CHAPTER 0

PRELIMINARIES

We include in this preliminary chapter some of the very basic concepts and results of set theory, linear algebra, and topology. We do this so that precise definitions and theorems will be at hand for reference. The exercises given here contain some of the main results. Although they should be routine for the student of this subject, we recommend that they be done carefully. The main theorems of Functional Analysis frequently rely on the Axiom of Choice, and in some cases are equivalent to this axiom from abstract set theory. The version of the Axiom of Choice that is ordinarily used in Functional Analysis is the Hausdorff maximality principle, which we state here without proof.

HAUSDORFF MAXIMALITY PRINCIPLE. *Let S be a nonempty set, and let $<$ denote a partial ordering on S, i.e., a transitive relation on S. Then there exists a maximal linearly ordered subset of S.*

Frequently encountered in our subject is the notion of an infinite product.

DEFINITION. Let I be a set, and for each $i \in I$ let X_i be a set. By the *Cartesian product* of the sets $\{X_i\}$, we mean the set of all functions f defined on I for which $f(i) \in X_i$ for each $i \in I$. We denote this set of functions by $\prod_{i\in I} X_i$ or simply by $\prod X_i$.

Ordinarily, a function $f \in \prod_{i\in I} X_i$ is denoted by $\{x_i\}$, where $x_i = f(i)$.

Fundamental to Functional Analysis are the notions of vector spaces and linear transformations.

DEFINITION. Let F denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A *vector space* over F is an additive abelian group X, on which the elements of F act by scalar multiplication:

(1) $a(x+y) = ax + ay$, $a(bx) = (ab)x$, and $(a+b)x = ax + bx$ for all $x, y \in X$ and $a, b \in F$.
(2) $1x = x$ for all $x \in X$.

The elements of F are called *scalars*. If $F = \mathbb{R}$, then X is called a *real vector space*, and if $F = \mathbb{C}$, then X is called a *complex vector space*. Obviously, a complex vector space can also be regarded as a real vector space, but not every real vector space is a complex vector space. See Exercise 0.1 below.

A subset Y of a vector space X is called a *subspace* if it is closed under addition and scalar multiplication.

A nonempty finite set $\{x_1, \ldots, x_n\}$ of nonzero elements of a vector space X is called *linearly dependent* if there exist elements $\{a_1, \ldots, a_n\}$ of F, not all 0, such that $\sum_{i=1}^n a_i x_i = 0$. An arbitrary set S of nonzero elements of X is called *linearly dependent* if some nonempty finite subset of S is linearly dependent. A subset $S \subseteq X$ of nonzero vectors is called *linearly independent* if it is not linearly dependent.

A subset B of a vector space X is said to be a *spanning set* for X if every element of X is a finite linear combination of elements of B. A *basis* of X is a linearly independent spanning subset of X.

EXERCISE 0.1. (a) Prove that every nontrivial vector space has a basis. HINT: The Hausdorff maximality principle.

(b) If B is a basis of a vector space X, show that each element $x \in X$ can be written uniquely as a finite linear combination $x = \sum_{i=1}^n a_i x_i$, where each $x_i \in B$.

(c) Show that any two bases of a vector space have the same cardinality, i.e., they can be put into 1-1 correspondence.

(d) Show that the set F^n of all n-tuples $(x_1, x_2, \ldots, x_n)$ of elements of F is a vector space with respect to coordinatewise addition and scalar multiplication.

(e) Prove that every complex vector space is automatically a real vector space. On the other hand, show that $\mathbb{R}^3$ is a real vector space but that scalar multiplication cannot be extended to $\mathbb{C}$ so that $\mathbb{R}^3$ is a complex vector space. HINT: What could ix possibly be?

DEFINITION. The *dimension* of a vector space X is the cardinality of a basis of X.

DEFINITION. Let I be a set, and let $\{X_i\}$, for $i \in I$, be a collection of vector spaces over F. By the *vector space direct product* $\prod_{i\in I} X_i$, we mean the cartesian product of the sets $\{X_i\}$, together with the operations:

(1) $\{x_i\} + \{y_i\} = \{x_i + y_i\}$
(2) $a\{x_i\} = \{ax_i\}$.

The (algebraic) *direct sum*

$$\bigoplus_{i\in I} X_i$$

is defined to be the subset of $\prod_{i\in I} X_i$ consisting of the elements $\{x_i\}$ for which $x_i = 0$ for all but a finite number of i's.

EXERCISE 0.2. (a) Prove that $\prod_{i\in I} X_i$ is a vector space.
(b) Show that $\bigoplus_{i\in I} X_i$ is a subspace of $\prod_{i\in I} X_i$.
(c) Prove that

$$F^n = \prod_{i\in\{1,\dots,n\}} F = \sum_{i\in\{1,\dots,n\}} F.$$

DEFINITION. A *linear transformation* from a vector space X into a vector space Y is a function $T : X \to Y$ for which

$$T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$$

for all $x_1, x_2 \in X$ and $a_1, a_2 \in F$.

A linear transformation $T : X \to Y$ is called a *linear isomorphism* if it is 1-1 and onto.

By the *kernel* $\ker(T)$ of a linear transformation T, we mean the set of all $x \in X$ for which $T(x) = 0$, and by the *range* of T we mean the set of all elements of Y of the form $T(x)$.

EXERCISE 0.3. Let X and Y be vector spaces, and let$B$ be a basis for X.

(a) Suppose T and S are linear transformations of X into Y. Show that $T = S$ if and only if $T(x) = S(x)$ for every $x \in B$.

(b) For each $b \in B$ let y_b be an element of Y. Show that there exists a (unique) linear transformation $T : X \to Y$ satisfying $T(b) = y_b$ for all $b \in B$.

(c) Let T be a linear transformation of X into Y. Prove that the kernel of T is a subspace of X and the range of T is a subspace of Y.

(d) Let $T : X \to Y$ be a linear isomorphism. Prove that $T^{-1} : Y \to X$ is a linear isomorphism.

DEFINITION. A *linear functional* on a vector space X over F is a linear transformation of X into $F \equiv F^1$.

EXERCISE 0.4. Let f be a linear functional on a vector space X, and let M be the kernel of f.

(a) If x is an element of X, which is not in M, show that every element $y \in X$ can be written uniquely as $y = m + ax$, where $m \in M$ and $a \in F$.

(b) Let f and g be linear functionals on X. Show that f is a nonzero multiple $f = ag$ of g if and only if $\ker(f) = \ker(g)$.

(c) Let T be a linear transformation of a vector space X onto the vector space F^n. Show that there exist elements $\{x_1, \dots , x_n\}$ of X, none of which belongs to $\ker(T)$, such that each element $y \in X$ can be written uniquely as $y = m + \sum_{i=1}^n a_i x_i$, where $m \in \ker(T)$ and each $a_i \in F$.

DEFINITION. If X is a vector space and M is a subspace of X, we define the *quotient space* X/M to be the set of all cosets $x + M$ of M together with the following operations:

$$(x + M) + (y + M) = (x + y) + M,$$

and

$$a(x + M) = ax + M$$

for all $x, y \in X$ and $a \in F$.

EXERCISE 0.5. Let M be a subspace of a vector space X.

(a) Prove that the quotient space X/M is a vector space.

(b) Define $\pi : X \to X/M$ by $\pi(x) = x + M$. Show that π is a linear transformation from X onto X/M. This transformation π is called the *natural map* or *quotient map* of X onto X/M.

(c) If T is a linear transformation of X into a vector space Y, and if $M \subseteq \ker(T)$, show that there exists a unique linear transformation $S : X/M \to Y$ such that $T = S \circ \pi$, where π is the natural map of X onto X/M.

Perhaps the most beautiful aspect of Functional Analysis is in its combining of linear algebra and topology. We give next the fundamental topological ideas that we will need.

DEFINITION. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X satisfying:

(1) $X \in \mathcal{T}$.
(2) $\emptyset \in \mathcal{T}$.
(3) The intersection of any finite number of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
(4) The union of an arbitrary collection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

The set X, or the pair $(X, \mathcal{T})$, is called a *topological space*.

The elements of a topology $\mathcal{T}$ are called *open* subsets of X, and their complements are called *closed* sets. An open set containing a point $x \in X$ is called an *open neighborhood* of x, and any set that contains an open neighborhood of x is itself called a *neighborhood* of x.

If A is a subset of a topological space $(X, \mathcal{T})$ and x is a point of A, then x is called an *interior point* of A if A contains a neighborhood of x. The *interior* of A is the set of all interior points of A.

If Y is a subset of a topological space $(X, \mathcal{T})$, then the *relative topology on* Y is the collection $\mathcal{T}'$ of subsets of Y obtained by intersecting the elements of $\mathcal{T}$ with Y. The collection $\mathcal{T}'$ is a topology on Y, and the pair $(Y, \mathcal{T}')$ is called a *topological subspace* of X.

A subset $\mathcal{B}$ of a topology $\mathcal{T}$ is called a *base* for $\mathcal{T}$ if each element $U \in \mathcal{T}$ is a union of elements of $\mathcal{B}$.

A topological space $(X, \mathcal{T})$ is called *second countable* if there exists a countable base $\mathcal{B}$ for $\mathcal{T}$.

A topological space $(X, \mathcal{T})$ is called a *Hausdorff* space if for each pair of distinct points $x, y \in X$ there exist open sets $U, V \in \mathcal{T}$ such that $x \in U,\ y \in V$, and $U \cap V = \emptyset$. X is called a *regular* topological space if, for each closed set $A \subseteq X$ and each point $x \notin A$, there exist open sets U and V such that $A \subseteq U,\ x \in V$, and $U \cap V = \emptyset$. X is called a *normal* topological space if, for each pair A, B of disjoint closed subsets of X, there exist open sets U, V such that $A \subseteq U,\ B \subseteq V$, and $U \cap V = \emptyset$.

By an *open cover* of a subset Y of a topological space X, we mean a collection $\mathcal{U}$ of open subsets of X for which $Y \subseteq \cup_{U \in \mathcal{U}} U$. A subset Y of a topological space X is called *compact* if every open cover $\mathcal{U}$ of Y has a finite subcover; i.e., there exist finitely many elements $U_1, \ldots, U_n$ of $\mathcal{U}$ such that $Y \subseteq \cup_1^n U_i$.

A topological space X is called *σ-compact* if it is a countable union of compact subsets.

A topological space X is called *locally compact* if, for every $x \in X$ and every open set U containing x, U contains a compact neighborhood of x.

A function F from one topological space X into another topological space Y is called *continuous* if $f^{-1}(U)$ is an open subset of X whenever U is an open subset of Y.

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies:

(1) $d(x, y) \geq 0$ for all $x, y \in X$.
(2) $d(x, y) = 0$ if and only if $x = y$.
(3) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

If X is a set on which a metric d is defined, then X (or the pair (X, d)) is called a *metric space*.

If d is a metric on a set X, x is an element of X, and $\epsilon > 0$, then the *ball* $B_\epsilon(x)$ of radius ϵ around x is defined to be the set of all $y \in X$ for which $d(x, y) < \epsilon$. A point x is called an *interior* point of a subset A of a metric space (X, d) if there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$, and a set A is called *open* relative to a metric d if every point of A is an interior point of A.

The topological space $(X, \mathcal{T})$ is called *metrizable* if there exists a metric d on X for which the elements of $\mathcal{T}$ coincide with the sets that are open sets relative to the metric d.

EXERCISE 0.6. (a) Let $\mathcal{A}$ be a collection of subsets of a set X. Prove that there is a smallest topology $\mathcal{T}$ on X that contains $\mathcal{A}$, and verify that a base for this topology consists of the collection of all sets B of the form

$$B = \cap_{i=1}^{n} A_i,$$

where each $A_i \in \mathcal{A}$.

(b) Let A be a subset of a topological space $(X, \mathcal{T})$. Prove that the interior of A is an open set. Prove that the intersection of all closed sets containing A is closed. This closed set is called the *closure* of A and is denoted by $\bar{A}$.

(c) Let Y be a subset of a topological space $(X, \mathcal{T})$, and write $\mathcal{T}'$ for the collection of subsets V of Y of the form $V = U \cap Y$ for $U \in \mathcal{T}$. Prove that $\mathcal{T}'$ is a topology on Y.

(d) Let d be a metric on a set X. Show that the collection of all sets that are open relative to d forms a topology on X.

(e) Let X and Y be topological spaces. Prove that a function $f : X \to Y$ is continuous if and only if for every open set $U \subseteq Y$ and every $x \in f^{-1}(U)$ there exists an open set $V \subseteq X$ such that $x \in V$ and $f(V) \subseteq U$.

EXERCISE 0.7. Let X be a set, and let $\{X_i\}$, for i in a set I, be a collection of topological spaces. For each i, let f_i be a map of X into X_i.

(a) Prove that there exists a smallest topology $\mathcal{T}$ on X for which each function f_i is continuous.

(b) Let $\mathcal{T}$ be as in part a. Show that, for each index i and each open subset $U_i \subseteq X_i$, the set $f_i^{-1}(U_i)$ belongs to $\mathcal{T}$.

(c) Let $\mathcal{T}$ be as in part a. Show that, for each finite set $i_1, \ldots, i_n$ of elements of I, and for each n-tuple $U_{i_1}, \ldots, U_{i_n}$, for U_{i_j} an open subset of X_{i_j}, the set

$$\cap_{j=1}^n f_{i_j}^{-1}(U_{i_j})$$

is in $\mathcal{T}$.

(d) Let $\mathcal{T}$ be as in part a. Show that each element of $\mathcal{T}$ is a union of sets of the form described in part c; i.e., the sets described in part c form a base for $\mathcal{T}$.

DEFINITION. Let X be a set, and for each i in a set I let f_i be a function from X into a topological space X_i. The smallest topology on X, for which each f_i is continuous, is called the *weak topology generated by the* f_i's.

If $\{X_i\}$, for $i \in I$, is a collection of topological spaces, write

$$X = \prod_{i \in I} X_i,$$

and define $f_i : X \to X_i$ by

$$f_i(\{x_j\}) = x_i.$$

The *product topology* on $X = \prod_{i \in I} X_i$ is defined to be the weak topology generated by the f_i's.

EXERCISE 0.8. Let X be a set, let $\{X_i\}$ for $i \in I$, be a collection of topological spaces, and for each $i \in I$ let f_i be a map of X into X_i. Let $\mathcal{T}$ denote the weak topology on X generated by the f_i's.

(a) Prove that $\mathcal{T}$ is Hausdorff if each X_i is Hausdorff and the functions $\{f_i\}$ separate the points of X. (The f_i's *separate* the points of X if $x \neq y \in X$ implies that there exists an $i \in I$ such that $f_i(x) \neq f_i(y)$.)

(b) Show that $\mathcal{T}$ is second countable if the index set I is countable and each topological space X_i is second countable.

(c) Conclude that the product space $Y = \prod_{i \in I} X_i$ is second countable if I is countable and each X_i is second countable.

(d) Suppose the index set I is countable, that the f_i's separate the points of X, and that each X_i is metrizable. Prove that $(X, \mathcal{T})$ is metrizable. HINT: Identify I with the set $\{1, 2, \dots\}$. If d_i denotes the metric on X_i, define d on X by

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \min(1, d_i(f_i(x), f_i(y))),$$

and show that d is a metric whose open sets coincide with the elements of $\mathcal{T}$.

(e) Let Y be the topological product space $Y = \prod_{i \in I} X_i$, and define $F : X \to Y$ by $[F(x)]_i = f_i(x)$. Suppose that the f_i's separate the points of X. Prove that F is a homeomorphism of $(X, \mathcal{T})$ into Y.

EXERCISE 0.9. (a) Prove that a topological space X is compact if and only if it satisfies the *finite intersection property*; i.e., if $\mathcal{F}$ is a collection of closed subsets of X, for which the intersection of any finite number of elements of $\mathcal{F}$ is nonempty, then the intersection of all the elements of $\mathcal{F}$ is nonempty.

(b) Prove that a compact Hausdorff space is normal.

(c) Prove that a regular space, having a countable base, is normal.

(d) Prove Urysohn's Lemma: If X is a normal topological space, and if A and B are nonempty disjoint closed subsets of X, then there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

(e) Let X be a regular space having a countable base. Show that there exists a sequence $\{f_n\}$ of continuous real-valued functions on X, such that for each closed set $A \subseteq X$ and each point $x \notin A$, there exists an n for which $f_n(x) \notin f_n(A)$. HINT: For each pair U, V of elements of the countable base, for which $U \subseteq \bar{U} \subset V$, use Urysohn's lemma on the sets $\bar{U}$ and $\tilde{V}$, where $\tilde{V}$ denotes the complement of V. Conclude that the topology on X coincides with the weak topology generated by the resulting f_n's.

(f) Prove that a regular space X, having a countable base, is metrizable. HINT: Use part e to construct a homeomorphism between X and a subset of a countable product of real lines.

(g) Prove that a locally compact Hausdorff space is regular and hence that a locally compact, second countable, Hausdorff space is metrizable.

DEFINITION. Let $(X, \mathcal{T})$ be a topological space, and let f be a function from X onto a set Y. The largest topology $\mathcal{Q}$ on Y for which f is continuous is called the *quotient topology* on Y.

EXERCISE 0.10. Let $(X, \mathcal{T})$ be a topological space, let $f : X \to Y$ be a map of X onto a set Y, and let $\mathcal{Q}$ be the quotient topology on Y.

(a) Prove that a subset $U \subseteq Y$ belongs to $\mathcal{Q}$ if and only if $f^{-1}(U)$ belongs to $\mathcal{T}$. That is, $\mathcal{Q} = \{U \subseteq Y : f^{-1}(U) \in \mathcal{T}\}$.

(b) Suppose Z is a topological space and that g is a function from $(Y, \mathcal{Q})$ into Z. Prove that g is continuous if and only if $g \circ f$ is continuous from $(X, \mathcal{T})$ into Z.