CHAPTER X

THE SPECTRAL THEOREM OF GELFAND

DEFINITION A *Banach algebra* is a complex Banach space A on which there is defined an associative multiplication $\times$ for which:

(1) $x \times (y + z) = x \times y + x \times z$ and $(y + z) \times x = y \times x + z \times x$ for all $x, y, z \in A$.
(2) $x \times (\lambda y) = \lambda x \times y = (\lambda x) \times y$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$.
(3) $\|x \times y\| \leq \|x\|\|y\|$ for all $x, y \in A$.

We call the Banach algebra *commutative* if the multiplication in A is commutative.

An *involution* on a Banach algebra A is a map $x \to x^*$ of A into itself that satisfies the following conditions for all $x, y \in A$ and $\lambda \in \mathbb{C}$.

(1) $(x + y)^* = x^* + y^*$.
(2) $(\lambda x)^* = \overline{\lambda} x^*$.
(3) $(x^*)^* = x$.
(4) $(x \times y)^* = y^* \times x^*$.
(5) $\|x^*\| = \|x\|$.

We call x^* the *adjoint* of x. A subset $S \subseteq A$ is called *selfadjoint* if $x \in S$ implies that $x^* \in S$.

A Banach algebra A on which there is defined an involution is called a *Banach *-algebra*.

An element of a Banach *-algebra is called *selfadjoint* if $x^* = x$. If a Banach *-algebra A has an identity I, then an element $x \in A$, for which $x \times x^* = x^* \times x = I$, is called a *unitary element* of A. A selfadjoint element x, for which $x^2 = x$, is called a *projection* in A. An element x that commutes with its adjoint x^* is called a *normal element* of A.

A Banach algebra A is a *C^*-algebra* if it is a Banach *-algebra, and if the equation

$$\|x \times x^*\| = \|x\|^2$$

holds for all $x \in A$. A *sub C^*-algebra* of a C^*-algebra A is a subalgebra B of A that is a closed subset of the Banach space A and is also closed under the adjoint operation.

REMARK. We ordinarily write xy instead of $x \times y$ for the multiplication in a Banach algebra. It should be clear that the axioms for a Banach algebra are inspired by the properties of the space $B(H)$ of bounded linear operators on a Hilbert space H.

EXERCISE 10.1. (a) Let A be the set of all $n \times n$ complex matrices, and for $M = [a_{ij}] \in A$ define

$$\|M\| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2}.$$

Prove that A is a Banach algebra with identity I. Verify that A is a Banach $*$-algebra if M^* is defined to be the complex conjugate of the transpose of M. Give an example to show that A is not a C^*-algebra.

(b) Suppose H is a Hilbert space. Verify that $B(H)$ is a C^*-algebra. Using as H the Hilbert space $\mathbb{C}^2$, give an example of an element $x \in B(H)$ for which $\|x^2\| \neq \|x\|^2$. Observe that this example is not the same as that in part a. (The norms are different.)

(c) Verify that $L^1(\mathbb{R})$ is a Banach algebra, where multiplication is defined to be convolution. Show further that, if $f^*(x)$ is defined to be $\overline{f(-x)}$, then $L^1(\mathbb{R})$ is a Banach *-algebra. Give an example to show that $L^1(\mathbb{R})$ is not a C^*-algebra.

(d) Verify that $C_0(\Delta)$ is a Banach algebra, where Δ is a locally compact Hausdorff space, the algebraic operations are pointwise, and the norm on $C_0(\Delta)$ is the supremum norm. Show further that $C_0(\Delta)$ is a C^*-algebra, if we define f^* to be $\overline{f}$. Show that $C_0(\Delta)$ has an identity if and only if Δ is compact.

(e) Let A be an arbitrary Banach algebra. Prove that the map $(x, y) \to xy$ is continuous from $A \times A$ into A.

(f) Let A be a Banach algebra. Suppose $x \in A$ satisfies $\|x\| < 1$. Prove that $0 = \lim_n x^n$.

(g) Let M be a closed subspace of a Banach algebra A, and assume that M is a two-sided ideal in (the ring) A; i.e., $xy \in M$ and $yx \in M$ if $x \in A$ and $y \in M$. Prove that the Banach space A/M is a Banach algebra and that the natural map $\pi : A \to A/M$ is a continuous homomorphism of the Banach algebra A onto the Banach algebra A/M.

(h) Let A be a Banach algebra with identity I and let x be an element of A. Show that the smallest subalgebra B of A that contains x coincides with the set of all polynomials in x, i.e., the set of all elements y of the form $y = \sum_{j=0}^{n} a_j x^j$, where each a_j is a complex number and $x^0 = I$. We denote this subalgebra by $[x]$ and call it the subalgebra of A *generated by x*.

(i) Let A be a Banach *-algebra. Show that each element $x \in A$ can be written uniquely as $x = x_1 + ix_2$, where x_1 and x_2 are selfadjoint. Show further that if A contains an identity I, then $I^* = I$. If A is a C^*-algebra with identity, and if U is a unitary element in A, show that $\|U\| = 1$.

(j) Let x be a selfadjoint element of a C^*-algebra A. Prove that $\|x^n\| = \|x\|^n$ for all nonnegative integers n. HINT: Do this first for $n = 2^k$.

EXERCISE 10.2. (Adjoining an Identity) Let A be a Banach algebra, and let B be the complex vector space $A \times \mathbb{C}$. Define a multiplication on B by

$$(x, \lambda) \times (x', \lambda') = (xx' + \lambda x' + \lambda' x\ ,\ \lambda\lambda'),$$

and set $\|(x, \lambda)\| = \|x\| + |\lambda|$.

(a) Prove that B is a Banach algebra with identity.

(b) Show that the map $x \to (x, 0)$ is an isometric isomorphism of the Banach algebra A onto an ideal M of B. Show that M is of codimension 1; i.e., the dimension of B/M is 1. (This map $x \to (x, 0)$ is called the *canonical isomorphism* of A into B.)

(c) Conclude that every Banach algebra is isometrically isomorphic to an ideal of codimension 1 in a Banach algebra with identity.

(d) Suppose A is a Banach algebra with identity, and let B be the Banach algebra $A \times \mathbb{C}$ constructed above. What is the relationship, if any, between the identity in A and the identity in B?

(e) If A is a Banach $*$-algebra, can A be imbedded isometrically and isomorphically as an ideal of codimension 1 in a Banach $*$-algebra?

THEOREM 10.1. *Let x be an element of a Banach algebra A with identity I, and suppose that $\|x\| = \alpha < 1$. Then the element $I - x$ is* *invertible in A and*

$$(I-x)^{-1}=\sum_{n=0}^{\infty}x^n.$$

PROOF. The sequence of partial sums of the infinite series $\sum_{n=0}^{\infty}x^n$ forms a Cauchy sequence in A, for

$$\begin{aligned}\|\sum_{n=0}^{j}x^n-\sum_{n=0}^{k}x^n\| &= \|\sum_{n=k+1}^{j}x^n\|\\ &\leq \sum_{n=k+1}^{j}\|x^n\|\\ &\leq \sum_{n=k+1}^{j}\|x\|^n\\ &= \sum_{n=k+1}^{j}\alpha^n.\end{aligned}$$

We write

$$y=\sum_{n=0}^{\infty}x^n=\lim_j\sum_{n=0}^{j}x^n=\lim_j S_j.$$

Then

$$\begin{aligned}(I-x)y &= \lim_j(I-x)S_j\\ &= \lim_j(I-x)\sum_{n=0}^{j}x^n\\ &= \lim_j(I-x^{j+1})\\ &= I,\end{aligned}$$

by part f of Exercise 10.1, showing that y is a right inverse for $I-x$. That y also is a left inverse follows similarly, whence $y=(I-x)^{-1}$, as desired.

EXERCISE 10.3. Let A be a Banach algebra with identity I.

(a) If $x\in A$ satisfies $\|x\|<1$, show that $I+x$ is invertible in A.

(b) Suppose $y\in A$ is invertible, and set $\delta=1/\|y^{-1}\|$. Prove that x is invertible in A if $\|x-y\|<\delta$. HINT: Write $x=y(I+y^{-1}(x-y))$.

(c) Conclude that the set of invertible elements in A is a nonempty, proper, open subset of A.

(d) Prove that the map $x \to x^{-1}$ is continuous on its domain. HINT: $y^{-1} - x^{-1} = y^{-1}(x-y)x^{-1}$.

(e) Let x be an element of A. Show that the infinite series

$$\sum_{n=0}^{\infty} x^n/n!$$

converges to an element of A. Define

$$e^x = \sum_{n=0}^{\infty} x^n/n!.$$

Show that

$$e^{x+y} = e^x e^y$$

if $xy = yx$. Compare with part c of Exercise 8.13.

(f) Suppose in addition that A is a Banach *-algebra and that x is a selfadjoint element of A. Prove that e^{ix} is a unitary element of A. Compare with part d of Exercise 8.13.

THEOREM 10.2. (Mazur's Theorem) *Let A be a Banach algebra with identity I, and assume further that A is a division ring, i.e., that every nonzero element of A has a multiplicative inverse. Then A consists of the complex multiples λI of the identity I, and the map $\lambda \to \lambda I$ is a topological isomorphism of $\mathbb{C}$ onto A.*

PROOF. Assume false, and let x be an element of A that is not a complex multiple of I. This means that each element $x_\lambda = x - \lambda I$ has an inverse.

Let f be an arbitrary element of the conjugate space A^* of A, and define a function F of a complex variable λ by

$$F(\lambda) = f(x_\lambda^{-1}) = f((x - \lambda I)^{-1}).$$

We claim first that F is an entire function of λ. Thus, let λ be fixed. We use the factorization formula

$$y^{-1} - z^{-1} = y^{-1}(z-y)z^{-1}.$$

We have

$$\begin{aligned} F(\lambda+h)-F(\lambda) &= f(x_{\lambda+h}^{-1})-f(x_\lambda^{-1}) \\ &= f(x_{\lambda+h}^{-1}(x_\lambda - x_{\lambda+h})x_\lambda^{-1}) \\ &= hf(x_{\lambda+h}^{-1}x_\lambda^{-1}). \end{aligned}$$

So,

$$\lim_{h\to 0}\frac{F(\lambda+h)-F(\lambda)}{h} = f(x_\lambda^{-2}),$$

and F is differentiable everywhere. See part d of Exercise 10.3.

Next, observe that

$$\begin{aligned} \lim_{\lambda\to\infty} F(\lambda) &= \lim_{\lambda\to\infty} f((x-\lambda I)^{-1}) \\ &= \lim_{\lambda\to\infty}(1/\lambda)f(((x/\lambda)-I)^{-1}) \\ &= 0. \end{aligned}$$

Therefore, F is a bounded entire function, and so by Liouville's Theorem, $F(\lambda)=0$ identically. Consequently, $f(x_0^{-1}) = f(x^{-1}) = 0$ for all $f\in A^*$. But this would imply that $x^{-1}=0$, which is a contradiction.

We introduce next a dual object for Banach algebras that is analogous to the conjugate space of a Banach space.

DEFINITION. Let A be a Banach algebra. By the *structure space* of A we mean the set Δ of all nonzero continuous algebra homomorphisms (linear and multiplicative) $\phi : A\to\mathbb{C}$. The structure space is a (possibly empty) subset of the conjugate space A^*, and we think of Δ as being equipped with the inherited weak* topology.

THEOREM 10.3. *Let A be a Banach algebra, and let Δ denote its structure space. Then Δ is locally compact and Hausdorff. Further, if A is a separable Banach algebra, then Δ is second countable and metrizable. If A contains an identity I, then Δ is compact.*

PROOF. Δ is clearly a Hausdorff space since the weak* topology on A^* is Hausdorff.

Observe next that if $\phi\in\Delta$, then $\|\phi\|\le 1$. Indeed, for any $x\in A$, we have

$$|\phi(x)| = |\phi(x^n)|^{1/n} \le \|\phi\|^{1/n}\|x\| \to \|x\|,$$

implying that $\|\phi\|\le 1$, as claimed. It follows then that Δ is contained in the closed unit ball $\overline{B_1}$ of A^*. Since the ball $\overline{B_1}$ in A^* is by Alaoglu's

Theorem compact in the weak* topology, we could show that Δ is compact by verifying that it is closed in $\overline{B_1}$. This we can do if A contains an identity I. Thus, let $\{\phi_\alpha\}$ be a net of elements of Δ that converges in the weak* topology to an element $\phi \in \overline{B_1}$. Since this convergence is pointwise convergence on A, it follows that $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in A$, whence ϕ is a homomorphism of the algebra A into $\mathbb{C}$. Also, since every nonzero homomorphism of A must map I to 1, it follows that $\phi(I) = 1$, whence ϕ is not the 0 homomorphism. Hence, $\phi \in \Delta$, as desired.

We leave the proof that Δ is always locally compact to the exercises.

Of course, if A is separable, then the weak* topology on $\overline{B_1}$ is compact and metrizable, so that Δ is second countable and metrizable in this case, as desired.

EXERCISE 10.4. Let A be a Banach algebra.

(a) Suppose that the elements of the structure space Δ of A separate the points of A. Prove that A is commutative.

(b) Suppose A is the algebra of all $n \times n$ complex matrices as defined in part a of Exercise 10.1. Prove that the structure space Δ of A is the empty set if $n > 1$.

(c) If A has no identity, show that Δ is locally compact. HINT: Show that the closure of Δ in $\overline{B_1}$ is contained in the union of Δ and $\{0\}$, whence Δ is an open subset of a compact Hausdorff space.

(d) Let B be the Banach algebra with identity constructed from A as in Exercise 10.2, and identify A with its canonical isomorphic image in B. Prove that every element ϕ in the structure space Δ_A of A has a unique extension to an element ϕ' in the structure space Δ_B of B. Show that there exists a unique element $\phi_0 \in \Delta_B$ whose restriction to A is identically 0. Show further that the above map $\phi \to \phi'$ is a homeomorphism of Δ_A onto $\Delta_B - \{\phi_0\}$.

DEFINITION. Let A be a Banach algebra and let Δ be its structure space. For each $x \in A$, define a function $\hat{x}$ on Δ by

$$\hat{x}(\phi) = \phi(x).$$

The map $x \to \hat{x}$ is called the *Gelfand transform* of A, and the function $\hat{x}$ is called the *Gelfand transform of x.*

EXERCISE 10.5. Let A be the Banach algebra $L^1(\mathbb{R})$ of part c of Exercise 10.1, and let Δ be its structure space.

(a) If λ is any real number, define $\phi_\lambda : A \to \mathbb{C}$ by

$$\phi_\lambda(f) = \int f(x)e^{-2\pi i\lambda x}\,dx.$$

Show that ϕ_λ is an element of Δ.

(b) Let ϕ be an element of Δ, and let h be the L^∞ function satisfying

$$\phi(f) = \int f(x)\overline{h(x)}\,dx.$$

Prove that $h(x+y) = h(x)h(y)$ for almost all pairs $(x,y) \in \mathbb{R}^2$. HINT: Show that

$$\int\int f(x)g(y)\overline{h}(x+y)\,dydx = \int\int f(x)g(y)\overline{h(x)h(y)}\,dydx$$

for all $f, g \in L^1(\mathbb{R})$.

(c) Let ϕ and h be as in part b, and let f be an element of $L^1(\mathbb{R})$ for which $\phi(f) \neq 0$. Write f_x for the function defined by $f_x(y) = f(x+y)$. Show that the map $x \to \phi(f_x)$ is continuous, and that

$$h(x) = \overline{\phi(f_{-x})/\phi(f)}$$

for almost all x. Conclude that h may be chosen to be a continuous function in $L^\infty(\mathbb{R})$, in which case $h(x+y) = h(x)h(y)$ for all $x, y \in \mathbb{R}$.

(d) Suppose h is a bounded continuous map of $\mathbb{R}$ into $\mathbb{C}$, which is not identically 0 and which satisfies $h(x+y) = h(x)h(y)$ for all x and y. Show that there exists a real number λ such that $h(x) = e^{2\pi i\lambda x}$ for all x. HINT: If h is not identically 1, show that there exists a smallest positive number δ for which $h(\delta) = 1$. Show then that $h(\delta/2) = -1$ and $h(\delta/4) = \pm i$. Conclude that $\lambda = \pm(1/\delta)$ depending on whether $h(\delta/4) = i$ or $-i$.

(e) Conclude that the map $\lambda \to \phi_\lambda$ of part a is a homeomorphism between $\mathbb{R}$ and the structure space Δ of $L^1(\mathbb{R})$. HINT: To prove that the inverse map is continuous, suppose that $\{\lambda_n\}$ does not converge to λ. Show that there exists an $f \in L^1(\mathbb{R})$ such that $\int f(x)e^{-2\pi i\lambda_n x}\,dx$ does not approach $\int f(x)e^{-2\pi i\lambda x}\,dx$.

(f) Show that, using the identification of Δ with $\mathbb{R}$ in part e, that the Gelfand transform on $L^1(\mathbb{R})$ and the Fourier transform on $L^1(\mathbb{R})$ are identical. Conclude that the Gelfand transform is 1-1 on $L^1(\mathbb{R})$.

THEOREM 10.4. *Let A be a Banach algebra. Then the Gelfand transform of A is a norm-decreasing homomorphism of A into the Banach algebra $C(\Delta)$ of all continuous complex-valued functions on Δ.*

EXERCISE 10.6. (a) Prove Theorem 10.4.

(b) If A is a Banach algebra without an identity, show that each function $\hat{x}$ in the range of the Gelfand transform is an element of $C_0(\Delta)$. HINT: The closure of Δ in $\overline{B_1}$ is contained in the union of Δ and $\{0\}$.

DEFINITION. Let A be a Banach algebra with identity I, and let x be an element of A. By the *resolvent* of x we mean the set $\mathrm{res}_A(x)$ of all complex numbers λ for which $\lambda I - x$ has an inverse in A. By the *spectrum* $\mathrm{sp}_A(x)$ of x we mean the complement of the resolvent of x; i.e., $\mathrm{sp}_A(x)$ is the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - x$ does not have an inverse in A. We write simply $\mathrm{res}(x)$ and $\mathrm{sp}(x)$ when it is unambiguous what the algebra A is.

By the *spectral radius* (relative to A) of x we mean the extended real number $\|x\|_{\mathrm{sp}}$ defined by

$$\|x\|_{\mathrm{sp}} = \sup_{\lambda \in \mathrm{sp}_A(x)} |\lambda|.$$

EXERCISE 10.7. Let A be a Banach algebra with identity I, and let x be an element of A.

(a) Show that the resolvent $\mathrm{res}_A(x)$ of x is open in $\mathbb{C}$, whence the spectrum $\mathrm{sp}_A(x)$ of x is closed.

(b) Show that the spectrum of x is nonempty, whence the spectral radius of x is nonnegative. HINT: Make an argument similar to the proof of Mazur's theorem.

(c) Show that $\|x\|_{\mathrm{sp}} \leq \|x\|$, whence the spectrum of x is compact. HINT: If $\lambda \neq 0$, then $\lambda I - x = \lambda(I - (x/\lambda))$.

(d) Show that there exists a $\lambda \in \mathrm{sp}_A(x)$ such that $\|x\|_{\mathrm{sp}} = |\lambda|$; i.e., the spectral radius is attained.

(e) (Spectral Mapping Theorem) If $p(z)$ is any complex polynomial, show that

$$\mathrm{sp}_A(p(x)) = p(\mathrm{sp}_A(x));$$

i.e., $\mu \in \mathrm{sp}_A(p(x))$ if and only if there exists a $\lambda \in \mathrm{sp}_A(x)$ such that $\mu = p(\lambda)$. HINT: Factor the polynomial $p(z) - \mu$ as

$$p(z) - \mu = c \prod_{i=1}^{n} (z - \lambda_i),$$

whence

$$p(x) - \mu I = c\prod_{i=1}^{n}(x - \lambda_i I).$$

Now, the left hand side fails to have an inverse if and only if some one of the factors on the right hand side fails to have an inverse.

THEOREM 10.5. *Let A be a commutative Banach algebra with identity I, and let x be an element of A. Then the spectrum $sp_A(x)$ of x coincides with the range of the Gelfand transform $\hat{x}$ of x. Consequently, we have*

$$\|x\|_{\mathrm{sp}} = \|\hat{x}\|_\infty.$$

PROOF. If there exists a ϕ in the structure space Δ of A for which $\hat{x}(\phi) = \lambda$, then

$$\phi(\lambda I - x) = \lambda - \phi(x) = \lambda - \hat{x}(\phi) = 0,$$

from which it follows that $\lambda I - x$ cannot have an inverse. Hence, the range of $\hat{x}$ is contained in $\mathrm{sp}(x)$.

Conversely, let λ be in the spectrum of x. Let J be the set of all multiples $(\lambda I - x)y$ of $\lambda I - x$ by elements of A. Then J is an ideal in A, and it is a proper ideal since $\lambda I - x$ has no inverse (I is not in J). By Zorn's Lemma, there exists a maximal proper ideal Mcontaining J. Now the closure of M is an ideal. If this closure of M is all of A, then there must exist a sequence $\{m_n\}$ of elements of M that converges to I. But, since the set of invertible elements in A is an open set, it must be that some m_n is invertible. But then M would not be a proper ideal. Therefore, $\overline{M}$ is proper, and since M is maximal it follows that M is itself closed.

Now A/M is a Banach algebra by part g of Exercise 10.1. Also, since M is maximal, we have that A/M is a field. By Mazur's Theorem (Theorem 10.2), we have that A/M is topologically isomorphic to the set of complex numbers. The natural map $\pi : A \to A/M$ is then a continuous nonzero homomorphism of A onto $\mathbb{C}$, i.e., π is an element of Δ. Further, $\pi(\lambda I - x) = 0$ since $\lambda I - x \in J \subseteq M$. Hence, $\hat{x}(\pi) = \lambda$, showing that λ belongs to the range of $\hat{x}$.

EXERCISE 10.8. Suppose A is a commutative Banach algebra with identity I, and let Δ be its structure space. Assume that x is an element of A for which the subalgebra $[x]$ generated by x is dense in A. (See part h of Exercise 10.1.) Prove that $\hat{x}$ is a homeomorphism of Δ onto the spectrum $\mathrm{sp}_A(x)$ of x.

THEOREM 10.6. *Let A be a commutative C^*-algebra with identity I. Then, for each $x \in A$, we have $\hat{x^*} = \overline{\hat{x}}$.*

PROOF. The theorem will follow if we show that $\hat{x}$ is real-valued if x is selfadjoint. (Why?) Thus, if x is selfadjoint, and if $U = e^{ix} = \sum_{n=0}^{\infty}(ix)^n/n!$, then we have seen in part f of Exercise 10.2 and part i of Exercise 10.1 that U is unitary and that $\|U\| = \|U^{-1}\| = 1$. Therefore, if ϕ is an element of the structure space Δ of A, then $|\phi(U)| \leq 1$ and $1/|\phi(U)| = |\phi(U^{-1})| \leq 1$, and this implies that $|\phi(U)| = 1$. On the other hand,

$$\phi(U) = \sum_{n=0}^{\infty}(i\phi(x))^n/n! = e^{i\phi(x)}.$$

But $|e^{it}| = 1$ if and only if t is real. Hence, $\hat{x}(\phi) = \phi(x)$ is real for every $\phi \in \Delta$.

The next result is an immediate consequence of the preceding theorem.

THEOREM 10.7. *If x is a selfadjoint element of a commutative C^*-algebra A with identity, then the spectrum $sp_A(x)$ of x is contained in the set of real numbers.*

EXERCISE 10.9. (A Formula for the Spectral Radius) Let A be a Banach algebra with identity I, and let x be an element of A. Write $\mathrm{sp}(x)$ for $\mathrm{sp}_A(x)$.

(a) If n is any positive integer, show that $\mu \in \mathrm{sp}(x^n)$ if and only if there exists a $\lambda \in \mathrm{sp}(x)$ such that $\mu = \lambda^n$, whence

$$\|x\|_{\mathrm{sp}} = \|x^n\|_{\mathrm{sp}}^{1/n}.$$

Conclude that

$$\|x\|_{\mathrm{sp}} \leq \liminf \|x^n\|^{1/n}.$$

(b) If f is an element of A^*, show that the function $\lambda \to f((\lambda I - x)^{-1})$ is analytic on the (open) resolvent $\mathrm{res}(x)$ of x. Show that the resolvent contains all λ for which $|\lambda| > \|x\|_{\mathrm{sp}}$.

(c) Let f be in A^*. Show that the function $F(\mu) = \mu f((I - \mu x)^{-1})$ is analytic on the disk of radius $1/\|x\|_{\mathrm{sp}}$ around 0 in $\mathbb{C}$. Show further that

$$F(\mu) = \sum_{n=0}^{\infty} f(x^n)\mu^{n+1}$$

on the disk of radius $1/\|x\|$ and hence also on the (possibly) larger disk of radius $1/\|x\|_{\mathrm{sp}}$.

(d) Using the Uniform Boundedness Principle, show that if $|\mu| < 1/\|x\|_{\mathrm{sp}}$, then the sequence $\{\mu^{n+1}x^n\}$ is bounded in norm, whence

$$\limsup \|x^n\|^{1/n} \leq 1/|\mu|$$

for all such μ. Show that this implies that

$$\limsup \|x^n\|^{1/n} \leq \|x\|_{\mathrm{sp}}.$$

(e) Derive the spectral radius formula:

$$\|x\|_{\mathrm{sp}} = \lim \|x^n\|^{1/n}.$$

(f) Suppose that A is a C^*-algebra and that x is a selfadjoint element of A. Prove that

$$\|x\| = \sup_{\lambda \in \mathrm{sp}(x)} |\lambda| = \|x\|_{\mathrm{sp}}.$$

THEOREM 10.8. (Gelfand's Theorem) *Let A be a commutative C^*-algebra with identity I. Then the Gelfand transform is an isometric isomorphism of the Banach algebra A onto $C(\Delta)$, where Δ is the structure space of A.*

PROOF. We have already seen that $x \to \hat{x}$ is a norm-decreasing homomorphism of A into $C(\Delta)$. We must show that the transform is an isometry and is onto.

Now it follows from part f of Exercise 10.9 and Theorem 10.4 that $\|x\| = \|\hat{x}\|_\infty$ whenever x is selfadjoint. For an arbitrary x, write $y = x^*x$. Then

$$\begin{aligned}
\|x\| &= \sqrt{\|y\|} \\
&= \sqrt{\|\hat{y}\|_\infty} \\
&= \sqrt{\|\widehat{x^*x}\|_\infty} \\
&= \sqrt{\|\widehat{x^*}\hat{x}\|_\infty} \\
&= \sqrt{\||\hat{x}|^2\|_\infty} \\
&= \sqrt{\|\hat{x}\|_\infty^2} \\
&= \|\hat{x}\|_\infty,
\end{aligned}$$

showing that the Gelfand transform is an isometry.

By Theorem 10.6, we see that the range $\hat{A}$ of the Gelfand transform is a subalgebra of $C(\Delta)$ that separates the points of Δ and is closed under complex conjugation. Then, by the Stone-Weierstrass Theorem, $\hat{A}$ must be dense in $C(\Delta)$. But, since A is itself complete, and the Gelfand transform is an isometry, it follows that $\hat{A}$ is closed in $C(\Delta)$, whence is all of $C(\Delta)$.

EXERCISE 10.10. Let A be a commutative C^*-algebra with identity I, and let Δ denote its structure space. Verify the following properties of the Gelfand transform on A.

(a) x is invertible if and only if $\hat{x}$ is never 0.

(b) $x = yy^*$ if and only if $\hat{x} \geq 0$.

(c) x is a unitary element of A if and only if $|\hat{x}| \equiv 1$.

(d) A contains a nontrivial projection if and only if Δ is not connected.

EXERCISE 10.11. Let A and B be commutative C^*-algebras, each having an identity, and let Δ_A and Δ_B denote their respective structure spaces. Suppose T is a (not a priori continuous) homomorphism of the algebra A into the algebra B. If ϕ is any linear functional on B, define $T'(\phi)$ on A by

$$T'(\phi) = \phi \circ T.$$

(a) Suppose ϕ is a *positive* linear functional on B; i.e., $\phi(xx^*) \geq 0$ for all $x \in B$. Show that ϕ is necessarily continuous.

(b) Prove that T' is a continuous map of Δ_B into Δ_A.

(c) Show that $\hat{x}(T'(\phi)) = \widehat{T(x)}(\phi)$ for each $x \in A$.

(d) Show that $\|T(x)\| \leq \|x\|$ and conclude that T is necessarily continuous.

(e) Prove that T' is onto if and only if T is 1-1. HINT: T is not 1-1 if and only if there exists a nontrivial continuous function on Δ_A that is identically 0 on the range of T'.

(f) Prove that T' is 1-1 if and only if T is onto.

(g) Prove that T' is a homeomorphism of Δ_B onto Δ_A if and only if T is an isomorphism of A onto B.

EXERCISE 10.12. (Independence of the Spectrum)

(a) Suppose B is a commutative C^*-algebra with identity I, and that A is a sub-C^*-algebra of B containing I. Let x be an element of A. Prove that $\mathrm{sp}_A(x) = \mathrm{sp}_B(x)$. HINT: Let T be the injection map of A into B.

(b) Suppose C is a (not necessarily commutative) C^*-algebra with identity I, and let x be a normal element of C. Suppose A is the smallest sub-C^*-algebra of C that contains x, x^*, and I. Prove that $\mathrm{sp}_A(x) = \mathrm{sp}_C(x)$. HINT: If $\lambda \in \mathrm{sp}_A(x)$, and $\lambda I - x$ has an inverse in C, let B be the smallest sub-C^*-algebra of C containing x, I, and $(\lambda I - x)^{-1}$. Then use part a.

(c) Let H be a separable Hilbert space, and let T be a normal element of $B(H)$. Let A be the smallest sub-C^*-algebra of $B(H)$ containing T, T^*, and I. Show that the spectrum $\mathrm{sp}(T)$ of the operator T coincides with the spectrum $\mathrm{sp}_A(T)$ of T thought of as an element of A.

THEOREM 10.9. (Spectral Theorem) *Let H be a separable Hilbert space, let A be a separable, commutative, sub-C^*-algebra of $B(H)$ that contains the identity operator I, and let Δ denote the structure space of A. Write $\mathcal{B}$ for the σ-algebra of Borel subsets of Δ. Then there exists a unique H-projection-valued measure p on $(\Delta, \mathcal{B})$ such that for every operator $S \in A$ we have*

$$S = \int \hat{S}\, dp.$$

That is, the inverse of the Gelfand transform is the integral with respect to p.

PROOF. Since A contains I, we know that Δ is compact and metrizable. Since the inverse T of the Gelfand transform is an isometric isomorphism of the Banach algebra $C(\Delta)$ onto A, we see that T satisfies the three conditions of Theorem 9.7.

(1) $T(fg) = T(f)T(g)$ for all $f, g \in C(\Delta)$.
(2) $T(\overline{f}) = [T(f)]^*$ for all $f \in C(\Delta)$.
(3) $T(1) = I$.

The present theorem then follows immediately from Theorem 9.7.

THEOREM 10.10. (Spectral Theorem for a Bounded Normal Operator) *Let T be a bounded normal operator on a separable Hilbert space H. Then there exists a unique H-projection-valued measure p on $(\mathbb{C}, \mathcal{B})$ such that*

$$T = \int f\, dp = \int f(\lambda)\, dp(\lambda),$$

where $f(\lambda) = \lambda$. (We also use the notation $T = \int \lambda\, dp(\lambda)$.) Furthermore, $p_{\mathrm{sp}(T)} = I$; i.e., p is supported on the spectrum of T.

PROOF. Let A_0 be the set of all elements $S \in B(H)$ of the form

$$S = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} T^i T^{*j},$$

where each $a_{ij} \in \mathbb{C}$, and let A be the closure in $B(H)$ of A_0. We have that A is the smallest sub-C^*-algebra of $B(H)$ that contains T, T^*, and I. It follows that A is a separable commutative sub-C^*-algebra of $B(H)$ that contains I. If Δ denotes the structure space of A, then, by Theorem 10.9, there exists a unique projection-valued measure q on $(\Delta, \mathcal{B})$ such that

$$S = \int \hat{S}\, dq = \int \hat{S}(\phi)\, dq(\phi)$$

for every $S \in A$.

Note next that the function $\hat{T}$ is 1-1 on Δ. For, if $\hat{T}(\phi_1) = \hat{T}(\phi_2)$, then $\widehat{T^*}(\phi_1) = \widehat{T^*}(\phi_2)$, and hence $\hat{S}(\phi_1) = \hat{S}(\phi_2)$ for every $S \in A_0$. Therefore, $\hat{S}(\phi_1) = \hat{S}(\phi_2)$ for every $S \in A$, showing that $\phi_1 = \phi_2$. Hence, $\hat{T}$ is a homeomorphism of Δ onto the subset $\mathrm{sp}_A(T)$ of $\mathbb{C}$. By part c of Exercise 10.12, $\mathrm{sp}_A(T) = \mathrm{sp}(T)$.

Define a projection-valued measure $p = \hat{T}_* q$ on $\mathrm{sp}(T)$ by

$$p_E = \hat{T}_* q_E = q_{\hat{T}^{-1}(E)}.$$

See part c of Exercise 9.3. Then p is a projection-valued measure on $(\mathbb{C}, \mathcal{B})$, and p is supported on $\mathrm{sp}(T)$.

Now, let f be the identity function on $\mathbb{C}$, i.e., $f(\lambda) = \lambda$. Then, by Exercise 9.13, we have that

$$\begin{aligned} \int \lambda\, dp(\lambda) &= \int f\, dp \\ &= \int (f \circ \hat{T})\, dq \\ &= \int \hat{T}\, dq \\ &= T, \end{aligned}$$

as desired.

Finally, let us show that the projection-valued measure p is unique. Suppose p' is another projection-valued measure on $(\mathbb{C}, \mathcal{B})$, supported on $\mathrm{sp}(T)$, such that

$$T = \int \lambda\, dp'(\lambda) = \int \lambda\, dp(\lambda).$$

It follows also that

$$T^* = \int \bar{\lambda}\, dp'(\lambda) = \int \bar{\lambda}\, dp(\lambda).$$

Then, for every function P of the form

$$P(\lambda) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \lambda^i \bar{\lambda}^j,$$

we have

$$\int P(\lambda)\, dp'(\lambda) = \int P(\lambda)\, dp(\lambda).$$

Whence, by the Stone-Weierstrass Theorem,

$$\int f(\lambda)\, dp'(\lambda) = \int f(\lambda)\, dp(\lambda)$$

for every continuous complex-valued function f on $\mathrm{sp}(T)$. If $q' = \hat{T}_*^{-1} p'$ is the projection-valued measure on Δ defined by

$$q'_E = p'_{\hat{T}(E)},$$

then, for any continuous function g on Δ, we have

$$\begin{aligned} \int g\, dq' &= \int (g \circ \hat{T}^{-1})\, dp' \\ &= \int (g \circ \hat{T}^{-1})\, dp \\ &= \int (g \circ \hat{T}^{-1} \circ \hat{T})\, dq \\ &= \int g\, dq. \end{aligned}$$

So, by the uniqueness assertion in the general spectral theorem, we have that $q' = q$. But then

$$p' = \hat{T}_* q' = \hat{T}_* q = p,$$

and the uniqueness is proved.

DEFINITION. The projection-valued measure p, associated as in the above theorem to a normal operator T, is called the *spectral measure* for T.

The next result is an immediate consequence of the preceding theorem.

THEOREM 10.11. (Spectral Theorem for a Bounded Selfadjoint Operator) *Let H be a separable Hilbert space, and let T be a selfadjoint element in $B(H)$. Then there exists a unique projection-valued measure p on $(\mathbb{R}, \mathcal{B})$ for which $T = \int \lambda \, dp(\lambda)$. Further, p is supported on the spectrum of T.*

REMARK. A slightly different notation is frequently used to indicate the spectral measure for a selfadjoint operator. Instead of writing $T = \int \lambda \, dp(\lambda)$, one often writes $T = \int \lambda \, dE_\lambda$. Also, such a projection-valued measure is sometimes referred to as a *resolution of the identity.*

EXERCISE 10.13. Let T be a normal operator in $B(H)$ and let p be its spectral measure.

(a) If U is a nonempty (relatively) open subset of $\mathrm{sp}(T)$, show that $p_U \neq 0$. If U is an infinite set, show that the range of p_U is infinite dimensional.

(b) Show that if E is a closed subset of $\mathbb{C}$ for which $p_E = I$, then E contains $\mathrm{sp}(T)$. Conclude that the smallest closed subset of $\mathbb{C}$ that supports p is the spectrum of T.

(c) If T is invertible, show that the function $1/\lambda$ is bounded on $\mathrm{sp}(T)$ and that $T^{-1} = \int (1/\lambda) \, dp(\lambda)$.

(d) If $\mathrm{sp}(T)$ contains at least two distinct points, show that $T = T_1 + T_2$, where T_1 and T_2 are both nonzero normal operators and $T_1 \circ T_2 = 0$.

(e) Suppose S is a bounded operator on H that commutes with both T and T^*. Prove that S commutes with every projection p_E for E a Borel subset of $\mathrm{sp}(T)$. HINT: Do this first for open subsets of $\mathrm{sp}(T)$, and then consider the collection of all sets E for which $p_E S = S p_E$. (It is a monotone class.)

(f) Suppose S is a bounded operator that commutes with T. Let $E = \mathrm{sp}(T) \cap B_\epsilon(\lambda_0)$, where $\epsilon > 0$ and λ_0 is a complex number. Show that, if x belongs to the range of p_E, then $S(x)$ also belongs to the range of p_E, implying that S commutes with p_E. (Use part b of Exercise 9.11.) Deduce the Fuglede-Putnam-Rosenbloom Theorem: If a bounded operator S commutes with a bounded normal operator T, then S also commutes with T^*.

EXERCISE 10.14. Let T be a normal operator on a separable Hilbert space H, let A be a sub-C^*-algebra of $B(H)$ that contains T and I, let f be a continuous complex-valued function on the spectrum $\mathrm{sp}(T)$ of T, and suppose S is an element of A for which $\hat{S} = f \circ \hat{T}$.

(a) Show that the spectrum $\mathrm{sp}(S)$ of S equals $f(\mathrm{sp}(T))$. Compare this result with the spectral mapping theorem (part e of Exercise 10.7).

(b) Let p^T denote the spectral measure for T and p^S denote the spectral measure for S. In the notation of Exercises 9.3 and 9.13, show that

$$p^S = f_*(p^T).$$

HINT: Show that $S = \int \lambda \, df_*(p^T)(\lambda)$, and then use the uniqueness assertion in the Spectral Theorem for a normal operator.

(c) Apply parts a and b to describe the spectral measures for $S = q(T)$ for q a polynomial and $S = e^T$.

EXERCISE 10.15. Let p be an H-projection-valued measure on the Borel space $(S, \mathcal{B})$. If f is an element of $L^\infty(p)$, define the *essential range* of f to be the set of all $\lambda \in \mathbb{C}$ for which

$$p_{f^{-1}(B_\epsilon(\lambda))} \neq 0$$

for every $\epsilon > 0$.

(a) Let f be an element of $L^\infty(p)$. If T is the bounded normal operator $\int f \, dp$, show that the spectrum of T coincides with the essential range of f. See part e of Exercise 9.10.

(b) Let f be an element of $L^\infty(p)$, and let $T = \int f \, dp$. Prove that the spectral measure q for T is the projection-valued measure $f_* p$. See Exercises 9.3 and 9.13.

EXERCISE 10.16. Let (S, μ) be a σ-finite measure space. For each $f \in L^\infty(\mu)$, let m_f denote the multiplication operator on $L^2(\mu)$ given by $m_f g = fg$. Let p denote the canonical projection-valued measure on $L^2(\mu)$.

(a) Prove that the operator m_f is a normal operator and that

$$m_f = \int f \, dp.$$

Find the spectrum $\mathrm{sp}(m_f)$ of m_f.

(b) Using $S = [0, 1]$ and μ as Lebesgue measure, find the spectrum and spectral measures for the following m_f's:

(1) $f = \chi_{[0,1/2]}$,
(2) $f(x) = x$,
(3) $f(x) = x^2$,
(4) $f(x) = \sin(2\pi x)$, and
(5) f is a step function $f = \sum_{i=1}^n a_i \chi_{I_i}$, where the a_i's are complex numbers and the I_i's are disjoint intervals.

(c) Let S and μ be as in part b. Compute the spectrum and spectral measure for m_f if f is the Cantor function.

DEFINITION. We say that an operator $T \in B(H)$ is *diagonalizable* if it can be represented as the integral of a function with respect to a projection-valued measure. That is, if there exists a Borel space $(S, \mathcal{B})$ and an H-projection-valued measure p on $(S, \mathcal{B})$ such that $T = \int f\, dp$ for some bounded $\mathcal{B}$-measurable function f. A collection B of operators is called *simultaneously diagonalizable* if there exists a projection-valued measure p on a Borel space $(S, \mathcal{B})$ such that each element of B can be represented as the integral of a function with respect to p.

REMARK. Theorem 10.11 and Theorem 10.10 show that selfadjoint and normal operators are diagonalizable. It is also clear that simultaneously diagonalizable operators commute.

EXERCISE 10.17. (a) Let H be a separable Hilbert space. Suppose B is a commuting, separable, selfadjoint subset of $B(H)$. Prove that the elements of B are simultaneously diagonalizable.

(b) Let H be a separable Hilbert space. Show that a separable, selfadjoint collection S of operators in $B(H)$ is simultaneously diagonalizable if and only if S is contained in a commutative sub-C^*-algebra of $B(H)$.

(c) Let A be an $n \times n$ complex matrix for which $a_{ij} = \overline{a_{ji}}$. Use the Spectral Theorem to show that there exists a unitary matrix U such that UAU^{-1} is diagonal. That is, use the Spectral Theorem to prove that every Hermitian matrix can be diagonalized.

One of the important consequences of the spectral theorem is the following:

THEOREM 10.12. (Stone's Theorem) *Let $t \to U_t$ be a map of $\mathbb{R}$ into the set of unitary operators on a separable Hilbert space H, and suppose that this map satisfies:*

(1) $U_{t+s} = U_t \circ U_s$ *for all* $t, s \in \mathbb{R}$.
(2) *The map* $t \to (U_t(x), y)$ *is continuous for every pair* $x, y \in H$.

Then there exists a unique projection-valued measure p on $(\mathbb{R}, \mathcal{B})$ such that

$$U_t = \int e^{-2\pi i \lambda t}\, dp(\lambda)$$

for each $t \in \mathbb{R}$.

PROOF. For each $f \in L^1(\mathbb{R})$, define a map L_f from $H \times H$ into $\mathbb{C}$ by

$$L_f(x, y) = \int_{\mathbb{R}} f(s)(U_s(x), y)\, ds.$$

It follows from Theorem 8.5 (see the exercise below) that for each $f \in L^1(\mathbb{R})$ there exists a unique element $T_f \in B(H)$ such that

$$L_f(x, y) = (T_f(x), y)$$

for all $x, y \in H$. Let B denote the set of all operators on H of the form T_f for $f \in L^1(\mathbb{R})$. Again by the exercise below, it follows that B is a separable commutative selfadjoint subalgebra of $B(H)$.

We claim first that the subspace H_0 spanned by the vectors of the form $y = T_f(x)$, for $f \in L^1(\mathbb{R})$ and $x \in H$, is dense in H. Indeed, if $z \in H$ is orthogonal to every element of H_0, then

$$\begin{aligned} 0 &= (T_f(z), z) \\ &= \int_{\mathbb{R}} f(s)(U_s(z), z)\, ds \end{aligned}$$

for all $f \in L^1(\mathbb{R})$, whence

$$(U_s(z), z) = 0$$

for almost all $s \in \mathbb{R}$. But, since this is a continuous function of s, it follows that

$$(U_s(z), z) = 0$$

for all s. In particular,

$$(z, z) = (U_0(z), z) = 0,$$

proving that H_0 is dense in H as claimed.

We let A denote the smallest sub-C^*-algebra of $B(H)$ that contains B and the identity operator I, and we denote by Δ the structure space of A. We see that A is the closure in $B(H)$ of the set of all elements of the form $\lambda I + T_f$, for $\lambda \in \mathbb{C}$ and $f \in L^1(\mathbb{R})$. So A is a separable commutative C^*-algebra. Again, by Exercise 10.18 below, we have that the map T that sends $f \in L^1(\mathbb{R})$ to the operator T_f is a norm-decreasing homomorphism of the Banach $*$-algebra $L^1(\mathbb{R})$ into the C^*-algebra A. Recall from Exercise 10.5 that the structure space of the Banach algebra

$L^1(\mathbb{R})$ is identified, specifically as in that exercise, with the real line $\mathbb{R}$. With this identification, we define $T' : \Delta \to \mathbb{R}$ by

$$T'(\phi) = \phi \circ T.$$

Because the topologies on the structures spaces of A and $L^1(\mathbb{R})$ are the weak* topologies, it follows directly that T' is continuous. For each $f \in L^1(\mathbb{R})$ we have the formula

$$\hat{f}(T'(\phi)) = [T'(\phi)](f) = \phi(T_f) = \widehat{T_f}(\phi).$$

By the general Spectral Theorem, we let q be the unique projection-valued measure on Δ for which

$$S = \int \hat{S}(\phi)\, dq(\phi)$$

for all $S \in A$, and we set $p = T'_* q$. Then p is a projection-valued measure on $(\mathbb{R}, \mathcal{B})$, and we have

$$\begin{aligned}\int \hat{f}\, dp &= \int (\hat{f} \circ T')\, dq \\ &= \int \hat{f}(T'(\phi))\, dq(\phi) \\ &= \int \widehat{T_f}(\phi)\, dq(\phi) \\ &= T_f\end{aligned}$$

for all $f \in L^1(\mathbb{R})$.

Now, for each $f \in L^1(\mathbb{R})$ and each real t we have

$$\begin{aligned}(U_t(T_f(x)), y) &= \int_{\mathbb{R}} f(s)(U_t(U_s(x)), y)\, ds \\ &= \int_{\mathbb{R}} f(s)(U_{t+s}(x), y)\, ds \\ &= \int_{\mathbb{R}} f_{-t}(s)(U_s(x), y)\, ds \\ &= (T_{f_{-t}}(x), y) \\ &= ([\int \widehat{f_{-t}}(\lambda)\, dp(\lambda)](x), y) \\ &= ([\int e^{-2\pi i \lambda t} \hat{f}(\lambda)\, dp(\lambda)](x), y) \\ &= ([\int e^{-2\pi i \lambda t}\, dp(\lambda)](T_f(x)), y),\end{aligned}$$

where f_{-t} is defined by $f_{-t}(x) = f(x-t)$. So, because the set H_0 of all vectors of the form $T_f(x)$ span a dense subspace of H,

$$U_t = \int e^{-2\pi i\lambda t}\,dp(\lambda),$$

as desired.

We have left to prove the uniqueness of p. Suppose $\tilde{p}$ is a projection-valued measure on $(\mathbb{R},\mathcal{B})$ for which $U_t = \int e^{-2\pi i\lambda t}\,d\tilde{p}(\lambda)$ for all t. Now for each vector $x \in H$, define the two measures μ_x and $\tilde{\mu}_x$ by

$$\mu_x(E) = (p_E(x), x)$$

and

$$\tilde{\mu}_x(E) = (\tilde{p}_E(x), x).$$

Our assumption on $\tilde{p}$ implies then that

$$\int e^{-2\pi i\lambda t}\,d\mu_x(\lambda) = \int e^{-2\pi i\lambda t}\,d\tilde{\mu}_x(\lambda)$$

for all real t. Using Fubini's theorem we then have for every $f \in L^1(\mathbb{R})$ that

$$\begin{aligned}\int \hat{f}(\lambda)\,d\mu_x(\lambda) &= \int\int f(t)e^{-2\pi i\lambda t}\,dtd\mu_x(\lambda)\\ &= \int f(t)\int e^{-2\pi i\lambda t}\,d\mu_x(\lambda)\,dt\\ &= \int f(t)\int e^{-2\pi i\lambda t}\,d\tilde{\mu}_x(\lambda)\,dt\\ &= \int \hat{f}(\lambda)\,d\tilde{\mu}_x(\lambda).\end{aligned}$$

Since the set of Fourier transforms of L^1 functions is dense in $C_0(\mathbb{R})$, it then follows that

$$\int g\,d\mu_x = \int g\,d\tilde{\mu}_x$$

for every $g \in C_0(\mathbb{R})$. Therefore, by the Riesz representation theorem, $\mu_x = \tilde{\mu}_x$. Consequently, $p = \tilde{p}$ (see part d of Exercise 9.2), and the proof is complete.

EXERCISE 10.18. Let the map $t \to U_t$ be as in the theorem above.

(a) Prove that U_0 is the identity operator on H and that $U_t^* = U_{-t}$ for all t.

(b) If $f \in L^1(\mathbb{R})$, show that there exists a unique element $T_f \in B(H)$ such that

$$\int_{\mathbb{R}} f(s)(U_s(x), y)\, ds = (T_f(x), y)$$

for all $x, y \in H$. HINT: Use Theorem 8.5.

(c) Prove that the assignment $f \to T_f$ defined in part b satisfies

$$\|T_f\| \leq \|f\|_1$$

for all $f \in L^1(\mathbb{R})$,

$$T_{f*g} = T_f \circ T_g$$

for all $f, g \in L^1(\mathbb{R})$ and

$$T_{f^*} = T_f^*$$

for all $f \in L^1(\mathbb{R})$, where

$$f^*(s) = \overline{f(-s)}.$$

(d) Conclude that the set of all T_f's, for $f \in L^1(\mathbb{R})$, is a separable commutative selfadjoint algebra of operators.

EXERCISE 10.19. Let H be a separable Hilbert space, let A be a separable, commutative, sub-C^*-algebra of $B(H)$, assume that A contains the identity operator I, and let Δ denote the structure space of A. Let x be a vector in H, and let M be the closure of the set of all vectors $T(x)$, for $T \in A$. That is, M is a *cyclic subspace* for A. Prove that there exists a finite Borel measure μ on Δ and a unitary operator U of $L^2(\mu)$ onto M such that

$$U^{-1} \circ T \circ U = m_{\hat{T}}$$

for every $T \in A$. HINT: Let G denote the inverse of the Gelfand transform of A. Define a positive linear functional L on $C(\Delta)$ by $L(f) = ([G(f)](x), x)$, use the Riesz Representation Theorem to get a measure μ, and then define $U(f) = [G(f)](x)$ on the dense subspace $C(\Delta)$ of $L^2(\mu)$.