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New operations for interval-valued Pythagorean fuzzy set

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Abstract. Interval-Valued Pythagorean Fuzzy Set (IVPFS), originally proposed by Peng and Yang, is a novel tool to deal with vagueness and incertitude. As a generalized set, IVPFS has a close relationship with Interval-Valued Intuitionistic Fuzzy Set (IVIFS). IVPFS can be reduced to IVIFS, satisfying the condition $\mu^+ + \nu^+ \leq 1$. However, the related operations of IVPFS do not take different conditions into consideration. In this paper, we initiate some new interval-valued Pythagorean fuzzy operators ($\diamondsuit, \square, \spadesuit, \clubsuit$, ✠, $\rightarrow$, \$) and discuss their properties in relation with some existing operators ($\cup, \cap, \oplus, \otimes$) in detail. It will promote the development of interval-valued Pythagorean fuzzy operators. Later, an algorithm is proposed to deal with Multi-Criteria Decision-Making (MCDM) problem based on the proposed $\spadesuit$ operator. Finally, the effectiveness and feasibility of the proposed algorithm is demonstrated by mine emergency decision-making example.

1. Introduction

Intuitionistic Fuzzy Set (IFS), initiated by Atanassov [1], is an extension of fuzzy set theory [2]. IFS is characterized by a membership degree and a non-membership degree and, hence, can depict the fuzzy character of data more comprehensively and detailedly. The prominent characteristic of IFS is that it assigns a membership degree and a nonmembership degree to each element with their sum equal to or less than one. However, in some practical decision-making processes, the sum of the membership degree and the nonmembership degree to which an alternative that satisfies the expert recommended criterion is provided, may be larger than one; however, their square sum is equal to or less than one.

Hence, Yager [3] examined Pythagorean Fuzzy Set (PFS) characterized by a membership degree and a nonmembership degree that satisfies the case in which the square sum of its membership degree and nonmembership degree is less than or equal to one. Yager and Abbasov [4] gave an example to illustrate this situation: an expert giving his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his nonmembership is $\frac{1}{2}$. Since the sum of the two values is bigger than 1, they are not available for IFS, yet are feasible for PFS because:

$$(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \leq 1.$$

Obviously, PFS is more effective than IFS in modeling the vagueness of the practical Multi-Criteria Decision-Making (MCDM) problems.

The PFS has been investigated from different perspectives, including decision-making technologies [5-13], aggregation operators [14-22], information measures [23-25], the extensions of PFS [26-31], and fundamental properties [32-34]. In particular, an extension of PFS, named Interval-Valued Pythagorean Fuzzy Set (IVPFS) [10,21], is a hot topic at present [35].

Peng and Yang [21] proposed some new operations ($\cup, \cap, \oplus, \otimes$) for IVPFS and discussed their properties

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in detail. Meanwhile, they studied two interval-valued Pythagorean fuzzy aggregation operators for integrating the interval-valued Pythagorean fuzzy information, such as IVPFWA and IVPFWG operators, and presented an interval-valued Pythagorean fuzzy elimination and choice translating reality method (ELECTRE) to solve Multi-Criteria Group Decision Making (MCGDM) problem with uncertainty. Liang et al. [17] conceived the maximizing deviation method based on interval-valued Pythagorean fuzzy weighted aggregating operator for MCGDM problem. Garg [36] introduced a novel interval-valued Pythagorean fuzzy accuracy function for solving MCDM problem. Rahman et al. [37] discussed interval-valued Pythagorean fuzzy geometric aggregation operators and their application to MCGDM problem. Garg [38] proposed a new improved score function of an interval-valued Pythagorean fuzzy set-based TOPSIS method. Chen [39] pioneered the IVPF outranking algorithm with a closeness-based assignment model for MCDM.

In order to enrich the operations of IVPFS, we define some new interval-valued Pythagorean fuzzy operators ($\diamondsuit, \square, \spadesuit, \clubsuit, ✠, \rightarrow, \$$) and discuss their properties with some existing operators ($\cup, \cap, \oplus, \otimes$) in detail.

To facilitate our discussion, the remainder of this paper is organized as follows. In Section 2, we review some fundamental conceptions of IVIFS and IVPFS. In Section 3, we propose some new operations for IVPFS and present some interesting properties. Meanwhile, some new operations and the existing operators are compared in detail. In Section 4, a new decision-making method based on $\spadesuit$ operator is proposed and a comparison is constructed. The paper is concluded in Section 5.

2. Preliminaries

This section presents the basic notions, definitions, and properties of IVIFS and IVPFS.

Definition 1. [40] Let Int([0, 1]) denote the set of all closed subintervals of [0, 1]. Let X be a universe of discourse. An IVIFS $\widetilde{I}$ in X is given by:

$$\widetilde{I} = \left\{ < x, \mu_{\widetilde{I}}(x), \nu_{\widetilde{I}}(x) > \mid x \in X \right\}, \tag{1}$$

where the functions $\mu_{\widetilde{I}} : X \to \text{Int}([0,1])(x \in X \to \mu_{\widetilde{I}}(x) \subseteq [0,1])$ and $\nu_{\widetilde{I}} : X \to \text{Int}([0,1])(x \in X \to \nu_{\widetilde{I}}(x) \subseteq [0,1])$ denote the membership degree and non-membership degree of the element $x \in X$ to set $\widetilde{I}$, respectively, and for every $x \in X$, $0 \leq \sup\{\mu_{\widetilde{I}}(x)\} + \sup\{\nu_{\widetilde{I}}(x)\} \leq 1$. Also, for each $x \in X$, $\mu_{\widetilde{I}}(x)$ and $\nu_{\widetilde{I}}(x)$ are closed intervals and their lower and upper bounds are denoted by $\mu^-_{\widetilde{I}}(x), \mu^+_{\widetilde{I}}(x), \nu^-_{\widetilde{I}}(x), \nu^+_{\widetilde{I}}(x)$, respectively. Therefore, $\widetilde{I}$ can also express another style as follows:

$$\widetilde{I} = \{< x, [\mu^-_{\widetilde{I}}(x), \mu^+_{\widetilde{I}}(x)], [\nu^-_{\widetilde{I}}(x), \nu^+_{\widetilde{I}}(x)] > | x \in X\}, \tag{2}$$

whose expression is subject to the condition $0 \leq \mu^+_{\widetilde{I}}(x) + \nu^+_{\widetilde{I}}(x) \leq 1$. The degree of indeterminacy $\pi_{\widetilde{I}}(x) = [\pi^-_{\widetilde{I}}(x), \pi^+_{\widetilde{I}}(x)] = [1 - \mu^+_{\widetilde{I}}(x) - \nu^+_{\widetilde{I}}(x), 1 - \mu^-_{\widetilde{I}}(x) - \nu^-_{\widetilde{I}}(x)]$. For convenience, Xu [41] called $\tilde{i} = ([\mu^-_{\tilde{i}}, \mu^+_{\tilde{i}}][\nu^-_{\tilde{i}}, \nu^+_{\tilde{i}}])$ an Interval-Valued Intuitionistic Fuzzy Number (IVIFN).

Definition 2. [10] Let Int([0, 1]) denote the set of all closed subintervals of [0, 1], and X be a universe of discourse. An IVPFS $\widetilde{P}$ in X is given by:

$$\widetilde{P} = \{< x, \mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x) > \mid x \in X\}, \tag{3}$$

whose functions $\mu_{\widetilde{P}} : X \to \text{Int}([0,1])(x \in X \to \mu_{\widetilde{P}}(x) \subseteq [0,1])$ and $\nu_{\widetilde{P}} : X \to \text{Int}([0,1])(x \in X \to \nu_{\widetilde{P}}(x) \subseteq [0,1])$ denote the membership degree and non-membership degree of the element $x \in X$ to the set $\widetilde{P}$, respectively, and for every $x \in X$, $0 \leq \sup\{(\mu_{\widetilde{P}}(x))^2\} + \sup\{(\mu_{\widetilde{P}}(x))^2\} \leq 1$. Also, for each $x \in X$, $\mu_{\widetilde{P}}(x)$ and $\nu_{\widetilde{P}}(x)$ are closed intervals and their lower and upper bounds are denoted by $\mu^-_{\widetilde{P}}(x), \mu^+_{\widetilde{P}}(x), \nu^-_{\widetilde{P}}(x)$, and $\nu^+_{\widetilde{P}}(x)$, respectively. So, $\widetilde{P}$ can also be expressed in another style as follows:

$$\widetilde{P} = \{< x, [\mu^-_{\widetilde{P}}(x), \mu^+_{\widetilde{P}}(x)], [\nu^-_{\widetilde{P}}(x), \nu^+_{\widetilde{P}}(x)] > | x \in X\}, \tag{4}$$

whose expression is subject to the condition $0 \leq (\mu^+_{\widetilde{P}}(x))^2 + (\nu^+_{\widetilde{P}}(x))^2 \leq 1$. The degree of indeterminacy $\pi_P(x)$ is shown as follows:

$$\pi_{\widetilde{P}}(x) = [\pi^-_{\widetilde{P}}(x), \pi^+_{\widetilde{P}}(x)]$$

$$= \left[\sqrt{1 - (\mu^+_{\widetilde{P}}(x))^2 - (\nu^+_{\widetilde{P}}(x))^2}, \sqrt{1 - (\mu^-_{\widetilde{P}}(x))^2 - (\nu^-_{\widetilde{P}}(x))^2}\right].$$

For convenience, Zhang [10] called:

$$\widetilde{p} = ([\mu^-_{\widetilde{p}}, \mu^+_{\widetilde{p}}], [\nu^-_{\widetilde{p}}, \nu^+_{\widetilde{p}}]),$$

an Interval-Valued Pythagorean Fuzzy Number (IVPFN). It is easily known that IVPFS reduces to PFS when the boundary is the same.

The main difference between IVPFNs and IVIFNs is their corresponding constraint cases, as easily shown in Figure 1 [10].

Definition 3. [21] Let:

$$\widetilde{p}_1 = ([\mu^-_1, \mu^+_1], [\nu^-_1, \nu^+_1]),$$

and:

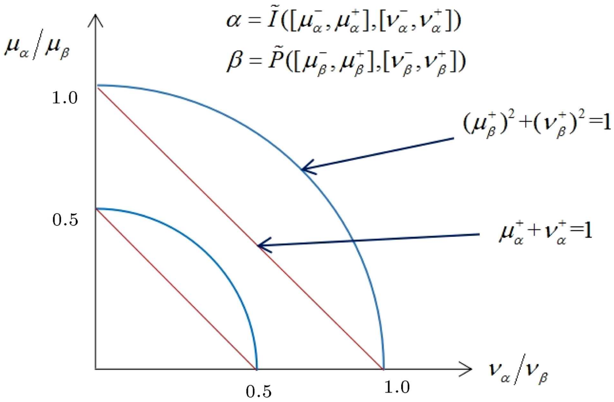


Figure 1. Comparison of spaces of the IVPFNs and IVIFNs.

$$\tilde{p}_2 = ([\mu_2^-, \mu_2^+], [\nu_2^-, \nu_2^+]),$$

be two IVPFNs, and then their relations are defined as follows:

1. $\tilde{p}_1 = \tilde{p}_2$ iff $\mu_1^- = \mu_2^-, \mu_1^+ = \mu_2^+, \nu_1^- = \nu_2^-$, and $\nu_1^+ = \nu_2^+$;
2. $\tilde{p}_1 \prec \tilde{p}_2$ iff $\mu_1^- \le \mu_2^-, \mu_1^+ \le \mu_2^+, \nu_1^- \ge \nu_2^-$, and $\nu_1^+ \ge \nu_2^+$.

Definition 4. [21] For any IVPFN $\tilde{p} = ([\mu_{\tilde{p}}^-, \mu_{\tilde{p}}^+], [\nu_{\tilde{p}}^-, \nu_{\tilde{p}}^+])$, the score function of $\tilde{p}$ is defined as follows:

$$s(\tilde{p}) = \frac{1}{2}[(\mu_{\tilde{p}}^-)^2 + (\mu_{\tilde{p}}^+)^2 - (\nu_{\tilde{p}}^-)^2 - (\nu_{\tilde{p}}^+)^2],$$

$$s(\tilde{p}) \in [-1, 1]. \tag{5}$$

Definition 5. [21] For any IVPFN $\tilde{p} = ([\mu_{\tilde{p}}^-, \mu_{\tilde{p}}^+], [\nu_{\tilde{p}}^-, \nu_{\tilde{p}}^+])$, the accuracy function of $\tilde{p}$ is defined as follows:

$$a(\tilde{p}) = \frac{1}{2}[(\mu_{\tilde{p}}^-)^2 + (\mu_{\tilde{p}}^+)^2 + (\nu_{\tilde{p}}^-)^2 + (\nu_{\tilde{p}}^+)^2],$$

$$a(\tilde{p}) \in [0, 1]. \tag{6}$$

For any two IVPFNs, $\tilde{p}_1, \tilde{p}_2$, the comparison rule is defined as follows:

1. if $s(\tilde{p}_1) > s(\tilde{p}_2)$, then $\tilde{p}_1 \succ \tilde{p}_2$;
2. if $s(\tilde{p}_1) = s(\tilde{p}_2)$, then:
   (a) if $a(\tilde{p}_1) > a(\tilde{p}_2)$, then $\tilde{p}_1 \succ \tilde{p}_2$;
   (b) if $a(\tilde{p}_1) = a(\tilde{p}_2)$, then $\tilde{p}_1 = \tilde{p}_2$.

3. Interval-valued Pythagorean fuzzy operators

This section reviews some existing interval-valued Pythagorean fuzzy operators and proposes some new interval-valued Pythagorean fuzzy operators.

3.1. Some existing interval-valued Pythagorean fuzzy operators

Definition 6. [21] Let $\tilde{P}, \tilde{P}_1$, and $\tilde{P}_2$, be three IVPFSs, and $\lambda > 0$. Then, their operations are defined as follows:

1. $\tilde{P}_1 \cup \tilde{P}_2 = \{< x, [\max\{\mu_1^-(x), \mu_2^-(x)\}, \max\{\mu_1^+(x), \mu_2^+(x)\}], [\min\{\nu_1^-(x), \nu_2^-(x)\}, \min\{\nu_1^+(x), \nu_2^+(x)\}] >| x \in X\}$;
2. $\tilde{P}_1 \cap \tilde{P}_2 = \{< x, [\min\{\mu_1^-(x), \mu_2^-(x)\}, \min\{\mu_1^+(x), \mu_2^+(x)\}], [\max\{\nu_1^-(x), \nu_2^-(x)\}, \max\{\nu_1^+(x), \nu_2^+(x)\}] >| x \in X\}$;
3. $\tilde{P}_1 \oplus \tilde{P}_2 = \Big\{ < x, \Big[\sqrt{(\mu_1^-(x))^2 + (\mu_2^-(x))^2 - (\mu_1^-(x))^2(\mu_2^-(x))^2}, \sqrt{(\mu_1^+(x))^2 + (\mu_2^+(x))^2 - (\mu_1^+(x))^2(\mu_2^+(x))^2}\Big], \Big[\nu_1^-(x)\nu_2^-(x), \nu_1^+(x)\nu_2^+(x)\Big] > \Big| x \in X \Big\}$;
4. $\tilde{P}_1 \otimes \tilde{P}_2 = \Big\{ < x, \left[\mu_1^-(x)\mu_2^-(x), \mu_1^+(x)\mu_2^+(x)\right], \Big[\sqrt{(\nu_1^-(x))^2 + (\nu_2^-(x))^2 - (\nu_1^-(x))^2(\nu_2^-(x))^2}, \sqrt{(\nu_1^+(x))^2 + (\nu_2^+(x))^2 - (\nu_1^+(x))^2(\nu_2^+(x))^2}\Big] \Big| x \in X \Big\}$;
5. $\lambda\tilde{P} = \Big\{ < x, \Big[\sqrt{1 - (1 - (\mu_P^-(x))^2)^\lambda}, \sqrt{1 - (1 - (\mu_P^+(x))^2)^\lambda}\Big], [(\nu_P^-(x))^\lambda, (\nu_P^+(x))^\lambda] >| x \in X \Big\}$;
6. $\tilde{P}^\lambda = \Big\{ < x, [(\mu_P^-(x))^\lambda, (\mu_P^+(x))^\lambda], \Big[\sqrt{1 - (1 - (\nu_P^-(x))^2)^\lambda}, \sqrt{1 - (1 - (\nu_P^+(x))^2)^\lambda}\Big] \Big| x \in X \Big\}$;
7. $P_1 \subseteq P_2$ iff $\mu_1^-(x) \le \mu_2^-(x), \mu_1^+(x) \le \mu_2^+(x), \nu_1^-(x) \ge \nu_2^-(x), \nu_1^+(x) \ge \nu_2^+(x)$, for $\forall x \in X$;
8. $P^c = \{< x, [\nu_P^-(x), \nu_P^+(x)], [\mu_P^-(x), \mu_P^+(x)] >| x \in X\}$.

3.2. Some new interval-valued Pythagorean fuzzy operators

Definition 7. The necessity operation on an IVPFS P is denoted by $\Box P$ and is defined as:

$$\Box P = \left\{ \left\langle x, [\mu_P^-(x), \mu_P^+(x)], \left[\sqrt{1 - (\mu_P^+(x))^2}, \sqrt{1 - (\mu_P^-(x))^2}\right] \right\rangle \middle| x \in X \right\}.$$

Example 1. Let P be an IVPFS over X such that:

$$P=\{<x_1,[0.2,0.3],[0.4,0.6]>,$$

$$<x_2,[0.2,0.4],[0.5,0.8]>,$$

$$<x_3,[0.4,0.5],[0.6,0.7]>,$$

$$<x_4,[0.5,0.6],[0.7,0.8]>\}.$$

Then, we can easily compute $\Box P$ by the above definition shown as follows:

$$\Box P=\{<x_1,[0.2,0.3],[0.9539,0.9798]>,$$

$$<x_2,[0.2,0.4],[0.9165,0.9798]>,$$

$$<x_3,[0.4,0.5],[0.8660,0.9165]>,$$

$$<x_4,[0.5,0.6],[0.8,0.8660]>\}.$$

Definition 8. The possibility operation on an IVPFS P is denoted by $\Diamond P$ and is defined as:

$$\Diamond P=\Bigg\{<x,[\sqrt{1-(\nu_P^+(x))^2},\sqrt{1-(\nu_P^-(x))^2}],$$

$$[\nu_P^-(x),\nu_P^+(x)]>|\, x\in X\Bigg\}.$$

Example 2. If we continue to use Example 1, then we can easily compute $\Diamond P$ by above definition shown as follows:

$$\Diamond P=\{<x_1,[0.2,0.3],[0.8000,0.9165]>,$$

$$<x_2,[0.2,0.4],[0.6000,0.8660]>,$$

$$<x_3,[0.4,0.5],[0.7141,0.8000]>,$$

$$<x_4,[0.5,0.6],[0.6000,0.7141]>\}.$$

Two new relations are defined by $P\subseteq_\Box Q$ iff $\mu_P^-(x)\leq\mu_Q^-(x),\mu_P^+(x)\leq\mu_Q^+(x)$ for $\forall x\in X$ and $P\subseteq_\Diamond Q$ iff $\nu_P^-(x)\geq\nu_Q^-(x),\nu_P^+(x)\geq\nu_Q^+(x)$ for $\forall x\in X$.

Definition 9. Let P and Q be two IVPFSs. For two IVPFSs P and Q, the operations are defined as follows:

1. $P\spadesuit Q=$
$$\left\{\left\langle x,\left[\sqrt{\frac{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}{2}},\sqrt{\frac{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}{2}}\ \right],\left[\sqrt{\frac{(\nu_P^-(x))^2+(\nu_Q^-(x))^2}{2}},\sqrt{\frac{(\nu_P^+(x))^2+(\nu_Q^+(x))^2}{2}}\ \right]\right\rangle\middle| x\in X\right\};$$
2. $P\to Q=$
$$\left\{\left\langle x,\left[\max\left\{\nu_P^-(x),\mu_Q^-(x)\right\},\max\left\{\nu_P^+(x),\mu_Q^+(x)\right\}\right],\left[\min\left\{\mu_P^-(x),\nu_Q^-(x)\right\},\min\left\{\mu_P^+(x),\nu_Q^+(x)\right\}\right]\right\rangle\middle| x\in X\right\};$$
3. $P\$Q=$
$$\left\{<x,\left[\sqrt{\mu_P^-(x)\mu_Q^-(x)},\sqrt{\mu_P^+(x)\mu_Q^+(x)}\right],\left[\sqrt{\nu_P^-(x)\nu_Q^-(x)},\sqrt{\nu_P^+(x)\nu_Q^+(x)}\right]>|x\in X\right\};$$
4. $P\clubsuit Q=$
$$\left\{\left\langle x,\left[\frac{\sqrt{2}\mu_P^-(x)\mu_Q^-(x)}{\sqrt{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}},\frac{\sqrt{2}\mu_P^+(x)\mu_Q^+(x)}{\sqrt{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}}\right],\left[\frac{\sqrt{2}\nu_P^-(x)\nu_Q^-(x)}{\sqrt{(\nu_P^-(x))^2+(\nu_Q^-(x))^2}},\frac{\sqrt{2}\nu_P^+(x)\nu_Q^+(x)}{\sqrt{(\nu_P^+(x))^2+(\nu_Q^+(x))^2}}\right]\right\rangle\middle| x\in X\right\};$$
5. $P\maltese Q=$
$$\left\{\left\langle x,\left[\sqrt{\frac{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}{2((\mu_P^-(x))^2+(\mu_Q^-(x))^2+1)}},\sqrt{\frac{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}{2((\mu_P^+(x))^2+(\mu_Q^+(x))^2+1)}}\right],\left[\sqrt{\frac{(\nu_P^-(x))^2+(\nu_Q^-(x))^2}{2((\nu_P^-(x))^2+(\nu_Q^-(x))^2+1)}},\sqrt{\frac{(\nu_P^+(x))^2+(\nu_Q^+(x))^2}{2((\nu_P^+(x))^2+(\nu_Q^+(x))^2+1)}}\right]\right\rangle\middle| x\in X\right\}.$$

Example 3. Let P and Q be two IVPFSs over X such that:

$$P=\{<x_1,[0.2,0.3],[0.4,0.6]>,$$

$$<x_2,[0.2,0.4],[0.5,0.8]>,$$

$$<x_3,[0.4,0.5],[0.6,0.7]>,$$

$$<x_4,[0.5,0.6],[0.7,0.8]>\},$$

and:

$$Q=\{<x_1,[0.2,0.4],[0.4,0.9]>,$$

$$<x_2,[0.3,0.4],[0.5,0.8]>,$$

$$<x_3,[0.5,0.6],[0.6,0.7]>,$$

$$<x_4,[0.5,0.7],[0.6,0.7]>\}.$$

Then, we can easily compute $P\spadesuit Q$, $P\to Q$, $P\$Q$, $P\clubsuit Q$ and $P\maltese Q$ by the above definitions shown in Table 1.

Lemma 1. For any three numbers $x,y,z\in[0,1]$, then:

$$z^2(2-x^2-y^2-z^2)+x^2y^2\geq 0. \tag{7}$$

Proof

(1) If $xy\geq z^2$, then:

$$z^2(2-x^2-y^2-z^2)+x^2y^2=z^2(2-x^2-y^2)$$

$$-z^4+x^2y^2\geq -w^4+x^2y^2\geq 0.$$

(2) If $xy\leq z^2$, then:

Table 1. The results of $\spadesuit, \rightarrow, \$, \clubsuit, \maltese$ operations on P and Q.

Operations	Results
$P\spadesuit Q$	$\{< x_1, [0.2, 0.3536], [0.4, 0.7649] >, < x_2, [0.2550, 0.4], [0.5, 0.8] >,$ $< x_3, [0.4528, 0.5523], [0.6, 0.7] >, < x_4, [0.5, 0.6519], [0.6519, 0.7517] >\}$
$P \rightarrow Q$	$\{< x_1, [0.4, 0.6], [0.2, 0.3] >, < x_2, [0.5, 0.8], [0.2, 0.4] >,$ $< x_3, [0.6, 0.7], [0.4, 0.5] >, < x_4, [0.7, 0.8], [0.5, 0.6] >\}$
$P\$Q$	$\{< x_1, [0.2, 0.3464], [0.4, 0.7348] >, < x_2, [0.2449, 0.4], [0.5, 0.8] >,$ $< x_3, [0.4472, 0.5477], [0.6, 0.7] >, < x_4, [0.5, 0.6481], [0.6481, 0.7483] >\}$
$P\clubsuit Q$	$\{< x_1, [0.2, 0.3394], [0.4, 0.7060] >, < x_2, [0.2353, 0.4], [0.5, 0.8] >,$ $< x_3, [0.4417, 0.5432], [0.6, 0.7] >, < x_4, [0.5, 0.6443], [0.6443, 0.7450] >\}$
$P\maltese Q$	$\{< x_1, [0.1925, 0.3162], [0.3482, 0.5192] >, < x_2, [0.2398, 0.3482], [0.4082, 0.5298] >,$ $< x_3, [0.3813, 0.4352], [0.4575, 0.4975] >, < x_4, [0.4082, 0.4793], [0.4793, 0.5150] >\}$

$$z^2(2-x^2-y^2-z^2)+x^2y^2 \geq xy(2-x^2-y^2-z^2)$$

$$+x^2y^2 = xy(1-x^2-y^2+1-z^2+xy)$$

$$\geq xy(1-x^2-y^2+xy)$$

$$=\begin{cases} xy(1-x^2+y(x-y)) \geq 0, & \text{iff } x \geq y \\ xy(1-y^2+x(y-x)) \geq 0, & \text{iff } x \leq y \end{cases}$$

Combining (1) with (2), we can have $z^2(2-x^2-y^2-z^2)+x^2y^2 \geq 0$.

The proof is completed.

Lemma 2. For any two numbers $x, y \in [0,1]$:

$$0 \leq x+y-2xy \leq 1. \tag{8}$$

Theorem 1. Let X be a nonempty set. For IVPFSs P, Q, and R in X:

(1) $(P \oplus Q)\spadesuit R \subseteq (P\spadesuit R) \oplus (Q\spadesuit R)$;

(2) $(P \otimes Q)\spadesuit R \supseteq (P\spadesuit R) \otimes Q\spadesuit R)$;

(3) $(P \oplus Q) \rightarrow R \subseteq (P \rightarrow R) \oplus (Q \rightarrow R)$;

(4) $(P \otimes Q) \rightarrow R \supseteq (P \rightarrow R) \otimes Q \rightarrow R)$;

(5) $(P \oplus Q)\$R \subseteq (P\$R) \oplus (Q\$R)$;

(6) $(P \otimes Q)\$R \supseteq (P\$R) \otimes Q\$R)$;

(7) $(P \oplus Q)\maltese R \subseteq (P\maltese R) \oplus (Q\maltese R)$;

(8) $(P \otimes Q)\maltese R \supseteq (P\maltese R) \otimes Q\maltese R)$.

Proof. We only prove (1) and (3) in detail. Statements (2), (4)-(8) can be proved in a similar way.

1. Let P, Q, and R be three given IVPFSs, and then $(P \oplus Q)\spadesuit R$ and $(P\spadesuit R) \oplus (Q\spadesuit R)$ are calculated as shown in Box I.

   Let $f(x)$ be calculated as shown in Box II. According to Lemma 1, we can have $f(x) \geq 0$.

   Furthermore, we can have the relation shown in Box III. Let $g(x)$ be calculated as shown in Box IV. According to Lemma 1, we can have $g(x) \geq 0$. Furthermore, we can have:

$$\sqrt{\frac{(\nu_P^-(x))^2(\nu_Q^-(x))^2+(\nu_R^-(x))^2}{2}}$$

$$\geq \sqrt{\frac{(\nu_P^-(x))^2+(\nu_R^-(x))^2}{2} * \frac{(\nu_Q^-(x))^2+(\nu_R^-(x))^2}{2}}.$$

   Similarly we can have the relation shown in Box V. According to Eq. (5), we can have $(P \oplus Q)\spadesuit R \subseteq (P\spadesuit R) \oplus (Q\spadesuit R)$.

3. Let P, Q, and R be three given IVPFSs, then $(P \oplus Q) \rightarrow R$ and $(P \rightarrow R) \oplus (Q \rightarrow R)$ be calculated as shown in Box VI.

   Let $f(x)$ be calculated as shown in Box VII, then we can discuss the four cases in the following:

   Case 1: If $\nu_P^-(x) \geq \nu_Q^-(x) \geq \mu_R^-(x)$ or $\nu_Q^-(x) \geq \nu_P^-(x) \geq \mu_R^-(x)$, then:

   Case 1.1: If $\nu_P^-(x)\nu_Q^-(x) \geq \mu_R^-(x)$, then:

$$f(x) = (\nu_P^-(x))^2(\nu_Q^-(x))^2 - ((\nu_P^-(x))^2$$

$$+(\nu_Q^-(x))^2 - (\nu_P^-(x))^2(\nu_Q^-(x))^2)$$

$$=2(\nu_P^-(x))^2(\nu_Q^-(x))^2 - (\nu_P^-(x))^2$$

$$-(\nu_Q^-(x))^2 \leq 0 \quad \text{(Lemma 2)}.$$

$$(P \oplus Q)\spadesuit R = \left\{ \left\langle x, \left[\sqrt{\frac{\left(\mu_P^-(x)\right)^2 + \left(\mu_Q^-(x)\right)^2 - \left(\mu_P^-(x)\right)^2 \left(\mu_Q^-(x)\right)^2 + \left(\mu_R^-(x)\right)^2}{2}}, \sqrt{\frac{\left(\mu_P^+(x)\right)^2 + \left(\mu_Q^+(x)\right)^2 - \left(\mu_P^+(x)\right)^2 (\mu_Q^+(x))^2 + \left(\mu_R^+(x)\right)^2}{2}} \right], \left[\sqrt{\frac{\left(\nu_P^-(x)\right)^2 \left(\nu_Q^-(x)\right)^2 + \left(\nu_R^-(x)\right)^2}{2}}, \sqrt{\frac{\left(\nu_P^+(x)\right)^2 \left(\nu_Q^+(x)\right)^2 + \left(\nu_R^+(x)\right)^2}{2}} \right] \right\rangle \middle| x \in X \right\},$$

$$(P\spadesuit R) \oplus (Q\spadesuit R)$$

$$= \left\{ \left\langle x, \left[\sqrt{\frac{(\mu_P^-(x))^2+(\mu_R^-(x))^2}{2} + \frac{(\mu_Q^-(x))^2+(\mu_R^-(x))^2}{2} - \frac{(\mu_P^-(x))^2+(\mu_R^-(x))^2}{2} * \frac{(\mu_Q^-(x))^2+(\mu_R^-(x))^2}{2}}, \sqrt{\frac{(\mu_P^+(x))^2 + (\mu_R^+(x))^2}{2} + \frac{(\mu_Q^+(x))^2 + (\mu_R^+(x))^2}{2} - \frac{(\mu_P^+(x))^2 + (\mu_R^+(x))^2}{2} * \frac{(\mu_Q^+(x))^2 + (\mu_R^+(x))^2}{2}} \right], \left[\sqrt{\frac{(\nu_P^-(x))^2+(\nu_R^-(x))^2}{2} * \frac{(\nu_Q^-(x))^2+(\nu_R^-(x))^2}{2}}, \sqrt{\frac{(\nu_P^+(x))^2+(\nu_R^+(x))^2}{2} * \frac{(\nu_Q^+(x))^2+(\nu_R^+(x))^2}{2}} \right] \right\rangle \middle| x \in X \right\}.$$

Box I

$$f(x) = \left(\sqrt{\frac{(\mu_P^-(x))^2 + (\mu_R^-(x))^2}{2} + \frac{(\mu_Q^-(x))^2 + (\mu_R^-(x))^2}{2} - \frac{(\mu_P^-(x))^2 + (\mu_R^-(x))^2}{2} * \frac{(\mu_Q^-(x))^2 + (\mu_R^-(x))^2}{2}} \right)^2 - \left(\sqrt{\frac{\left(\mu_P^-(x)\right)^2 + \left(\mu_Q^-(x)\right)^2 - \left(\mu_P^-(x)\right)^2 \left(\mu_Q^-(x)\right)^2 + \left(\mu_R^-(x)\right)^2}{2}} \right)^2$$

$$= \frac{\left(\mu_P^-(x)\right)^2 \left(\mu_Q^-(x)\right)^2 + \left(\mu_R^-(x)\right)^2 \left(2 - (\mu_P^-(x))^2 - (\mu_Q^-(x))^2 - (\mu_R^-(x))^2\right)}{2}.$$

Box II

$$\sqrt{\frac{(\mu_P^-(x))^2 + (\mu_R^-(x))^2}{2} + \frac{(\mu_Q^-(x))^2 + (\mu_R^-(x))^2}{2} - \frac{(\mu_P^-(x))^2 + (\mu_R^-(x))^2}{2} * \frac{(\mu_Q^-(x))^2 + (\mu_R^-(x))^2}{2}}$$

$$\geq \sqrt{\frac{(\mu_P^-(x))^2 + (\mu_Q^-(x))^2 - (\mu_P^-(x))^2(\mu_Q^-(x))^2 + (\mu_R^-(x))^2}{2}}.$$

Box III

$$g(x)=\left(\sqrt{\frac{(\nu_P^-(x))^2(\nu_Q^-(x))^2+(\nu_R^-(x))^2}{2}}\right)^2-\left(\sqrt{\frac{(\nu_P^-(x))^2+(\nu_R^-(x))^2}{2}*\frac{(\nu_Q^-(x))^2+(\nu_R^-(x))^2}{2}}\right)^2$$
$$=\frac{(\nu_P^-(x))^2(\nu_Q^-(x))^2+(\nu_R^-(x))^2(2-(\nu_P^-(x))^2-(\nu_Q^-(x))^2-(\nu_R^-(x))^2)}{2}.$$

Box IV

$$\sqrt{\frac{(\mu_P^+(x))^2+(\mu_R^+(x))^2}{2}+\frac{(\mu_Q^+(x))^2+(\mu_R^+(x))^2}{2}-\frac{(\mu_P^+(x))^2+(\mu_R^+(x))^2}{2}*\frac{(\mu_Q^+(x))^2+(\mu_R^+(x))^2}{2}}$$
$$\geq\sqrt{\frac{(\mu_P^+(x))^2+(\mu_Q^+(x))^2-(\mu_P^+(x))^2(\mu_Q^+(x))^2+(\mu_R^+(x))^2}{2}},\sqrt{\frac{(\nu_P^+(x))^2(\nu_Q^+(x))^2+(\nu_R^+(x))^2}{2}}$$
$$\geq\sqrt{\frac{(\nu_P^+(x))^2+(\nu_R^+(x))^2}{2}*\frac{(\nu_Q^+(x))^2+(\nu_R^+(x))^2}{2}}.$$

Box V

$$(P\oplus Q)\to R=\left\{\left\langle x,\left[\max\left\{\nu_P^-(x)\nu_Q^-(x),\mu_R^-(x)\right\},\max\left\{\nu_P^+(x)\nu_Q^+(x),\mu_R^+(x)\right\}\right],\right.\right.$$
$$\left[\min\left\{\sqrt{(\mu_P^-(x))^2+(\mu_Q^-(x))^2-(\mu_P^-(x))^2(\mu_Q^-(x))^2},\nu_R^-(x)\right\},\right.$$
$$\left.\left.\left.\min\left\{\sqrt{(\mu_P^+(x))^2+(\mu_Q^+(x))^2-(\mu_P^+(x))^2(\mu_Q^+(x))^2},\nu_R^+(x)\right\}\right]\right\rangle\right|x\in X\right\},$$
$$(P\to R)\oplus(Q\to R)=\left\{\left\langle x,\left[\sqrt{\begin{array}{l}\max\{(\nu_P^-(x))^2,(\mu_R^-(x))^2\}+\max\{(\nu_Q^-(x))^2,(\mu_R^-(x))^2\}\\ \quad-\max\{(\nu_P^-(x))^2,(\mu_R^-(x))^2\}*\max\{(\nu_Q^-(x))^2,(\mu_R^-(x))^2\}\end{array}},\right.\right.\right.$$
$$\left.\sqrt{\begin{array}{l}\max\{(\nu_P^+(x))^2,(\mu_R^+(x))^2\}+\max\{(\nu_Q^+(x))^2,(\mu_R^+(x))^2\}-\max\left\{(\nu_P^+(x))^2,(\mu_R^+(x))^2\right\}\\ \quad*\max\left\{(\nu_Q^+(x))^2,(\mu_R^+(x))^2\right\}\end{array}}\right],$$
$$\left.\left.[\min\{\mu_P^-(x),\nu_R^-(x)\}*\min\{\mu_Q^-(x),\nu_R^-(x)\},\min\{\mu_P^+(x),\nu_R^+(x)\}*\min\{\mu_Q^+(x),\nu_R^+(x)\}\ ]\right\rangle\right|x\in X\right\}.$$

Box VI

$$f(x)=\left(\max\{\nu_P^-(x)\nu_Q^-(x),\mu_R^-(x)\}\right)^2-\left(\sqrt{\begin{array}{l}\max\{(\nu_P^-(x))^2,(\mu_R^-(x))^2\}+\max\{(\nu_Q^-(x))^2,(\mu_R^-(x))^2\}\\ \quad-\max\{(\nu_P^-(x))^2,(\mu_R^-(x))^2\}*\max\{(\nu_Q^-(x))^2,(\mu_R^-(x))^2\}\end{array}}\right)^2$$
$$=\max\{\nu_P^-(x)\nu_Q^-(x),\mu_R^-(x)\}-(\max\{(\nu_P^-(x))^2,(\mu_R^-(x))^2\}+\max\{(\nu_Q^-(x))^2,(\mu_R^-(x))^2\}$$
$$-\max\{(\nu_P^-(x))^2,(\mu_R^-(x))^2\}*\max\{(\nu_Q^-(x))^2,(\mu_R^-(x))^2\}).$$

Box VII

Case 1.2: If $\nu_P^-(x)\nu_Q^-(x) \leq \mu_R^-(x)$, then:

$$\begin{aligned} f(x) =&(\mu_R^-(x))^2 - ((\nu_P^-(x))^2 + (\nu_Q^-(x))^2 \\ &- (\nu_P^-(x))^2(\nu_Q^-(x))^2) \\ =&((\mu_R^-(x))^2 - (\nu_P^-(x))^2) \\ &+ (\nu_Q^-(x))^2((\nu_P^-(x))^2 - 1) \\ &\leq 0((\mu_R^-(x))^2 - (\nu_P^-(x))^2 \\ &\leq 0, (\nu_P^-(x))^2 - 1 \leq 0). \end{aligned}$$

Case 2: If $\nu_P^-(x) \geq \mu_R^-(x) \geq \nu_Q^-(x)$, then:

Case 2.1: If $\nu_P^-(x)\nu_Q^-(x) \geq \mu_R^-(x)$, then:

$$\begin{aligned} f(x) =&(\nu_P^-(x))^2(\nu_Q^-(x))^2 - ((\nu_P^-(x))^2 \\ &+ (\mu_R^-(x))^2 - (\nu_P^-(x))^2(\mu_R^-(x))^2) \\ =&(\nu_P^-(x))^2((\nu_Q^-(x))^2 - 1) \\ &+ (\mu_R^-(x))^2((\nu_P^-(x))^2 - 1) \leq 0. \end{aligned}$$

Case 2.2: If $\nu_P^-(x)\nu_Q^-(x) \leq \mu_R^-(x)$, then:

$$\begin{aligned} f(x) =&(\mu_R^-(x))^2 - ((\nu_P^-(x))^2 + (\mu_R^-(x))^2 \\ &- (\nu_P^-(x))^2(\mu_R^-(x))^2) \\ =&(\nu_P^-(x))^2((\mu_R^-(x))^2 - 1) \leq 0. \end{aligned}$$

Case 3: If $\nu_Q^-(x) \geq \mu_R^-(x) \geq \nu_P^-(x)$, then:

Case 3.1: If $\nu_P^-(x)\nu_Q^-(x) \geq \mu_R^-(x)$, then:

$$\begin{aligned} f(x) =&(\nu_P^-(x))^2(\nu_Q^-(x))^2 - ((\mu_R^-(x))^2 \\ &+ (\nu_Q^-(x))^2 - (\mu_R^-(x))^2(\nu_Q^-(x))^2) \\ =&(\nu_Q^-(x))^2((\nu_P^-(x))^2 - 1) \\ &+ (\mu_R^-(x))^2((\nu_Q^-(x))^2 - 1) \leq 0. \end{aligned}$$

Case 3.2: If $\nu_P^-(x)\nu_Q^-(x) \leq \mu_R^-(x)$, then:

$$\begin{aligned} f(x) =&(\mu_R^-(x))^2 - ((\mu_R^-(x))^2 + (\nu_Q^-(x))^2 \\ &- (\mu_R^-(x))^2(\nu_Q^-(x))^2) \\ =&(\nu_Q^-(x))^2((\mu_R^-(x))^2 - 1) \leq 0. \end{aligned}$$

Case 4: If $\mu_R^-(x) \geq \nu_P^-(x) \geq \nu_Q^-(x)$ or $\mu_R^-(x) \geq \nu_Q^-(x) \geq \nu_P^-(x)$, then:

Case 4.1: If $\nu_P^-(x)\nu_Q^-(x) \geq \mu_R^-(x)$, then:

$$\begin{aligned} f(x) =& (\nu_P^-(x))^2(\nu_Q^-(x))^2 - ((\mu_R^-(x))^2 \\ &+ (\mu_R^-(x))^2 - (\mu_R^-(x))^2(\mu_R^-(x))^2) \\ =& (\nu_P^-(x))^2(\nu_Q^-(x))^2 + (\mu_R^-(x))^2(\mu_R^-(x))^2 \\ &- 2(\mu_R^-(x))^2 \leq (\mu_R^-(x))^2(\mu_R^-(x))^2 \\ &+ (\mu_R^-(x))^2(\mu_R^-(x))^2 - 2(\mu_R^-(x))^2(\nu_P^-(x) \\ &\leq \mu_R^-(x), \nu_Q^-(x) \leq \mu_R^-(x)) \\ =& 2(\mu_R^-(x))^2((\mu_R^-(x))^2 - 1) \leq 0. \end{aligned}$$

Case 4.2: If $\nu_P^-(x)\nu_Q^-(x) \leq \mu_R^-(x)$, then:

$$\begin{aligned} f(x) =&(\mu_R^-(x))^2 - ((\mu_R^-(x))^2 + (\mu_R^-(x))^2 \\ &- (\mu_R^-(x))^2(\mu_R^-(x))^2) \\ =&(\mu_R^-(x))^2((\mu_R^-(x))^2 - 1) \leq 0. \end{aligned}$$

From above four cases, we can have $f(x) \leq 0$, i.e.:

$$\sqrt{\begin{aligned} &\max\{(\nu_P^-(x))^2, (\mu_R^-(x))^2\} \\ &+ \max\{(\nu_Q^-(x))^2, (\mu_R^-(x))^2\} \\ &- \max\{(\nu_P^-(x))^2, (\mu_R^-(x))^2\} \\ &* \max\{(\nu_Q^-(x))^2, (\mu_R^-(x))^2\} \end{aligned}} \\ \geq \max\{\nu_P^-(x)\nu_Q^-(x), \mu_R^-(x)\}.$$

Similarly:

$$\sqrt{\begin{aligned} &\max\{(\nu_P^+(x))^2, (\mu_R^+(x))^2\} \\ &+ \max\{(\nu_Q^+(x))^2, (\mu_R^+(x))^2\} \\ &- \max\{(\nu_P^+(x))^2, (\mu_R^+(x))^2\} \\ &* \max\{(\nu_Q^+(x))^2, (\mu_R^+(x))^2\} \end{aligned}} \\ \geq \max\{\nu_P^+(x)\nu_Q^+(x), \mu_R^+(x)\}.$$

Let:

$$\begin{aligned} g(x) = \min\{&(\mu_P^-(x))^2 + (\mu_Q^-(x))^2 \\ &- (\mu_P^-(x))^2(\mu_Q^-(x))^2, (\nu_R^-(x))^2\} \\ &- \min\{(\mu_P^-(x))^2, (\nu_R^-(x))^2\} \\ &\quad \min\{(\mu_Q^-(x))^2, (\nu_R^-(x))^2\}, \end{aligned}$$

then:

Case 1: If $\mu_P^-(x) \geq \mu_Q^-(x) \geq \nu_R^-(x)$ or $\mu_Q^-(x) \geq \mu_P^-(x) \geq \nu_R^-(x)$, then:

$$\begin{aligned}(\mu_P^-(x))^2 &+ (\mu_Q^-(x))^2 - (\mu_P^-(x))^2(\mu_Q^-(x))^2 \\ &- (\nu_R^-(x))^2 = (\mu_P^-(x))^2 - (\nu_R^-(x))^2 \\ &+ (\mu_Q^-(x))^2(1-(\mu_P^-(x))^2) \\ &\geq 0(\mu_P^-(x) \geq \nu_R^-(x), 1-(\mu_P^-(x))^2 \geq 0).\end{aligned}$$

Hence:

$$\begin{aligned}g(x) =&(\nu_R^-(x))^2 - (\nu_R^-(x))^4 \\ =&(\nu_R^-(x))^2(1-(\nu_R^-(x))^2) \geq 0.\end{aligned}$$

Case 2: If $\mu_P^-(x) \geq \nu_R^-(x) \geq \mu_Q^-(x)$, then:

$$\begin{aligned}g(x) =&(\nu_R^-(x))^2 - (\nu_R^-(x))^2(\mu_Q^-(x))^2 \\ =&(\nu_R^-(x))^2(1-(\mu_Q^-(x))^2) \geq 0.\end{aligned}$$

Case 3: If $\mu_Q^-(x) \geq \nu_R^-(x) \geq \mu_P^-(x)$, then:

$$\begin{aligned}(\mu_P^-(x))^2 &+ (\mu_Q^-(x))^2 - (\mu_P^-(x))^2(\mu_Q^-(x))^2 \\ &- (\nu_R^-(x))^2 = (\mu_Q^-(x))^2 - (\nu_R^-(x))^2 \\ &+ (\mu_P^-(x))^2(1-(\mu_Q^-(x))^2) \\ &\geq 0(\mu_Q^-(x) \geq \nu_R^-(x), 1-(\mu_Q^-(x))^2 \geq 0).\end{aligned}$$

Hence:

$$\begin{aligned}g(x) =&(\nu_R^-(x))^2 - (\nu_R^-(x))^2(\mu_P^-(x))^2 \\ =&(\nu_R^-(x))^2(1-(\mu_P^-(x))^2) \geq 0.\end{aligned}$$

Case 4: If $\nu_R^-(x) \geq \mu_Q^-(x) \geq \mu_P^-(x)$ or $\nu_R^-(x) \geq \mu_P^-(x) \geq \mu_Q^-(x)$, then:

Case 4.1: If $(\mu_P^-(x))^2 + (\mu_Q^-(x))^2 - (\mu_P^-(x))^2 (\mu_Q^-(x))^2 \leq (\nu_R^-(x))^2$, then:

$$\begin{aligned}g(x) =&(\mu_P^-(x))^2+(\mu_Q^-(x))^2-(\mu_P^-(x))^2(\mu_Q^-(x))^2 \\ &- (\mu_P^-(x))^2(\mu_Q^-(x))^2 = (\mu_P^-(x))^2 \\ &+ (\mu_Q^-(x))^2 - 2(\mu_P^-(x))^2(\mu_Q^-(x))^2 \geq 0.\end{aligned}$$

(Lemma 2).

Case 4.2: If $(\mu_P^-(x))^2 + (\mu_Q^-(x))^2 - (\mu_P^-(x))^2 (\mu_Q^-(x))^2 \geq (\nu_R^-(x))^2$, then:

$$\begin{aligned}g(x) =&(\nu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_Q^-(x))^2 \\ &\geq 0(\nu_R^-(x) > \mu_P^-(x), \nu_R^-(x) > \mu_Q^-(x)).\end{aligned}$$

From the above four cases, we can have $g(x) \geq 0$, i.e.:

$$\min\left\{\sqrt{(\mu_P^-(x))^2+(\mu_Q^-(x))^2-(\mu_P^-(x))^2(\mu_Q^-(x))^2}, \nu_R^-(x)\right\}$$
$$\geq \min\{\mu_P^-(x), \nu_R^-(x)\} * \min\{\mu_Q^-(x), \nu_R^-(x)\}.$$

Similarly:

$$\min\left\{\sqrt{(\mu_P^+(x))^2+(\mu_Q^+(x))^2-(\mu_P^+(x))^2(\mu_Q^+(x))^2}, \nu_R^+(x)\right\}$$
$$\geq \min\{\mu_P^+(x), \nu_R^+(x)\} * \min\{\mu_Q^+(x), \nu_R^+(x)\}.$$

Hence, according to Eq. (5), we can have:

$$(P \oplus Q) \rightarrow R \subseteq (P \rightarrow R) \oplus (Q \rightarrow R).$$

Theorem 2. Let X be a nonempty set. For IVPFSs P, Q and R in X, then:

(1) $(P♠Q) \oplus R = (P \oplus R)♠(Q \oplus R)$;
(2) $(P♠Q) \otimes R = (P \otimes R)♠(Q \otimes R)$;
(3) $(P\$Q) \oplus R \subseteq (P \oplus R)\$(Q \oplus R)$;
(4) $(P\$Q) \otimes R \supseteq (P \otimes R)\$(Q \otimes R)$;
(5) $(P♣Q) \oplus R \supseteq (P \oplus R)♣(Q \oplus R)$;
(6) $(P♣Q) \otimes R \subseteq (P \otimes R)♣(Q \otimes R)$;
(7) $(P✠Q) \oplus R \supseteq (P \oplus R)✠(Q \oplus R)$;
(8) $(P✠Q) \otimes R \subseteq (P \otimes R)✠(Q \otimes R)$.

Proof. We only prove (1) and (5) in detail, and Statements (2)-(4) and (6)-(8) can be proved in a similar way.

(1) Let P, Q and R be three given IVPFSs, then $(P♠Q) \oplus R$ and $(P \oplus R)♠(Q \oplus R)$ are calculated as shown in Box VIII.

(5) Let P, Q, and R be three given IVPFSs, then $(P♣Q) \oplus R$ and $(P \oplus R)♣(Q \oplus R)$ are calculated as shown in Box IX.

Let $f(x)$ be calculated as shown in Box X; then, it is calculated as shown in Boxes XI and XII.

Hence, we can have the relations shown in Box XIII. Therefore, according to Eq. (5), we can have:

$$(P♣Q) \oplus R \subseteq (P \oplus R)♣(Q \oplus R).$$

Definition 10. Let us define the generalized operator $\overset{n}{\underset{i=1}{♠}}$ over the IVPFS $P_i (i = 1, 2, \cdots, n)$ by:

$$(P\spadesuit Q)\oplus R=\Bigg\{\Bigg\langle x,\Bigg[\sqrt{\frac{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}{2}+(\mu_R^-(x))^2-\frac{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}{2}(\mu_R^-(x))^2},$$
$$\sqrt{\frac{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}{2}+(\mu_R^+(x))^2-\frac{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}{2}(\mu_R^+(x))^2}\Bigg],$$
$$\Bigg[\sqrt{\frac{(\nu_P^-(x))^2+(\nu_Q^-(x))^2}{2}}\nu_R^-(x),\sqrt{\frac{(\nu_P^+(x))^2+(\nu_Q^+(x))^2}{2}}\nu_R^+(x)\Bigg]\Bigg\rangle\Bigg|x\in X\Bigg\},$$

$$(P\oplus R)\spadesuit(Q\oplus R)$$
$$=\Bigg\{\Bigg\langle x,\Bigg[\sqrt{\frac{(\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2+(\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2}{2}},$$
$$\sqrt{\frac{(\mu_P^+(x))^2+(\mu_R^+(x))^2-(\mu_P^+(x))^2(\mu_R^+(x))^2+(\mu_Q^+(x))^2+(\mu_R^+(x))^2-(\mu_Q^+(x))^2(\mu_R^+(x))^2}{2}}\Bigg],$$
$$\Bigg[\sqrt{\frac{(\nu_P^-(x))^2(\nu_Q^-(x))^2+(\nu_R^-(x))^2(\nu_Q^-(x))^2}{2}},\sqrt{\frac{(\nu_P^+(x))^2(\nu_Q^+(x))^2+(\nu_R^+(x))^2(\nu_Q^+(x))^2}{2}}\Bigg]\Bigg\rangle\Bigg|x\in X\Bigg\}$$
$$=\Bigg\{\Bigg\langle x,\Bigg[\sqrt{\frac{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}{2}+(\mu_R^-(x))^2-\frac{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}{2}(\mu_R^-(x))^2},$$
$$\sqrt{\frac{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}{2}+(\mu_R^+(x))^2-\frac{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}{2}(\mu_R^+(x))^2}\Bigg],$$
$$\Bigg[\sqrt{\frac{(\nu_P^-(x))^2+(\nu_Q^-(x))^2}{2}}\nu_R^-(x),\sqrt{\frac{(\nu_P^+(x))^2+(\nu_Q^+(x))^2}{2}}\nu_R^+(x)\Bigg]\Bigg\rangle\Bigg|x\in X\Bigg\}=(P\oplus R)\spadesuit(Q\oplus R).$$

Box VIII

$$\mathop{\spadesuit}_{i=1}^{n}P_i=\Bigg\{\Bigg\langle x,\Bigg[\sqrt{\frac{\sum_{i=1}^{n}(\mu_{P_i}^-(x))^2}{n}},\sqrt{\frac{\sum_{i=1}^{n}(\mu_{P_i}^+(x))^2}{n}}\Bigg],$$
$$\Bigg[\sqrt{\frac{\sum_{i=1}^{n}(\nu_{P_i}^-(x))^2}{n}},\sqrt{\frac{\sum_{i=1}^{n}(\nu_{P_i}^+(x))^2}{n}}\Bigg]\Bigg\rangle\Bigg|x\in X\Bigg\}. \quad (9)$$

When $n=2$, it reduces to $\mathop{\spadesuit}_{i=1}^{2}P_i=P_1\spadesuit P_2$ defined above.

Theorem 3. For every IVPFS P_i $(i=1,2,\cdots,n)$ and Q:

(1) $\left(\mathop{\spadesuit}_{i=1}^{n}P_i^c\right)^c=\mathop{\spadesuit}_{i=1}^{n}P_i$;

(2) $\mathop{\spadesuit}_{i=1}^{n}P_i\oplus Q=\mathop{\spadesuit}_{i=1}^{n}(P_i\oplus Q)$;

(3) $\mathop{\spadesuit}_{i=1}^{n}P_i\otimes Q=\mathop{\spadesuit}_{i=1}^{n}(P_i\otimes Q)$;

(4) $\Box\left(\mathop{\spadesuit}_{i=1}^{n}P_i\right)=\mathop{\spadesuit}_{i=1}^{n}\Box P_i$;

(5) $\Diamond\left(\mathop{\spadesuit}_{i=1}^{n}P_i\right)=\mathop{\spadesuit}_{i=1}^{n}\Diamond P_i$.

Proof. It is trivial.

$$(P\clubsuit Q)\oplus R$$
$$=\left\{\left\langle x,\left[\sqrt{\frac{2(\mu_P^-(x))^2(\mu_Q^-(x))^2+(\mu_P^-(x))^2(\mu_R^-(x))^2+(\mu_Q^-(x))^2(\mu_R^-(x))^2-2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2}{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}},\right.\right.\right.$$
$$\left.\sqrt{\frac{2(\mu_P^+(x))^2(\mu_Q^+(x))^2+(\mu_P^+(x))^2(\mu_R^+(x))^2+(\mu_Q^+(x))^2(\mu_R^+(x))^2-2(\mu_P^+(x))^2(\mu_Q^+(x))^2(\mu_R^+(x))^2}{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}}\right],$$
$$\left.\left.\left[\frac{\sqrt{2}\nu_P^-(x)\nu_Q^-(x)\nu_R^-(x)}{\sqrt{(\nu_P^-(x))^2+(\nu_Q^-(x))^2}},\frac{\sqrt{2}\nu_P^+(x)\nu_Q^+(x)\nu_R^+(x)}{\sqrt{(\nu_P^+(x))^2+(\nu_Q^+(x))^2}}\right]\right\rangle\right| x\in X\right\},$$
$$(P\oplus R)\clubsuit(Q\oplus R)$$
$$=\left\{\left\langle x,\left[\sqrt{\frac{2((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}{(\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2}},\right.\right.\right.$$
$$\left.\sqrt{\frac{2((\mu_P^+(x))^2+(\mu_R^+(x))^2-(\mu_P^+(x))^2(\mu_R^+(x))^2)((\mu_Q^+(x))^2+(\mu_R^+(x))^2-(\mu_Q^+(x))^2(\mu_R^+(x))^2)}{(\mu_P^+(x))^2+(\mu_Q^+(x))^2+2(\mu_R^+(x))^2-(\mu_P^+(x))^2(\mu_R^+(x))^2-(\mu_Q^+(x))^2(\mu_R^+(x))^2}}\right],$$
$$\left.\left.\left[\frac{\sqrt{2}\nu_P^-(x)\nu_Q^-(x)\nu_R^-(x)}{\sqrt{(\nu_P^-(x))^2+(\nu_Q^-(x))^2}},\frac{\sqrt{2}\nu_P^+(x)\nu_Q^+(x)\nu_R^+(x)}{\sqrt{(\nu_P^+(x))^2+(\nu_Q^+(x))^2}}\right]\right\rangle\right| x\in X\right\}.$$

Box IX

$$f(x)=\left(\sqrt{\frac{2(\mu_P^-(x))^2(\mu_Q^-(x))^2+(\mu_P^-(x))^2(\mu_R^-(x))^2+(\mu_Q^-(x))^2(\mu_R^-(x))^2-2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2}{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}}\right)^2$$
$$-\left(\sqrt{\frac{2((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}{(\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2}}\right)^2.$$

Box X

Definition 11. For every $P\in$ IVPFS(X) and for every $m,n\in N$, we define:

$$P^{(m,n)}=\left\{\left\langle x,\left[\sqrt{\frac{((\mu_P^-(x))^2)^m}{n}},\sqrt{\frac{((\mu_P^+(x))^2)^m}{n}}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{1-\frac{(1-(\nu_P^-(x))^2)^m}{n}},\sqrt{1-\frac{(1-(\nu_P^+(x))^2)^m}{n}}\right]\right| x\in X\right\rangle\right\}.$$

It can be known that $P^{(m,n)}$ is still an IVPFN. From this definition, we can have the following:

(1) If $m\le m_1$, then $P^{(m,n)}\supseteq P^{(m_1,n)}$;

(2) If $n\le n_1$, then $P^{(m,n)}\supseteq P^{(m,n_1)}$;

(3) If $P\subseteq Q$, then $P^{(m,n)}\subseteq Q^{(m,n)}$.

$$f(x) = \frac{2(\mu_P^-(x))^2(\mu_Q^-(x))^2 + (\mu_P^-(x))^2(\mu_R^-(x))^2 + (\mu_Q^-(x))^2(\mu_R^-(x))^2 - 2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2}{(\mu_P^-(x))^2 + (\mu_Q^-(x))^2}$$

$$- \frac{2((\mu_P^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2)((\mu_Q^-(x))^2 + (\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2)}{(\mu_P^-(x))^2 + (\mu_Q^-(x))^2 + 2(\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2}$$

$$= \frac{1}{((\mu_P^-(x))^2 + (\mu_Q^-(x))^2)((\mu_P^-(x))^2 + (\mu_Q^-(x))^2 + 2(\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2)}$$

$$\Big[\Big(2(\mu_P^-(x))^2(\mu_Q^-(x))^2 + (\mu_P^-(x))^2(\mu_R^-(x))^2 + (\mu_Q^-(x))^2(\mu_R^-(x))^2 - 2(\mu_P^-(x))^2(\mu_Q^-(x))^2(mu_R^-(x))^2\Big)$$

$$\Big((\mu_P^-(x))^2 + (\mu_Q^-(x))^2 + (\mu_R^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$- 2\left((\mu_P^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2\right)\Big((\mu_Q^-(x))^2 + (\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$\Big((\mu_P^-(x))^2 + (\mu_Q^-(x))^2\Big)\Big]$$

$$= \frac{1}{((\mu_P^-(x))^2 + (\mu_Q^-(x))^2)((\mu_P^-(x))^2 + (\mu_Q^-(x))^2 + 2(\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2)}$$

$$\Big[\Big(2(\mu_P^-(x))^2(\mu_Q^-(x))^2 + (\mu_P^-(x))^2(\mu_R^-(x))^2 + (\mu_Q^-(x))^2(\mu_R^-(x))^2 - 2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$\Big((\mu_P^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2 + (\mu_Q^-(x))^2 + (\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$- 2\left((\mu_P^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2\right)\Big((\mu_Q^-(x))^2 + (\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$\Big((\mu_P^-(x))^2 + (\mu_Q^-(x))^2\Big)\Big]$$

$$= \frac{1}{((\mu_P^-(x))^2 + (\mu_Q^-(x))^2)((\mu_P^-(x))^2 + (\mu_Q^-(x))^2 + 2(\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2)}$$

$$\Big[\Big(2(\mu_P^-(x))^2(\mu_Q^-(x))^2 + (\mu_P^-(x))^2(\mu_R^-(x))^2 + (\mu_Q^-(x))^2(\mu_R^-(x))^2 - 2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$\left((\mu_P^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2\right) + \Big(2(\mu_P^-(x))^2(\mu_Q^-(x))^2 + (\mu_P^-(x))^2(\mu_R^-(x))^2$$

$$+(\mu_Q^-(x))^2(\mu_R^-(x))^2 - 2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)\Big((\mu_Q^-(x))^2 + (\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$- \left((\mu_P^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2\right)\Big((\mu_Q^-(x))^2 + (\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)$$

$$\Big((\mu_P^-(x))^2 + (\mu_Q^-(x))^2\Big) - \left((\mu_P^-(x))^2 + (\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2\right)$$

$$\Big((\mu_Q^-(x))^2 + (\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)\Big((\mu_P^-(x))^2 + (\mu_Q^-(x))^2\Big)\Big]$$

$$= \frac{1}{((\mu_P^-(x))^2 + (\mu_Q^-(x))^2)((\mu_P^-(x))^2 + (\mu_Q^-(x))^2 + 2(\mu_R^-(x))^2 - (\mu_P^-(x))^2(\mu_R^-(x))^2 - (\mu_Q^-(x))^2(\mu_R^-(x))^2)}$$

Box XI

Continuation of Box XI:

$$
\begin{aligned}
&\Big[((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)\Big(2(\mu_P^-(x))^2(\mu_Q^-(x))^2+(\mu_P^-(x))^2(\mu_R^-(x))^2\\
&(\mu_Q^-(x))^2(\mu_R^-(x))^2-2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2-((\mu_Q^-(x))^2+(\mu_R^-(x))^2\\
&-(\mu_Q^-(x))^2(\mu_R^-(x))^2)((\mu_P^-(x))^2+(\mu_Q^-(x))^2)\Big)+\Big((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)\\
&\Big(2(\mu_P^-(x))^2(\mu_Q^-(x))^2+(\mu_P^-(x))^2(\mu_R^-(x))^2+(\mu_Q^-(x))^2(\mu_R^-(x))^2-2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2\\
&-((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)((\mu_P^-(x))^2+(\mu_Q^-(x))^2)\Big)\Big]\\
=&\frac{1}{((\mu_P^-(x))^2+(\mu_Q^-(x))^2)((\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}\\
&\Big[((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)\Big((\mu_P^-(x))^2(\mu_Q^-(x))^2-(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2\\
&-(\mu_Q^-(x))^4+(\mu_Q^-(x))^4(\mu_R^-(x))^2\Big)+\Big((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)\\
&\Big((\mu_P^-(x))^2(\mu_Q^-(x))^2-(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2-(\mu_P^-(x))^4+(\mu_P^-(x))^4(\mu_R^-(x))^2\Big)\Big]\\
=&\frac{1}{((\mu_P^-(x))^2+(\mu_Q^-(x))^2)((\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}\\
&\Big[((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)\left(1-(\mu_R^-(x))^2\right)\Big((\mu_P^-(x))^2(\mu_Q^-(x))^2-(\mu_Q^-(x))^4\Big)\\
&+\Big((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)\left(1-(\mu_R^-(x))^2\right)\Big((\mu_P^-(x))^2(\mu_Q^-(x))^2-(\mu_P^-(x))^4\Big)\Big]\\
=&\frac{1}{((\mu_P^-(x))^2+(\mu_Q^-(x))^2)((\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}\\
&\Big[((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)\left(1-(\mu_R^-(x))^2\right)(\mu_Q^-(x))^2\Big((\mu_P^-(x))^2-(\mu_Q^-(x))^2\Big)\\
&+\Big((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2\Big)\left(1-(\mu_R^-(x))^2\right)(\mu_P^-(x))^2\Big((\mu_Q^-(x))^2-(\mu_P^-(x))^2\Big)\\
=&\frac{1}{((\mu_P^-(x))^2+(\mu_Q^-(x))^2)((\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}\\
&\Big[\left(1-(\mu_R^-(x))^2\right)\Big((\mu_P^-(x))^2-(\mu_Q^-(x))^2\Big)\Big((\mu_Q^-(x))^2((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)\\
&-(\mu_P^-(x))^2((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)\Big)\Big]\\
=&-\frac{\left(1-(\mu_R^-(x))^2\right)\Big((\mu_P^-(x))^2-(\mu_Q^-(x))^2\Big)^2(\mu_R^-(x))^2}{((\mu_P^-(x))^2+(\mu_Q^-(x))^2)((\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}\\
&\leq 0.
\end{aligned}
$$

Box XII

$$\sqrt{\frac{2(\mu_P^-(x))^2(\mu_Q^-(x))^2+(\mu_P^-(x))^2(\mu_R^-(x))^2+(\mu_Q^-(x))^2(\mu_R^-(x))^2-2(\mu_P^-(x))^2(\mu_Q^-(x))^2(\mu_R^-(x))^2}{(\mu_P^-(x))^2+(\mu_Q^-(x))^2}}$$

$$\leq\sqrt{\frac{2((\mu_P^-(x))^2+(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2)((\mu_Q^-(x))^2+(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2)}{(\mu_P^-(x))^2+(\mu_Q^-(x))^2+2(\mu_R^-(x))^2-(\mu_P^-(x))^2(\mu_R^-(x))^2-(\mu_Q^-(x))^2(\mu_R^-(x))^2}}.$$

Similarly:

$$\sqrt{\frac{2(\mu_P^+(x))^2(\mu_Q^+(x))^2+(\mu_P^+(x))^2(\mu_R^+(x))^2+(\mu_Q^+(x))^2(\mu_R^+(x))^2-2(\mu_P^+(x))^2(\mu_Q^+(x))^2(\mu_R^+(x))^2}{(\mu_P^+(x))^2+(\mu_Q^+(x))^2}}$$

$$\leq\sqrt{\frac{2((\mu_P^+(x))^2+(\mu_R^+(x))^2-(\mu_P^+(x))^2(\mu_R^+(x))^2)((\mu_Q^+(x))^2+(\mu_R^+(x))^2-(\mu_Q^+(x))^2(\mu_R^+(x))^2)}{(\mu_P^+(x))^2+(\mu_Q^+(x))^2+2(\mu_R^+(x))^2-(\mu_P^+(x))^2(\mu_R^+(x))^2-(\mu_Q^+(x))^2(\mu_R^+(x))^2}}.$$

Box XIII

Example 4. Let P be an IVPFS over X such that:

$$P=\{< x_1,[0.2,0.3],[0.4,0.6]>,$$
$$< x_2,[0.2,0.4],[0.5,0.8]>,$$
$$< x_3,[0.4,0.5],[0.6,0.7]>,$$
$$< x_4,[0.5,0.6],[0.7,0.8]>\}.$$

Then, we can easily compute $P^{(m,n)}$ by the above definition shown in Table 2.

For a better understanding of the trend of $P^{(m,n)}$ where m and n are of different values, we give its score function with the figure form shown in Figure 2. Based on Figure 2, we can find that it is decreasing when m (or n) is increasing when n (or m) stays at the same level. Meanwhile, when m or n is gradually decreasing, the tread of its score function becomes slight.

Theorem 4. Let X be a nonempty set. For every IVPFS P in X and $m,n\in N$:

(1) $(P^{(m,1)})^{(n,1)}=P^{(mn,1)}=(P^{(n,1)})^{(m,1)}$;
(2) $(P^{(1,m)})^{(1,n)}=P^{(1,mn)}=(P^{(1,n)})^{(1,m)}$;
(3) $(P^{(m,1)})^{(1,n)}=P^{(m,n)}$;
(4) $(P^{(1,m)})^{(n,1)}=P^{(n,m^n)}$.

Proof. It is trivial.

Theorem 5. Let X be a nonempty set. For every IVPFS P in X and $m,m_1,n,n_1\in N$:

(1) $\Box(P^{(m,n)})=(\Box P)^{(m,n)}$;
(2) $\Diamond(P^{(m,n)})=(\Diamond P)^{(m,n)}$;
(3) $(\Box(P^{(m,n)}))^{(m_1,n_1)}=\Box(P^{(mm_1,n_1n^{m_1})})$;
(4) $(\Diamond(P^{(m,n)}))^{(m_1,n_1)}=\Diamond(P^{(mm_1,n_1n^{m_1})})$;
(5) $(\Box(P^{(m,n)}))^{(m_1,n_1)}\otimes(\Box(P^{(n,m)}))^{(n_1,m_1)}=\Box(P^{(mm_1+nn_1,m_1n_1n^{m_1}m^{n_1})})$;
(6) $(\Diamond(P^{(m,n)}))^{(m_1,n_1)}\otimes(\Diamond(P^{(n,m)}))^{(n_1,m_1)}=\Diamond(P^{(mm_1+nn_1,m_1n_1n^{m_1}m^{n_1})})$.

Proof. It is trivial.

3.3. Comparison with new interval-valued Pythagorean fuzzy operators and the existing ones

3.3.1. Comparison with some existing Pythagorean fuzzy operators

The prominent characteristic of interval-valued Pythagorean fuzzy operators is that they can permit the membership and nonmembership degrees for a given set to have an interval value. This kind of situation is more or less similar to that encountered in intuitionistic fuzzy environments, where the concept of IFSs has been extended to that of interval-valued IFSs to describe the case of interval values in which the membership and nonmembership degrees of an element are assigned to a set. It should be noted that when the upper and lower limits of the interval values are identical, IVPFS becomes PFS, indicating that the latter is a special case of the former. Hence, our proposed interval-valued Pythagorean fuzzy operators are more suitable in solving real problems compared with [3,5].

Table 2. The results of operation $P^{(m,n)}$ when m and n are with different values.

$\boldsymbol{m, n}$	**Results**
$m=1, n=1$	$\{< x_1, [0.2, 0.3], [0.4, 0.6] >, < x_2, [0.2, 0.4], [0.5, 0.8] >, < x_3, [0.4, 0.5], [0.6, 0.7] >,$ $< x_4, [0.5, 0.6], [0.7, 0.8] >\}$
$m=1, n=2$	$\{< x_1, [0.1414, 0.2121], [0.7616, 0.8246] >, < x_2, [0.1414, 0.2828], [0.7906, 0.9055] >,$ $< x_3, [0.2828, 0.3536], [0.8246, 0.8631] >, < x_4, [0.3536, 0.4243], [0.8631, 0.9055] >\}$
$m=1, n=3$	$\{< x_1, [0.1155, 0.1732], [0.8485, 0.8869] >, < x_2, [0.1155, 0.2309], [0.8660, 0.9381] >,$ $< x_3, [0.2309, 0.2887], [0.8869, 0.9110] >, < x_4, [0.2887, 0.3464], [0.9110, 0.9381] >\}$
$m=1, n=4$	$\{< x_1, [0.1000, 0.1500], [0.8888, 0.9165] >, < x_2, [0.1000, 0.2000], [0.9014, 0.9539] >,$ $< x_3, [0.2000, 0.2500], [0.9165, 0.9341] >, < x_4, [0.2500, 0.3000], [0.9341, 0.9539] >\}$
$m=1, n=5$	$\{< x_1, [0.0894, 0.1342], [0.9121, 0.9338] >, < x_2, [0.0894, 0.1789], [0.9220, 0.9633] >,$ $< x_3, [0.1789, 0.2236], [0.9338, 0.9476] >, < x_4, [0.2236, 0.2683], [0.9476, 0.9633] >\}$
$m=1, n=6$	$\{< x_1, [0.0816, 0.1225], [0.9274, 0.9452] >, < x_2, [0.0816, 0.1633], [0.9354, 0.9695] >,$ $< x_3, [0.1633, 0.2041], [0.9452, 0.9566] >, < x_4, [0.2041, 0.2449], [0.9566, 0.9695] >\}$
$m=1, n=7$	$\{< x_1, [0.0756, 0.1134], [0.9381, 0.9532] >, < x_2, [0.0756, 0.1512], [0.9449, 0.9739] >,$ $< x_3, [0.1512, 0.1890], [0.9532, 0.9629] >, < x_4, [0.1890, 0.2268], [0.9629, 0.9739] >\}$
$m=1, n=8$	$\{< x_1, [0.0707, 0.1061], [0.9460, 0.9592] >, < x_2, [0.0707, 0.1414], [0.9520, 0.9772] >,$ $< x_3, [0.1414, 0.1768], [0.9592, 0.9676] >, < x_4, [0.1768, 0.2121], [0.9676, 0.9772] >\}$
$m=1, n=9$	$\{< x_1, [0.0667, 0.1000], [0.9522, 0.9638] >, < x_2, [0.0667, 0.1333], [0.9574, 0.9798] >,$ $< x_3, [0.1333, 0.1667], [0.9638, 0.9713] >, < x_4, [0.1667, 0.2000], [0.9713, 0.9798] >\}$
$m=1, n=10$	$\{< x_1, [0.0632, 0.0949], [0.9571, 0.9675] >, < x_2, [0.0632, 0.1265], [0.9618, 0.9818] >,$ $< x_3, [0.1265, 0.1581], [0.9675, 0.9742] >, < x_4, [0.1581, 0.1897], [0.9742, 0.9818] >\}$
$m=2, n=1$	$\{< x_1, [0.0400, 0.0900], [0.5426, 0.7684] >, < x_2, [0.0400, 0.1600], [0.6614, 0.9330] >,$ $< x_3, [0.1600, 0.2500], [0.7684, 0.8602] >, < x_4, [0.2500, 0.3600], [0.8602, 0.9330] >\}$
$m=2, n=2$	$\{< x_1, [0.0283, 0.0636], [0.8045, 0.8917] >, < x_2, [0.0283, 0.1131], [0.8478, 0.9671] >,$ $< x_3, [0.1131, 0.1768], [0.8917, 0.9327] >, < x_4, [0.1768, 0.2546], [0.9327, 0.9671] >\}$
$m=2, n=3$	$\{< x_1, [0.0231, 0.0520], [0.8745, 0.9292] >, < x_2, [0.0231, 0.0924], [0.9014, 0.9782] >,$ $< x_3, [0.0924, 0.1443], [0.9292, 0.9557] >, < x_4, [0.1443, 0.2078], [0.9557, 0.9782] >\}$
$m=2, n=4$	$\{< x_1, [0.0200, 0.0450], [0.9075, 0.9474] >, < x_2, [0.0200, 0.0800], [0.9270, 0.9837] >,$ $< x_3, [0.0800, 0.1250], [0.9474, 0.9669] >, < x_4, [0.1250, 0.1800], [0.9669, 0.9837] >\}$
$m=2, n=5$	$\{< x_1, [0.0179, 0.0402], [0.9268, 0.9582] >, < x_2, [0.0179, 0.0716], [0.9421, 0.9870] >,$ $< x_3, [0.0716, 0.1118], [0.9582, 0.9736] >, < x_4, [0.1118, 0.1610], [0.9736, 0.9870] >\}$
$m=2, n=6$	$\{< x_1, [0.0163, 0.0367], [0.9394, 0.9653] >, < x_2, [0.0163, 0.0653], [0.9520, 0.9891] >,$ $< x_3, [0.0653, 0.1021], [0.9653, 0.9781] >, < x_4, [0.1021, 0.1470], [0.9781, 0.9891] >\}$

Table 2. The results of operation $P^{(m,n)}$ when m and n are with different values (continued).

m, n	Results
$m=2, n=7$	$\{<x_1,[0.0151,0.0340],[0.9483,0.9703]>,<x_2,[0.0151,0.0605],[0.9590,0.9907]>,$ $<x_3,[0.0605,0.0945],[0.9703,0.9812]>,<x_4,[0.0945,0.1361],[0.9812,0.9907]>\}$
$m=2, n=8$	$\{<x_1,[0.0141,0.0318],[0.9549,0.9741]>,<x_2,[0.0141,0.0566],[0.9642,0.9919]>,$ $<x_3,[0.0566,0.0884],[0.9741,0.9836]>,<x_4,[0.0884,0.1273],[0.9836,0.9919]>\}$
$m=2, n=9$	$\{<x_1,[0.0133,0.0300],[0.9600,0.9770]>,<x_2,[0.0133,0.0533],[0.9682,0.9928]>,$ $<x_3,[0.0533,0.0833],[0.9770,0.9854]>,<x_4,[0.0833,0.1200],[0.9854,0.9928]>\}$
$m=2, n=10$	$\{<x_1,[0.0126,0.0285],[0.9641,0.9793]>,<x_2,[0.0126,0.0506],[0.9715,0.9935]>,$ $<x_3,[0.0506,0.0791],[0.9793,0.9869]>,<x_4,[0.0791,0.1138],[0.9869,0.9935]>\}$
$m=3, n=1$	$\{<x_1,[0.0080,0.0270],[0.6382,0.8590]>,<x_2,[0.0080,0.0640],[0.7603,0.9764]>,$ $<x_3,[0.0640,0.1250],[0.8590,0.9313]>,<x_4,[0.1250,0.2160],[0.9313,0.9764]>\}$
$m=3, n=2$	$\{<x_1,[0.0057,0.0191],[0.8388,0.9322]>,<x_2,[0.0057,0.0453],[0.8883,0.9883]>,$ $<x_3,[0.0453,0.0884],[0.9322,0.9663]>,<x_4,[0.0884,0.1527],[0.9663,0.9883]>\}$
$m=3, n=3$	$\{<x_1,[0.0046,0.0156],[0.8958,0.9553]>,<x_2,[0.0046,0.0370],[0.9270,0.9922]>,$ $<x_3,[0.0370,0.0722],[0.9553,0.9776]>,<x_4,[0.0722,0.1247],[0.9776,0.9922]>\}$
$m=3, n=4$	$\{<x_1,[0.0040,0.0135],[0.9229,0.9667]>,<x_2,[0.0040,0.0320],[0.9458,0.9942]>,$ $<x_3,[0.0320,0.0625],[0.9667,0.9833]>,<x_4,[0.0625,0.1080],[0.9833,0.9942]>\}$
$m=3, n=5$	$\{<x_1,[0.0036,0.0121],[0.9389,0.9734]>,<x_2,[0.0036,0.0286],[0.9569,0.9953]>,$ $<x_3,[0.0286,0.0559],[0.9734,0.9866]>,<x_4,[0.0559,0.0966],[0.9866,0.9953]>\}$
$m=3, n=6$	$\{<x_1,[0.0033,0.0110],[0.9493,0.9779]>,<x_2,[0.0033,0.0261],[0.9642,0.9961]>,$ $<x_3,[0.0261,0.0510],[0.9779,0.9889]>,<x_4,[0.0510,0.0882],[0.9889,0.9961]>\}$
$m=3, n=7$	$\{<x_1,[0.0030,0.0102],[0.9567,0.9811]>,<x_2,[0.0030,0.0242],[0.9694,0.9967]>,$ $<x_3,[0.0242,0.0472],[0.9811,0.9905]>,<x_4,[0.0472,0.0816],[0.9905,0.9967]>\}$
$m=3, n=8$	$\{<x_1,[0.0028,0.0095],[0.9622,0.9835]>,<x_2,[0.0028,0.0226],[0.9733,0.9971]>,$ $<x_3,[0.0226,0.0442],[0.9835,0.9917]>,<x_4,[0.0442,0.0764],[0.9917,0.9971]>\}$
$m=3, n=9$	$\{<x_1,[0.0027,0.0090],[0.9665,0.9853]>,<x_2,[0.0027,0.0213],[0.9763,0.9974]>,$ $<x_3,[0.0213,0.0417],[0.9853,0.9926]>,<x_4,[0.0417,0.0720],[0.9926,0.9974]>\}$
$m=3, n=10$	$\{<x_1,[0.0025,0.0085],[0.9699,0.9868]>,<x_2,[0.0025,0.0202],[0.9787,0.9977]>,$ $<x_3,[0.0202,0.0395],[0.9868,0.9933]>,<x_4,[0.0395,0.0683],[0.9933,0.9977]>\}$
$m=4, n=1$	$\{<x_1,[0.0016,0.0081],[0.7086,0.9123]>,<x_2,[0.0016,0.0256],[0.8268,0.9916]>,$ $<x_3,[0.0256,0.0625],[0.9123,0.9656]>,<x_4,[0.0625,0.1296],[0.9656,0.9916]>\}$

Table 2. The results of operation $P^{(m,n)}$ when m and n are with different values (continued).

m, n	Results
$m=4, n=2$	$\{< x_1, [0.0011, 0.0057], [0.8666, 0.9571] >, < x_2, [0.0011, 0.0181], [0.9175, 0.9958] >,$ $< x_3, [0.0181, 0.0442], [0.9571, 0.9829] >, < x_4, [0.0442, 0.0916], [0.9829, 0.9958] >\}$
$m=4, n=3$	$\{< x_1, [0.0009, 0.0047], [0.9133, 0.9716] >, < x_2, [0.0009, 0.0148], [0.9458, 0.9972] >,$ $< x_3, [0.0148, 0.0361], [0.9716, 0.9887] >, < x_4, [0.0361, 0.0748], [0.9887, 0.9972] >\}$
$m=4, n=4$	$\{< x_1, [0.0008, 0.0041], [0.9357, 0.9788] >, < x_2, [0.0008, 0.0128], [0.9596, 0.9979] >,$ $< x_3, [0.0128, 0.0313], [0.9788, 0.9915] >, < x_4, [0.0313, 0.0648], [0.9915, 0.9979] >\}$
$m=4, n=5$	$\{< x_1, [0.0007, 0.0036], [0.9489, 0.9831] >, < x_2, [0.0007, 0.0114], [0.9678, 0.9983] >,$ $< x_3, [0.0114, 0.0280], [0.9831, 0.9932] >, < x_4, [0.0280, 0.0580], [0.9932, 0.9983] >\}$
$m=4, n=6$	$\{< x_1, [0.0007, 0.0033], [0.9576, 0.9859] >, < x_2, [0.0007, 0.0105], [0.9733, 0.9986] >,$ $< x_3, [0.0105, 0.0255], [0.9859, 0.9943] >, < x_4, [0.0255, 0.0529], [0.9943, 0.9986] >\}$
$m=4, n=7$	$\{< x_1, [0.0006, 0.0031], [0.9638, 0.9879] >, < x_2, [0.0006, 0.0097], [0.9771, 0.9988] >,$ $< x_3, [0.0097, 0.0236], [0.9879, 0.9952] >, < x_4, [0.0236, 0.0490], [0.9952, 0.9988] >\}$
$m=4, n=8$	$\{< x_1, [0.0006, 0.0029], [0.9684, 0.9895] >, < x_2, [0.0006, 0.0091], [0.9800, 0.9989] >,$ $< x_3, [0.0091, 0.0221], [0.9895, 0.9958] >, < x_4, [0.0221, 0.0458], [0.9958, 0.9989] >\}$
$m=4, n=9$	$\{< x_1, [0.0005, 0.0027], [0.9719, 0.9906] >, < x_2, [0.0005, 0.0085], [0.9823, 0.9991] >,$ $< x_3, [0.0085, 0.0208], [0.9906, 0.9962] >, < x_4, [0.0208, 0.0432], [0.9962, 0.9991] >\}$
$m=4, n=10$	$\{< x_1, [0.0005, 0.0026], [0.9748, 0.9916] >, < x_2, [0.0005, 0.0081], [0.9841, 0.9992] >,$ $< x_3, [0.0081, 0.0198], [0.9916, 0.9966] >, < x_4, [0.0198, 0.0410], [0.9966, 0.9992] >\}$
$m=5, n=1$	$\{< x_1, [0.0003, 0.0024], [0.7628, 0.9448] >, < x_2, [0.0003, 0.0102], [0.8733, 0.9970] >,$ $< x_3, [0.0102, 0.0313], [0.9448, 0.9826] >, < x_4, [0.0313, 0.0778], [0.9826, 0.9970] >\}$
$m=5, n=2$	$\{< x_1, [0.0002, 0.0017], [0.8893, 0.9728] >, < x_2, [0.0002, 0.0072], [0.9388, 0.9985] >,$ $< x_3, [0.0072, 0.0221], [0.9728, 0.9913] >, < x_4, [0.0221, 0.0550], [0.9913, 0.9985] >\}$
$m=5, n=3$	$\{< x_1, [0.0002, 0.0014], [0.9277, 0.9819] >, < x_2, [0.0002, 0.0059], [0.9596, 0.9990] >,$ $< x_3, [0.0059, 0.0180], [0.9819, 0.9942] >, < x_4, [0.0180, 0.0449], [0.9942, 0.9990] >\}$
$m=5, n=4$	$\{< x_1, [0.0002, 0.0012], [0.9463, 0.9865] >, < x_2, [0.0002, 0.0051], [0.9699, 0.9992] >,$ $< x_3, [0.0051, 0.0156], [0.9865, 0.9957] >, < x_4, [0.0156, 0.0389], [0.9957, 0.9992] >\}$
$m=5, n=5$	$\{< x_1, [0.0001, 0.0011], [0.9573, 0.9892] >, < x_2, [0.0001, 0.0046], [0.9760, 0.9994] >,$ $< x_3, [0.0046, 0.0140], [0.9892, 0.9965] >, < x_4, [0.0140, 0.0348], [0.9965, 0.9994] >\}$
$m=5, n=6$	$\{< x_1, [0.0001, 0.0010], [0.9645, 0.9910] >, < x_2, [0.0001, 0.0042], [0.9800, 0.9995] >,$ $< x_3, [0.0042, 0.0128], [0.9910, 0.9971] >, < x_4, [0.0128, 0.0317], [0.9971, 0.9995] >\}$

Table 2. The results of operation $P^{(m,n)}$ when m and n are with different values (continued).

$\boldsymbol{m, n}$	**Results**
$m=5, n=7$	$\{< x_1, [0.0001, 0.0009], [0.9697, 0.9923] >, < x_2, [0.0001, 0.0039], [0.9829, 0.9996] >, < x_3, [0.0039, 0.0118], [0.9923, 0.9975] >, < x_4, [0.0118, 0.0294], [0.9975, 0.9996] >\}$
$m=5, n=8$	$\{< x_1, [0.0001, 0.0009], [0.9735, 0.9933] >, < x_2, [0.0001, 0.0036], [0.9851, 0.9996] >, < x_3, [0.0036, 0.0110], [0.9933, 0.9978] >, < x_4, [0.0110, 0.0275], [0.9978, 0.9996] >\}$
$m=5, n=9$	$\{< x_1, [0.0001, 0.0008], [0.9765, 0.9940] >, < x_2, [0.0001, 0.0034], [0.9867, 0.9997] >, < x_3, [0.0034, 0.0104], [0.9940, 0.9981] >, < x_4, [0.0104, 0.0259], [0.9981, 0.9997] >\}$
$m=5, n=10$	$\{< x_1, [0.0001, 0.0008], [0.9789, 0.9946] >, < x_2, [0.0001, 0.0032], [0.9881, 0.9997] >, < x_3, [0.0032, 0.0099], [0.9946, 0.9983] >, < x_4, [0.0099, 0.0246], [0.9983, 0.9997] >\}$
$m=6, n=1$	$\{< x_1, [0.0001, 0.0007], [0.8054, 0.9650] >, < x_2, [0.0001, 0.0041], [0.9067, 0.9989] >, < x_3, [0.0041, 0.0156], [0.9650, 0.9912] >, < x_4, [0.0156, 0.0467], [0.9912, 0.9989] >\}$
$m=6, n=2$	$\{< x_1, [0.0000, 0.0005], [0.9079, 0.9827] >, < x_2, [0.0000, 0.0029], [0.9545, 0.9995] >, < x_3, [0.0029, 0.0110], [0.9827, 0.9956] >, < x_4, [0.0110, 0.0330], [0.9956, 0.9995] >\}$
$m=6, n=3$	$\{< x_1, [0.0000, 0.0004], [0.9396, 0.9885] >, < x_2, [0.0000, 0.0024], [0.9699, 0.9996] >, < x_3, [0.0024, 0.0090], [0.9885, 0.9971] >, < x_4, [0.0090, 0.0269], [0.9971, 0.9996] >\}$
$m=6, n=4$	$\{< x_1, [0.0000, 0.0004], [0.9551, 0.9914] >, < x_2, [0.0000, 0.0020], [0.9775, 0.9997] >, < x_3, [0.0020, 0.0078], [0.9914, 0.9978] >, < x_4, [0.0078, 0.0233], [0.9978, 0.9997] >\}$
$m=6, n=5$	$\{< x_1, [0.0000, 0.0003], [0.9642, 0.9931] >, < x_2, [0.0000, 0.0018], [0.9820, 0.9998] >, < x_3, [0.0018, 0.0070], [0.9931, 0.9982] >, < x_4, [0.0070, 0.0209], [0.9982, 0.9998] >\}$
$m=6, n=6$	$\{< x_1, [0.0000, 0.0003], [0.9703, 0.9943] >, < x_2, [0.0000, 0.0017], [0.9851, 0.9998] >, < x_3, [0.0017, 0.0064], [0.9943, 0.9985] >, < x_4, [0.0064, 0.0190], [0.9985, 0.9998] >\}$
$m=6, n=7$	$\{< x_1, [0.0000, 0.0003], [0.9746, 0.9951] >, < x_2, [0.0000, 0.0015], [0.9872, 0.9998] >, < x_3, [0.0015, 0.0059], [0.9951, 0.9987] >, < x_4, [0.0059, 0.0176], [0.9987, 0.9998] >\}$
$m=6, n=8$	$\{< x_1, [0.0000, 0.0003], [0.9778, 0.9957] >, < x_2, [0.0000, 0.0014], [0.9888, 0.9999] >, < x_3, [0.0014, 0.0055], [0.9957, 0.9989] >, < x_4, [0.0055, 0.0165], [0.9989, 0.9999] >\}$
$m=6, n=9$	$\{< x_1, [0.0000, 0.0002], [0.9803, 0.9962] >, < x_2, [0.0000, 0.0014], [0.9901, 0.9999] >, < x_3, [0.0014, 0.0052], [0.9962, 0.9990] >, < x_4, [0.0052, 0.0156], [0.9990, 0.9999] >\}$
$m=6, n=10$	$\{< x_1, [0.0000, 0.0002], [0.9823, 0.9966] >, < x_2, [0.0000, 0.0013], [0.9911, 0.9999] >, < x_3, [0.0013, 0.0049], [0.9966, 0.9991] >, < x_4, [0.0049, 0.0148], [0.9991, 0.9999] >\}$
$m=7, n=1$	$\{< x_1, [0.0000, 0.0002], [0.8396, 0.9778] >, < x_2, [0.0000, 0.0016], [0.9309, 0.9996] >, < x_3, [0.0016, 0.0078], [0.9778, 0.9955] >, < x_4, [0.0078, 0.0280], [0.9955, 0.9996] >\}$

Table 2. The results of operation $P^{(m,n)}$ when m and n are with different values (continued).

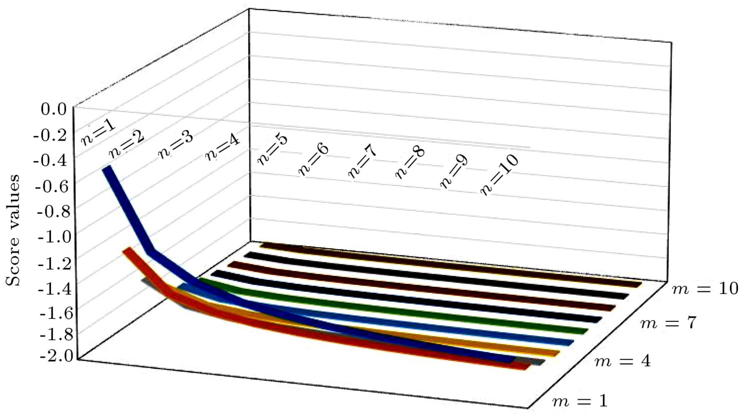
$\boldsymbol{m, n}$	**Results**
$m = 7, n = 2$	$\{< x_1, [0.0000, 0.0002], [0.9233, 0.9889] >, < x_2, [0.0000, 0.0012], [0.9661, 0.9998] >, < x_3, [0.0012, 0.0055], [0.9889, 0.9978] >, < x_4, [0.0055, 0.0198], [0.9978, 0.9998] >\}$
$m = 7, n = 3$	$\{< x_1, [0.0000, 0.0001], [0.9495, 0.9926] >, < x_2, [0.0000, 0.0009], [0.9775, 0.9999] >, < x_3, [0.0009, 0.0045], [0.9926, 0.9985] >, < x_4, [0.0045, 0.0162], [0.9985, 0.9999] >\}$
$m = 7, n = 4$	$\{< x_1, [0.0000, 0.0001], [0.9624, 0.9945] >, < x_2, [0.0000, 0.0008], [0.9832, 0.9999] >, < x_3, [0.0008, 0.0039], [0.9945, 0.9989] >, < x_4, [0.0039, 0.0140], [0.9989, 0.9999] >\}$
$m = 7, n = 5$	$\{< x_1, [0.0000, 0.0001], [0.9700, 0.9956] >, < x_2, [0.0000, 0.0007], [0.9866, 0.9999] >, < x_3, [0.0007, 0.0035], [0.9956, 0.9991] >, < x_4, [0.0035, 0.0125], [0.9991, 0.9999] >\}$
$m = 7, n = 6$	$\{< x_1, [0.0000, 0.0001], [0.9751, 0.9963] >, < x_2, [0.0000, 0.0007], [0.9888, 0.9999] >, < x_3, [0.0007, 0.0032], [0.9963, 0.9993] >, < x_4, [0.0032, 0.0114], [0.9993, 0.9999] >\}$
$m = 7, n = 7$	$\{< x_1, [0.0000, 0.0001], [0.9787, 0.9969] >, < x_2, [0.0000, 0.0006], [0.9904, 0.9999] >, < x_3, [0.0006, 0.0030], [0.9969, 0.9994] >, < x_4, [0.0030, 0.0106], [0.9994, 0.9999] >\}$
$m = 7, n = 8$	$\{< x_1, [0.0000, 0.0001], [0.9814, 0.9972] >, < x_2, [0.0000, 0.0006], [0.9916, 1.0000] >, < x_3, [0.0006, 0.0028], [0.9972, 0.9994] >, < x_4, [0.0028, 0.0099], [0.9994, 1.0000] >\}$
$m = 7, n = 9$	$\{< x_1, [0.0000, 0.0001], [0.9835, 0.9976] >, < x_2, [0.0000, 0.0005], [0.9926, 1.0000] >, < x_3, [0.0005, 0.0026], [0.9976, 0.9995] >, < x_4, [0.0026, 0.0093], [0.9995, 1.0000] >\}$
$m = 7, n = 10$	$\{< x_1, [0.0000, 0.0001], [0.9851, 0.9978] >, < x_2, [0.0000, 0.0005], [0.9933, 1.0000] >, < x_3, [0.0005, 0.0025], [0.9978, 0.9996] >, < x_4, [0.0025, 0.0089], [0.9996, 1.0000] >\}$
$m = 8, n = 1$	$\{< x_1, [0.0000, 0.0001], [0.8673, 0.9858] >, < x_2, [0.0000, 0.0007], [0.9486, 0.9999] >, < x_3, [0.0007, 0.0039], [0.9858, 0.9977] >, < x_4, [0.0039, 0.0168], [0.9977, 0.9999] >\}$
$m = 8, n = 2$	$\{< x_1, [0.0000, 0.0000], [0.9360, 0.9929] >, < x_2, [0.0000, 0.0005], [0.9747, 0.9999] >, < x_3, [0.0005, 0.0028], [0.9929, 0.9989] >, < x_4, [0.0028, 0.0119], [0.9989, 0.9999] >\}$
$m = 8, n = 3$	$\{< x_1, [0.0000, 0.0000], [0.9578, 0.9953] >, < x_2, [0.0000, 0.0004], [0.9832, 1.0000] >, < x_3, [0.0004, 0.0023], [0.9953, 0.9992] >, < x_4, [0.0023, 0.0097], [0.9992, 1.0000] >\}$
$m = 8, n = 4$	$\{< x_1, [0.0000, 0.0000], [0.9685, 0.9965] >, < x_2, [0.0000, 0.0003], [0.9874, 1.0000] >, < x_3, [0.0003, 0.0020], [0.9965, 0.9994] >, < x_4, [0.0020, 0.0084], [0.9994, 1.0000] >\}$
$m = 8, n = 5$	$\{< x_1, [0.0000, 0.0000], [0.9749, 0.9972] >, < x_2, [0.0000, 0.0003], [0.9899, 1.0000] >, < x_3, [0.0003, 0.0017], [0.9972, 0.9995] >, < x_4, [0.0017, 0.0075], [0.9995, 1.0000] >\}$
$m = 8, n = 6$	$\{< x_1, [0.0000, 0.0000], [0.9791, 0.9977] >, < x_2, [0.0000, 0.0003], [0.9916, 1.0000] >, < x_3, [0.0003, 0.0016], [0.9977, 0.9996] >, < x_4, [0.0016, 0.0069], [0.9996, 1.0000] >\}$

Table 2. The results of operation $P^{(m,n)}$ when m and n are with different values (continued).

$\boldsymbol{m, n}$	**Results**
$m = 8, n = 7$	$\{< x_1, [0.0000, 0.0000], [0.9821, 0.9980] >, < x_2, [0.0000, 0.0002], [0.9928, 1.0000] >, < x_3, [0.0002, 0.0015], [0.9980, 0.9997] >, < x_4, [0.0015, 0.0063], [0.9997, 1.0000] >\}$
$m = 8, n = 8$	$\{< x_1, [0.0000, 0.0000], [0.9844, 0.9982] >, < x_2, [0.0000, 0.0002], [0.9937, 1.0000] >, < x_3, [0.0002, 0.0014], [0.9982, 0.9997] >, < x_4, [0.0014, 0.0059], [0.9997, 1.0000] >\}$
$m = 8, n = 9$	$\{< x_1, [0.0000, 0.0000], [0.9861, 0.9984] >, < x_2, [0.0000, 0.0002], [0.9944, 1.0000] >, < x_3, [0.0002, 0.0013], [0.9984, 0.9997] >, < x_4, [0.0013, 0.0056], [0.9997, 1.0000] >\}$
$m = 8, n = 10$	$\{< x_1, [0.0000, 0.0000], [0.9875, 0.9986] >, < x_2, [0.0000, 0.0002], [0.9950, 1.0000] >, < x_3, [0.0002, 0.0012], [0.9986, 0.9998] >, < x_4, [0.0012, 0.0053], [0.9998, 1.0000] >\}$
$m = 9, n = 1$	$\{< x_1, [0.0000, 0.0000], [0.8898, 0.9910] >, < x_2, [0.0000, 0.0003], [0.9617, 0.9999] >, < x_3, [0.0003, 0.0020], [0.9910, 0.9988] >, < x_4, [0.0020, 0.0101], [0.9988, 0.9999] >\}$
$m = 9, n = 2$	$\{< x_1, [0.0000, 0.0000], [0.9465, 0.9955] >, < x_2, [0.0000, 0.0002], [0.9810, 1.0000] >, < x_3, [0.0002, 0.0014], [0.9955, 0.9994] >, < x_4, [0.0014, 0.0071], [0.9994, 1.0000] >\}$
$m = 9, n = 3$	$\{< x_1, [0.0000, 0.0000], [0.9647, 0.9970] >, < x_2, [0.0000, 0.0002], [0.9874, 1.0000] >, < x_3, [0.0002, 0.0011], [0.9970, 0.9996] >, < x_4, [0.0011, 0.0058], [0.9996, 1.0000] >\}$
$m = 9, n = 4$	$\{< x_1, [0.0000, 0.0000], [0.9736, 0.9977] >, < x_2, [0.0000, 0.0001], [0.9906, 1.0000] >, < x_3, [0.0001, 0.0010], [0.9977, 0.9997] >, < x_4, [0.0010, 0.0050], [0.9997, 1.0000] >\}$
$m = 9, n = 5$	$\{< x_1, [0.0000, 0.0000], [0.9790, 0.9982] >, < x_2, [0.0000, 0.0001], [0.9925, 1.0000] >, < x_3, [0.0001, 0.0009], [0.9982, 0.9998] >, < x_4, [0.0009, 0.0045], [0.9998, 1.0000] >\}$
$m = 9, n = 6$	$\{< x_1, [0.0000, 0.0000], [0.9825, 0.9985] >, < x_2, [0.0000, 0.0001], [0.9937, 1.0000] >, < x_3, [0.0001, 0.0008], [0.9985, 0.9998] >, < x_4, [0.0008, 0.0041], [0.9998, 1.0000] >\}$
$m = 9, n = 7$	$\{< x_1, [0.0000, 0.0000], [0.9850, 0.9987] >, < x_2, [0.0000, 0.0001], [0.9946, 1.0000] >, < x_3, [0.0001, 0.0007], [0.9987, 0.9998] >, < x_4, [0.0007, 0.0038], [0.9998, 1.0000] >\}$
$m = 9, n = 8$	$\{< x_1, [0.0000, 0.0000], [0.9869, 0.9989] >, < x_2, [0.0000, 0.0001], [0.9953, 1.0000] >, < x_3, [0.0001, 0.0007], [0.9989, 0.9999] >, < x_4, [0.0007, 0.0036], [0.9999, 1.0000] >\}$
$m = 9, n = 9$	$\{< x_1, [0.0000, 0.0000], [0.9884, 0.9990] >, < x_2, [0.0000, 0.0001], [0.9958, 1.0000] >, < x_3, [0.0001, 0.0007], [0.9990, 0.9999] >, < x_4, [0.0007, 0.0034], [0.9999, 1.0000] >\}$
$m = 9, n = 10$	$\{< x_1, [0.0000, 0.0000], [0.9895, 0.9991] >, < x_2, [0.0000, 0.0001], [0.9962, 1.0000] >, < x_3, [0.0001, 0.0006], [0.9991, 0.9999] >, < x_4, [0.0006, 0.0032], [0.9999, 1.0000] >\}$
$m = 10, n = 1$	$\{< x_1, [0.0000, 0.0000], [0.9083, 0.9942] >, < x_2, [0.0000, 0.0001], [0.9714, 1.0000] >, < x_3, [0.0001, 0.0010], [0.9942, 0.9994] >, < x_4, [0.0010, 0.0060], [0.9994, 1.0000] >\}$

Table 2. The results of operation $P^{(m,n)}$ when m and n are with different values (continued).

m, n	Results
$m = 10, n = 2$	$\{< x_1, [0.0000, 0.0000], [0.9553, 0.9971] >, < x_2, [0.0000, 0.0001], [0.9858, 1.0000] >, < x_3, [0.0001, 0.0007], [0.9971, 0.9997] >, < x_4, [0.0007, 0.0043], [0.9997, 1.0000] >\}$
$m = 10, n = 3$	$\{< x_1, [0.0000, 0.0000], [0.9704, 0.9981] >, < x_2, [0.0000, 0.0001], [0.9906, 1.0000] >, < x_3, [0.0001, 0.0006], [0.9981, 0.9998] >, < x_4, [0.0006, 0.0035], [0.9998, 1.0000] >\}$
$m = 10, n = 4$	$\{< x_1, [0.0000, 0.0000], [0.9779, 0.9986] >, < x_2, [0.0000, 0.0001], [0.9929, 1.0000] >, < x_3, [0.0001, 0.0005], [0.9986, 0.9999] >, < x_4, [0.0005, 0.0030], [0.9999, 1.0000] >\}$
$m = 10, n = 5$	$\{< x_1, [0.0000, 0.0000], [0.9824, 0.9988] >, < x_2, [0.0000, 0.0000], [0.9944, 1.0000] >, < x_3, [0.0000, 0.0004], [0.9988, 0.9999] >, < x_4, [0.0004, 0.0027], [0.9999, 1.0000] >\}$
$m = 10, n = 6$	$\{< x_1, [0.0000, 0.0000], [0.9853, 0.9990] >, < x_2, [0.0000, 0.0000], [0.9953, 1.0000] >, < x_3, [0.0000, 0.0004], [0.9990, 0.9999] >, < x_4, [0.0004, 0.0025], [0.9999, 1.0000] >\}$
$m = 10, n = 7$	$\{< x_1, [0.0000, 0.0000], [0.9874, 0.9992] >, < x_2, [0.0000, 0.0000], [0.9960, 1.0000] >, < x_3, [0.0000, 0.0004], [0.9992, 0.9999] >, < x_4, [0.0004, 0.0023], [0.9999, 1.0000] >\}$
$m = 10, n = 8$	$\{< x_1, [0.0000, 0.0000], [0.9890, 0.9993] >, < x_2, [0.0000, 0.0000], [0.9965, 1.0000] >, < x_3, [0.0000, 0.0003], [0.9993, 0.9999] >, < x_4, [0.0003, 0.0021], [0.9999, 1.0000] >\}$
$m = 10, n = 9$	$\{< x_1, [0.0000, 0.0000], [0.9902, 0.9994] >, < x_2, [0.0000, 0.0000], [0.9969, 1.0000] >, < x_3, [0.0000, 0.0003], [0.9994, 0.9999] >, < x_4, [0.0003, 0.0020], [0.9999, 1.0000] >\}$
$m = 10, n = 10$	$\{< x_1, [0.0000, 0.0000], [0.9912, 0.9994] >, < x_2, [0.0000, 0.0000], [0.9972, 1.0000] >, < x_3, [0.0000, 0.0003], [0.9994, 0.9999] >, < x_4, [0.0003, 0.0019], [0.9999, 1.0000] >\}$



	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
$m = 1$	-0.5	-1.1367	-1.3492	-1.4623	-1.5343	-1.5849	-1.6228	-1.6525	-1.6765	-1.6964
$m = 2$	-1.2443	-1.5702	-1.6807	-1.7389	-1.7757	-1.8013	-1.8204	-1.8352	-1.8472	-1.857
$m = 3$	-1.5737	-1.7625	-1.8263	-1.8596	-1.8804	-1.8948	-1.9055	-1.9137	-1.9203	-1.9257
$m = 4$	-1.7393	-1.8579	-1.8977	-1.9183	-1.931	-1.9398	-1.9462	-1.9512	-1.9551	-1.9583
$m = 5$	-1.8302	-1.9094	-1.9358	-1.9493	-1.9576	-1.9632	-1.9674	-1.9705	-1.973	-1.9751
$m = 6$	-1.8838	-1.9391	-1.9574	1.9667	-1.9724	-1.9763	-1.9791	-1.9812	-1.9829	-1.9842
$m = 7$	-1.9173	-1.9573	-1.9705	-1.9771	-1.9812	-1.9839	-1.9859	-1.9874	-1.9886	-1.9895
$m = 8$	-1.9392	-1.9689	-1.9787	-1.9837	-1.9867	-1.9887	-1.9901	-1.9912	-1.9921	-1.9927
$m = 9$	-1.9541	-1.9768	-1.9842	-1.988	-1.9902	-1.9917	-1.9928	-1.9936	-1.9943	-1.9948
$m = 10$	-1.9647	-1.9822	-1.988	-1.9909	-1.9926	-1.9938	-1.9946	-1.9952	-1.9957	-1.9961

Figure 2. The score function with taking different m and n.

Table 3. Comparison of the different operators.

Operations	Examples	Single operation	Binary operation	Adjustable parameter	Easy to calculate	Aggregation
$\cup$ [21]	$P\cup Q$	No	Yes	No	Yes	No
$\cap$ [21]	$P\cap Q$	No	Yes	No	Yes	No
$\oplus$ [21]	$P\oplus Q$	No	Yes	No	No	Yes
$\otimes$ [21]	$P\otimes Q$	No	Yes	No	No	Yes
λ [21]	P^{λ} or λP	Yes	No	No	No	No
$\subseteq$ [21]	$P\subseteq Q$	No	Yes	No	Yes	No
c [21]	P^c	Yes	No	No	Yes	No
$\Box$	$\Box P$	Yes	No	No	Yes	No
$\Diamond$	$\Diamond P$	Yes	No	No	No	No
$\spadesuit$	$P\spadesuit Q$	No	Yes	No	No	Yes
$\rightarrow$	$P\rightarrow Q$	No	Yes	No	Yes	No
\$	$P\$Q$	No	Yes	No	No	No
$\clubsuit$	$P\clubsuit Q$	No	Yes	No	No	No
✠	P✠Q	No	Yes	No	No	No
(m,n)	$P^{(m,n)}$	Yes	No	Yes	No	No

3.3.2. Comparison with some existing interval-valued Pythagorean fuzzy operators

The proposed interval-valued Pythagorean fuzzy operators promote the development of fundamental properties of IVPFS. For the sake of clarity, several comparative analyses are conducted to demonstrate the advantages and disadvantages of the new operators with existing operators [5,21]. Table 3 presents further details.

4. Novel approach to MCDM with interval-valued Pythagorean fuzzy information based on new operator

4.1. Problem description

This section introduces a novel approach to MCDM under the interval-valued Pythagorean fuzzy environment. A typical MCDM problem with the interval-valued Pythagorean fuzzy information can be described as follows. Let $A = A_1, A_2, \cdots, A_m$ be a set of alternatives and $C = \{C_1, C_2, \cdots, C_n\}$ be a set of attributes with the weight vector being $W = \{w_1, w_2, \cdots, w_n\}$, thereby satisfying $w_i \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. The expert is organized to decide on alternatives. For attribute C_j $(j = 1, 2, \cdots, n)$ of alternative A_i $(i = 1, 2, \cdots, m)$, the decision-maker is required to use IVPFNs to express their preference information, which can be denoted as

$$p_{ij} = \left([\mu_{ij}^-, \mu_{ij}^+], [\nu_{ij}^-, \nu_{ij}^+]\right) \quad (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n).$$

Therefore, an interval-valued Pythagorean fuzzy decision matrix can be obtained by $P = (p_{ij})_{m\times n}$. A novel approach based on the interval-valued Pythagorean fuzzy operator $\spadesuit$ is introduced to solve this problem.

Step 1. Transform matrix $P = (P_{ij})_n$ into a normalized interval-valued Pythagorean fuzzy matri $P' = (p'_{ij})_{m\times n}$ by Eq. (10);

$$p'_{ij} = \left(\left[(\mu_{ij}^-)', (\mu_{ij}^+)'\right], \left[(\nu_{ij}^-)', (\nu_{ij}^+)'\right]\right) = \begin{cases} ([\mu_{ij}^-, \mu_{ij}^+], [\nu_{ij}^-, \nu_{ij}^+]), & C_j \text{ is benefit criteria,} \\ ([\nu_{ij}^-, \nu_{ij}^+], [\mu_{ij}^-, \mu_{ij}^+]), & C_j \text{ is cost criteria.} \end{cases} \tag{10}$$

Step 2. For alternative A_i, we utilize the proposed $\spadesuit$ operators defined in Eq. (11) to aggregate all the attribute values:

$$p_i = \spadesuit_{j=1}^{n} w_j p_{ij} = \left(\left[\sqrt{\frac{\sum_{j=1}^{n}(1-(1-(\mu_{ij}^-)^2)^{w_j})}{n}}, \sqrt{\frac{\sum_{j=1}^{n}(1-(1-(\mu_{ij}^+)^2)^{w_j})}{n}}\right], \left[\sqrt{\frac{\sum_{j=1}^{n}((\nu_{ij}^-)^{2w_j})}{n}}, \sqrt{\frac{\sum_{j=1}^{n}((\nu_{ij}^+)^{2w_j})}{n}}\right]\right). \tag{11}$$

Table 4. The tabular form of interval-valued Pythagorean fuzzy decision matrix in Example 5.

	C_1	C_2	C_3	C_4	C_5
A_1	$([0.3, 0.4], [0.3, 0.4])$	$([0.5, 0.7], [0.6, 0.7])$	$([0.5, 0.7], [0.3, 0.7])$	$([0.4, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$
A_2	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$
A_3	$([0.7, 0.8], [0.1, 0.2])$	$([0.8, 0.9], [0.1, 0.2])$	$([0.0, 0.0], [0.2, 0.3])$	$([0.5, 0.6], [0.1, 0.2])$	$([0.7, 0.8], [0.1, 0.2])$
A_4	$([0.3, 0.4], [0.5, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$	$([0.3, 0.5], [0.6, 0.7])$	$([0.4, 0.6], [0.1, 0.2])$	$([0.5, 0.6], [0.1, 0.2])$
A_5	$([0.5, 0.6], [0.6, 0.7])$	$([0.4, 0.7], [0.1, 0.2])$	$([0.3, 0.4], [0.6, 0.7])$	$([0.5, 0.6], [0.3, 0.4])$	$([0.4, 0.6], [0.1, 0.2])$

Therefore, we can obtain a series of overall values $p_i (i = 1, 2, \cdots, m)$ of alternatives;

Step 3. Rank p_i based on score function $s(p_i)$ by Eq. (5);

Step 4. The alternative A_i $(i = 1, 2, \cdots, m)$ is ranked based the corresponding overall values.

4.2. A case study in mine emergency decision making

We will consider the emergency decision-making problems of mine accidents by employing the proposed decision-making algorithm based on new operator ♠. The mine explosion is one of the most hazardous dangers in mine accidents. The mine explosion enormously threatens the safety of workers and their lives and imperils the safety production of mine. Since the explosion accidents often occur unexpectedly and suddenly, it is not easy to predict the accident and have enough preparations and emergency actions ahead of time. Therefore, the emergency response plans and the simulations of the accidents are a requisite approach in disaster preparedness and appropriate responses. The high quality and feasibility of the emergency plans will directly influence the later emergency actions and affect the evolution of disasters. Consequently, the evaluation and decision of the given emergency plans with simulations is considered essential for the disaster management of mine accidents [42].

Example 5. Assume that there are five emergency plans $A = \{A_1, A_2, A_3, A_4, A_5\}$ to be considered for an explosion accident in the coal mine. The expert chooses decision parameters set $C = \{C_1, C_2, C_3, C_4\}$ to be the noxious gas concentration C_1 (denoted as gas), reducing casualty of current events C_2 (denoted as casualty), the smoke and the dust level C_3 (denoted as smoke), the feasibility of rescue operations C_4 (denoted as feasibility), and repairing facility damages caused by emergency C_5 (denoted as facility). Based on the general evolving principle and the characteristics of the mine accidents, we can determine that all attributes are benefit attributes. Suppose that the expert has the following prior weight set given by his/her prior experience or preference:

$$w = (w_1, w_2, w_3, w_4, w_5) = (0.2, 0.2, 0.1, 0.3, 0.2).$$

Table 4 gives the assessment of emergency plans arising from questionnaire investigation by the expert and constructing an interval-valued Pythagorean fuzzy decision matrix with its tabular form.

In what follows, we utilize the algorithm proposed above to select emergency plans under interval-valued Pythagorean fuzzy information.

Step 1. The decision matrix does not require being normalized because all the attributes are benefit attributes;

Step 2. For the alternative, A_i $(i = 1, 2, 3, 4, 5)$ is utilized to aggregate the attribute values. Therefore, we can obtain a set of overall values:

$$p_1 = ([0.2049, 0.2953], [0.8684, 0.9128]),$$

$$p_2 = ([0.2361, 0.2916], [0.9038, 0.9316]),$$

$$p_3 = ([0.3223, 0.3943], [0.6588, 0.7406]),$$

$$p_4 = ([0.1954, 0.2679], [0.7906, 0.8451]),$$

$$p_5 = ([0.2081, 0.2964], [0.7746, 0.8278]).$$

Step 3. The scores of p_i $(i = 1, 2, 3, 4, 5)$ are calculated based on Definition 6 to obtain:

$$s(p_1) = -1.2809, \qquad s(p_2) = -1.3077,$$

$$s(p_3) = -0.6828, \qquad s(p_4) = -1.1725,$$

$$s(p_5) = -1.0978.$$

Therefore, the rank of the overall values is:

$$s(p_3) > s(p_5) > s(p_4) > s(p_1) > s(p_2).$$

Step 4. The alternative A_i $(i = 1, 2, 3, 4, 5)$ is ranked based on $s(p_i) (i = 1, 2, 3, 4, 5)$ to obtain $A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$. Therefore, A_3 is the best alternative.

4.3. A comparison analysis

In order to further verify the practicability of the proposed multi-attribute decision making approach based on the proposed ♠ operator under IVPFS environment, a comparison study with some existing methods [17,32,37] is now conducted.

First, we give two core definitions related to interval-valued Pythagorean fuzzy aggregation operators in the following.

Definition 12. [**32**] Let $p_i = ([\mu_i^-, \mu_i^+], [\nu_i^-, \nu_i^+])$ $(i = 1, 2, \cdots, m)$ be a collection of IVPFNs. Then, the aggregated value by using an Interval-valued Pythagorean Fuzzy Weighted Average (IPFWA) operator is defined as:

$$\mathrm{IPFWA}(p_1, p_2, \cdots, p_n) = \overset{n}{\underset{i=1}{\oplus}} w_i p_i$$

$$= \left(\left[\sqrt{1 - \prod_{i=1}^{n} (1 - (\mu_i^-)^2)^{w_i}}, \sqrt{1 - \prod_{i=1}^{n} (1 - (\mu_i^+)^2)^{w_i}} \right], \left[\prod_{i=1}^{n} (\nu_i^-)^{w_i}, \prod_{i=1}^{n} (\nu_i^+)^{w_i} \right] \right). \quad (12)$$

However, we can see that the IPFWA operator has drawbacks in some cases described as follows. Let p_j $(j = 1, 2, \cdots, n)$ be a collection of IVPFNs. If there is i such that $p_i = [1, 1], [0, 0]$, then, based on Eq. (12), we can have:

$$\mathrm{IPFWA}(p_1, p_2, \cdots, p_n) = ([1, 1], [0, 0]).$$

This result may cause counter-intuitive phenomena in MCDM. In other words, it is only determined by p_i to make a decision, and the decision information of others can be neglected.

Moreover, based on Eq. (12), if there is an IVPFN such that $p_i = ([\mu_i^-, \mu_i^+], [0, 0])$, the aggregated value is:

$$\mathrm{IPFWA}(p_1, p_2, \cdots, p_n) = ([\mu^-, \mu^+], [0, 0]).$$

In other words, the non-membership degree of aggregated value must be zero. This result may cause counter-intuitive phenomena in some cases. Hence, it is unreasonable and unsuitable to apply Eq. (12) to aggregate the information in MCDM when meeting the special cases mentioned above.

Definition 13. [**32**] Let $p_i = ([\mu_i^-, \mu_i^+], [\nu_i^-, \nu_i^+])$ $(i = 1, 2, \cdots, m)$ be a collection of IVPFNs, and then the aggregated value by using an Interval-valued Pythagorean Fuzzy Weighted Geometric (IPFWG) operator is defined as:

$$\mathrm{IPFWG}(p_1, p_2, \cdots, p_n) = \overset{n}{\underset{i=1}{\otimes}} (p_i)^{w_i}$$

$$= \left(\left[\prod_{i=1}^{n} (\mu_i^-)^{w_i}, \prod_{i=1}^{n} (\mu_i^+)^{w_i} \right], \left[\sqrt{1 - \prod_{i=1}^{n} (1 - (\nu_i^-)^2)^{w_i}}, \sqrt{1 - \prod_{i=1}^{n} (1 - (\nu_i^+)^2)^{w_i}} \right] \right). \quad (13)$$

However, we can see that the IPFWA operator also has drawbacks in some cases, described as follows. $p_j (j = 1, 2, \cdots, n)$ is a collection of IVPFNs. If there is i such that $p_i = [0, 0], [1, 1]$, then, based on Eq. (13), we can have:

$$\mathrm{IPFWA}(p_1, p_2, \cdots, p_n) = ([0, 0], [1, 1]).$$

This result may cause counter-intuitive phenomena in MCDM. In other words, it is only determined by p_i to make a decision, and the decision information of others can be neglected.

Moreover, based on Eq. (13), if there is an IVPFN such that $p_i = ([0, 0], [\nu_i^-, \nu_i^+])$, the aggregated value is

$$\mathrm{IPFWG}(p_1, p_2, \cdots, p_n) = ([0, 0], [\nu^-, \nu^+]).$$

In other words, the membership degree of aggregated value must be zero. This result may cause counter-intuitive phenomena in some cases. Hence, it is unreasonable and unsuitable to apply Eq. (13) to aggregate the information in MCDM when meeting the special cases mentioned above.

4.3.1. A comparison analysis from Example 5

We take Example 5, and the interval-valued Pythagorean fuzzy decision matrix show in Table 4.

If the existing methods in Garg [32], Rahman et al. [37], Liang et al. [17], and the proposed method are applied to solve the MCDM problem in Example 5, then the results can be obtained, as shown in Table 5.

Table 5. A comparison study with some existing methods in Example 5.

Method	The final ranking	The optimal alternative
Garg: IPFWA [32]	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$	A_3
Garg: IPFWG [32]	$A_5 \succ A_4 \succ A_1 \succ A_2 \succ A_3$	A_5
Rahman et al.: IPFWG [37]	$A_5 \succ A_4 \succ A_1 \succ A_2$	A_5
Liang et al.: IPFWA [17]	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$	A_3
Proposed algorithm	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$	A_3

Table 6. The tabular form of interval-valued Pythagorean fuzzy decision matrix in Example 5.

	C_1	C_2	C_3	C_4	C_5
A_1	$([0.3, 0.4], [0.3, 0.4])$	$([0.5, 0.7], [0.6, 0.7])$	$([0.5, 0.7], [0.3, 0.7])$	$([0.4, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$
A_2	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$	$([0.5, 0.6], [0.6, 0.7])$
A_3	$([0.1, 0.2], [0.1, 0.2])$	$([0.1, 0.1], [0.1, 0.2])$	$([0.9, 1.0], [0.0, 0.0])$	$([0.2, 0.3], [0.6, 0.7])$	$([0.4, 0.4], [0.1, 0.2])$
A_4	$([0.8, 0.9], [0.1, 0.2])$	$([0.6, 0.1], [0.1, 0.2])$	$([0.7, 0.8], [0.2, 0.3])$	$([0.6, 0.8], [0.1, 0.2])$	$([0.7, 0.9], [0.1, 0.2])$
A_5	$([0.5, 0.6], [0.6, 0.7])$	$([0.4, 0.7], [0.1, 0.2])$	$([0.3, 0.4], [0.6, 0.7])$	$([0.5, 0.6], [0.3, 0.4])$	$([0.4, 0.6], [0.1, 0.2])$

Table 7. A comparison study with some existing methods in Example 6.

Method	The final ranking	The optimal alternative
Garg: IPFWA [32]	$A_3 \succ A_4 \succ A_5 \succ A_1 \succ A_2$	A_3
Garg: IPFWG [32]	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$	A_4
Rahman et al.: IPFWG [37]	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$	A_4
Liang et al.: IPFWA [17]	$A_3 \succ A_4 \succ A_5 \succ A_1 \succ A_2$	A_3
Proposed algorithm	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$	A_4

According to the above results shown in Table 5, the ranking order of the four alternatives and optimal alternative is in agreement with the results of [17,32]. For Garg [32] (IPFWG) and Rahman et al. [37] (IPFWG), the optimal alternative is A_5, which is different from other methods, and it is unreasonable due to its drawback discussed in Definition 13.

4.3.2. A comparison analysis from Example 6

Example 6 is the continuation of Example 5. Suppose that the another expert has the following prior weight set given by his/her prior experience or preference:

$$w = (w_1, w_2, w_3, w_4, w_5) = (0.2, 0.2, 0.1, 0.3, 0.2).$$

Table 6 shows the assessment of emergency plans arising from questionnaire investigation by the expert and constructing an interval-valued Pythagorean fuzzy decision matrix with its tabular form.

If the existing methods in Garg [32], Rahman et al. [37], Liang et al. [17], and the proposed method are applied to solve the MCDM problem in Example 6, then the results can be obtained, as shown in Table 7.

According to the above results shown in Table 7, the ranking order of the four alternatives and optimal alternative is in agreement with the results of [32,37]. For Garg [32] (IPFWA) and Liang et al. [17] (IPFWA), the optimal alternative is A_3, which is different from other methods, and it is unreasonable due to its drawback discussed in Definition 12.

5. Conclusions

This paper presented some novel interval-valued Pythagorean fuzzy operators ($\diamondsuit, \square, \spadesuit, \clubsuit$, ✠, $\rightarrow$, \$) and discussed their properties with some existing operators ($\cup, \cap, \oplus, \otimes$) in detail. Meanwhile, a new decision-making method based on $\spadesuit$ operator was given. In the future, we will apply the proposed operators ($\diamondsuit, \square, \spadesuit, \clubsuit$, ✠, $\rightarrow$, \$) to other fuzzy environment [26,43-57].

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