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On a Problem of Oppenheim concerning "Factorisatio Numerorum"

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Let $f(n)$ denote the number of factorizations of the natural number n into factors larger than 1 where the order of the factors does not count. We say n is "highly factorable" if $f(m) < f(n)$ for all $m < n$. We prove that $f(n) = n \cdot L(n)^{-1+o(1)}$ for n highly factorable, where $L(n) = \exp\{\log n \operatorname{logloglog} n / \operatorname{loglog} n\}$. This result corrects the 1926 paper of Oppenheim where it is asserted that $f(n) = n \cdot L(n)^{-2+o(1)}$. Some results on the multiplicative structure of highly factorable numbers are proved and a table of them up to 10^9 is provided. Of independent interest, a new lower bound is established for the function $\Psi(x, y)$, the number of $n \leqslant x$ free of prime factors exceeding y.

1. INTRODUCTION

Let $f(n)$ denote the number of factorizations of the natural number n into factors larger than 1, where the order of the factors does not count. Also let $f(1) = 1$. Thus, for example, $f(12) = 4$ since 12 has the factorizations

$$12, \quad 2 \cdot 6, \quad 3 \cdot 4, \quad 2 \cdot 2 \cdot 3.$$

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In this paper we establish a rather accurate estimate for the maximal order of $f(n)$. Roughly, we show that this maximal order is $n \cdot L(n)^{-1+o(1)}$, where

$$L(n) = \exp(\log n \cdot \log_3 n / \log_2 n)$$

and $\log_k n$ denotes the k-fold iteration of the natural logarithm. For a more explicit determination of the "$o(1)$," see our theorems in Sections 2, 4, and 5.

In [13], Oppenheim also considered the problem of the maximal order of $f(n)$, but he erroneously claimed that it was $n \cdot L(n)^{-2+o(1)}$. His error arose when he assumed uniformity in k for his estimation of the maximal order of the Piltz divisor function $d_k(n)$, the number of factorizations of n into exactly k positive factors with order counting.

We present two different proofs that there is an infinite set of n with $f(n) \geqslant n \cdot L(n)^{-1+o(1)}$. In the first proof (Theorem 2.1), we show that the average value of $f(n)$ for $n \leqslant x$ with n divisible by only very small prime factors is $x \cdot L(x)^{-1+o(1)}$. Our proof requires an accurate lower bound for the function $\Psi(z, y)$ when y is about $e^{\sqrt{\log z}}$. Here

$$\Psi(z, y) = \#\{n\colon 1 \leqslant n \leqslant z, P(n) \leqslant y\},$$

where $P(n)$ denotes the largest prime factor of n when $n > 1$, $P(1) = 1$, and where $\#A$ denotes the cardinality of the set A. Although there is a large literature on $\Psi(z, y)$, little is known about lower bounds when

$$e^{(\log z)^{\varepsilon}} < y < e^{(\log z)^{5/8}}.$$

In Section 3 we establish a lower bound for $\Psi(z, y)$ that agrees closely with the known upper bound if $y > (\log z)^{1+\varepsilon}$.

In Section 4 we present a second proof that the maximal order of $f(n)$ is at least $n \cdot L(n)^{-1+o(1)}$. We accomplish this by explicitly exhibiting integers with many factorizations. These integers have a somewhat prohibitive structure. More "natural" candidates, like the product of the primes up to k, or $k!$, or the least common multiple of the integers up to k, do not work. (We can show $f(n) = n \cdot L(n)^{-2+o(1)}$ for the first and last sequences. For $n = k!$, we have $f(n) = n \cdot L(n)^{(-1+o(1))\log_3 n}$.) To get lower estimates for $f(n)$, we use the relationship, also exploited by Oppenheim, between $f(n)$ and $d_k(n)$. While Theorem 4.1 has the advantage of being constructive, Theorem 2.1 has its own advantage in that the result holds for the smaller function $f_0(n)$ which counts only factorizations of n into distinct factors.

In Section 5 we show that $f(n) \leqslant n \cdot L(n)^{-1+o(1)}$ for all n. Our proof employs a common trick that Rankin [15] and de Bruijn [2, Part II] also used to study $\Psi(x, y)$. The proof also uses the formula

$$\sum_{P(n) \leqslant y} f(n)\, n^{-s} = \prod_{\substack{P(n) \leqslant y \\ n > 1}} (1 - n^{-s})^{-1}, \tag{1.1}$$

which is a generalization of a formula of McMahon [11] who had no restriction on $P(n)$ on either side of the equation. Our formula is certainly valid for all s in the half plane $\operatorname{Re} s > 0$, but we shall only use it for s real and $\frac{1}{2} < s < 1$.

We say that a natural number n is *highly factorable* if $f(m) < f(n)$ for all m, $1 \leqslant m < n$. There is an obvious analogy with the highly composite numbers n of Ramanujan [14] which satisfies $d(m) < d(n)$ for all m, $1 \leqslant m < n$. It is obvious that if $n > 1$ is highly factorable, then there is some $t \geqslant 1$ with

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}, \qquad a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t \geqslant 1,$$

where p_i denotes the ith prime. In Section 6 we show that $p_t > (\log n)^{1-\delta}$ for any $\delta > 0$ and all sufficiently large highly factorable n. It follows, of course, from the prime number theorem that $p_t \leqslant (1 + o(1)) \log n$. We also show that $p_t^2 \nmid n$, if n is sufficiently large.

It is not particularly easy to compute $f(n)$. For example, to find that $f(1800) = 137$ takes some work. In Section 7 we present an algorithm for the computation of $f(n)$. We have used this algorithm (on a computer) to find all of the highly factorable numbers below 10^9. These numbers are listed in Table I.

We are able to show that the number of values of $f(n)$ that do not exceed x is $x^{o(1)}$, but we do not include the details here.

We now mention some related results. Oppenheim [13] also considered the average value of $f(n)$, showing

$$\frac{1}{x} \sum_{n \leqslant x} f(n) \sim \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{3/4}}.$$

This result was independently obtained by Szekeres and Turán [17].

There is a second function connected with the name "Factorisatio Numerorum," namely $F(n)$, the number of factorizations of n into factors larger than 1, where now different permutations of the same factorization are counted as different factorizations. Thus $F(12) = 8$ since 12 has the factorizations

$$12, \quad 2 \cdot 6, \quad 3 \cdot 4, \quad 4 \cdot 3, \quad 6 \cdot 2, \quad 2 \cdot 2 \cdot 3, \quad 2 \cdot 3 \cdot 2, \quad 3 \cdot 2 \cdot 2.$$

Kalmár [9] showed that

$$\sum_{n \leqslant x} F(n) \sim \frac{-x^{\rho}}{\rho \zeta'(\rho)},$$

where $\zeta(s)$ is the Riemann zeta functions and $\rho > 1$ is such that $\zeta(\rho) = 2$. Other papers on $F(n)$ are by Erdös [3], Evans [4], Hille [7], Ikehara [8], and Kalmár [9].

TABLE I: HIGHLY FACTORABLE INTEGERS BELOW 10^9

n	number of factorizations of n	exponents in the prime decomposition of n
1	1	none
4	2	2
8	3	3
12	4	2 1
16	5	4
24	7	3 1
36	9	2 2
48	12	4 1
72	16	3 2
96	19	5 1
120	21	3 1 1
144	29	4 2
192	30	6 1
216	31	3 3
240	38	4 1 1
288	47	5 2
360	52	3 2 1
432	57	4 3
480	64	5 1 1
576	77	6 2
720	98	4 2 1
960	105	6 1 1
1080	109	3 3 1
1152	118	7 2
1440	171	5 2 1
2160	212	4 3 1
2880	289	6 2 1
4320	382	5 3 1
5040	392	4 2 1 1
5760	467	7 2 1
7200	484	5 2 2
8640	662	6 3 1
10080	719	5 2 1 1
11520	737	8 2 1
12960	783	5 4 1
14400	843	6 2 2
15120	907	4 3 1 1
17280	1097	7 3 1
20160	1261	6 2 1 1
25920	1386	6 4 1
28800	1397	7 2 2
30240	1713	5 3 1 1
34560	1768	8 3 1

TABLE I: HIGHLY FACTORABLE INTEGERS BELOW 10^9

n	number of factorizations of n	exponents in the prime decomposition of n
40320	2116	7 2 1 1
50400	2179	5 2 2 1
51840	2343	7 4 1
60480	3079	6 3 1 1
80640	3444	8 2 1 1
90720	3681	5 4 1 1
100800	3930	6 2 2 1
120960	5288	7 3 1 1
151200	5413	5 3 2 1
161280	5447	9 2 1 1
172800	5653	8 3 2
181440	6756	6 4 1 1
201600	6767	7 2 2 1
241920	8785	8 3 1 1
302400	10001	6 3 2 1
362880	11830	7 4 1 1
453600	12042	5 4 2 1
483840	14166	9 3 1 1
604800	17617	7 3 2 1
725760	20003	8 4 1 1
907200	22711	6 4 2 1
1088640	24270	7 5 1 1
1209600	29945	8 3 2 1
1451520	32789	9 4 1 1
1814400	40774	7 4 2 1
2177280	41702	8 5 1 1
2419200	49320	9 3 2 1
2903040	52412	10 4 1 1
3326400	54613	6 3 2 1 1
3628800	70520	8 4 2 1
4838400	79177	10 3 2 1
5322240	79459	9 3 1 1 1
5443200	86222	7 5 2 1
6652800	99235	7 3 2 1 1
7257600	118041	9 4 2 1
9676800	124207	11 3 2 1
9979200	129296	6 4 2 1 1
10886400	151500	8 5 2 1
13305600	173377	8 3 2 1 1
14515200	192371	10 4 2 1
18144000	199668	8 4 3 1
19958400	239312	7 4 2 1 1
21772800	257381	9 5 2 1
25401600	259906	8 4 2 2

TABLE I: HIGHLY FACTORABLE INTEGERS BELOW 10^9

n	number of factorizations of n	exponents in the prime decomposition of n
26611200	292951	9 3 2 1 1
29030400	306091	11 4 2 1
31933440	313907	10 4 1 1 1
36288000	340413	9 4 3 1
39916800	425240	8 4 2 1 1
43545600	425254	10 5 2 1
50803200	443995	9 4 2 2
53222400	481392	10 3 2 1 1
59875200	525030	7 5 2 1 1
72576000	564234	10 4 3 1
76204800	574761	8 5 2 2
79833600	729916	9 4 2 1 1
101606400	737393	10 4 2 2
106444800	771932	11 3 2 1 1
119750400	947375	8 5 2 1 1
152409600	996347	9 5 2 2
159667200	1217160	10 4 2 1 1
199584000	1262260	8 4 3 1 1
217728000	1279554	10 5 3 1
239500800	1649624	9 5 2 1 1
279417600	1653287	8 4 2 2 1
304819200	1677259	10 5 2 2
319334400	1978932	11 4 2 1 1
399168000	2205059	9 4 3 1 1
479001600	2787810	10 5 2 1 1
558835200	2894057	9 4 2 2 1
638668800	3148035	12 4 2 1 1
718502400	3470553	9 6 2 1 1
798336000	3737489	10 4 3 1 1
838252800	3786089	8 5 2 2 1
958003200	4590111	11 5 2 1 1

The function $f(n)$ is related to the concept of partitions of a multiset (or multipartite partitions). For example, $f(2^n) = p(n)$, the number of numerical partitions of n, and $f(p_1 p_2 \dots p_n) = B_n$, the nth Bell number, that is, the number of partitions of an n-element set. In general $f(p_1^{a_1} p_2^{a_2} \dots p_n^{a_n})$ is the number of partitions of the multiset which has a_i copies of p_i for each i (or equivalently, the number of partitions of the vector $(a_1, \dots, a_n)$ into lattice point summands $(b_1, \dots, b_n)$ with each $b_i \geqslant 0$). There is a large literature on the subject of partitions of a multiset. The interested reader is referred to Section P64 of W. J. Leveque's "Reviews in Number Theory." Our algorithm in Section 7 for the computation of $f(n)$ appears to be the first practical algorithm for computing the number of partitions of a multiset.

Throughout the paper the letters p and q always denote primes. Also we shall let $\log_k^j x$ denote $(\log_k x)^j$, where $\log_k$ represents the k-fold iteration of the natural logarithm. We shall continue to let $P(n)$ denote the largest prime factor of n if $n > 1$ and $P(1) = 1$.

2. A LOWER BOUND FOR THE MAXIMAL ORDER OF $f_0(n)$

Recall that $f_0(n)$ denotes the number of factorizations of n into distinct factors greater than 1, order of factors not counting.

THEOREM 2.1. *There is a constant C such that for infinitely many n,*

$$f_0(n) \geqslant n \cdot \exp\left\{-\frac{\log n}{\log_2 n}\left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + C\,\frac{\log_4^2 n}{\log_3^2 n}\right)\right\}.$$

Proof. Let x be large and let A denote the set of integers a, $1 < a \leqslant \exp(\log_2^2 x)$ with $P(a) \leqslant \log x/\log_2 x$. Then from the Corollary to Theorem 3.1 we have

$$\begin{aligned} \#A &= \Psi(\exp(\log_2^2 x), \log x/\log_2 x) - 1 \\ &= \exp\left\{\log_2^2 x - \log_2 x\left(\log_3 x + \log_4 x - 1 + \frac{\log_4 x - 1}{\log_3 x} + O\left(\frac{\log_4^2 x}{\log_3^2 x}\right)\right)\right\}. \end{aligned}$$

Let $k = [\log x/\log_2^2 x]$ and let B denote the set of k-element subsets of A. Then

$$\begin{aligned} \#B &= \binom{\#A}{k} \geqslant \left(\frac{\#A}{k}\right)^k \\ &> \frac{1}{\#A}\left(\frac{\#A}{\log x/\log_2^2 x}\right)^{\log x/\log_2^2 x} \\ &= x \cdot \exp\left\{-\frac{\log x}{\log_2 x}\left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O\left(\frac{\log_4^2 x}{\log_3^2 x}\right)\right)\right\}. \end{aligned}$$

Consider the mapping $\Pi: B \to \mathbb{Z}$, where if $S \in B$, then $\Pi(S)$ is the product of the members of S. Note that

$$0 < \Pi(S) \leqslant x \qquad \text{and} \qquad P(\Pi(S)) \leqslant \log x/\log_2 x.$$

Moreover S corresponds to a factorization of $\Pi(S)$ into exactly k distinct factors. Thus

$$\sum_{\substack{n \leqslant x \\ P(n) \leqslant \log x/\log_2 x}} f_0(n) \geqslant \sum_{n \in \Pi(B)} f_0(n) \geqslant \#B.$$

We conclude that there is an $n \leqslant x$ with

$$f_0(n) \geqslant \#B / \Psi(x, \log x/\log_2 x).$$

But Theorem 1 in de Bruijn [2, Part II] contains the assertion that

$$\Psi(x, \log x/\log_2 x) = \exp\{(1 + o(1)) \log x \cdot \log_3 x/\log_2^2 x\}.$$

Thus there is an $n \leqslant x$ with

$$f_0(n) \geqslant x \cdot \exp \left\{ -\frac{\log x}{\log_2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O\left(\frac{\log_4^2 x}{\log_3^2 x} \right) \right) \right\}, \tag{2.1}$$

which proves the theorem.

3. Integers Free of Large Prime Factors

If $u \geqslant 1$ is fixed, it is well known that

$$\lim_{x \to \infty} \frac{1}{x} \Psi(x, x^{1/u}) = \rho(u) > 0, \tag{3.1}$$

where $\rho(u)$ is the Dickman–de Bruijn function. The best result in this direction is that if $x^2 + u^2 \to \infty$ subject to the constraint $1 \leqslant u \leqslant (\log x)^{3/8 - \varepsilon}$, then $\Psi(x, x^{1/u}) \sim x\rho(u)$ (de Bruijn [2, Part I] plus the best known results on the error term in the prime number theorem). From de Bruijn [1] we have for $u \geqslant 3$

$$\rho(u) = \exp \left\{ -u \left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u} \right) \right) \right\}. \tag{3.2}$$

For each $u \geqslant 1$, let

$$D(u) = \inf_{x \geqslant 1} \frac{1}{x} \Psi(x, x^{1/u}).$$

Thus from (3.1) it follows that $0 < D(u) \leqslant \rho(u)$. We shall show in this section that the right side of (3.2) is also a valid estimation for $D(u)$.

There are at least two other papers where a lower bound for $\Psi(x, x^{1/u})$ is established. In [5], Fainleib shows that

$$\frac{1}{x} \Psi(x, x^{1/u}) \geqslant \exp \left\{ -u \left(\log u + \log_2 u - 1 + c \frac{\log_2 u}{\log u} \right) \right\}$$

for some absolute constant c and for $3 \leqslant u < \log x / \log_2 x$. His method is to use an asymptotic result (stated without proof) for certain differential delay equations that are similar to equations studied by Levin. In [6], Halberstam uses the Buchstab identity and an induction argument to show that for $3 \leqslant u < u_0(x)$

$$\frac{1}{x} \Psi(x, x^{1/u}) \geqslant 2e^{-10} \cdot \exp\{-u(\log u + \log_2 u + \eta(u))\},$$

where $\eta(u)$ is an explicit function that is asymptotic to $\log_2 u / \log u$. The function $u_0(x)$ is not explicitly given, but tracing it through the proof, we find that the Halberstam inequality is claimed only for a region where the asymptotic relation (3.1) is already known. However, it is possible to tighten the estimates in Halberstam's proof and establish his inequality for the larger region $3 \leqslant u \leqslant c \log x / (\log_2 x)^{5/3+\varepsilon}$.

Our method of proof is to produce a succession of increasingly sharp estimates for $D(u)$ using the inequality

$$\Psi(x, x^{1/u}) \geqslant \sum_i \Psi(x/m_i, w),$$

where the m_i run over certain integers composed solely of primes in the interval $(w, x^{1/u}]$ and where $w \approx x^{(1-\varepsilon)/u}$. We begin with a crude estimate that is essentially implicit in de Bruijn [2, Part II].

LEMMA. *There is a constant c_1 such that if $u \geqslant c_1$ and $x \geqslant 1$, then*

$$\Psi(x, x^{1/u}) > x/u^{3u}.$$

Proof. Since $\Psi(x, x^{1/u}) \geqslant 1$, the result is trivial if $u^{3u} > x$. So assume $x \geqslant u^{3u}$. From what we have said above, we also may assume $u > (\log x)^{3/8-\varepsilon}$ (if u is sufficiently large).

Thus, we suppose $c_1 \leqslant u$, $(\log x)^{3/8-\varepsilon} < u$, $u^{3u} \leqslant x$. Then $x^{1/u} \geqslant c_1^3$, so that

$$\pi(x^{1/u}) > ux^{1/u}/(2 \log x),$$

if c_1 is large enough. Let $\pi'(y)$ denote $\pi(y)$ if $y \geqslant 2$ and $\pi'(y) = 1$ otherwise. Let $u = m + \theta$, where $m = [u]$. We evidently have

$$\begin{aligned}\Psi(x, x^{1/u}) &\geqslant \pi(x^{1/u})^m \, \pi'(x^{\theta/u})/(m+1)! \\ &> \left(\frac{ux^{1/u}}{2 \log x}\right)^m \left(\frac{x^{\theta/u}}{2 \log x}\right)/u^m \\ &= x/(2 \log x)^{m+1} \\ &\geqslant x \cdot \exp\{-(u+1)(\log_2 x + \log 2)\} \\ &> x \cdot \exp\{-3u \log u\} = x/u^{3u},\end{aligned}$$

where the last inequality is valid for $u > (\log x)^{3/8-\varepsilon}$ and u sufficiently large.

THEOREM 3.1. *If $x \geqslant 1$ and $u \geqslant 3$, we have*

$$\Psi(x, x^{1/u}) \geqslant x \cdot \exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + C\frac{\log_2^2 u}{\log^2 u}\right)\right\},$$

where C is an absolute constant.

Proof. It suffices to show the theorem for all $u \geqslant c_2$, where c_2 is an arbitrary absolute constant. Since $\Psi(x, x^{1/u}) \geqslant 1$, we may assume

$$x > u^u. \tag{3.3}$$

Consider the intervals

$$I_j = (x^{(1/u)(1-(k+1-j)/\log^3 u)}, x^{(1/u)(1-(k-j)/\log^3 u)}]$$

for $j = 1,..., k$, where $k = [\log^2 u \log_2 u]$. Let

$$\alpha_j = \frac{\exp(1/\log^2 u) - 1}{\exp(k/\log^2 u) - 1} \exp((j-1)/\log^2 u)$$

for $j = 1,..., k$. Note that

$$\begin{aligned}\exp(k/\log^2 u) &= \exp(\log_2 u + O(1/\log^2 u)) \\ &= \log u + O(1/\log u).\end{aligned} \tag{3.4}$$

Let $m_{j,1}, m_{j,2},...,$ denote the integers composed of exactly $[\alpha_j u]$ primes (not necessarily distinct) from I_j. Let $m_1, m_2, \ldots$, denote the integers of the form $m_{1,i_1} m_{2,i_2} \ldots m_{k,i_k}$. Then we evidently have

$$\Psi(x, x^{1/u}) \geqslant \sum_i \Psi(x/m_i, w), \qquad w = x^{(1/u)(1-(k/\log^3 u))}. \tag{3.5}$$

Note that for each m_i,

$$\frac{\log m_i}{\log x} \geqslant \sum_{j=1}^{k} \frac{[\alpha_j u]}{u}\left(1 - \frac{k+1-j}{\log^3 u}\right) = \sum_j \alpha_j \left(1 - \frac{k+1-j}{\log^3 u}\right) + O\left(\frac{k}{u}\right). \tag{3.6}$$

Now

$$\sum_j \alpha_j = 1 \tag{3.7}$$

and from (3.4),

$$\begin{aligned}
\sum_j \alpha_j (k+1-j) &= \alpha_1 \sum_j (k+1-j) \exp((j-1)/\log^2 u) \\
&= \alpha_1 \left\{ \exp\left(\frac{1}{\log^2 u}\right) \cdot \left(\exp\left(\frac{k}{\log^2 u}\right) - 1\right) \right. \\
&\quad \left. - k\left(\exp\left(\frac{1}{\log^2 u}\right) - 1\right) \right\} \Big/ \left\{ \exp\left(\frac{1}{\log^2 u}\right) - 1 \right\}^2 \\
&= \frac{\exp(1/\log^2 u)}{\exp(1/\log^2 u) - 1} - \frac{k}{\exp(k/\log^2 u) - 1} \\
&= \log^2 u \cdot \left(1 + O\left(\frac{1}{\log^2 u}\right)\right) - \frac{\log^2 u \log_2 u}{\log u - 1 + O(1/\log u)} \\
&= \log^2 u - \log u \log_2 u + O(\log_2 u).
\end{aligned} \tag{3.8}$$

Thus from (3.6)–(3.8) we have

$$\frac{\log m_i}{\log x} \geqslant 1 - \frac{1}{\log u} + \frac{\log_2 u}{\log^2 u} + O\left(\frac{\log_2 u}{\log^3 u}\right).$$

Since $x/m_i > 1$, we may define v_i so that $w = (x/m_i)^{1/v_i}$, that is,

$$\begin{aligned}
v_i &= \frac{\log(x/m_i)}{\log w} \\
&\leqslant \left\{ \frac{1}{\log u} - \frac{\log_2 u}{\log^2 u} + O\left(\frac{\log_2 u}{\log^3 u}\right) \right\} \Big/ \left\{ \frac{1}{u}\left(1 - \frac{\log_2 u}{\log u} + O\left(\frac{1}{\log^3 u}\right)\right) \right\}
\end{aligned}$$

$$= \frac{u}{\log u} \left\{1 - \frac{\log_2 u}{\log u} + O\left(\frac{\log_2 u}{\log^2 u}\right)\right\} \cdot \left\{1 + \frac{\log_2 u}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right\}$$

$$= \frac{u}{\log u} \left(1 + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right).$$

Thus if we let $v = \max\{v_i\}$, we have

$$v \leqslant \frac{u}{\log u} \left(1 + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right). \tag{3.9}$$

Since $w \geqslant (x/m_i)^{1/v}$, we have from (3.5) that

$$\Psi(x, x^{1/u}) \geqslant \sum_i \Psi(x/m_i, (x/m_i)^{1/v}) \geqslant xD(v) \sum_i 1/m_i. \tag{3.10}$$

It remains to estimate $D(v)$ and $\sum 1/m_i$. For the latter, note that

$$\sum_i \frac{1}{m_i} = \prod_{j=1}^{k} \sum_i \frac{1}{m_{j,i}} \geqslant \prod_j \left(\left(\sum_{p \in I_j} \frac{1}{p}\right)^{[\alpha_j u]} / [\alpha_j u]!\right). \tag{3.11}$$

Now from (3.3)

$$\sum_{p \in I_j} \frac{1}{p} = \operatorname{loglog} x^{(1/u)(1-(k-j)/\log^3 u)} - \operatorname{loglog} x^{(1/u)(1-(k+1-j)/\log^3 u)}$$

$$+ O\left(\left(\frac{u}{\log x}\right)^{10}\right)$$

$$= \log\left(1 - \frac{k-j}{\log^3 u}\right) - \log\left(1 - \frac{k+1-j}{\log^3 u}\right) + O\left(\frac{1}{\log^{10} u}\right)$$

$$= \log\left\{\left(1 - \frac{k+1-j}{\log^3 u} + \frac{1}{\log^3 u}\right)\Big/\left(1 - \frac{k+1-j}{\log^3 u}\right)\right\}$$

$$+ O\left(\frac{1}{\log^{10} u}\right)$$

$$= \frac{1}{\log^3 u}\left(1 + \frac{k+1-j}{\log^3 u}\right) + O\left(\frac{\log_2^2 u}{\log^5 u}\right).$$

Then using (3.7) and (3.8) we have

$$\begin{aligned}
\log &\left(\prod_j \left(\sum_{p\in I_j} \frac{1}{p}\right)^{[\alpha_j u]}\right)\\
&= \sum_j [\alpha_j u] \log \left(\sum_{p\in I_j} \frac{1}{p}\right)\\
&= \sum_j [\alpha_j u] \left(-3 \log_2 u + \frac{k+1-j}{\log^3 u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right)\\
&= u \sum_j \alpha_j \left(-3 \log_2 u + \frac{k+1-j}{\log^3 u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right) + O(k \log_2 u)\\
&= u \left(-3 \log_2 u + \frac{1}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right). \qquad (3.12)
\end{aligned}$$

From (3.7) and Stirling's formula, we have

$$\begin{aligned}
\log \left(\prod_j [\alpha_j u]!\right) &= \sum_j [\alpha_j u](\log[\alpha_j u] - 1) + O(k \log u)\\
&= \sum_j \alpha_j u(\log(\alpha_j u) - 1) + O(k \log u)\\
&= u \left(\log u - 1 + \sum_j \alpha_j \log \alpha_j\right) + O(k \log u). \quad (3.13)
\end{aligned}$$

To estimate this last sum, we use (3.4), (3.7), (3.8) to get

$$\begin{aligned}
\sum_j \alpha_j \log \alpha_j &= \sum_j \alpha_j \left(\log \alpha_1 + \frac{j-1}{\log^2 u}\right)\\
&= \log \alpha_1 - \frac{1}{\log^2 u} \sum_j \alpha_j(k+1-j) + \frac{k}{\log^2 u} \sum_j \alpha_j\\
&= \log \alpha_1 - 1 + \frac{\log_2 u}{\log u} + O\left(\frac{\log_2 u}{\log^2 u}\right) + \frac{k}{\log^2 u}\\
&= \log \left(\exp\left(\frac{1}{\log^2 u}\right) - 1\right) - \log \left(\exp\left(\frac{k}{\log^2 u}\right) - 1\right)\\
&\quad - 1 + \frac{\log_2 u}{\log u} + \log_2 u + O\left(\frac{\log_2 u}{\log^2 u}\right)\\
&= -2 \log_2 u - \log_2 u + \frac{1}{\log u} - 1 + \frac{\log_2 u}{\log u} + \log_2 u + O\left(\frac{\log_2 u}{\log^2 u}\right)\\
&= -2 \log_2 u - 1 + \frac{\log_2 u + 1}{\log u} + O\left(\frac{\log_2 u}{\log^2 u}\right).
\end{aligned}$$

With this result and (3.13), we have

$$\log\left(\prod_j [\alpha_j u]!\right) = u\left(\log u - 2\log_2 u - 2 + \frac{\log_2 u + 1}{\log u} + O\left(\frac{\log_2 u}{\log^2 u}\right)\right). \tag{3.14}$$

Thus following from (3.10)–(3.12) and (3.14) we have

$$\Psi(x, x^{1/u}) \geqslant xD(v)\cdot\exp\left\{-u\left(\log u + \log_2 u - 2 + \frac{\log_2 u}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right)\right\}. \tag{3.15}$$

From the lemma and (3.9) we have for large u

$$\log D(v) \geqslant -3v\log v \geqslant -3u,$$

so that (3.15) becomes

$$\Psi(x, x^{1/u}) \geqslant x\cdot\exp\{-u\,(\log u + \log_2 u + O(1))\}.$$

Using this result with (3.9) we have

$$\begin{aligned}\log D(v) &\geqslant -v(\log v + \log_2 v + O(1))\\ &\geqslant -\frac{u}{\log u}\left(1 + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right)(\log u + O(1))\\ &= -u\left(1 + O\left(\frac{1}{\log u}\right)\right),\end{aligned}$$

so that from (3.15) we now obtain

$$\Psi(x, x^{1/u}) \geqslant x\cdot\exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u}{\log u} + O\left(\frac{1}{\log u}\right)\right)\right\}.$$

We iterate our procedure one more time using this last result with (3.9) to get

$$\begin{aligned}\log D(v) &\geqslant -v\left(\log v + \log_2 v - 1 + O\left(\frac{\log_2 v}{\log v}\right)\right)\\ &\geqslant -\frac{u}{\log u}\left(1 + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right)\left(\log u - 1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\\ &= -u\left(1 - \frac{1}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right),\end{aligned}$$

so that (3.15) at last gives

$$\Psi(x, x^{1/u}) \geqslant x \cdot \exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right)\right\},$$

which was to be shown.

COROLLARY. *If* $\varepsilon > 0$ *is arbitrary and* $3 \leqslant u \leqslant (1-\varepsilon)\log x/\log_2 x$, *then*

$$\Psi(x, x^{1/u}) = x \cdot \exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + E(x, u)\right)\right\},$$

where

$$|E(x, u)| \leqslant c_\varepsilon \frac{\log_2^2 u}{\log^2 u},$$

where c_ε *is a constant that depends only on the choice of* ε.

Proof. Theorem 3.1 is half of the corollary. The other half follows from Theorem 2 in de Bruijn [2, Part II].

4. AN EXPLICIT EXAMPLE

In this section we explicitly describe an infinite set of integers, each of which has many factorizations.

THEOREM 4.1. *Let* x *be large and let*

$$\varepsilon = \frac{1}{\log_2 x}\left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x}\right),$$

$$t = (1 + \varepsilon \log_2^2 x)^{1/\varepsilon},\ k = \log x/\log_2^2 x,$$

$$n = \prod_{p \leqslant t} p^{[kp^{\varepsilon - 1}]}.$$

Then there is an absolute constant C such that

$$f(n) \geqslant n \cdot \exp\left\{-\frac{\log n}{\log_2 n}\left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + C\frac{\log_4 n}{\log_3^2 n}\right)\right\}.$$

Proof. We first show that $\log n$ cannot be too much bigger then $\log x$. In fact, we show

$$\log n \leqslant \log x + O(\log x/\log_2^2 x). \tag{4.1}$$

To see this, note that

$$\log n \leqslant \sum_{p \leqslant t} kp^{\varepsilon-1} \log p. \tag{4.2}$$

Now if we let $\pi(s) = li(s) + \Delta(s)$, then

$$\begin{aligned} \sum_{p \leqslant t} p^{\varepsilon-1} \log p &= \int_{2-}^{t} s^{\varepsilon-1} \log s \, d\pi(s) \\ &= \int_{2}^{t} s^{\varepsilon-1} \, ds + \int_{2-}^{t} s^{\varepsilon-1} \log s \, d\Delta(s). \end{aligned} \tag{4.3}$$

We shall show that the last integral in (4.3) is $O(1)$. First note that

$$\begin{aligned} \log t &= \frac{1}{\varepsilon} \left(\log \varepsilon + 2 \log_3 x + O\left(\frac{1}{\log_2 x \log_3 x}\right)\right) \\ &= \frac{\log_2 x}{\log_3 x + \log_4 x + \log_4 x/\log_3 x} \\ &\quad \times \left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x} - \frac{\log_4^2 x}{2 \log_3^2 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right) \\ &= \log_2 x \left(1 - \frac{\log_4^2 x}{2 \log_3^3 x} + O\left(\frac{\log_4 x}{\log_3^3 x}\right)\right). \end{aligned} \tag{4.4}$$

With this estimate and the fact that $t^{\varepsilon} \sim \log_2 x \log_3 x$, we have for $2 \leqslant s \leqslant t$ and t large,

$$s^{\varepsilon} = (\log s)^{\varepsilon \log s/\log\log s} \leqslant (\log s)^{\varepsilon \log t/\log\log t} = (\log s)^{1+o(1)}.$$

Also using $|\Delta(s)| \ll s/\log^4 s$, we have

$$\begin{aligned} \int_{2-}^{t} s^{\varepsilon-1} \log s \, d\Delta(s) &= t^{\varepsilon-1} \log t \, \Delta(t) - 2^{\varepsilon-1} \log 2 \, \Delta(2) \\ &\quad - \int_{2}^{t} s^{\varepsilon-2}((\varepsilon-1) \log s + 1) \, \Delta(s) \, ds \\ &= O(1) + O\left(\int_{2}^{t} \frac{s^{\varepsilon}}{s \log^3 s} \, ds\right) \\ &= O\left(\int_{2}^{t} \frac{1}{s \log^{3/2} s} \, ds\right) \\ &= O(1). \end{aligned} \tag{4.5}$$

Using (4.5) in (4.3) we have

$$\begin{aligned}\sum_{p\leqslant t} p^{\varepsilon-1}\log p &= \int_2^t s^{\varepsilon-1}\,ds + O(1)\\ &= \frac{1}{\varepsilon}t^{\varepsilon} - \frac{1}{\varepsilon}2^{\varepsilon} + O(1)\\ &= \frac{1}{\varepsilon}(t^{\varepsilon}-1) + O(1)\\ &= \log_2^2 x + O(1).\end{aligned} \tag{4.6}$$

Thus (4.1) follows from (4.2) and (4.6).

Recall that the Piltz divisor function $d_l(n)$ counts the number of factorizations of n into exactly l positive factors, where 1 is allowed as a factor and different permutations of a single factorization count separately. It is easily shown that $d_l(n)$ is multiplicative and that

$$d_l(p^a) = \binom{l+a-1}{a-1}.$$

Moreover, we evidently have for any choice of l that

$$f(n) \geqslant d_l(n)/l!.$$

Thus

$$\begin{aligned}\log f(n) &\geqslant \log d_{[k]}(n) - \log[k]!\\ &= \sum_{p\leqslant t}\log\binom{[k]+[kp^{\varepsilon-1}]-1}{[kp^{\varepsilon-1}]-1} - \log[k]!.\end{aligned} \tag{4.7}$$

Now if $a, b \geqslant 2$, then

$$\log\binom{[a]+[b]-1}{[b]-1} = (a+b)\log(a+b) - a\log a - b\log b + O(\log(a+b))$$

so that

$$\begin{aligned}&\log\binom{[k]+[kp^{\varepsilon-1}]-1}{[kp^{\varepsilon-1}]-1}\\ &\quad= k(1+p^{\varepsilon-1})(\log k + \log(1+p^{\varepsilon-1})) - k\log k\\ &\qquad - kp^{\varepsilon-1}(\log k + (\varepsilon-1)\log p) + O(\log k)\\ &\quad= k(1+p^{\varepsilon-1})\log(1+p^{\varepsilon-1}) + k(1-\varepsilon)\,p^{\varepsilon-1}\log p + O(\log k).\end{aligned} \tag{4.8}$$

Now

$$\sum_{p \leqslant t} (1 + p^{\varepsilon - 1}) \log(1 + p^{\varepsilon - 1}) = \sum_{p \leqslant t} p^{\varepsilon - 1} + O(1)$$
$$= \int_{2^-}^{t} s^{\varepsilon - 1} \, d\pi(s) + O(1)$$
$$= \int_{2}^{t} \frac{s^{\varepsilon - 1}}{\log s} \, ds + \int_{2^-}^{t} s^{\varepsilon - 1} \, d\Delta(s) + O(1). \quad (4.9)$$

The last integral is

$$t^{\varepsilon - 1} \Delta(t) - 2^{\varepsilon - 1} \Delta(2) - \int_{2}^{t} (\varepsilon - 1) \, s^{\varepsilon - 2} \Delta(s) \, ds$$
$$= O(1) + O\left(\int_{2}^{t} \frac{s^{\varepsilon}}{s \log^4 s} \, ds\right) = O(1)$$

by (4.5). Also

$$\int_{2}^{t} \frac{s^{\varepsilon - 1}}{\log s} \, ds = \int_{2^{\varepsilon}}^{t^{\varepsilon}} \frac{du}{\log u} - li(t^{\varepsilon}) + O\left(\int_{2^{\varepsilon}}^{2} \frac{du}{\log u}\right)$$
$$= \frac{t^{\varepsilon}}{\varepsilon \log t - 1} \left(1 + O\left(\frac{1}{\varepsilon^2 \log^2 t}\right)\right) + O(|\log \varepsilon|).$$

Thus using (4.4), we have

$$\sum_{p \leqslant t} (1 + p^{\varepsilon - 1}) \log(1 + p^{\varepsilon - 1})$$
$$= \frac{\log_2 x (\log_3 x + \log_4 x + \log_4 x / \log_3 x)(1 + O(1/\log_3^2 x))}{\log_3 x + \log_4 x + \log_4 x / \log_3 x - 1 + O(\log_4^2 x / \log_3^2 x)}$$
$$= \log_2 x \left(1 + \frac{1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right)\left(1 + O\left(\frac{1}{\log_3^2 x}\right)\right)$$
$$= \log_2 x \left(1 + \frac{1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right). \quad (4.10)$$

Thus from (4.1), (4.6)–(4.8), and (4.10), we have

$$\log f(n) \geqslant \frac{\log x}{\log_2 x} \left(1 + \frac{1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right)$$
$$+ \log x - \frac{\log x}{\log_2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x}\right) + O\left(\frac{\log x}{\log_2^2 x}\right)$$

$$- \frac{\log x}{\log_2^2 x} (\log_2 x + O(\log_3 x))$$

$$= \log x - \frac{\log x}{\log_2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right)$$

$$\geqslant \log n - \frac{\log n}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right)\right),$$

which proves the theorem.

5. An Upper Bound for $f(n)$

In this section, to get an upper bound for $f(n)$, we employ a formula of MacMahon and a method that Rankin and de Bruijn used to get upper bounds for $\Psi(x, y)$.

Theorem 5.1. *There is a constant C such that for all large n*

$$f(n) \leqslant n \cdot \exp \left\{ - \frac{\log n}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + C \frac{\log_4^2 n}{\log_3^2 n}\right) \right\}.$$

Proof. Since $f(n)$ depends only on the array of exponents in the prime factorization of n and not on the choice of the primes themselves, to prove the theorem it is sufficient to consider only integers n that are divisible by all the primes up to some point. Let $l(n) = \log n + \log n / \log_2^{10} n$. Since

$$\sum_{p \leqslant l(n)} \log p > \log n$$

for all large n, we may assume $P(n) \leqslant l(n)$. From (1.1) we have for any choice of $c > 0$,

$$f(n) \leqslant n^c \sum_{P(m) \leqslant l(n)} f(m)/m^c = n^c \prod_{\substack{P(m) \leqslant l(n) \\ m > 1}} (1 - m^{-c})^{-1}. \tag{5.1}$$

We shall choose

$$c = 1 - \frac{1}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} - \frac{\log_4^2 n}{\log_3^2 n}\right).$$

Thus to prove the theorem it is sufficient to show that

$$A \stackrel{\text{def}}{=} \log \prod_{\substack{P(m) \leqslant l(n) \\ m > 1}} (1 - m^{-c})^{-1} = O\left(\frac{\log n}{\log_2 n} \cdot \frac{\log_4^2 n}{\log_3^2 n}\right). \tag{5.2}$$

Now

$$A = \sum_{P(m) \leqslant l(n)} m^{-c} + O(1) = \prod_{p \leqslant l(n)} (1 - p^{-c})^{-1} + O(1),$$

and

$$B \stackrel{\text{def}}{=} \log \prod_{p \leqslant l(n)} (1 - p^{-c})^{-1} = \sum_{p \leqslant l(n)} p^{-c} + O(1).$$

By an argument similar to (4.9) and the subsequent calculations we have

$$\begin{aligned}
B &= \frac{l(n)^{1-c}}{(1-c)\log l(n) - 1} \{1 + O((1-c)^{-2} \log^{-2} l(n))\} + O(|\log(1-c)|) \\
&= \frac{\exp\{\log_3 n + \log_4 n + ((\log_4 n - 1)/\log_3 n) - (\log_4^2 n/\log_3^2 n)\}}{\log_3 n + \log_4 n - 1 + ((\log_4 n - 1)/\log_3 n) - (\log_4^2 n/\log_3^2 n)} \\
&\quad \times \left\{1 + O\left(\frac{1}{\log_3^2 n}\right)\right\} \\
&= \log_2 n \left[\exp\left\{\frac{\log_4 n - 1}{\log_3 n} - \frac{\log_4^2 n}{\log_3^2 n}\right\} \Big/ 1 + \frac{\log_4 n - 1}{\log_3 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right)\right] \\
&= \log_2 n \left\{1 - \frac{\log_4^2 n}{2 \log_3^2 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right)\right\} \\
&< \log_2 n - \log_2^{1/2} n,
\end{aligned}$$

for all large n. Thus

$$A = e^B + O(1) \leqslant (\log n)\, e^{-\log_2^{1/2} n} + O(1) = o\left(\frac{\log n \log_4^2 n}{\log_2 n \log_3^2 n}\right)$$

which establishes (5.2) and thus the theorem.

6. The Largest Prime Factor of a Highly Factorable Number and Other Problems

If n is highly factorable (that is, $f(m) < f(n)$ for all m, $1 \leqslant m < n$) and n is large, then we saw in the proof of Theorem 5.1 that $P(n) < \log n +$

$\log n/\log_2^{10} n$. In this section we use Theorem 2.1 and the method of Theorem 5.1 to show that for each $\delta > 0$ we have $P(n) > (\log n)^{1-\delta}$ for all sufficiently large highly factorable numbers n.

THEOREM 6.1. *For all large highly factorable numbers n we have*

$$P(n) > (\log n)^{1-(\log_3 n)^{-2}} \tag{6.1}$$

Proof. Our strategy is to get an upper bound result for $f(n)$ for those n which do not satisfy (6.1). This upper bound will be smaller than our lower bound result for highly factorable numbers (Theorem 2.1). We then conclude that these n are not highly factorable.

Let $l(n) = (\log n)^{1-(\log_3 n)^{-2}}$. If $P(n) < l(n)$, then the argument of (5.1) shows that

$$f(n) \leqslant n^c \prod_{\substack{P(m) \leqslant l(n) \\ m>1}} (1-m^{-c})^{-1}$$

for any $c > 0$. We shall choose

$$c = 1 - \frac{1}{\log_2 n}\left(\log_3 n + \log_4 n + \frac{\log_4 n}{\log_3 n} - \frac{1}{2\log_3 n}\right).$$

It thus follows from the proof of Theorem 2.1 (by applying (2.1) with x replaced by n) that n will not be highly factorable if

$$A \overset{\text{def}}{=} \log \prod_{\substack{P(m) \leqslant l(n) \\ m>1}} (1-m^{-c})^{-1} = o\left(\frac{\log n}{\log_2 n \log_3 n}\right). \tag{6.2}$$

As in Section 5 we may argue that

$$\log A = \frac{l(n)^{1-c}}{(1-c)\log l(n) - 1}\left(1 + O((1-c)^{-2}\log^{-2} l(n))\right) + O(|\log(1-c)|). \tag{6.3}$$

Now

$$(1-c)\log l(n) = \left(1 - \frac{1}{\log_3^2 n}\right)\left(\log_3 n + \log_4 n + \frac{\log_4 n}{\log_3 n} - \frac{1}{2\log_3 n}\right),$$

so that

$$l(n)^{1-c} = \frac{\log_2 n \log_3 n(1 + \log_4 n/\log_3 n + 1/2\log_3 n + O(\log_4^2 n/\log_3^2 n))}{(1 + 1/\log_3 n + O(\log_4 n/\log_3^2 n))}.$$

Thus from (6.3) we have

$$\log A = \frac{\log_2 n(\log_3 n + \log_4 n - \frac{1}{2} + O(\log_4^2 n/\log_3 n))}{\log_3 n + \log_4 n + O(\log_4 n/\log_3 n)} \left(1 + O\left(\frac{1}{\log_3^2 n}\right)\right)$$
$$= \log_2 n \left(1 - \frac{1}{2\log_3 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right)\right),$$

which gives (6.2).

The following lemma will help us prove that $P(n)\| n$ if n is a large highly factorable number.

LEMMA. *Suppose p, q are primes and n is an integer with $p^2 | n$, $p^2 \neq n$, $q \nmid n$. Then $f(qn/p) \geqslant \frac{6}{5} f(n)$.*

Proof. Let $\mathscr{F}(n)$ denote the set of factorizations of n. Thus an element $\varphi \in \mathscr{F}(n)$ is a multiset of integers exceeding 1 whose product is n. If $\varphi \in \mathscr{F}(n)$, let $|\varphi|_p$ denote the number of unequal factors in φ which are multiples of p. For example, if $p = 2$ and $\varphi = \{4, 4, 6, 11\}$ is a factorization of 1056, then $|\varphi|_2 = 2$.

Given $\varphi \in \mathscr{F}(n)$ we can transform φ into a factorization of qn/p by changing one p to a q. Thus φ corresponds to $|\varphi|_p$ different factorization of qn/p. Moreover, every factorization of qn/p arises in exactly one way in this fashion. Thus

$$f(qn/p) = \sum_{\varphi \in \mathscr{F}(n)} |\varphi|_p.$$

Let $f_p(n) = \#\{\varphi \in \mathscr{F}(n) \colon |\varphi|_p = 1\}$. Thus

$$f(qn/p) = f_p(n) + \sum_{\substack{\varphi \in \mathscr{F}(n) \\ |\varphi|_p \geqslant 2}} |\varphi|_p \geqslant f_p(n) + \sum_{\substack{\varphi \in \mathscr{F}(n) \\ |\varphi|_p \geqslant 2}} 2$$
$$= 2f(n) - f_p(n). \tag{6.4}$$

Hence to prove the lemma it suffices to show $f_p(n) \leqslant \frac{4}{5} f(n)$.

Say $p^k \| n$. If $\varphi \in \mathscr{F}(n)$ and $|\varphi|_p = 1$, then for some $j | k$ and some d, φ contains k/j copies of the factor $p^j d$. Let A, B, C, D respectively denote the number of $\varphi \in \mathscr{F}(n)$ with $|\varphi|_p = 1$ and

for A: φ contains k/j copies of $p^j d$, where $j > 1$ and $p^k d \neq p^2$,

for B: φ contains k copies of pd, where $p^k d \neq p^2$,

for C: φ contains p^2,

for D: φ contains two copies of p.

Then $A + B + C + D = f_p(n)$. Note that $C = D = 0$ unless $k = 2$. We now show that each of B, C, D is at most A.

If $\varphi \in \mathscr{E}(n)$ is counted by B, we can let $\varphi' \in \mathscr{E}(n)$ be the same factorization except that the pd's are consolidated into one factor $p^k d^k$. Then φ' is counted by A and the mapping $\varphi \to \varphi'$ is one to one, so $B \leqslant A$.

Suppose now $k = 2$ so that $C, D > 0$. Each type C factorization can have the p^2 consolidated with one of the other factors in φ (using $n \neq p^2$) to form a type A factorization. Thus $C \leqslant A$. Obviously $C = D$, so $D \leqslant A$ as well.

We now show that $A \leqslant f(n) - f_p(n)$. Indeed, if $\varphi \in \mathscr{E}(n)$ is counted by A, we let $\varphi' \in \mathscr{E}(n)$ be the same factorization except that one of the factors $p^j d$ is split into p, $p^{j-1}d$. It is evident that the mapping $\varphi \to \varphi'$ is one to one. Moreover, $|\varphi'|_p \geqslant 2$. For if $p = p^{j-1}d$, then $p^j d = p^2$ occurs at least twice in φ (if not, then φ would be a type C factorization), so that p^2 occurs at least once in φ'.

Thus

$$f_p(n) = A + B + C + D \leqslant 4A \leqslant 4f(n) - 4f_p(n),$$

so that $f_p(n) \leqslant \frac{4}{5} f(n)$ and $f(qn/p) \geqslant \frac{6}{5} f(n)$ from (6.4).

THEOREM 6.2. *There is an $\varepsilon > 0$ such that if n is a large highly factorable number and $(1 - \varepsilon) P(n) < p \leqslant P(n)$, then $p \| n$.*

Proof. Say n is a large highly factorable number with the prime factorization

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}.$$

Say for some p_s, $(1 - \varepsilon) p_t < p_s \leqslant p_t$, we have $a_s \geqslant 2$. Let $k = [6 \log_2 n]$ and let

$$\gamma_k = \frac{p_{t+1} p_{t+2} \cdots p_{t+k}}{p_s p_{s-1} \cdots p_{s-k+1}}.$$

We now estimate γ_k. From Theorem 6.1, we have $p_t > (\log n)^{1-\delta}$, where $\delta > 0$ is small. Thus from the prime number theorem with error term, we have

$$p_{t+k} < p_t(1 + 1/\log_2 n), \qquad p_{s-k} > (1 - \varepsilon - 1/\log_2 n)\, p_t.$$

Thus

$$\begin{aligned} \log \gamma_k &< k(\log(1 + 1/\log_2 n) - \log(1 - \varepsilon - 1/\log_2 n)) \\ &= k(-\log(1 - \varepsilon) + O(1/\log_2 n)) \\ &\leqslant -6 \log(1 - \varepsilon) \log_2 n + O(1). \end{aligned}$$

We now choose $\varepsilon = \frac{1}{7}$. Thus for large n we have

$$\gamma_k < (\log n)^{13/14} < (1-\varepsilon)\, p_t < p_s.$$

Thus the integer $n' = n\gamma_k/p_s$ is smaller than n.

We now show $f(n') > f(n)$, thus contradicting the choice of n as a highly factorable number. Indeed, using the lemma k times we have

$$f(n\gamma_k) \geqslant (\tfrac{6}{5})^k f(n).$$

Also, if $|\varphi|$ denotes the number of unequal factors in the factorization φ, we have, using the notation of the lemma,

$$\begin{aligned} f(n\gamma_k) &\leqslant \sum_{\varphi \in \mathscr{F}(n')} (|\varphi| + 1) \\ &\leqslant f(n') \cdot \max\{|\varphi| + 1 : \varphi \in \mathscr{F}(n')\} \\ &< f(n') \cdot \left(\frac{\log n}{\log 2} + 1\right). \end{aligned}$$

Thus

$$f(n') > \left(\frac{\log n}{\log 2} + 1\right)^{-1} f(n\gamma_k) \geqslant \left(\frac{\log n}{\log 2} + 1\right)^{-1} \left(\frac{6}{5}\right)^{[6 \log_2 n]} f(n) > f(n).$$

This contradiction proves the theorem.

Remark. Our proof has us taking $\varepsilon = \frac{1}{7}$. Being a little more careful, we could actually choose any $\varepsilon < \frac{1}{6}$. Proving a better lemma will allow even larger choices for ε. Indeed, with more effort it is possible to replace the $\frac{6}{5}$ of the lemma with $2 - \delta$, where $\delta > 0$ is arbitrarily small, provided we assume m has many prime factors. With such an improved lemma, we could then prove Theorem 6.2 for any $\varepsilon < \frac{1}{2}$. We conjecture that this result is best possible, that is, that asymptotically 50% of the primes in a highly factorable number appear with exponent one.

We next might ask how many primes, if any, appear with exponent 2, 3, etc. We can prove that if $p^2 | m$, $q \| m$, then $f(qm) > (\frac{3}{2} - \delta) f(pm)$ provided m has many prime factors. If our conjecture that asymptotically $\frac{1}{2}$ of the exponents are 1 is correct, then we can argue similarly as in Theorem 6.2 to show that there are asymptotically at least (and we conjecture at most) $\frac{1}{6}$ of the exponents equal to 2. Continuing with such a chain of conjectures, we conjecture that for each fixed k there are asymptotically exactly $1/k(k+1)$ of the exponents equal to k. Note that numbers of the form $n!$ also have this property. Also note that in Table I there are many numbers of the form $n!$ which are highly factorable, namely for $n = 1, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

However, this is only a temporary phenomenon; that is, if n is sufficiently large, then $n!$ is not highly factorable. We know this, since we can show, using inequality (1.52) in Oppenheim [9], that

$$\log f(n!) = n \log n - (1 + o(1))\, n \log_2^2 n,$$

while if $n!$ were highly factorable, then we would have

$$\log f(n!) = n \log n - (1 + o(1))\, n \log_2 n.$$

It is somewhat a mystery to us why $n!$ has so few factorizations. Indeed if m is the product of the primes up to $n \log n - 2n$, then $m < n!$, m is of course square-free, and yet m has far more factorizations than $n!$ (if n is large). Probably the "fault" with $n!$ is that the exponents on the small primes are wastefully large. Another possibility is that our conjecture above that a large highly factorable number has asymptotically $1/k(k+1)$ of the exponents equal to k is wrong.

We now mention a few additional problems.

(1) From Table I we see that if n is highly factorable and $4 < n < 10^9$, then there is a prime p with n/p highly factorable. Does this remain true for all highly factorable numbers $n > 4$? For infinitely many? See Robin [16] for examples of highly composite numbers n such that n/p is not highly composite for all primes p.

(2) Let $N(x)$ denote the number of highly factorable numbers $n \leqslant x$. It is easy to see that $N(x) \gg \log x$ since if $n > 1$ is highly factorable and if n' is the next highly factorable number, then $n' \leqslant 2n$. Does $\log N(x)/\text{loglog}\, x$ tend to a limit larger than 1? Can it at least be shown that there are quantities α, β with $1 < \alpha \leqslant \beta < \infty$ such that

$$\alpha < \log N(x)/\text{loglog}\, x < \beta$$

for all large x?

(3) If n, n' are consecutive highly factorable numbers, does $n'/n \to 1$? Does $f(n')/f(n) \to 1$?

(4) Find asymptotic formulas for the exponents on the small primes of a highly factorable number.

(5) A highly factorable number is a "champion" for the function $f(n)$. What do the champions for $f_0(n)$ or $F(n)$ look like? What is the maximal order of $F(n)$? Some work has been done on this: see Erdös [3], Evans [4], Hille [7], and Kalmár [9].

7. Calculation of Table I.

In this section, we describe the algorithm used to determine the values displayed in Table I. If $n = \Pi p_i^{a_i}$ (with $p_1 = 2$, $p_2 = 3$, etc.) is highly factorable, then we must have $a_1 \geqslant a_2 \geqslant \ldots$. We calculated (by computer) $f(n)$ for each of the 1274 values of $n \leqslant 10^9$ whose prime exponents are monotone nonincreasing. Table I shows the 118 numbers found to be highly factorable. We have suppressed the values of $f(n)$ for n not highly factorable. We shall gladly send these values to any interested reader. (Knowing $f(n)$ for n satisfying $a_1 \geqslant a_2 \geqslant \ldots$ and $n \leqslant 10^9$ allows one to readily determine $f(n)$ for any $n \leqslant 10^9$ and for infinitely many other n.)

The computational problem then is how to determine the number of partitions of a multiset $\mathscr{M}$ having a_i copies of i, for $1 \leqslant i \leqslant k$. Our solution is to systematically generate each such partition, and count them in the process. To make the generation process systematic, we impose the structure of a rooted tree on the collection of all partitions of $\mathscr{M}$. The partitions are then enumerated by a standard tree-traversal algorithm of computer science called "preorder traversal"; for a description of this algorithm, see, for example, [10, p. 334]. Thus, our algorithm is specified by describing how the tree structure is imposed.

First, if B_1 and B_2 are submultisets of $\mathscr{M}$ let us write "$B_1 \geqslant B_2$" to mean that B_1 is lexicographically larger than B_2, where B_1 and B_2 themselves are written with their elements in decreasing order. We agree to always write a partition π of $\mathscr{M}$ with the blocks in order

$$\pi = (B_1, B_2, \ldots, B_l), \qquad B_1 \geqslant B_2 \geqslant \cdots \geqslant B_l.$$

In the case where $\mathscr{M}$ contains simply a_1 copies of 1, a partition is the usual notion of "numerical partition of the integer a_1," and the above convention agrees with the traditional way of writing numerical partitions.

Now let $\pi = (B_1, B_2, \ldots, B_l)$ and $\pi' = (B_1', B_2', \ldots, B_{l-1}')$ be two partitions of $\mathscr{M}$ with l and $l-1$ blocks, respectively. Let us say that π' is an immediate offspring of π (or that π is the parent of π') provided these conditions are met: for some $j < l$,

(i) $B_i' = B_i$ for all $i < j$,

(ii) each of B_i' and B_i contains exactly one element for all $i > j$,

(iii) $B_j' = B_j \cup B_k$ for some $k > j$, and the unique element of B_k is the smallest element of B_j'.

We check that with this definition every partition has a unique parent with one exception, namely the partition whose every block contains one element. This latter partition is the root of our tree. Finding a partition's parent is simple: with the blocks written in lexicographically decreasing order, remove the smallest element from the rightmost nonsingleton block and let it become a singleton. Thus, for example, with $\mathscr{M} = \{3, 3, 3, 3, 2, 2, 2, 1, 1, 1\}$ the unique path from π to the root is given as follows:

$$\begin{aligned} \pi = {} & \{3,3,1\}\ \{3,2,1\}\ \{3,2\}\ \{2,1\} \\ & \{3,3,1\}\ \{3,2,1\}\ \{3,2\}\ \{2\}\ \{1\} \\ & \{3,3,1\}\ \{3,2,1\}\ \{3\}\ \{2\}\ \{2\}\ \{1\} \\ & \{3,3,1\}\ \{3,2\}\ \{3\}\ \{2\}\ \{2\}\ \{1\}\ \{1\} \\ & \{3,3,1\}\ \{3\}\ \{3\}\ \{2\}\ \{2\}\ \{2\}\ \{1\}\ \{1\} \\ & \{3,3\}\ \{3\}\ \{3\}\ \{2\}\ \{2\}\ \{2\}\ \{1\}\ \{1\}\ \{1\} \\ & \{3\}\ \{3\}\ \{3\}\ \{3\}\ \{2\}\ \{2\}\ \{2\}\ \{1\}\ \{1\}\ \{1\}. \end{aligned}$$

In [12], MacMahon presents a table of values of $f(n)$ for those n which divide one of $2^{10} \cdot 3^8$, $2^{10} \cdot 3 \cdot 5$, $2^9 \cdot 3^2 \cdot 5$, $2^8 \cdot 3^3 \cdot 5$, $2^6 \cdot 3^2 \cdot 5^2$, $2^5 \cdot 3^3 \cdot 5^2$. There are four values of $f(n)$ which disagree with our computations. We double-checked our computations for these numbers by a different algorithm and have come to the conclusion that MacMahon's figures are in error. Specifically

$$\begin{aligned} f(2^{10} \cdot 3^5) &= 3804, && \text{not } 3737, \\ f(2^9 \cdot 3^8) &= 13715, && \text{not } 13748, \\ f(2^{10} \cdot 3^8) &= 21893, && \text{not } 21938, \\ f(2^4 \cdot 3 \cdot 5) &= 38, && \text{not } 28. \end{aligned}$$

The latter two discrepancies could have been typographical errors. MacMahon does not state how he prepared his table. He states as his "Cardinal Theorem" a formula for the generating function

$$\sum_{n_1=0}^{\infty} f(2^{n_1} 3^{n_2} \cdots p_s^{n_s})\, x^{n_1}$$

(the exponent on 2 is variable, all others fixed). The formula involves a summation over all factorizations of $3^{n_2} \cdots p_s^{n_s}$, so it is probably not a better means of enumeration than what we have done.

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