R}^n$.

EXERCISE 6.19. (The Green's Function for the Laplacian) Let $L$ denote the Laplacian on $\mathbb{R}^n$; i.e.,

$$L(u) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2},$$

for $u$ any almost everywhere twice differentiable function on $\mathbb{R}^n$. Let $D$ be the space of all functions $u \in L^2(\mathbb{R}^n)$, all of whose first and second order partial derivatives are continuous and belong to $L^2(\mathbb{R}^n)$. Think of $L$ as a mapping of $D$ into $L^2(\mathbb{R}^n)$. Let $\tilde{D}$ be the set of all $f \in L^2(\mathbb{R}^n)$ such that $\|\xi\|^2 U(f)(\xi)$ belongs to $L^2(\mathbb{R}^n)$, and define $\tilde{L} : \tilde{D} \to L^2(\mathbb{R}^n)$ by $\tilde{L}(f) = U^{-1}(mU(f))$, where $U$ denotes the $L^2$ Fourier transform on $\mathbb{R}^n$, and $m$ is the function defined by $m(\xi) = -4\pi^2\|\xi\|^2$.

(a) Show that $D \subseteq \tilde{D}$, that $\tilde{L}$ is an extension of $L$, and that the graph of $\tilde{L}$ is closed.

(b) Assume that $n \geq 5$. Find a Green's function $g$ for $\tilde{L}$, and observe that $g$ is also a Green's function for $L$. HINT: Set

$$p(x) = c/(1 + \|x\|^2)^{(n+2)/2},$$

find a $u_0 \in D$ such that $L(u_0) = \tilde{L}(u_0) = p$. Then use Exercises 6.7 and 6.12.

(c) Extrapolating from the results in part b, find a Green's function for the Laplacian in $\mathbb{R}^3$ and $\mathbb{R}^4$.

(d) Find a Green's function for the Laplacian in $\mathbb{R}^2$ and in $\mathbb{R}$. HINT: Notice that the Green's functions in parts b and c satisfy $L(g) = 0$ except at the origin.

**Hilbert Transform on the Line**

If $m$ is a bounded measurable function on $\mathbb{R}$, we may define a bounded operator $M$ on $L^2$ by

$$M(f) = U^{-1}(mU(f)),$$

where $U$ denotes the $L^2$ Fourier transform. Such an operator $M$ is called a *multiplier operator* or simply a *multiplier*.

EXERCISE 6.20. Suppose $m = \hat{f}$ for some $L^1$ function $f$. Show that the multiplier operator $M$ is given by

$$M(g) = f * g.$$

Note, therefore, that multipliers are generalizations of $L^1$ convolution operators.

REMARK. Recall from Theorem 6.2 that $L^1$ convolution operators determine bounded operators on every $L^p$ space $(1 \leq p \leq \infty)$. If $m$ is not the Fourier transform of an $L^1$ function, then the multiplier $M$ (a priori a bounded operator on $L^2(\mathbb{R})$) may or may not have extensions to bounded operators on $L^p$ spaces other than $p = 2$, and it is frequently important to know when it does have such extensions.

EXERCISE 6.21. Let $m$ be a bounded measurable function on $\mathbb{R}$.

(a) Suppose that the multiplier $M$, corresponding to the function $m$, determines a bounded operator from $L^p(\mathbb{R})$ into itself for every $1 < p < \infty$. Show that the multiplier corresponding to the function $\overline{m}$ is the adjoint $M^*$ of $M$, and hence is a bounded operator from $L^q(\mathbb{R})$ into itself for every $1 < q < \infty$.

(b) Prove that the multiplier $M$, corresponding to the function $m$, determines a bounded operator from $L^p(\mathbb{R})$ into itself, for some $1 < p < \infty$, if and only if $M$ is a bounded operator from $L^{p'}(\mathbb{R})$ into itself, where $1/p + 1/p' = 1$.

Perhaps the most important example of a nontrivial multiplier is the following.

DEFINITION. Let $h$ denote the function $-i\mathrm{sgn}$, where sgn is the signum function. The *Hilbert transform* is the multiplier operator $H$ corresponding to the function $h$; i.e., on $L^2(\mathbb{R})$ we have

$$H(f) = U^{-1}(-i\mathrm{sgn}U(f)).$$

REMARK. In view of the results in Exercises 6.15 and 6.20, we might expect the Hilbert transform to correspond somehow to convolution by the nonintegrable function $1/\pi x$. Indeed, this is what we shall see below.

EXERCISE 6.22. (a) Show that the Hilbert transform has no extension to a bounded operator on $L^1$. HINT: For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have that $U(f) = \hat{f}$ is continuous.

(b) Suppose $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and $\hat{f} \in L^1(\mathbb{R})$. Verify the following sequence of equalities:

$$\begin{aligned}
[H(f)](x) &= [U^{-1}(-i\mathrm{sgn}\hat{f})](x) \\
&= (1/\pi) \lim_{\delta \to 0} [U^{-1}(U(f_\delta)\hat{f})](x) \\
&= (1/\pi) \lim_{\delta \to 0} \int_{-\infty}^{\infty} \hat{f}(\xi)[U(f_\delta)](\xi) e^{2\pi i x\xi} \, d\xi \\
&= (1/\pi) \lim_{\delta \to 0} \int_{-\infty}^{\infty} f(x+t) f_\delta^*(t) \, dt \\
&= \lim_{\delta \to 0} \int_{|t| \geq \delta} f(x-t)/\pi t \, dt,
\end{aligned}$$

where $f_\delta$ is the function from part e of Exercise 6.15. Note that this shows that the operator $H$ can be thought of as a generalization of convolution, in this case by the nonintegrable function $1/\pi x$.

(c) Verify that if $f$ is a real-valued function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and if $\hat{f}$ is in $L^1(\mathbb{R})$, then $H(f)$ also is real-valued.

EXERCISE 6.23. (a) For each positive real number $y$, define the function $g_y$ by

$$g_y(\xi) = e^{-2\pi y|\xi|}.$$

Show that

$$\operatorname{sgn}(\xi) g_y(\xi) = (-1/2\pi y) g_y'(\xi)$$

for every $y > 0$ and every $\xi \neq 0$.

(b) Let $f$ be a Schwartz function. For any real $x$, let $f_x$ denote the function defined by

$$f_x(y) = f(x + y).$$

Verify the following sequence of equalities:

$$\begin{aligned}
[H(f)](x) &= \lim_{y \to 0} (i/2\pi y)[U^{-1}(g_y' \hat{f})](x) \\
&= \lim_{y \to 0} (i/2\pi y) \int_{-\infty}^{\infty} g_y'(\xi) \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi \\
&= \lim_{y \to 0} (-i/2\pi y) \int_{-\infty}^{\infty} g_y(\xi) \widehat{f_x}'(\xi) \, d\xi \\
&= \lim_{y \to 0} (-i/2\pi y) \int_{-\infty}^{\infty} \widehat{g_y}(t)(-2\pi i t) f_x(t) \, dt \\
&= \lim_{y \to 0} \int_{-\infty}^{\infty} \frac{t/\pi}{t^2 + y^2} f(x - t) \, dt.
\end{aligned}$$

Note again that the Hilbert transform can be regarded as a kind of convolution by $1/\pi x$.

THEOREM 6.9. *The Hilbert transform determines a bounded operator from $L^p$ into itself, for each $1 < p < \infty$.*

PROOF. Given a $1 < p < \infty$, it will suffice to prove that there exists a positive constant $c_p$ such that

$$\|H(f)\|_p \leq c_p \|f\|_p$$

for all real-valued, $C^\infty$ functions $f$ having compact support. (Why?) First, let $n$ be a positive integer, and let $p = 2n$. For such a fixed real-valued, $C^\infty$ function $f$ having compact support, define a function $F$ of a complex variable $z = x + iy$ by

$$F(z) = (1/\pi i) \int_{-\infty}^{\infty} f(t)/(t - z) \, dt.$$

Then $F$ is analytic at each point $z = x + iy$ for $y > 0$ (it has a derivative there). It follows easily that there exists a constant $c$ for which

$$|F(x + iy)| \leq c/y \tag{6.4}$$

for all $x$ and all $y > 0$, and

$$|F(x + iy)| \leq c/|x| \tag{6.5}$$

for all $y > 0$ and all sufficiently large $x$. (See Exercise 6.24 below.)

If we write $F = U + iV$, then since $f$ is real-valued we have

$$U(x + iy) = \int_{-\infty}^{\infty} \frac{y/\pi}{y^2 + (x - t)^2} f(t)\, dt$$

and

$$V(x + iy) = \int_{-\infty}^{\infty} \frac{(x - t)/\pi}{(x - t)^2 + y^2} f(t)\, dt.$$

Then, by Exercises 6.7 and 6.23, we have that for every real $x$

$$f(x) = \lim_{y \to 0} U(x + iy),$$

and

$$[H(f)](x) = \lim_{y \to 0} V(x + iy).$$

We fix a sequence $\{y_j\}$ converging to 0 and define $U_j(x) = U(x + iy_j)$ and $V_j(x) = V(x + iy_j)$. Then $f = \lim U_j$ and $H(f) = \lim V_j$.

Because $F$ is analytic in the upper half plane, and because of inequalities (6.4) and (6.5), we have that

$$\int_{-\infty}^{\infty} F^{2n}(x + iy)\, dx = 0 \tag{6.6}$$

for each positive $y$. (See Exercise 6.24.) Hence

$$\int_{-\infty}^{\infty} \Re(F^{2n})(x + iy)\, dx = 0$$

for every positive $y$.

From trigonometry, we see that there exist positive constants $a_n$ and $b_n$ such that

$$\sin^{2n}(\theta) \leq a_n \cos^{2n}(\theta) + (-1)^n b_n \cos(2n\theta)$$

for all real $\theta$. Indeed, choose $b_n$ so that this is true for $\theta$ near $\pi/2$ and then choose $a_n$ so the inequality holds for other $\theta$'s. It follows then that for any complex number $z$ we have

$$\Im(z)^{2n} \leq a_n \Re(z)^{2n} + (-1)^n b_n \Re(z^{2n}).$$

So, we have that

$$V(x+iy)^{2n} \leq a_n U(x+iy)^{2n} + (-1)^n b_n \Re(F^{2n}(x+iy)),$$

whence

$$\int_{-\infty}^{\infty} V(x+iy)^{2n}\,dx \leq a_n \int_{-\infty}^{\infty} U(x+iy)^{2n}\,dx$$

implying that

$$\int_{-\infty}^{\infty} |V_j(x)|^{2n}\,dx \leq a_n \int_{-\infty}^{\infty} |U_j(x)|^{2n}\,dx$$

for all $j$. So, by the dominated convergence theorem and part b of Exercise 6.7,

$$\int_{-\infty}^{\infty} |[H(f)](x)|^p\,dx \leq a_n \int_{-\infty}^{\infty} |f(x)|^p\,dx,$$

and

$$\|H(f)\|_p \leq a_n^{1/p}\|f\|_p,$$

where $p = 2n$.

We have thus shown that the Hilbert transform determines a bounded operator from $L^p$ into itself, for $p$ of the form $2n$. By the Riesz Interpolation Theorem, it follows then that the Hilbert transform determines a bounded operator from $L^p$ into itself, for $2 \leq p < \infty$. The proof can now be completed by appealing to Exercise 6.21 for the cases $1 < p < 2$.

EXERCISE 6.24. (a) Show that any constant $c \geq \int |f(t)|\,dt$ will satisfy inequality (6.4). Supposing that $f$ is supported in the interval $[-a, a]$, show that any constant $c \geq 2\int |f(t)|\,dt$ will satisfy inequality (6.5) if $|x| \geq 2a$.

(b) Establish Equation (6.6) by integrating around a large square contour.

(c) Let $m$ be the characteristic function of an open interval $(a, b)$, where $-\infty \leq a < b \leq \infty$. Prove that the multiplier $M$, corresponding to $m$ determines a bounded operator on every $L^p$ for $1 < p < \infty$. HINT: Write $m$ as a finite linear combination of translates of $-i\mathrm{sgn}$.

(d) Let $m$ be the characteristic function of the set $E = \cup_n[2n, 2n+1]$. Verify that the multiplier $M$ corresponding to $m$ has no bounded extension to any $L^p$ space for $p \neq 2$.