CHAPTER II

THE HAHN-BANACH EXTENSION THEOREMS AND EXISTENCE OF LINEAR FUNCTIONALS

In this chapter we deal with the problem of extending a linear functional on a subspace Y to a linear functional on the whole space X. The quite abstract results that the Hahn-Banach Theorem comprises (Theorems 2.1, 2.2, 2.3, and 2.6) are, however, of significant importance in analysis, for they provide existence proofs. Applications are made already in this chapter to deduce the existence of remarkable mathematical objects known as Banach limits and translation-invariant measures. One may wish to postpone these applications as well as Theorems 2.4 and 2.5 to a later time. However, the set of exercises concerning convergence of nets should not be omitted, for they will be needed later on.

Let X be a real vector space and let $B = \{x_\alpha\}$, for α in an index set $\mathcal{A}$, be a basis for X. Given any set $\{t_\alpha\}$ of real numbers, also indexed by $\mathcal{A}$, we may define a linear transformation $\phi : X \to \mathbb{R}$ by

$$\phi(x) = \phi(\sum c_\alpha x_\alpha) = \sum c_\alpha t_\alpha,$$

where $x = \sum c_\alpha x_\alpha$. Note that the sums above are really finite sums, since only finitely many of the coefficients c_α are nonzero for any given x. This ϕ is a linear functional.

EXERCISE 2.1. Prove that if x and y are distinct vectors in a real vector space X, then there exists a linear functional ϕ such that

$\phi(x) \neq \phi(y)$. That is, there exist enough linear functionals on X to separate points.

More interesting than the result of the previous exercise is whether there exist linear functionals with some additional properties such as positivity, continuity, or multiplicativity. As we proceed, we will make precise what these additional properties should mean. We begin, motivated by the Riesz representation theorems of the preceding chapter, by studying the existence of positive linear functionals. See Theorem 2.1 below. To do this, we must first make sense of the notion of positivity in a general vector space.

DEFINITION. Let X be a real vector space. By a *cone* or *positive cone* in X we shall mean a subset P of X satisfying

(1) If x and y are in P, then $x + y$ is in P.
(2) If x is in P and t is a positive real number, then tx is in P.

Given vectors $x_1, x_2 \in X$, we say that $x_1 \geq x_2$ if $x_1 - x_2 \in P$.

Given a positive cone $P \subseteq X$, we say that a linear functional f on X is *positive*, if $f(x) \geq 0$ whenever $x \in P$.

EXERCISE 2.2. (a) Prove that the set of nonnegative functions in a vector space of real-valued functions forms a cone.

(b) Show that the set of nonpositive functions in a vector space of real-valued functions forms a cone.

(c) Let P be the set of points (x, y, z) in $\mathbb{R}^3$ for which $x > \sqrt{y^2 + z^2}$. Prove that P is a cone in the vector space $\mathbb{R}^3$.

(d) Let X be the vector space $\mathbb{R}^2$, and let P be the positive cone in X comprising the points (x, y) for $x \geq 0$ and $y \geq 0$. Suppose Y is the subspace of X comprising the points (t, t) for t real. Show that every linear functional on Y is a positive linear functional. Show also that there exists a linear functional f on Y for which no extension g of f to all of X is a positive linear functional.

THEOREM 2.1. (Hahn-Banach Theorem, Positive Cone Version) *Let P be a cone in a real vector space X, and let Y be a subspace of X having the property that for each $x \in X$ there exists a $y \in Y$ such that $y \geq x$; i.e., $y - x \in P$. Suppose f is a positive linear functional on Y, i.e., $f(y) \geq 0$ if $y \in P \cap Y$. Then there exists a linear functional g on X such that*

(1) *For each $y \in Y$, $g(y) = f(y)$; i.e., g is an extension of f.*
(2) *$g(x) \geq 0$ if $x \in P$; i.e., g is a positive linear functional on X.*

PROOF. Applying the hypotheses both to x and to $-x$, we see that: Given $x \in X$, there exists a $y \in Y$ such that $y - x \in P$, and there exists a $y' \in Y$ such that $y' - (-x) = y' + x \in P$. We will use the existence of these elements of Y later on.

Let S be the set of all pairs (Z, h), where Z is a subspace of X that contains Y, and where h is a positive linear functional on Z that is an extension of f. Since the pair (Y, f) is clearly an element of S, we have that S is nonempty.

Introduce a partial ordering on S by setting

$$(Z, h) \leq (Z', h')$$

if Z is a subspace of Z' and h' is an extension of h, that is $h'(z) = h(z)$ for all $z \in Z$. By the Hausdorff maximality principle, let $\{(Z_\alpha, h_\alpha)\}$ be a maximal linearly ordered subset of S. Clearly, $Z = \cup Z_\alpha$ is a subspace of X. Also, if $z \in Z$, then $z \in Z_\alpha$ for some α. Observe that if $z \in Z_\alpha$ and $z \in Z_\beta$, then, without loss of generality, we may assume that $(Z_\alpha, h_\alpha) \leq (Z_\beta, h_\beta)$. Therefore, $h_\alpha(z) = h_\beta(z)$, so that we may uniquely define a number $h(z) = h_\alpha(z)$, whenever $z \in Z_\alpha$.

We claim that the function h defined above is a linear functional on the subspace Z. Thus, let z and w be elements of Z. Then $z \in Z_\alpha$ and $w \in Z_\beta$ for some α and β. Since the set $\{(Z_\gamma, h_\gamma)\}$ is linearly ordered, we may assume, again without loss of generality, that $Z_\alpha \subseteq Z_\beta$, whence both z and w are in Z_β. Therefore,

$$h(tz + sw) = h_\beta(tz + sw) = th_\beta(z) + sh_\beta(w) = th(z) + sh(w),$$

showing that h is a linear functional.

Note that, if $y \in Y$, then $h(y) = f(y)$, so that h is an extension of f. Also, if $z \in Z \cap P$, then $z \in Z_\alpha \cap P$ for some α, whence

$$h(z) = h_\alpha(z) \geq 0,$$

showing that h is a positive linear functional on Z.

We prove next that Z is all of X, and this will complete the proof of the theorem. Suppose not, and let v be an element of X which is not in Z. We will derive a contradiction to the maximality of the linearly ordered subset $\{(Z_\alpha, h_\alpha)\}$ of the partially ordered set S. Let Z' be the set of all vectors in X of the form $z + tv$, where $z \in Z$ and $t \in \mathbb{R}$. Then Z' is a subspace of X which properly contains Z.

Let Z_1 be the set of all $z \in Z$ for which $z - v \in P$, and let Z_2 be the set of all $z' \in Z$ for which $z' + v \in P$. We have seen that both Z_1 and

Z_2 are nonempty. We make the following observation. If $z \in Z_1$ and $z' \in Z_2$, then $h(z') \geq -h(z)$. Indeed, $z + z' = z - v + z' + v \in P$. So,

$$h(z + z') = h(z) + h(z') \geq 0,$$

and $h(z') \geq -h(z)$, as claimed. Hence, we see that the set B of numbers $\{h(z')\}$ for which $z' \in Z_2$ is bounded below. In fact, any number of the form $-h(z)$ for $z \in Z_1$ is a lower bound for B. We write $b = \inf B$. Similarly, the set A of numbers $\{-h(z)\}$ for which $z \in Z_1$ is bounded above, and we write $a = \sup A$. Moreover, we see that $a \leq b$. Note that if $z \in Z_1$, then $h(z) \geq -a$.

Choose any c for which $a \leq c \leq b$, and define h' on Z' by

$$h'(z + tv) = h(z) - tc.$$

Clearly, h' is a linear functional on Z' that extends h and hence extends f. Let us show that h' is a positive linear functional on Z'. On the one hand, if $z + tv \in P$, and if $t > 0$, then $z/t \in Z_2$, and

$$h'(z + tv) = th'((z/t) + v) = t(h(z/t) - c) \geq t(b - c) \geq 0.$$

On the other hand, if $t < 0$ and $z + tv = |t|((z/|t|) - v) \in P$, then $z/(-t) = z/|t| \in Z_1$, and

$$h'(z + tv) = |t|h'((z/(-t)) - v) = |t|(h(z/(-t)) + c) \geq |t|(c - a) \geq 0.$$

Hence, h' is a positive linear functional, and therefore $(Z', h') \in S$. But since $(Z, h) \leq (Z', h')$, it follows that the set $\{(Z_\alpha, h_\alpha)\}$ together with (Z', h') constitutes a strictly larger linearly ordered subset of S, which is a contradiction. Therefore, Z is all of X, h is the desired extension g of f, and the proof is complete.

REMARK. The impact of the Hahn-Banach Theorem is the existence of linear functionals having specified properties. The above version guarantees the existence of many positive linear functionals on a real vector space X, in which there is defined a positive cone. All we need do is find a subspace Y, satisfying the condition in the theorem, and then any positive linear functional on Y has a positive extension to all of X.

EXERCISE 2.3. (a) Verify the details showing that the ordering $\leq$ introduced on the set S in the preceding proof is in fact a partial ordering.

(b) Verify that the function h' defined in the preceding proof is a linear functional on Z'.

(c) Suppose ϕ is a linear functional on the subspace Z' of the above proof. Show that, if ϕ is an extension of h and is a positive linear functional on Z', then the number $-\phi(v)$ must be between the numbers a and b of the preceding proof.

EXERCISE 2.4. Let X be a vector space of bounded real-valued functions on a set S. Let P be the cone of nonnegative functions in X. Show that any subspace Y of X that contains the constant functions satisfies the hypothesis of Theorem 2.1.

We now investigate linear functionals that are, in some sense, bounded.

DEFINITION. By a *seminorm* on a real vector space X, we shall mean a real-valued function ρ on X that satisfies:

(1) $\rho(x) \geq 0$ for all $x \in X$,
(2) $\rho(x+y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$, and
(3) $\rho(tx) = |t|\rho(x)$, for all $x \in X$ and all t.

If, in addition, ρ satisfies $\rho(x) = 0$ if and only if $x = 0$, then ρ is called a *norm*, and $\rho(x)$ is frequently denoted by $\|x\|$ or $\|x\|_\rho$. If X is a vector space on which a norm is defined, then X is called a *normed linear space*.

A weaker notion than that of a seminorm is that of a subadditive functional, which is the same as a seminorm except that we drop the nonnegativity condition (condition (1)) and weaken the homogeneity in condition (3). That is, a real-valued function ρ on a real vector space X is called a *subadditive functional* if:

(1) $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$, and
(2) $\rho(tx) = t\rho(x)$ for all $x \in X$ and $t \geq 0$.

EXERCISE 2.5. Determine whether or not the following are seminorms (subadditive functionals, norms) on the specified vector spaces.

(a) $X = L^p(\mathbb{R})$, $\rho(f) = \|f\|_p = (\int |f|^p)^{1/p}$, for $1 \leq p < \infty$.

(b) X any vector space, $\rho(x) = |f(x)|$, where f is a linear functional on X.

(c) X any vector space, $\rho(x) = \sup_\nu |f_\nu(x)|$, where $\{f_\nu\}$ is a collection of linear functionals on X.

(d) $X = C_0(\Delta)$, $\rho(f) = \|f\|_\infty$, where Δ is a locally compact Hausdorff topological space.

(e) X is the vector space of all infinitely differentiable functions on

$\mathbb{R}$, n, m, k are nonnegative integers, and

$$\rho(f) = \sup_{|x|\leq N} \sup_{0\leq j\leq k} \sup_{0\leq m\leq M} |x^m f^{(j)}(x)|.$$

(f) X is the set of all bounded real-valued functions on a set S, and $\rho(f) = \sup f(x)$.

(g) X is the space l^∞ of all bounded, real-valued sequences $\{a_1, a_2, ...\}$, and

$$\rho(\{a_n\}) = \limsup a_n.$$

REMARK. Theorem 2.2 below is perhaps the most familiar version of the Hahn-Banach theorem. So, although it can be derived as a consequence of Theorem 2.1 and is in fact equivalent to that theorem (see parts d and e of Exercise 2.6), we give here an independent proof.

THEOREM 2.2. (Hahn-Banach Theorem, Seminorm Version) *Let ρ be a seminorm on a real vector space X. Let Y be a subspace of X, let f be a linear functional on Y, and assume that*

$$f(y) \leq \rho(y)$$

for all $y \in Y$. Then there exists a linear functional g on X, which is an extension of f and which satisfies

$$g(x) \leq \rho(x)$$

for all $x \in X$.

PROOF. By analogy with the proof of Theorem 2.1, we let S be the set of all pairs (Z, h), where Z is a subspace of X containing Y, h is a linear functional on Z that extends f, and $h(z) \leq \rho(z)$ for all $z \in Z$. We give to S the same partial ordering as in the preceding proof. By the Hausdorff maximality principle, let $\{(Z_\alpha, h_\alpha)\}$ be a maximal linearly ordered subset of S. As before, we define $Z = \cup Z_\alpha$, and h on Z by $h(z) = h_\alpha(z)$ whenever $z \in Z_\alpha$. It follows as before that h is a linear functional on Z, that extends f, for which $h(z) \leq \rho(z)$ for all $z \in Z$, so that the proof will be complete if we show that $Z = X$.

Suppose that $Z \neq X$, and let v be a vector in X which is not in Z. Define Z' to be the set of all vectors of the form $z + tv$, for $z \in Z$ and $t \in \mathbb{R}$. We observe that for any z and z' in Z,

$$h(z) + h(z') = h(z + z') \leq \rho(z + v + z' - v) \leq \rho(z + v) + \rho(z' - v),$$

or that

$$h(z') - \rho(z' - v) \leq \rho(z + v) - h(z).$$

Let A be the set of numbers $\{h(z') - \rho(z' - v)\}$ for $z' \in Z$, and put $a = \sup A$. Let B be the numbers $\{\rho(z + v) - h(z)\}$ for $z \in Z$, and put $b = \inf B$. It follows from the calculation above that $a \leq b$. Choose c to be any number for which $a \leq c \leq b$, and define h' on Z' by

$$h'(z + tv) = h(z) + tc.$$

Obviously h' is linear and extends f. If $t > 0$, then

$$\begin{aligned}
h'(z + tv) &= t(h(z/t) + c) \\
&\leq t(h(z/t) + b) \\
&\leq t(h(z/t) + \rho((z/t) + v) - h(z/t)) \\
&= t\rho((z/t) + v) \\
&= \rho(z + tv).
\end{aligned}$$

And, if $t < 0$, then

$$\begin{aligned}
h'(z + tv) &= |t|(h(z/|t|) - c) \\
&\leq |t|(h(z/|t|) - a) \\
&\leq |t|(h(z/|t|) - h(z/|t|) + \rho((z/|t|) - v)) \\
&= |t|\rho((z/|t|) - v) \\
&= \rho(z + tv),
\end{aligned}$$

which proves that $h'(z + tv) \leq \rho(z + tv)$ for all $z + tv \in Z'$.

Hence, $(Z', h') \in S$, $(Z, h) \leq (Z', h')$, and the maximality of the linearly ordered set $\{(Z_\alpha, h_\alpha)\}$ is contradicted. This completes the proof.

THEOREM 2.3. (Hahn-Banach Theorem, Norm Version) *Let Y be a subspace of a real normed linear space X, and suppose that f is a linear functional on Y for which there exists a positive constant M satisfying $|f(y)| \leq M\|y\|$ for all $y \in Y$. Then there exists an extension of f to a linear functional g on X satisfying $|g(x)| \leq M\|x\|$ for all $x \in X$.*

EXERCISE 2.6. (a) Prove the preceding theorem.

(b) Let the notation be as in the proof of Theorem 2.2. Suppose ϕ is a linear functional on Z' that extends the linear functional h and for which $\phi(z') \leq \rho(z')$ for all $z' \in Z'$. Prove that $\phi(v)$ must satisfy $a \leq \phi(v) \leq b$.

(c) Show that Theorem 2.2 holds if the seminorm ρ is replaced by the weaker notion of a subadditive functional.

(d) Derive Theorem 2.2 as a consequence of Theorem 2.1. HINT: Let $X' = X \oplus \mathbb{R}$, Define P to be the set of all $(x,t) \in X'$ for which $\rho(x) \leq t$, let $Y' = Y \oplus \mathbb{R}$, and define f' on Y' by $f'(y,t) = t - f(y)$. Now apply Theorem 2.1.

(e) Derive Theorem 2.1 as a consequence of Theorem 2.2. HINT: Define ρ on X by $\rho(x) = \inf f(y)$, where the infimum is taken over all $y \in Y$ for which $y - x \in P$. Show that ρ is a subadditive functional, and then apply part c.

We devote the next few exercises to developing the notion of convergence of nets. This topological concept is of great use in functional analysis. The reader should notice how crucial the axiom of choice is in these exercises. Indeed, the Tychonoff theorem (Exercise 2.11) is known to be equivalent to the axiom of choice.

DEFINITION. A *directed set* is a nonempty set D, on which there is defined a transitive and reflexive partial ordering $\leq$, satisfying the following condition: If $\alpha, \beta \in D$, then there exists an element $\gamma \in D$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. That is, every pair of elements of D has an upper bound.

If C and D are two directed sets, and h is a mapping from C into D, then h is called *order-preserving* if $c_1 \leq c_2$ implies that $h(c_1) \leq h(c_2)$. An order-preserving map h of C into D is called *cofinal* if for each $\alpha \in D$ there exists a $\beta \in C$ such that $\alpha \leq h(\beta)$.

A *net* in a set X is a function f from a directed set D into X. A net f in X is frequently denoted, in analogy with a sequence, by $\{x_\alpha\}$, where $x_\alpha = f(\alpha)$.

If $\{x_\alpha\}$ denotes a net in a set X, then a *subnet* of $\{x_\alpha\}$ is determined by an order-preserving cofinal function h from a directed set C into D, and is the net g defined on C by $g(\beta) = x_{h(\beta)}$. The values $h(\beta)$ of the function h are ordinarily denoted by $h(\beta) = \alpha_\beta$, whence the subnet g takes the notation $g(\beta) = x_{\alpha_\beta}$.

A net $\{x_\alpha\}$, $\alpha \in D$, in a topological space X is said to *converge* to an element $x \in X$, and we write $x = \lim_\alpha x_\alpha$, if For each open set U containing x, there exists an $\alpha \in D$ such that $x_{\alpha'} \in U$ whenever $\alpha \leq \alpha'$.

EXERCISE 2.7. (a) Show that any linearly ordered set is a directed set.

(b) Let S be a set and let D be the set of all finite subsets F of S. Show that D is a directed set if the partial ordering on D is given by

$F_1 \leq F_2$ if and only if $F_1 \subseteq F_2$.

(c) Let x be a point in a topological space X, and let D be the partially-ordered set of all neighborhoods of x with the ordering $U \leq V$ if and only if $V \subseteq U$. Prove that D is a directed set.

(d) Let D and D' be directed sets. Show that $D \times D'$ is a directed set, where the ordering is given by $(\alpha, \alpha') \leq (\beta, \beta')$ if and only if $\alpha \leq \alpha'$ and $\beta \leq \beta'$.

(e) Verify that every sequence is a net.

(f) Let $\{x_n\}$ be a sequence. Show that there exist subnets of the net $\{x_n\}$ which are not subsequences.

EXERCISE 2.8. (a) (Uniqueness of Limits) Let $\{x_\alpha\}$ be a net in a Hausdorff topological space X. Suppose $x = \lim x_\alpha$ and $y = \lim x_\alpha$. Show that $x = y$.

(b) Suppose $\{x_\alpha\}$ and $\{y_\alpha\}$ are nets (defined on the same directed set D) in $\mathbb{C}$, and assume that $x = \lim x_\alpha$ and $y = \lim y_\alpha$. Prove that

$$x + y = \lim(x_\alpha + y_\alpha),$$

$$xy = \lim(x_\alpha y_\alpha),$$

and that if $a \leq x_\alpha \leq b$ for all α, then

$$a \leq x \leq b.$$

(c) Prove that if a net $\{x_\alpha\}$ converges to an element x in a topological space X, then every subnet $\{x_{\alpha_\beta}\}$ of $\{x_\alpha\}$ also converges to x.

(d) Prove that a net $\{x_\alpha\}$ in a topological space X converges to an element $x \in X$ if and only if every subnet $\{x_{\alpha_\beta}\}$ of $\{x_\alpha\}$ has in turn a subnet $\{x_{\alpha_{\beta_\gamma}}\}$ that converges to x. HINT: To prove the "if" part, argue by contradiction.

(e) Let A be a subset of a topological space X. We say that an element $x \in X$ is a *cluster point* of A if there exists a net $\{x_\alpha\}$ in A such that $x = \lim x_\alpha$. Prove that A is closed if and only if it contains all of its cluster points.

(f) Let f be a function from a topological space X into a topological space Y. Show that f is continuous at a point $x \in X$ if and only if for each net $\{x_\alpha\}$ that converges to $x \in X$, the net $\{f(x_\alpha)\}$ converges to $f(x) \in Y$.

EXERCISE 2.9. (a) Let X be a compact topological space. Show that every net in X has a convergent subnet. HINT: Let $\{x_\alpha\}$ be a net in X defined on a directed set D. For each $\alpha \in D$, define $V_\alpha \subseteq X$ to be the set of all $x \in X$ for which there exists a neighborhood U_x of X such that $x_\beta \notin U_x$ whenever $\alpha \leq \beta$. Show that, if $x \notin \cup V_\alpha$, then x is the limit of some subnet of $\{x_\alpha\}$. Now, argue by contradiction.

(b) Prove that a topological space X is compact if and only if every net in X has a convergent subnet. HINT: Let $\mathcal{F}$ be a collection of closed subsets of X for which the intersection of any finite number of elements of $\mathcal{F}$ is nonempty. Let D be the directed set whose elements are the finite subsets of $\mathcal{F}$.

(c) Let $\{x_\alpha\}$ be a net in a metric space X. Define what it means for the net $\{x_\alpha\}$ to be a *Cauchy* net. Show that, if X is a complete metric space, then a net $\{x_\alpha\}$ is convergent if and only if it is a Cauchy net.

EXERCISE 2.10. Let X be a set, let $\{f_i\}$, for i in an index set I, be a collection of real-valued functions on X, and let $\mathcal{T}$ be the weakest topology on X for which each f_i is continuous.

(a) Show that a net $\{x_\alpha\}$ in the topological space $(X, \mathcal{T})$ converges to an element $x \in X$ if and only if

$$f_i(x) = \lim_\alpha f_i(x_\alpha)$$

for every $i \in I$.

(b) Let X be a set, for each $x \in X$ let Y_x be a topological space, and let Y be the topological product space

$$Y = \prod_{x \in X} Y_x.$$

Prove that a net $\{y_\alpha\}$ in Y converges if and only if, for each $x \in X$, the net $\{y_\alpha(x)\}$ converges in Y_x.

EXERCISE 2.11. Prove the Tychonoff Theorem. That is, prove that if $X = \prod_{i \in I} X_i$, where each X_i is a compact topological space, then X is a compact topological space. HINT: Let $\{x_\alpha\}$ be a net in X, defined on a directed set D. Show that there exists a convergent subnet as follows:

(a) Let S be the set of all triples (J, C, h), where $J \subseteq I$, C is a directed set, and h is a cofinal, order-preserving map of C into D such that the subnet $x_{h(\beta)}$ satisfies $\{(x_{h(\beta)})_i\}$ converges for every $i \in J$. We say that

$$(J_1, C_1, h_1) \leq (J_2, C_2, h_2)$$

if $J_1 \subseteq J_2$ and the subnet determined by h_2 is itself a subnet of the subnet determined by h_1. Prove that S is a nonempty partially ordered set.

(b) Let $\{(J_\lambda, C_\lambda, h_\lambda)\}$ be a maximal linearly ordered subset of S, and set $I_0 = \cup_\lambda J_\lambda$. Prove that there exists a directed set C_0 and a cofinal map h_0 such that $(I_0, C_0, h_0) \in S$ and such that $(J_\lambda, C_\lambda, h_\lambda) \leq (I_0, C_0, h_0)$ for all λ.

(c) Let I_0 be as in part b. Prove that $I_0 = I$, and then complete the proof to Tychonoff's Theorem.

EXERCISE 2.12. (a) Suppose $\{f_\alpha\}$ is a net of linear functionals on a vector space X, and suppose that the net converges pointwise to a function f. Prove that f is a linear functional.

(b) Suppose ρ is a subadditive functional on a vector space X and that $x \in X$. Prove that $-\rho(-x) \leq \rho(x)$.

(c) Suppose ρ is a subadditive functional on a vector space X, and let F^ρ be the set of all linear functionals f on X for which $f(x) \leq \rho(x)$ for every $x \in X$. Let K be the compact Hausdorff space

$$K = \prod_{x \in X} [-\rho(-x), \rho(x)]$$

(thought of as a space of functions on X). Prove that F^ρ is a closed subset of K. Conclude that F^ρ is a compact Hausdorff space in the topology of pointwise convergence on X.

THEOREM 2.4. *Let ρ be a subadditive functional on a vector space X, and let g be a linear functional on X such that $g(x) \leq \rho(x)$ for all $x \in X$. Suppose γ is a linear transformation of X into itself for which $\rho(\gamma(x)) = \rho(x)$ for all $x \in X$. Then there exists a linear functional h on X satisfying:*

1. *$h(x) \leq \rho(x)$ for all $x \in X$.*
2. *$h(\gamma(x)) = h(x)$ for all $x \in X$.*
3. *If $x \in X$ satisfies $g(x) = g(\gamma^n(x))$ for all positive n, then $h(x) = g(x)$.*

PROOF. For each positive integer n, define

$$g_n(x) = (1/n) \sum_{i=1}^{n} g(\gamma^i(x)).$$

Let F^ρ and K be as in the preceding exercise. Then the sequence $\{g_n\}$ is a net in the compact Hausdorff space K, and consequently there exists a convergent subnet $\{g_{n_\alpha}\}$. By Exercise 2.12, we know then that the subnet $\{g_{n_\alpha}\}$ of the sequence (net) $\{g_n\}$ converges pointwise to a linear functional h on X and that $h(x) \leq \rho(x)$ for all $x \in X$.

Using the fact that $-\rho(x) \leq g(\gamma^i(x)) \leq \rho(x)$ for all $x \in X$ and all $i > 0$, and the fact that the cofinal map $\alpha \to n_\alpha$ diverges to infinity, we have that

$$\begin{aligned}
h(\gamma(x)) &= \lim_\alpha g_{n_\alpha}(\gamma(x)) \\
&= \lim_\alpha (1/n_\alpha) \sum_{i=1}^{n_\alpha} g(\gamma^{i+1}(x)) \\
&= \lim_\alpha (1/n_\alpha) \sum_{i=2}^{n_\alpha+1} g(\gamma^i(x)) \\
&= \lim_\alpha (1/n_\alpha) [\sum_{i=1}^{n_\alpha} g(\gamma^i(x)) + g(\gamma^{n_\alpha+1}(x)) - g(\gamma(x))] \\
&= \lim_\alpha (1/n_\alpha) \sum_{i=1}^{n_\alpha} g(\gamma^i(x)) \\
&= \lim_\alpha g_{n_\alpha}(x) \\
&= h(x),
\end{aligned}$$

which proves the second statement of the theorem.

Finally, if x is such that $g(\gamma^n(x)) = g(x)$ for all positive n, then $g_n(x) = g(x)$ for all n, whence $h(x) = g(x)$, and this completes the proof.

EXERCISE 2.13. (Banach Means) Let $X = l^\infty$ be the vector space of all bounded sequences $\{a_1, a_2, a_3, \ldots\}$ of real numbers. A *Banach mean* or *Banach limit* is a linear functional M on X such that for all $\{a_n\} \in X$ we have:

$$\inf a_n \leq M(\{a_n\}) \leq \sup a_n.$$

and

$$M(\{a_{n+1}\}) = M(\{a_n\}).$$

(a) Prove that there exists a Banach limit on X. HINT: Use Theorem 2.2, or more precisely part c of Exercise 2.6, with Y the subspace of constant sequences, f the linear functional sending a constant sequence to that constant, and ρ the subadditive functional given by $\rho(\{a_n\}) = \limsup a_n$. Then use Theorem 2.4 applied to the extension g of f. (Note that, since the proof to Theorem 2.4 depends on the Tychonoff theorem, the very existence of Banach means depends on the axiom of choice.)

(b) Show that any Banach limit M satisfies $M(\{a_n\}) = L$, if $L = \lim a_n$, showing that any Banach limit is a generalization of the ordinary notion of limit.

(c) Show that any Banach limit assigns the number 1/2 to the sequence $\{0, 1, 0, 1, \dots\}$.

(d) Construct a sequence $\{b_n\} \in X$ which does not converge but for which

$$\lim(b_{n+1} - b_n) = 0.$$

Show that any linear functional g on X, for which $g(\{a_n\}) \leq \limsup a_n$ for all $\{a_n\} \in X$, satisfies $g(\{b_n\}) = g(\{b_{n+1}\})$.

(e) Use the sequence $\{b_n\}$ of part d to prove that there exist uncountably many distinct Banach limits on X. HINT: Use the Hahn-Banach Theorem and Theorem 2.4 to find a Banach limit that takes the value r on this sequence, where r is any number satisfying $\liminf b_n \leq r \leq \limsup b_n$.

EXERCISE 2.14. Prove the following generalization of Theorem 2.4. Let ρ be a subadditive functional on a vector space X, and let g be a linear functional on X such that $g(x) \leq \rho(x)$ for all $x \in X$. Suppose $\gamma_1, \dots, \gamma_n$ are commuting linear transformations of X into itself for which $\rho(\gamma_i(x)) = \rho(x)$ for all $x \in X$ and all $1 \leq i \leq n$. Then there exists a linear functional h on X satisfying:

(1) $h(x) \leq \rho(x)$ for all $x \in X$.
(2) $h(\gamma_i(x)) = h(x)$ for all $x \in X$ and all $1 \leq i \leq n$.
(3) If $x \in X$ satisfies $g(x) = g(\gamma_i^k(x))$ for all positive k and all $1 \leq i \leq n$, then $h(x) = g(x)$.

HINT: Use the proof to Theorem 2.4 and mathematical induction.

THEOREM 2.5. (Hahn-Banach Theorem, Semigroup-Invariant Version) *Let ρ be a subadditive functional on a real vector space X, and let f be a linear functional on a subspace Y of X for which $f(y) \leq \rho(y)$ for all $y \in Y$. Suppose Γ is an abelian semigroup of linear transformations of X into itself for which:*

(1) *$\rho(\gamma(x)) = \rho(x)$ for all $\gamma \in \Gamma$ and $x \in X$; i.e., ρ is invariant under Γ.*
(2) *$\gamma(Y) \subseteq Y$ for all $\gamma \in \Gamma$; i.e., Y is invariant under Γ.*

(3) $f(\gamma(y)) = f(y)$ *for all* $\gamma \in \Gamma$ *and* $y \in Y$; *i.e.,* f *is invariant under* Γ.

Then there exists a linear functional g *on* X *for which*

(a) g *is an extension of* f.

(b) $g(x) \leq \rho(x)$ *for all* $x \in X$.

(c) $g(\gamma(x)) = g(x)$ *for all* $\gamma \in \Gamma$ *and* $x \in X$; *i.e.,* g *is invariant under* Γ.

PROOF. For A a finite subset of Γ, we use part c of Exercise 2.6 and then Exercise 2.14 to construct a linear functional g_A on X satisfying:

(1) g_A is an extension of f.

(2) $g_A(x) \leq \rho(x)$ for all $x \in X$.

(3) $g_A(\gamma(x)) = g_A(x)$ for all $x \in X$ and $\gamma \in A$.

If as in Exercise 2.12 $K = \prod_{x \in X}[-\rho(-x), \rho(x)]$, then $\{g_A\}$ can be regarded as a net in the compact Hausdorff space K. Let $\{g_{A_\beta}\}$ be a convergent subnet, and write $h = \lim_\beta g_{A_\beta}$. Then, h is a function on X, and is in fact the pointwise limit of a net of linear functionals, and so is itself a linear functional.

Clearly, $h(x) \leq \rho(x)$ for all $x \in X$, and h is an extension of f.

To see that $h(\gamma(x)) = h(x)$ for all $\gamma \in \Gamma$, fix a γ_0, and let $A_0 = \{\gamma_0\}$. By the definition of a subnet, there exists a β_0 such that if $\beta \geq \beta_0$ then $A_\beta \geq A_0$. Hence, if $\beta \geq \beta_0$, then $\{\gamma_0\} \subseteq A_\beta$. So, if $\beta \geq \beta_0$, then $g_{A_\beta}(\gamma_0(x)) = g_{A_\beta}(x)$ for all x. Hence,

$$h(\gamma_0(x)) = \lim_\beta g_{A_\beta}(\gamma_0(x)) = \lim_\beta g_{A_\beta}(x) = h(x),$$

as desired.

DEFINITION. Let S be a set. A *ring* of subsets of S is a collection $\mathcal{R}$ of subsets of S such that if $E, F \in \mathcal{R}$, then both $E \cup F$ and $E \Delta F$ are in $\mathcal{R}$, where $E \Delta F = (E \cap \tilde{F}) \cup (F \cap \tilde{E})$ is the symmetric difference of E and F. By a *finitely additive measure* on S, we mean an assignment $E \to \mu(E)$, of a ring $\mathcal{R}$ of subsets of S into the extended nonnegative real numbers, such that

$$\mu(\emptyset) = 0$$

and

$$\mu(E_1 \cup \ldots \cup E_n) = \mu(E_1) + \ldots + \mu(E_n)$$

whenever $\{E_1 \ldots, E_n\}$ is a pairwise disjoint collection of elements of $\mathcal{R}$.

EXERCISE 2.15. (Translation-Invariant Finitely Additive Measures) Let X be the vector space of all bounded functions on $\mathbb{R}$ with compact support, and let P be the positive cone of nonnegative functions in X.

(a) Let I be a positive linear functional on X. For each bounded subset $E \subset \mathbb{R}$, define $\mu(E) = I(\chi_E)$. Show that the set of all bounded subsets of $\mathbb{R}$ is a ring $\mathcal{R}$ of sets and that μ is a finitely additive measure on this ring.

(b) Show that there exists a finitely additive measure ν, defined on the ring of all bounded subsets of $\mathbb{R}$, such that $\nu(E)$ is the Lebesgue measure for every bounded Lebesgue measurable subset E of $\mathbb{R}$, and such that $\nu(E+x) = \nu(E)$ for all bounded subsets E of $\mathbb{R}$ and all real numbers x. (Such a measure is said to be *translation-invariant*.) HINT: Let Y be the subspace of X consisting of the bounded Lebesgue measurable functions of bounded support, let $I(f) = \int f$, and let Γ be the semigroup of linear transformations of X determined by the semigroup of all translations of $\mathbb{R}$. Now use Theorem 2.5.

(c) Let μ be the finitely additive measure of part b. For each subset E of $\mathbb{R}$, define $\nu(E) = \lim_n \mu(E \cap [-n, n])$. Prove that μ is a translation-invariant, finitely additive measure on the σ-algebra of all subsets of $\mathbb{R}$, and that μ agrees with Lebesgue measure on Lebesgue measurable sets.

(d) Prove that there exists no countably additive translation-invariant measure μ on the σ-algebra of all subsets of $\mathbb{R}$ that agrees with Lebesgue measure on Lebesgue measurable sets. HINT: Suppose μ is such a countably additive measure. Define an equivalence relation on $\mathbb{R}$ by setting $x \equiv y$ if $y - x \in \mathbb{Q}$, i.e., $y - x$ is a rational number. Let $E \subset (0,1)$ be a set of representatives of the equivalence classes of this relation. Show first that $\cup_{q \in \mathbb{Q} \cap (0,1)} E + q \subset (0,2)$, whence $\mu(E)$ must be 0. Then show that $(0,1) \subset \cup_{q \in \mathbb{Q}} E + q$, whence $\mu(E)$ must be positive.

DEFINITION. Let X be a complex vector space. A *seminorm* on X is a real-valued function ρ that is *subadditive* and *absolutely homogeneous*; i.e.,

$$\rho(x+y) \leq \rho(x) + \rho(y)$$

for all $x, y \in X$, and

$$\rho(\lambda x) = |\lambda| \rho(x)$$

for all $x \in X$ and $\lambda \in \mathbb{C}$. If, in addition, $x \neq 0$ implies that $\rho(x) > 0$, then ρ is called a *norm* on the complex vector space X.

THEOREM 2.6. (Hahn-Banach Theorem, Complex Version) *Let ρ be a seminorm on a complex vector space X. Let Y be a subspace of X,* *and let f be a complex-linear functional on Y satisfying $|f(y)| \leq \rho(y)$ for all $y \in Y$. Then there exists a complex-linear functional g on X satisfying g is an extension of f, and $|g(x)| \leq \rho(x)$ for all $x \in X$.*

EXERCISE 2.16. Prove Theorem 2.6 as follows:

(a) Use Theorem 2.2 to extend the real part u of f to a real linear functional a on X that satisfies $a(x) \leq \rho(x)$ for all $x \in X$.

(b) Use Exercise 1.11 and part a to define a complex linear functional g on X that extends f.

(c) For $x \in X$, choose a complex number λ of absolute value 1 such that $|g(x)| = \lambda g(x)$. Then show that

$$|g(x)| = g(\lambda x) = a(\lambda x) \leq \rho(x).$$

(d) State and prove a theorem for complex spaces that is analogous to Theorem 2.3.