Herbrand Universe and Herbrand Base

Summary

- Herbrand Universe [Chang-Lee Ch. 4.3]
- Herbrand Base [Chang-Lee Ch. 4.3]

Background

Desiderata

- We know how to prove a theorem by proving that a set of clauses S is inconsistent
- Given a set of clauses S to prove that S is inconsistent we need to check all interpretations over all possible domains: Unfeasible!
- Given Church and Turing we aim for a procedure that stops in a finite number of steps when S is unsatisfiable
  - This might not be good enough but it is the best we can have!
- How can we find such a procedure ?

Herbrand's Approach

General Idea

Given S set of clauses

- Focus on a unique domain (**Herbrand Universe**)
- Turn the S into a series of ground clauses over this domain (**Expansion**)
- S is unsatisfiable iff we can find an unsatisfiable set of ground clauses in a finite number of steps
- Herbrand proved we can actually do this

Hint on the Expansion Concept

On Infinite Domain

Quantifiers can not be reduced to a finite number of check

Example

$$\forall x \exists y G(x, y)$$

If domain is the the natural numbers and $G(x, y)$ represents that x is greater or equal than y, intuitively this proposition is true, but we can not really check for every instance

Hint on the Expansion Concept II

On Finite Domain

Quantifiers can be reduced to a finite number of check:

- $\forall x P(x) = P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$
- $\exists x P(x) = P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$

Example

$$\forall x \exists y G(x, y)$$

If domain is the the interval [0,10] we can actually check that this is true

Herbrand Universe

Definition (Herbrand Universe)

Given S set of clauses universally quantified (Skolem Standard Form)

- H_0 all constants in S. If no constant in S then $H_0 = a$
- For $i > 0$ $H_i = H_{i-1}$ united with the set of all terms $f^n(t_1, \cdots, t_n)$ for all functions f^n occurring in S where $t_j \in H_{i-1}$ $j = 1, \cdots, n$.
- H_i is called the ith level constant set of S
- H_∞ is called the herbrand universe of S

Example of Herbrand Universe I

Example (no function)

Let $S = \{P(x) \vee Q(x), R(z), T(y) \vee \neg W(y)\}$

Example of Herbrand Universe I

Example (no function)

Let $S = \{P(x) \vee Q(x), R(z), T(y) \vee \neg W(y)\}$

No function

1. $H_0 = \{a\}$ (no constant)
2. $H_1 = H_0$ (no function)
3. $\cdots$
4. $H_\infty = \{a\}$

Example of Herbrand Universe II

Example (one function)

Let $S = \{P(a), \neg P(x) \vee P(f(x))\}$

Example of Herbrand Universe II

Example (one function)

Let $S = \{P(a), \neg P(x) \vee P(f(x))\}$

1. $H_0 = \{a\}$ (all constants)
2. $H_1 = \{a, f(a)\}$ (all functions applied to all terms in H_0 union H_0)
3. $H_2 = \{a, f(a), f(f(a))\}$ (all functions applied to all terms in H_1 union H_1)
4. $\cdots$
5. $H_\infty = \{a, f(a), f(f(a)), f(f(f((a)))), \cdots\}$

Example of Herbrand Universe III

Example (two functions)

Let $S = \{P(f(x), a, g(y), b)\}$

Example of Herbrand Universe III

Example (two functions)

Let $S = \{P(f(x), a, g(y), b)\}$

1. $H_0 = \{a, b\}$ (all constants)
2. $H_1 = \{a, b, f(a), g(a), f(b), g(b)\}$ (all functions applied to all terms in H_0 union H_0)
3. $H_2 = \{a, b, f(a), g(a), f(b), g(b), f(f(a)), f(g(a)), f(f(b)), f(g(b)), g(f(a)), g(g(a)), g(f(b)), g(g(b))\}$ (all functions applied to all terms in H_1 union H_1)
4. $\cdots$

Exercise

Example

Let $F = \forall x(P(x) \rightarrow Q(x)) \wedge \exists y P(y) \wedge \forall z \neg Q(z)$

- $S = \neg P(x) \vee Q(x), P(a), \neg Q(z)$
- $S' = \neg P(x) \vee Q(x), P(f(x)), \neg Q(z)$

Exercise

Example

Let $F = \forall x(P(x) \rightarrow Q(x)) \land \exists y P(y) \land \forall z \neg Q(z)$

- $S = \neg P(x) \lor Q(x), P(a), \neg Q(z)$
- $S' = \neg P(x) \lor Q(x), P(f(x)), \neg Q(z)$

Herbrand Universe for S

1. $H_0 = H_1 = \cdots = H_\infty = \{a\}$.

Exercise

Example

Let $F = \forall x(P(x) \rightarrow Q(x)) \wedge \exists y P(y) \wedge \forall z \neg Q(z)$

- $S = \neg P(x) \vee Q(x), P(a), \neg Q(z)$
- $S' = \neg P(x) \vee Q(x), P(f(x)), \neg Q(z)$

Herbrand Universe for S

1. $H_0 = H_1 = \cdots = H_\infty = \{a\}$.

Herbrand Universe for S'

1. $H_0 = \{a\}$
2. $H_2 = \{a, f(a)\}$
3. $\cdots$
4. $H_\infty = \{a, f(a), f(f(a)), \cdots\}$.

Herbrand Base

Definition (Herbrand base)

Let S be a set of clauses. the Herbrand Base for S is the set of all ground atoms of the form $P^n(t_1, \cdots, t_n)$ for all $n - placed$ predicates occurring in S, where $t_j \; j = 1, \cdots, n$ are elements of the Herbrand universe of S.

Herbrand Base: Example

Example

Let $S = \{P(f(x)), Q(g(c))\}$

- The Herbrand Universe:
  $\{c, f(c), g(c), f(f(c)), f(g(c)), g(f(c)), g(g(c)), \cdots\}$
- The Herbrand Base:
  $\{P(c), Q(c), P(f(c)), P(g(c)), Q(f(c))$
  $Q(g(c)), P(f(f(c))), P(f(g(c))), \cdots\}$

Ground Instances for Clauses

Definition (Ground instances for an Herbrand Base)

A ground instance of a clause C of a set S of clauses is a clause obtained by replacing variables in C by members of the Herbrand universe

Example (Ground instances for an Herbrand Base)

Let $S = \{P(x), Q(f(y)) \vee R(y)\}$

1. $C = P(x)$ is a clause
2. $H = \{a, f(a), f(f(a)), \cdots\}$
3. $P(f(a))$ and $P(a)$ are ground instances of C.

Exercises

Exercise

- Given $S = \{P(f(x), a, g(f(x), b))\}$
  1. Find H_0 and H_1
  2. Find all the ground instances of S over H_0
  3. Find all the ground instances of S over H_1
- Let F be a formula and let S denote the standard form of $\neg F$. Find the necessary and sufficient condition for F such that the Herbrand Universe of S is finite.