A. Supplementary Information: relation between different relative error criteria

Our relative error criterion of $|\sin(\hat{W}, w)|$ differs somewhat from the criterion used in (Negahban et al., 2016), which was

$$\frac{||\hat{W} - w||_2}{||w||_2},$$

where both w and and $\hat{W}$ need to be normalized to sum to 1. To represent this compactly, we introduce the notation $D(x, y)$ for positive vectors x, y, defined as

$$D(x,y) = \frac{\left|\left| \frac{y}{||y||_1} - \frac{x}{||x||_1} \right|\right|_2}{\left|\left| \frac{y}{||y||_1} \right|\right|_2},$$

so that the criterion of (Negahban et al., 2016) can be written simply as $D(\hat{W}, w)$.

We will show that if $\hat{W}$ and w satisfy $\max_{i,j} w_i/w_j \leq b$ and $\max_{i,j} \hat{W}_i/\hat{W}_j \leq b$, then the two relative error criteria are within a multiplicative factor of $\sqrt{b}$. Thus, ignoring factors depending on the the skewness b, we may pass from one to the other at will.

The proof will require a sequence of lemmas, which we present next. The first lemma provides some inequalities satisfied by the the sine error measure.

Lemma A.1. *Let $x, y \in \Re^n$ and denote by $\sin(x, y)$ the sine of the angle made by these vectors. Then we have that*

$$|\sin(x,y)| = \min_\beta \frac{||\beta x - y||_2}{||y||_2} = \inf_{\alpha \neq 0} \frac{||x - \alpha y||_2}{||\alpha y||_2}$$

Moreover, if the angle between x and y is less than $\pi/2$ (which always holds when x and y are nonnegative), we have that

$$\frac{1}{\sqrt{2}} \left|\left| \frac{x}{||x||_2} - \frac{y}{||y||_2} \right|\right|_2 \leq |\sin(x,y)| \leq \left|\left| \frac{x}{||x||_2} - \frac{y}{||y||_2} \right|\right|_2. \tag{25}$$

Moreover, since $\sin(x, y) = \sin(y, x)$ the expressions remain valid if we permute x and y.

Proof. We begin with the first equality. Observe that $\min_\beta ||\beta x - y||_2$ is the distance between y and its orthogonal projection on the 1-dimensional subspace spanned by x; by definition of sine, this is also $||y||_2 |\sin(x, y)|$, which implies the equality sought.

The second equality directly follows from the change of variable $\alpha = 1/\beta$. Passing from min to inf is necessary is necessary in case the optimal β is 0, which happens when x and y are orthogonal.

Let now θ be the angle made by x and y. An analysis of the triangle defined by 0, $x/||x||_2$ and $y/||y||_2$ shows that $\sin(x, y) = \sin(\frac{\pi-\theta}{2}) \left|\left| \frac{x}{||x||_2} - \frac{y}{||y||_2} \right|\right|_2$, which implies (25) since $\theta \in [0, \frac{\pi}{2}]$. ■

We will also need the following lemma on the ratio between the 1- and 2- norms of vectors.

Lemma A.2. *Let $x \in \Re^n_+$ be such that $\max_{i,j} \frac{x_i}{x_j} \leq b$. Then*

$$\frac{||x||_2}{||x||_1} \leq \min\left(1, \sqrt{\frac{b}{n}}\right).$$

Proof. That $||x||_2 \leq ||x||_1 \cdot 1$ is well-known. To prove the same with 1 replaced by $\sqrt{\frac{b}{n}}$, we argue as follows. First, without loss of generality, we may assume $x_i \in [1, b]$ for all i. Let Z be any random variable supported on the interval $[1, b]$. Observe that

$$E[Z^2] \leq bE[Z] \leq bE[Z]^2,$$

where the first inequality follows because $Z \leq b$ and the second inequality follows because $E[Z] \geq 1$. We can rearrange this as

$$\frac{E[Z]^2}{E[Z^2]} \geq \frac{1}{b}.$$

Now let Z be uniform over $x_1, \ldots, x_n$. In this case, this last inequality specializes to

$$\frac{\left((1/n)\sum_{i=1}^n x_i\right)^2}{(1/n)\sum_{i=1}^n x_i^2} \geq \frac{1}{b},$$

or

$$\frac{||x||_1^2}{||x||_2^2} \geq \frac{n}{b},$$

and now, inverting both sides and taking square roots, we obtain what we need to show. ■

Lemma A.3. $|\sin(x, y)| \leq D(x, y)$ *holds for nonnegative $x, y \in \Re^n$.*

Proof.

$$\begin{aligned} D(x,y) &= \frac{\left|\left| \frac{x}{||x||_1} - \frac{y}{||y||_1} \right|\right|_2}{\left|\left| \frac{y}{||y||_1} \right|\right|_2} \\ &= \frac{\left|\left| x\frac{||y||_1}{||x||_1} - y \right|\right|_2}{||y||_2} \\ &\geq \inf_\beta \frac{||\beta x - y||_2}{||y||_2} \\ &= |\sin(x,y)|, \end{aligned}$$

where the last step used Lemma A.1. ■

Lemma A.4. *Suppose $x \in \Re^n_+$ and $\max_{i,j} \frac{x_i}{x_j} \le b$. Then there holds*

$$D(x,y) \le \min\left(1+\sqrt{n}, 1+\sqrt{b}\right)\sqrt{2}\sin(x,y)$$

Proof. Without loss of generality, we assume $||x|| = ||y|| = 1$, which means we can simplify $\left|\left|\frac{x}{||x||_2} - \frac{y}{||y||_2}\right|\right|_2$ as $||x-y||_2$. Since $||y||_3 = 1$, we have

$$\begin{aligned} D(x,y) &= \frac{\left|\left|\frac{x}{1^T x} - \frac{y}{1^T y}\right|\right|_2}{\left|\left|\frac{y}{1^T y}\right|\right|_2} \\ &\le \left|\left|\frac{1^T y}{1^T x} x - y\right|\right|_2 \\ &\le ||x-y||_2 + ||x||_2 \left|\frac{1^T y}{1^T x} - 1\right| \\ &= ||x-y||_2 + \frac{||x||_2}{1^T x}\left|1^T(y-x)\right| \\ &\le ||x-y||_2\left(1+\sqrt{n}\frac{||x||_2}{||x||_1}\right), \end{aligned}$$

where in the last inequality we have used

$$\left|1^T(y-x)\right| \le ||y-x||_1 \le \sqrt{n}\,||y-x||\,,$$

and $||y-x||_2 \le 1$ due to the positivity of x and y. Now using Lemma A.1 to bound $||x-y||_2 \le \sqrt{2}\sin(x,y)$, we have that the first part of the bound follows then from $||x||_2 \le ||x||_1$, and the second one from Lemma A.2. ■

B. Supplementary Information: proof of Theorem 2

Our starting point is a lemma from (Hajek & Raginsky), which we will use throughout the lower bound proofs, and which we introduce next.

Let $d(w,w')$ be a metric on $\mathcal{W}\times\mathcal{W}$. Let $P_w(y)$ be an indexed family of probability distributions on the observation space $\mathcal{Y}$. Let $\hat{w}(y)$ be an estimator based on observations $y \in \mathcal{Y}$ and let $\mathbf{Y}$ represent the random vector associated with the observations conditioned on w. We use $E_{\mathbf{Y}}[\cdot]$ to denote expectation with respect to the randomness in $\mathbf{Y}$.

We first lower bound the worst-case error by means of a Bayesian prior. Namely, we observe that if we generate w according to some distribution π, then using $E_\pi[\cdot]$ to denote expectation when w is generated this way, we have

$$\sup_{w\in\mathcal{W}} \mathbb{E}_{\mathbf{Y}}[d(w,\hat{w}(\mathbf{Y})] \ge \mathbb{E}_{\pi,\mathbf{Y}}[d(w,\hat{w}(\mathbf{Y})]] \qquad (26)$$

We will use [(Hajek & Raginsky) Chap. 13, Corollary 13.2] to obtain a lower bound on (components of) the latter quantity.

Lemma B.1. *Let π be any prior distribution on $\mathcal{W}$, and let μ be any joint probability distribution of a random pair $(w,w') \in \mathcal{W}\times\mathcal{W}$, such that the marginal distributions of both w and w' are equal to π. Then*

$$\mathbb{E}_{\pi,\mathbf{Y}}[d(w,\hat{w}(\mathbf{Y})]] \ge \mathbb{E}_\mu[d(w,w')(1-\|P_w - P_{w'}\|_{TV}]$$

where $\|\cdot\|_{\mathrm{TV}}$ represents the total-variation distance between distributions.

We will need a slight generalization of the Lemma for our purposes. In particular, we note that it is sufficient that the measure $d(w,w')$ satisfies a weak version of triangle inequality, i.e., $\gamma d(w_1,w_2) \le d(w_1,\hat{w}) + d(w_2,\hat{w})$ for some pre-specified constant γ. Following along the same lines as the proof of Le-Cam's two-point method in [(Hajek & Raginsky)] we get:

$$\sup_{w\in\mathcal{W}} \mathbb{E}_w[d(w,\hat{w})] \ge \gamma\mathbb{E}_\mu[d(w,w')(1-\|P_w - P_{w'}\|_{\mathrm{TV}}] \quad (27)$$

Next, to apply this lemma we need to associate the random variables of interest in our problem with the the measure P_w. The random variable Y_e and the corresponding observations y_e are associated with the edge $e \in E$ of our graph. In particular, let B_e be the eth row of B. Recall that BB^T is the graph Laplacian. For an edge $e = (ij)$, let $y_e = 1$ if i wins over j and -1 otherwise.

We now define our distribution π: Let $B = \sum_{i=1}^n \sigma_i u_i v_i^T$ be a singular decomposition of B. We augment the collection of singular vectors $\sigma_i, v_i, i = 1,2,\ldots,d$ with the constant vector $v_0 = \frac{1}{\sqrt{n}}\mathbf{1}$. We observe that this collection $V = [v_0, v_1\ldots,v_n]$ forms an orthonormal basis. We overload notation and collect the observations, y_e, $e\in E$ into a vector $\mathbf{y}$ and the corresponding random-variable $\mathbf{Y}$. We specify define $\pi(w)$ by placing a uniform distribution on the hypercube $\{-1,1\}^n$. We then let $z = (z_1,\ldots,z_n) \sim$ Unif$\{-1,1\}^n$ and write:

$$w_z = V\Lambda z = \sqrt{n}v_0 + \delta\sum_{i=1}^n \frac{z_i}{\sigma_i} v_i \qquad (28)$$

where, δ is a suitably small number to be specified later. So, in particular, $\lambda_0 = \sqrt{n}$ and $\lambda_i = \delta/\sigma_i$ for $i = 1,2,\ldots,n$. We note that the norm of w_z's defined this way are all equal, i.e.,

$$\begin{aligned} \|w_z\| &= \|V\Lambda z\| = \|\Lambda z\| \\ &= \sqrt{n + \delta^2\sum_{i=1}^n \frac{1}{\sigma_i^2}} \end{aligned} \qquad (29)$$

Our (square) error criterion $\sin^2(\hat{W}, w)$, is lower bounded by

$$\frac{1}{2}\rho(w,\hat{w}) := \frac{1}{2}\left\|\frac{w}{\|w\|} - \frac{\hat{w}}{\|\hat{w}\|}\right\|^2 = \rho(w,\hat{w}),$$

see Lemma A.1.

Next, we closely follow the argument in the proof of Assouad's lemma [(Hajek & Raginsky)]. To do this we need to express $\rho(w,\hat{w})$ as a decomposable metric. To this end, let $\hat{\alpha}(y) = V^T\hat{w}(y)$. We will suppress dependence on y when it is clear from the context. We write:

$$\min_{\hat{w}(\mathbf{Y})} \mathbb{E}_{\pi,\mathbf{Y}}[\rho(w,\hat{w}(\mathbf{Y}))] = \min_{\hat{w}} \mathbb{E}_{\pi,\mathbf{Y}} \left\| \frac{w}{\|w\|} - \frac{\hat{w}}{\|\hat{w}\|} \right\|^2$$

$$\begin{aligned} &= \min_{\hat{w}(\mathbf{Y})} \mathbb{E}_{\pi,\mathbf{Y}} \left\| V^T \left(\frac{w}{\|w\|} - \frac{\hat{w}}{\|\hat{w}\|} \right) \right\|^2 \\ &= \min_{\hat{\alpha}(\mathbf{Y})} \mathbb{E}_{\pi,\mathbf{Y}} \sum_{i=0}^{n} \left(\frac{\lambda_i z_i}{\|\Lambda z\|} - \frac{\hat{\alpha}_i}{\|\hat{\alpha}\|} \right)^2 \\ &\geq \sum_{i=1}^{n} \min_{\beta_i(\mathbf{Y})} \mathbb{E}_{\pi,\mathbf{Y}} \left(\frac{\lambda_i z_i}{\|\Lambda z\|} - \beta_i(\mathbf{Y}) \right)^2 \\ &= \sum_{i=1}^{n} \min_{\eta_i(\mathbf{Y})} \frac{\lambda_i^2}{\|\Lambda z\|^2} \mathbb{E}_{\pi,\mathbf{Y}} \left(z_i - \eta_i(\mathbf{Y}) \right)^2, \end{aligned} \quad (30)$$

where $\beta_i(\mathbf{Y}), \eta_i(\mathbf{Y})$ are estimators using the whole vector $\mathbf{Y}$ for each i, and the last equality follows from $\|\Lambda z\|$ being constant over the support of z. We are now going to apply the variation (27) of Lemma B.1 to each $\mathbb{E}_{\pi,\mathbf{Y}} d_i(z,\eta_i(\mathbf{Y})) := \mathbb{E}_{\pi,\mathbf{Y}} (z_i - \eta_i(\mathbf{Y}))^2$ individually. For this purpose, we define the distribution $\mu_i(z,z')$ by keeping z uniformly distributed in $\{-1,1\}^n$, and flipping the i^{th} bit to obtain z' (formally, $z'_i = -z_i$ and $z'_j = z_j$ for every $j \neq i$). Clearly, $\mathbb{E}_{\pi,\mathbf{Y}} d_i(z,z') = 4$. We next work on simplifying the total variation (TV) term in the expression of Lemma B.1. First, note that since we have k independent observations per-edge, we tensorize the probability distributions and denote it as $P_w^{\otimes k}$. By the Pinsker's lemma it follows that the total variation distance can be upperbounded by the the Kullback-Leibler Divergence [(Hajek & Raginsky)], and furthermore, it follows from standard algebraic manipulations (see [(Duchi) Example 3.4]) that,

$$\begin{aligned} \|P_w^{\otimes k} - P_{w'}^{\otimes k}\|_{\text{TV}}^2 &\leq \frac{1}{2} D_{KL}(P_w^{\otimes k} \| P_w'^{\otimes k}) \\ &\leq \frac{k}{4} \|B(\log(w) - \log(w'))\|^2. \end{aligned} \quad (31)$$

Indeed, recall that the probability of i winning over j is $\frac{w_i}{w_i+w_j} = \frac{1}{1+w_j/w_i}$, and observe that $B_e \log(w) = \log(w_i/w_j)$. Hence we can write

$$P_w(y_e) \triangleq \text{Prob}[Y_e = y_e \mid B_e, w] = \frac{1}{1+\exp(-y_e B_e \log(w))}.$$

Thus P_w and $P_{w'}$ satisfy the "logistic regression" distribution, and [(Duchi) Example 3.4]) derives Eq. (31) for total variation distance between such distributions.

Now we prove in Section B.1 below that for $\delta\sigma_{\max} n\Omega_{avg} \leq 1$ and $\delta^2 n\Omega_{\text{avg}}/2 \leq 1/4$, we have,

$$\|B(\log(w) - \log(w'))\|^2 \leq 16\delta^2. \quad (32)$$

Hence it follows from (27) that for every estimator $\eta_i(\mathbf{Y})$ and for such δ,

$$\mathbb{E}_{\pi,\mathbf{Y}} \left(z_i - \eta_i(\mathbf{Y}) \right)^2 \geq \gamma 4(1 - \sqrt{4k\delta^2}),$$

and then from (30) that

$$\begin{aligned} \min_{\hat{w}(\mathbf{Y})} \mathbb{E}_{\pi,\mathbf{Y}}[\rho(w,\hat{w}(\mathbf{Y}))] &\geq \gamma \sum_{i=1}^{n} \frac{\lambda_i^2}{\|\Lambda z\|^2} 4(1 - \sqrt{4k\delta^2}) \\ &\geq \gamma \sum_{i=1}^{n} \frac{4\delta^2(1 - \sqrt{4k\delta^2})}{\sigma_i^2 n} \\ &= 2\gamma\delta^2(1 - \sqrt{4k\delta^2}) \frac{n-1}{n} \Omega_{avg}, \end{aligned}$$

where we have used $\sum_i \frac{1}{\sigma_i^2} = \text{tr}(L^\dagger) = \frac{n-1}{2}\Omega_{\text{avg}}$. The result of Theorem 2 follows then from taking $\delta^2 = \frac{1}{16k}$. We need to make sure that the conditions $\delta\sigma_{\max} n\Omega_{avg} \leq 1$ and $\delta^2 n\Omega_{\text{avg}}/2 \leq 1/4$ are satisfied, and for that it suffices to take $k \geq c\sigma_{\max} n\Omega_{\text{avg}}$ for some absolute constant c. Finally, recall that $\sigma_{\max}$ is the largest singularvalue of B, and $L = BB^T$, so that $\sigma_{\max} = \sqrt{\lambda_{\max}(L)}$, so the condition we need can be written as $k \geq c\sqrt{\lambda_{\max}(L)} n\Omega_{\text{avg}}$.

B.1. Proof of Equation (32)

In this subsection, we complete the proof by providing a proof of Eq. (32). Our starting point is the observation that, $\log([w_z]_\ell) = \log(1 + \delta\sum_{j=1}^n v_{\ell j}\frac{z_j}{\sigma_j})$. Noting that by Cauchy-Schwartz inequality

$$\begin{aligned} \left| \delta \sum_{j=1}^{n} v_{\ell j} \frac{z_j}{\sigma_j} \right| &\leq \sqrt{\delta^2 \left(\sum_{j=1}^{n} \frac{1}{\sigma_j^2} \right)} \\ &= \sqrt{\delta^2 \frac{n-1}{2} \Omega_{avg}} \\ &\leq \sqrt{\delta^2 n \Omega_{avg}/2} \end{aligned} \quad (33)$$

we enforce the constraint that δ should be sufficiently small so

$$\delta^2 n\Omega_{avg}/2 \leq 1/4. \quad (34)$$

This constraint enables us to use a Taylor approximation for $\log([w_z]_\ell) - \log([w_{z'}]_\ell)$.

We use the Taylor's expansion

$$f(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(\xi)(x-1)^2,$$

for the function $f(x) = \log(x)$. This gives us

$$\log x = x - 1 + \frac{1}{2} f''(\xi)(x-1)^2,$$

where ξ belongs to the interval between 1 and x. In particular,

$$\begin{aligned} \log([w_z]_l) &= \log(1 + \delta \sum_j \frac{z_j}{\sigma_j}[v_j]_l) \\ &= \delta \sum_j \frac{z_j}{\sigma_j}[v_j]_l + C_l \delta^2 (\sum_j \frac{z_j}{\sigma_j}[v_j]_l)^2, \end{aligned}$$

where because of Eq. (33) and our bound on δ, we have that C_l is upper bounded by $(1/2) f''(1/2) = 2$.

Similarly,

$$\log([w_{z'}]_l = \delta \sum_j \frac{z'_j}{\sigma_j}[v_j]_l + C_l \delta^2 (\sum_j \frac{z'_j}{\sigma_j}[v_j]_l)^2,$$

where $C_{l'}$ is lower bounded by $(1/2) f''(3/2) = 2/9$.

Observe that, according to our joint distribution over the pair (w, w'), we have the bit i flipped, while all others remain the same, namely, $z_j = z'_j$ for $j \neq i$ and $z_i = -z'_i$. Thus

$$\log([w_z]_l) - \log([w_{z'}]_l = 2\delta \frac{z_i}{\sigma_i}[v_i]_l + (C_l - C_{l'}) \delta^2 (\sum_j \frac{z'_j}{\sigma_j}[v_j]_l)^2$$

We can write this as

$$\log w_z - \log w_{z'} = 2\delta \frac{z_i}{\sigma_i} v_i + \delta^2 h_z.$$

Recalling that V is the vector that stacks up the vectors v_i as columns, we then have

$$\begin{aligned} ||h_z||_2 &\leq ||h_z||_1 \\ &= \sum_l (2 - 2/9)(\sum_{j \neq i,0} \frac{z_j}{\sigma_j}[v_j]_l)^2 \\ &\leq \sum_l 2(\sum_{j \neq i,0} \frac{z_j}{\sigma_j} V_{lj})^2 \\ &= 2(\sum_{j \neq i,0} [V(\mathrm{diag}(\sigma)^{-1} z)]_j)^2 \\ &\leq 2||\mathrm{diag}(\sigma)^{-1} z||_2^2 \\ &= 2\sum_{j=1}^{n} \frac{1}{\sigma_j^2} \\ &= 2\mathrm{tr}(L^\dagger) \\ &\leq 2n\Omega_{\mathrm{avg}}. \end{aligned}$$

This leads us to:

$$\begin{aligned} \|B(\log(w_z) - \log(w_{z'}))\| &\leq \frac{2\delta}{\sigma_i} \|Bv_i\| + \delta^2 \|B(h_z - h_{z'})\| \\ &\leq 2\delta + 4\delta^2 \sigma_{\max} n \Omega_{avg}. \end{aligned}$$

Under the assumption that that δ is small enough so that

$$\delta \sigma_{\max} n \Omega_{\mathrm{avg}} \leq 1$$

we obtain that

$$\|B(\log(w_z) - \log(w_{z'}))\| \leq 4\delta,$$

which is what we needed to show.

C. Supplementary Information: proof of Lemma 1

We use the following version of Chernoff's inequality: if Y_l are are independent random variables with zero expectation, variances σ_l^2, and further satisfying $|Y_l| \leq 1$ almost surely, then

$$P\left(\left|\sum_{l=1}^{K} Y_l\right| \geq \lambda\sigma\right) \leq C \max\left(e^{-c\lambda^2}, e^{-c\lambda\sigma}\right), \quad (35)$$

for some absolute constants $C, c > 0$, where $\sigma^2 = \sum_{i=1}^{k} \sigma_i^2$ (see Theorem 2.1.3 of (Tao, 2012)). Note that when $\lambda \leq \sigma$, this reduces to

$$P\left(\left|\sum_{l=1}^{K} Y_l\right| \geq \lambda\sigma\right) \leq C e^{-c\lambda^2}. \quad (36)$$

Let X_{ij}^l be the outcome of the l'th coin toss comparing nodes i and j; that is, X_{ij}^l is an indicator variable equal to one if i wins the toss. We let $Y_l = X_{ij}^l - p_{ij}$. Then Y_l are independent random variables, $|Y_l| \leq 1$, and thus we can apply Eq. (35). Note that $\sigma_l^2 = 1/v_{ij}$ as shown in (6).

We apply Eq. (35) with the choice of $\lambda = \sqrt{C_{n,\delta}}$. Choosing $k \geq 4bC_{n,\delta}$, i.e. $c_2 \geq 4$ in view of Assumption 1, and using that $v_{ij} \leq 4b$, it follows that

$$\lambda^2 = C_{n,\delta} \leq \frac{k}{v_{ij}} = \sigma^2,$$

so that $\lambda \leq \sigma$. Thus Eq. (35) reduced to Eq. (36), which yields

$$P\left(|kF_{ij} - kp_{ij}| \geq \sqrt{C_{n,\delta}}\sqrt{k/v_{ij}}\right) \leq Ce^{-cC_{n,\delta}} \leq \frac{\delta}{n^2},$$

where this last inequality requires a suitable choice of the constant c_1, and we remind that kF_{ij} is the number of successes of i over j, and. Applying the union bound over the $|E| \leq n^2$ pairs i, j yields the result.

D. Supplementary Information on the experiments in Section 3

We first note that we implemented a minor modification of our algorithm: Our estimators (4) use $\log R_{ij}$, and are thus not defined when the ratio R_{ij} of wins is zero or infinite, i.e. when one agent wins no comparison with one of its neighbors. To avoid this problem, we artificially assign half a win to such agents. Note that these events are typically rare, and their joint probability tends to zero when k grows. Our error analysis can actually be shown to remain valid for our modified algorithm.

Each data point in the curves presented in Section 3 corresponds to the average error $|\sin(\hat{W}, w)|$ on a number N_{test} of independent trials, chosen sufficiently large so that the curves are stables. The weights w_i were independently randomly generated for each node i, with $\log w_i$ following a uniform distribution between 0 and $\log b$. For experiments on Erdos-Renyi graphs, a new graph was created at each trial. Disconnected graphs were discarded, so the results should be understood as conditional to the graph being connected.