## A. Proofs - Separable Objective

**Proof of Lemma 1: (Critical point structure of separable objective)**
Denoting by $\tanh(\frac{\boldsymbol{q}}{\mu})$ a vector in $\mathbb{R}^n$ with elements $\tanh(\frac{\boldsymbol{q}}{\mu})_i = \tanh(\frac{q_i}{\mu})$ we have

$$\mathrm{grad}[f_{Sep}](\boldsymbol{q})_i = (\boldsymbol{I} - \boldsymbol{q}\boldsymbol{q}^*)\tanh(\frac{\boldsymbol{q}}{\mu}).$$

Thus critical points are ones where either $\tanh(\frac{\boldsymbol{q}}{\mu}) = \boldsymbol{0}$ (which cannot happen on $\mathbb{S}^{n-1}$) or $\tanh(\frac{\boldsymbol{q}}{\mu})$ is in the nullspace of $(\boldsymbol{I} - \boldsymbol{q}\boldsymbol{q}^*)$, which implies $\tanh(\frac{\boldsymbol{q}}{\mu}) = b\boldsymbol{q}$ for some constant $b$. The equation $\tanh(\frac{x}{\mu}) = bx$ has either a single solution at the origin or 3 solutions at $\{0, \pm r(b)\}$ for some $r(b)$. Since this equation must be solved simultaneously for every element of $\boldsymbol{q}$, we obtain $\forall i \in [n] : q_i \in \{0, \pm r(b)\}$. To obtain solutions on the sphere, one then uses the freedom we have in choosing $b$ (and thus $r(b)$) such that $\|\boldsymbol{q}\| = 1$. The resulting set of critical points is thus

$$A = \mathcal{P}_{\mathbb{S}^{n-1}}\Big[\{-1, 0, 1\}^n \setminus \{\boldsymbol{0}\}\Big].$$

To prove the form of the stable manifolds, we first show that for $q_i$ such that $|q_i| = \|\boldsymbol{q}\|_\infty$ and any $q_j$ such that $|q_j| + \Delta = |q_i|$ and sufficiently small $\Delta > 0$, we have

$$-\mathrm{grad}[f_{Sep}](\boldsymbol{q})_i \mathrm{sign}(q_i) > -\mathrm{grad}[f_{Sep}](\boldsymbol{q})_j \mathrm{sign}(q_j) \tag{9}$$

For ease of notation we now assume $q_i, q_j > 0$ and hence $\Delta = q_i - q_j$, otherwise the argument can be repeated exactly with absolute values instead. The above inequality can then be written as

$$\underbrace{(q_i - q_j)\sum_{k=1}^{n}\tanh(\frac{q_k}{\mu})q_k - \tanh(\frac{q_i}{\mu}) + \tanh(\frac{q_j}{\mu})}_{\equiv h} > 0.$$

If we now define $s^2 = \sum\limits_{\substack{k=1 \\ k\neq i}}^{n-1} q_k^2$ and $q_n = \sqrt{1 - s^2 - (q_j+\Delta)^2}$ we have

$$h = \Delta\left(\begin{array}{c}\tanh(\frac{q_j+\Delta}{\mu})\,(q_j+\Delta) + \\ \tanh(\frac{\sqrt{1-s^2-(q_j+\Delta)^2}}{\mu})\sqrt{1-s^2-(q_j+\Delta)^2}\end{array}\right) \\ +\Delta\sum_{k\neq i,n}\tanh(\frac{q_k}{\mu})q_k - \tanh(\frac{q_j+\Delta}{\mu}) + \tanh(\frac{q_j}{\mu})$$

$$= \Delta\left(\begin{array}{c}\underbrace{\begin{array}{c}\sum\limits_{k\neq i,n}\tanh(\frac{q_k}{\mu})q_k + \tanh(\frac{q_j}{\mu})q_j \\ + \tanh(\frac{\sqrt{1-s^2-q_j^2}}{\mu})\sqrt{1-s^2-q_j^2}\end{array}}_{\equiv h_1} \\ \underbrace{-\mathrm{sech}^2(\frac{q_j}{\mu})\frac{1}{\mu}}_{\equiv h_2}\end{array}\right) + O(\Delta^2)$$

where the $O(\Delta^2)$ term is bounded. Defining a vector $\boldsymbol{r} \in \mathbb{R}^n$ by

$$k \neq i, n : r_k = q_k, r_i = q_j, r_n = \sqrt{1 - s^2 - q_j^2}$$

we have $\|\boldsymbol{r}\|^2 = 1$. Since $\tanh(x)$ is concave for $x > 0$, and $|r_i| \leq 1$, we find

$$h_1 = \sum_{k=1}^{n}\tanh(\frac{r_k}{\mu})r_k \geq \tanh(\frac{1}{\mu})\sum_{k=1}^{n}r_k^2 = \tanh(\frac{1}{\mu}).$$

From $|q_i| = \|\boldsymbol{q}\|_\infty$ it follows that $q_i \geq \frac{1}{\sqrt{n}}$ and thus $q_j \geq \frac{1}{\sqrt{n}} - \Delta$. Using this inequality and properties of the hyperbolic secant we obtain

$$h_2 \leq 4\exp(-2\frac{q_j}{\mu} - \log\mu) \leq \exp(\frac{2\Delta}{\mu} - \frac{2}{\mu\sqrt{n}} - \log\mu + \log 4)$$

and plugging in $\mu = \frac{c}{\sqrt{n}\log n}$ for some $c < 1$

$$\leq \exp(\frac{2\Delta}{\mu} - \frac{2\log n}{c} - \log c + \frac{1}{2}\log n + \log\log n + \log 4).$$

We can bound this quantity by a constant, say $h_2 \leq \frac{1}{2}$, by requiring

$$A \equiv \frac{2\Delta}{\mu} - \log c + (\frac{1}{2} - \frac{2}{c})\log n + \log\log n \leq -\log 8$$

and for and $c < 1$, using $-\log n + \log\log n < 0$ we have

$$A < \frac{2\Delta}{\mu} - \log c - (\frac{2}{c} - 1)\log n.$$

Since $\Delta$ can be taken arbitrarily small, it is clear that $c$ can be chosen in an $n$-independent manner such that $A \leq -\log 8$. We then find

$$h_1 - h_2 \geq \tanh(\frac{1}{\mu}) - \frac{1}{2} \geq \tanh(\sqrt{n}\log n) - \frac{1}{2} > 0$$

since this inequality is strict, $\Delta$ can be chosen small enough such that $\left|O(\Delta^2)\right| < \Delta(h_1 - h_2)$ and hence

$$h > 0,$$

proving 9.

It follows that under negative gradient flow, a point with $|q_j| < ||\boldsymbol{q}||_\infty$ cannot flow to a point $\boldsymbol{q}'$ such that $|q'_j| = ||\boldsymbol{q}'||_\infty$. From the form of the critical points, for every such $j$, $\boldsymbol{q}$ must thus flow to a point such that $q'_j = 0$ (the value of the $j$ coordinate cannot pass through 0 to a point where $|q'_j| = ||\boldsymbol{q}'||_\infty$ since from smoothness of the objective this would require passing some $\boldsymbol{q}''$ with $q''_j = 0$, at which point $\text{grad}\left[f_{\text{Sep}}\right](\boldsymbol{q}'')_j = 0$).

As for the maximal magnitude coordinates, if there is more than one coordinate satisfying $|q_{i_1}| = |q_{i_2}| = \|\boldsymbol{q}\|_\infty$, it is clear from symmetry that at any subsequent point $\boldsymbol{q}'$ along the gradient flow line $\left|q'_{i_1}\right| = \left|q'_{i_2}\right|$. These coordinates cannot change sign since from the smoothness of the objective this would require that they pass through a point where they have magnitude smaller than $1/\sqrt{n}$, at which point some other coordinate must have a larger magnitude (in order not to violate the spherical constraint), contradicting the above result for non-maximal elements. It follows that the sign pattern of these elements is preserved during the flow. Thus there is a single critical point to which any $\boldsymbol{q}$ can flow, and this is given by setting all the coordinates with $|q_j| < \|\boldsymbol{q}\|_\infty$ to 0 and multiplying the remaining coordinates by a positive constant to ensure the resulting vector is on $\mathbb{S}^n$. Denoting this critical point by $\boldsymbol{\alpha}$, there is a vector $\boldsymbol{b}$ such that $\boldsymbol{q} = \mathcal{P}_{\mathbb{S}^{n-1}}\left[\boldsymbol{a}(\boldsymbol{\alpha}) + \boldsymbol{b}\right]$ and $\text{supp}(\boldsymbol{a}(\boldsymbol{\alpha})) \cap \text{supp}(\boldsymbol{b}) = \varnothing$, $\|\boldsymbol{b}\|_\infty < 1$ with the form of $\boldsymbol{a}(\boldsymbol{\alpha})$ given by 5 . The collection of all such points defines the stable manifold of $\boldsymbol{\alpha}$.

$\square$

**Proof of Lemma 2: (Separable objective gradient projection).** i) We consider the $\text{sign}(w_i) = 1$ case; the $\text{sign}(w_i) = -1$ case follows directly. Recalling that $\boldsymbol{u}^{(i)*}\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w})) = \tanh\left(\frac{w_i}{\mu}\right) - \tanh\left(\frac{q_n}{\mu}\right)\frac{w_i}{q_n}$, we first prove

$$\tanh\left(\frac{w_i}{\mu}\right) - \tanh\left(\frac{q_n}{\mu}\right)\frac{w_i}{q_n} \geq c(q_n - w_i) \qquad (10)$$

for some $c > 0$ whose form will be determined later. The inequality clearly holds for $w_i = q_n$. To verify that it holds for smaller values of $w_i$ as well, we now show that

$$\frac{\partial}{\partial w_i}\left[\tanh\left(\frac{w_i}{\mu}\right) - \tanh\left(\frac{q_n}{\mu}\right)\frac{w_i}{q_n} - c(q_n - w_i)\right] < 0$$

which will ensure that it holds for all $w_i$. We define $s^2 = 1 - ||\boldsymbol{w}||^2 + w_i^2$ and denote $q_n = \sqrt{s^2 - w_i^2}$ to extract the $w_i$ dependence, giving

$$\frac{\partial}{\partial w_i}\left[\tanh\left(\frac{w_i}{\mu}\right) - \tanh\left(\frac{q_n}{\mu}\right)\frac{w_i}{q_n} - c(q_n - w_i)\right]$$

$$= \begin{array}{c} \frac{1}{\mu}\text{sech}^2\left(\frac{w_i}{\mu}\right) + \frac{1}{\mu}\text{sech}^2\left(\frac{\sqrt{s^2-w_i^2}}{\mu}\right)\frac{w_i^2}{s^2-w_i^2} \\ -\tanh\left(\frac{\sqrt{s^2-w_i^2}}{\mu}\right)\frac{s^2}{(s^2-w_i^2)^{3/2}} + c\left(\frac{w_i}{\sqrt{s^2-w_i^2}} + 1\right) \end{array}$$

$$\leq \begin{array}{c} \frac{4}{\mu}\left(e^{-2\frac{w_i}{\mu}} + e^{-2\frac{\sqrt{s^2-w_i^2}}{\mu}}\right) \\ -\tanh\left(\frac{\sqrt{s^2-w_i^2}}{\mu}\right)\frac{s^2}{(s^2-w_i^2)^{3/2}} + 2c \end{array}$$

Where in the last inequality we used properties of the sech function and $q_n \geq w_i$. We thus want to show

$$\frac{4}{\mu}\left(e^{-2\frac{w_i}{\mu}} + e^{-2\frac{q_n}{\mu}}\right) + 2c \leq \tanh\left(\frac{q_n}{\mu}\right)\frac{q_n^2 + w_i^2}{q_n^3}$$

and using $\log(\frac{1}{\mu})\mu \leq w_i \leq q_n$ and $c = \frac{\frac{1-\mu^2}{1+\mu^2} - 8\mu}{2}$ we have

$$\begin{aligned} &\frac{4}{\mu}\left(e^{-2\frac{w_i}{\mu}} + e^{-2\frac{q_n}{\mu}}\right) + 2c \\ &\leq \frac{8e^{-2\frac{w_i}{\mu}}}{\mu} + 2c \leq 8\mu + 2c \leq \frac{1-\mu^2}{1+\mu^2} \\ &= \tanh\left(\log(\frac{1}{\mu})\right) \leq \tanh\left(\frac{q_n}{\mu}\right)\frac{1}{q_n} \\ &< \tanh\left(\frac{q_n}{\mu}\right)\frac{q_n^2 + w_i^2}{q_n^3} \end{aligned}$$

and it follows that 10 holds. For $\mu < \frac{1}{16}$ we are guaranteed that $c > 0$.

From examining the RHS of 10 (and plugging in $q_n = \sqrt{s^2 - w_i^2}$) we see that any lower bound on the gradient of an element $w_j$ applies also to any element $|w_i| \leq |w_j|$. Since for $|w_j| = ||\boldsymbol{w}||_\infty$ we have $q_n - w_j = w_j\zeta$, for every $\log(\frac{1}{\mu})\mu \leq w_i$ we obtain the bound

$$\boldsymbol{u}^{(i)*}\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w})) \geq c\, \|\boldsymbol{w}\|_\infty \,\zeta$$

□

**Proof of Theorem 1: (Gradient descent convergence rate for separable function)**.

We obtain a convergence rate by first bounding the number of iterations of Riemannian gradient descent in $\mathcal{C}_{\zeta_0}\backslash\mathcal{C}_1$, and then considering $\mathcal{C}_1\backslash B_r^\infty$.

From Lemma 16 we obtain $\mathcal{C}_{\zeta_0}\backslash\mathcal{C}_1 \subseteq \mathcal{C}_{\zeta_0}\backslash B^\infty_{1/\sqrt{n+3}}$. Choosing $c_2$ so that $\mu < \frac{1}{2}$, we can apply Lemma 2, and for $\boldsymbol{u}$ defined in 7, we thus have

$$|w_i| > \mu\log(\frac{1}{\mu}) \Rightarrow \boldsymbol{u}^{(i)*}\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w})) > c\|\boldsymbol{w}\|_\infty\zeta_0.$$

Since from Lemma 7 the Riemannian gradient norm is bounded by $\sqrt{n}$, we can choose $c_1, c_2$ such that $\mu\log(\frac{1}{\mu}) < \frac{1}{2\sqrt{n+3}}, \eta < \frac{1}{6\sqrt{n^2+3n}}$. This choice of $\eta$ then satisfies the conditions of Lemma 17 with $r = \mu\log(\frac{1}{\mu}), b = \frac{1}{\sqrt{n+3}}, M = \sqrt{n}$, which gives that after a gradient step

$$\zeta' \geq \zeta\left(1 + \frac{c}{2}\sqrt{\frac{n}{n+3}}\eta\right) \geq \zeta\left(1 + \tilde{c}\eta\right) \tag{11}$$

for some suitably chosen $\tilde{c} > 0$. If we now define by $\boldsymbol{w}^{(t)}$ the $t$-th iterate of Riemannian gradient descent and $\zeta^{(t)} \equiv \frac{q_n^{(t)}}{\|\boldsymbol{w}^{(t)}\|_\infty} - 1, \zeta^{(0)} \equiv \zeta_0$, for iterations such that $\boldsymbol{w}^{(t)} \in \mathcal{C}_\zeta\backslash\mathcal{C}_1$ we find

$$\zeta^{(t)} \geq \zeta^{(t-1)}\left(1 + \tilde{c}\eta\right) \geq \zeta_0\left(1 + \tilde{c}\eta\right)^t$$

and the number of iterations required to exit $\mathcal{C}_{\zeta_0}\backslash\mathcal{C}_1$ is

$$t_1 = \frac{\log(\frac{1}{\zeta_0})}{\log(1 + \tilde{c}\eta)}. \tag{12}$$

To bound the remaining iterations, we use Lemma 2 to obtain that for every $\boldsymbol{w} \in \mathcal{C}_{\zeta_0}\backslash B_r^\infty$,

$$\|\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w}))\|^2 \geq \frac{\left\|\boldsymbol{u}^{(i)*}\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w}))\right\|^2}{||\boldsymbol{u}^{(i)}||^2} \geq \zeta_0^2c^2r^2$$

where we have used $||\boldsymbol{u}^{(i)}||^2 = 1 + \frac{w_i^2}{q_n^2} \leq 2$. We thus have

$$\sum_{i=0}^{T-1}\left\|\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w})^{(i)})\right\|^2$$

$$= \sum_{i=0}^{t_1-1}\left\|\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w})^{(i)})\right\|^2 + \sum_{i=t_1}^{T-1}\left\|\text{grad}[f_{\text{Sep}}](\boldsymbol{q}(\boldsymbol{w})^{(i)})\right\|^2$$

$$> \frac{\zeta_0^2c^2}{(n+3)}t_1 + (T - t_1)c^2r^2. \tag{13}$$

Choosing $\eta < \frac{1}{2L}$ where $L$ is the gradient Lipschitz constant of $f_s$, from Lemma 5 we obtain

$$\frac{2\left(f_{\text{Sep}}(\boldsymbol{q}^{(0)}) - f^*_{\text{Sep}}\right)}{\eta} > \sum_{i=0}^{T-1}\left\|\text{grad}[f_{\text{Sep}}](\boldsymbol{q}^{(i)})\right\|^2.$$

According to Lemma B, $L = 1/\mu$ and thus the above holds if we demand $\eta < \frac{\mu}{2}$. Combining 12 and 13 gives

$$T < \frac{2\left(f_{\text{Sep}}(\boldsymbol{q}^{(0)}) - f^*_{\text{Sep}}\right)}{\eta c^2r^2} + \frac{\left(1 - \frac{\zeta_0^2}{(n+3)r^2}\right)\log(\frac{1}{\zeta_0})}{\log(1 + \tilde{c}\eta)}.$$

To obtain the final rate, we use in $g(\boldsymbol{w}^0) - g^* \leq \sqrt{n}$ and $\tilde{c}\eta < 1 \Rightarrow \frac{1}{\log(1+\tilde{c}\eta)} < \frac{\tilde{C}}{\tilde{c}\eta}$ for some $\tilde{C} > 0$. Thus one can choose $C > 0$ such that

$$T < \frac{C}{\eta}\left(\frac{\sqrt{n}}{r^2} + \log(\frac{1}{\zeta_0})\right). \tag{14}$$

From Lemma 1 the ball $B_r^\infty$ contains a global minimizer of the objective, located at the origin.

The probability of initializing in $\bigcup\limits_{\breve{A}}\mathcal{C}_{\zeta_0}$ is simply given from Lemma 3 and by summing over the $2n$ possible choices of $\mathcal{C}_{\zeta_0}$, one for each global minimizer (corresponding to a single signed basis vector).

□

**Lemma 5** (Riemannian gradient descent iterate bound)**.** *For a Riemannian gradient descent algorithm on the sphere with step size $t_k < \frac{1}{2L}$, where L is a lipschitz constant for $\nabla f(\boldsymbol{q})$, one has*

$$\begin{aligned} f(\boldsymbol{q}_1) - f(\boldsymbol{q}^\star) &\geq f(\boldsymbol{q}_1) - f(\boldsymbol{q}_T) \\ &\geq \frac{t_k}{2}\|\text{grad}\,[f]\,(\boldsymbol{q}_k)\|^2. \end{aligned}$$

*Proof.* Just as in the euclidean setting, we can obtain a lower bound on progress in function values of iterates of the Riemannian gradient descent algorithm from a lower bound on the Riemannian gradient. Consider $f : S^{n-1} \to \mathbb{R}$, which has $L$-lipschitz gradient. Let $\boldsymbol{q}_k$ denote the current iterate of Riemannian gradient descent, and let $t_k > 0$ denote the step size. Then we can form the Taylor approximation to $f \circ \text{Exp}_{\boldsymbol{q}_k}(\boldsymbol{v})$ at $\boldsymbol{0}_{\boldsymbol{q}_k}$:

$$\hat{f} : B_1(\boldsymbol{0}_{\boldsymbol{q}_k}) \cap T_{\boldsymbol{q}_k}S^{n-1} \to \mathbb{R} : \boldsymbol{v} \mapsto f(\boldsymbol{q}_k) + \langle \boldsymbol{v}, \nabla f(\boldsymbol{q}_k)\rangle.$$

From Taylor's theorem, we have for any $\boldsymbol{v} \in B_1(\mathbf{0}_{\boldsymbol{q}_k}) \cap T_{\boldsymbol{q}_k} S^{n-1}$

$$|\hat{f}(\boldsymbol{v}) - f \circ \mathrm{Exp}_{\boldsymbol{q}_k}(\boldsymbol{v})| \leq \frac{1}{2} \|\mathrm{Hess}[f](\boldsymbol{q}_k)\| \|\boldsymbol{v} - \mathbf{0}_{\boldsymbol{q}_k}\|^2,$$

where the matrix norm is the operator norm on $\mathbb{R}^{n \times n}$. Using the gradient-lipschitz property of $f$, we readily compute

$$\begin{aligned} \|\mathrm{Hess}[f](\boldsymbol{q}_k)\| &\leq \|\nabla^2 f(\boldsymbol{q}_k)\| + |\langle \nabla f(\boldsymbol{q}_k), \boldsymbol{q}_k \rangle| \\ &\leq 2L, \end{aligned}$$

since $\nabla f(\mathbf{0}) = \mathbf{0}$ and $\boldsymbol{q}_k \in S^{n-1}$. We thus have

$$f \circ \mathrm{Exp}_{\boldsymbol{q}_k}(\boldsymbol{v}) \leq f(\boldsymbol{q}_k) + \langle \boldsymbol{v}, \nabla f(\boldsymbol{q}_k) \rangle + L \|\boldsymbol{v}\|^2.$$

If we put $\boldsymbol{v} = -t_k \mathrm{grad}[f](\boldsymbol{q}_k)$ and write $\boldsymbol{q}_{k+1} = \mathrm{Exp}_{\boldsymbol{q}_k}(-t_k \mathrm{grad}\,[f]\,(\boldsymbol{q}_k))$, the previous expression becomes

$$\begin{aligned} f(\boldsymbol{q}_{k+1}) &\leq f(\boldsymbol{q}_k) - t_k \|\mathrm{grad}\,[f]\,(\boldsymbol{q}_k)\|^2 + t_k^2 L \|\mathrm{grad}\,[f]\,(\boldsymbol{q}_k)\|^2 \\ &\leq f(\boldsymbol{q}_k) - \frac{t_k}{2} \|\mathrm{grad}\,[f]\,(\boldsymbol{q}_k)\|^2 \end{aligned}$$

if $t_k < \frac{1}{2L}$. Thus progress in objective value is guaranteed by lower-bounding the Riemannian gradient.

As in the euclidean setting, summing the previous expression over iterations $k$ now yields

$$\begin{aligned} \sum_{k=1}^{T-1} f(\boldsymbol{q}_k) - f(\boldsymbol{q}_{k+1}) &= f(\boldsymbol{q}_1) - f(\boldsymbol{q}_T) \\ &\geq \frac{t_k}{2} \sum_{k=1}^{T-1} \|\mathrm{grad}\,[f]\,(\boldsymbol{q}_k)\|^2; \end{aligned}$$

in addition, it holds $f(\boldsymbol{q}_1) - f(\boldsymbol{q}_T) \leq f(\boldsymbol{q}_1) - f(\boldsymbol{q}^\star)$. Plugging in a constant step size gives the desired result. □

**Lemma 6** (Lipschitz constant of $\nabla f$). *For any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, it holds*

$$\|\nabla f(\boldsymbol{x}_1) - \nabla f(\boldsymbol{x}_2)\| \leq \frac{1}{\mu} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|.$$

*Proof.* It will be enough to study a single coordinate function of $\nabla f$. Using a derivative given in section D.1, we have for $x \in \mathbb{R}$

$$\frac{d}{dx} \tanh(x/\mu) = \frac{1}{\mu} \mathrm{sech}^2 \left( \frac{x}{\mu} \right).$$

A bound on the magnitude of the derivative of this smooth function implies a lipschitz constant for $x \mapsto \tanh(x/\mu)$. To find the bound, we differentiate again and find the critical points of the function. We have, using the chain rule,

$$\begin{aligned} \frac{d}{dx} \left( \frac{1}{\mu} \mathrm{sech}^2 \left( \frac{x}{\mu} \right) \right) &= \frac{-4}{\mu} \mathrm{sech} \left( \frac{x}{\mu} \right) \cdot \frac{1}{(e^{x/\mu} + e^{-x/\mu})^2} \\ &\quad \cdot \left( \frac{1}{\mu} e^{x/\mu} - \frac{1}{\mu} e^{-x/\mu} \right) \\ &= -\frac{1}{\mu^2} \frac{e^{x/\mu} - e^{-x/\mu}}{(e^{x/\mu} + e^{-x/\mu})^3}. \end{aligned}$$

The denominator of this final expression vanishes nowhere. Hence, the only critical point satisfies $x/\mu = -x/\mu$, which implies $x = 0$. Therefore it holds

$$\frac{d}{dx} \tanh(x/\mu) \leq \frac{1}{\mu} \mathrm{sech}^2(0) = \frac{1}{\mu},$$

which shows that $\tanh(x/\mu)$ is $(1/\mu)$-lipschitz.

Now let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be any two points of $\mathbb{R}^n$. Then one has

$$\begin{aligned} \|\nabla f(\boldsymbol{x}_1) - \nabla f(\boldsymbol{x}_2)\| &= \left( \sum_i (\tanh(x_{1i}/\mu) - \tanh(x_{2i}/\mu))^2 \right)^{1/2} \\ &= \left( \sum_i |\tanh(x_{1i}/\mu) - \tanh(x_{2i}/\mu)|^2 \right)^{1/2} \\ &\leq \left( \sum_i \frac{1}{\mu} \left| \frac{x_{1i}}{\mu} - \frac{x_{2i}}{\mu} \right|^2 \right)^{1/2} \\ &= \frac{1}{\mu} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \end{aligned}$$

completing the proof. □

**Lemma 7** (Separable objective gradient bound). *The separable objective gradient obeys*

$$\|\nabla_{\boldsymbol{w}} g(\boldsymbol{w})\| \leq \sqrt{2n}$$

$$\|\mathrm{grad}[f](\boldsymbol{q})\| \leq \sqrt{n}$$

*Proof.* Recalling that the Euclidean gradient is given by $\nabla f_{\mathrm{Sep}}(\boldsymbol{q})_i = \tanh\left(\frac{q_i}{\mu}\right)$ we use Jensen's inequality, convexity of the $L^2$ norm and the triangle inequality to obtain

$$\|\nabla g_s(\boldsymbol{w})\|^2 \leq \|\nabla f_{\mathrm{Sep}}(\boldsymbol{q})\|^2 + \left| \tanh\left( \frac{q_n}{\mu} \right) \right|^2 \frac{\|\boldsymbol{w}\|^2}{q_n^2} \leq 2n$$

while

$$\|\mathrm{grad}[f_{\mathrm{Sep}}](\boldsymbol{q})\| = \|(\boldsymbol{I} - qq^*) \nabla f_{\mathrm{Sep}}(\boldsymbol{q})\| \leq \|\nabla f_{\mathrm{Sep}}(\boldsymbol{q})\| = \sqrt{n}$$

□

## B. Proofs - Dictionary Learning

**Proof of Lemma 4:(Dictionary learning population gradient)**
For simplicity we consider the case $\text{sign}(\boldsymbol{w}_i) = 1$. The converse follows by a similar argument. We have

$$\boldsymbol{u}^{(i)*}\text{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w})) =$$

$$\mathbb{E}_{\boldsymbol{x}}\left[\tanh\left(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}}{\mu}\right)\left(-x_n\frac{w_i}{q_n}+x_i\right)\right] \tag{15}$$

Following the notation of (Sun et al., 2017), we write $x_j = b_j v_j$ where $b_j \sim \text{Bern}(\theta), v_j \sim \mathcal{N}(0,1)$ and denote the vectors of these variables by $\mathcal{J}, v$ respectively. Defining $Y^{(n)} = \sum_{j\neq n} q(\boldsymbol{w})_j x_j, X^{(n)} = q_n v_n$, $Y$ is Gaussian conditioned on a certain setting of $\mathcal{J}$. Using Lemma 40 in (Sun et al., 2017) the first term in 15 is

$$-\frac{w_i\theta}{q_n^2}\mathbb{E}_{\boldsymbol{v},\mathcal{J}|b_n=1}\left[\tanh\left(\frac{Y^{(n)}+X^{(n)}}{\mu}\right)X^{(n)}\right]$$

$$= -\frac{w_i}{\mu}\theta\mathbb{E}_{\boldsymbol{v},\mathcal{J}|b_n=1}\left[\text{sech}^2\left(\frac{Y^{(n)}+X^{(n)}}{\mu}\right)\right]$$

and similarly the second term in 15 is, with $X^{(i)} = w_i v_i, Y^{(i)} = \sum_{j\neq i} q(\boldsymbol{w})_j x_j$

$$\frac{\theta}{w_i}\mathbb{E}_{\boldsymbol{v},\mathcal{J}|b_i=1}\left[\tanh\left(\frac{Y^{(i)}+X^{(i)}}{\mu}\right)X^{(i)}\right]$$

$$= \frac{w_i\theta}{\mu}\mathbb{E}_{\boldsymbol{v},\mathcal{J}|b_i=1}\left[\text{sech}^2\left(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}}{\mu}\right)\right]$$

if we now define $X = \sum_{j\neq n,i} q^*(\boldsymbol{w})_j x_j$ we have

$$\boldsymbol{u}^{(i)*}\text{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w})) =$$

$$= \frac{w_i\theta}{\mu}\left(\begin{array}{c}\mathbb{E}_{\boldsymbol{v},\mathcal{J}|b_i=1}\left[\text{sech}^2\left(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}}{\mu}\right)\right]\\ -\mathbb{E}_{\boldsymbol{v},\mathcal{J}|b_n=1}\left[\text{sech}^2\left(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}}{\mu}\right)\right]\end{array}\right)$$

$$= \frac{w_i\theta}{\mu}\mathbb{E}_{\boldsymbol{v},\mathcal{J}}\left[\begin{array}{c}\text{sech}^2\left(\frac{X+b_nq_nv_n+w_iv_i}{\mu}\right)\\ -\text{sech}^2\left(\frac{X+q_nv_n+w_ib_iv_i}{\mu}\right)\end{array}\right]$$

$$= \frac{w_i\theta(1-\theta)}{\mu}\mathbb{E}_{\boldsymbol{v},\mathcal{J}\setminus\{n,i\}}\left[\begin{array}{c}\text{sech}^2\left(\frac{X+w_iv_i}{\mu}\right)\\ -\text{sech}^2\left(\frac{X+q_nv_n}{\mu}\right)\end{array}\right] \tag{16}$$

### B.1. Bounds for $\mathbb{E}\left[\text{sech}^2(Y)\right]$

We already have a lower bound in Lemma 20 of (Sun et al., 2017) that we can use for the second term, so we need an upper bound for the first term. Following from p. 865, we define $Y \sim \mathcal{N}(0,\sigma_Y^2)$ , $Z = \exp\left(\frac{-2Y}{\mu}\right)$, and defining $\beta = 1 - \frac{1}{\sqrt{T}}$ for some $T > 1$ we have

$$\text{sech}^2(Y/\mu) = \frac{4Z}{(1+Z)^2} \leq \frac{4Z}{(1+\beta Z)^2} = \sum_{k=0}^{\infty} b_k Z^{k+1}$$

Where $b_k = (-\beta)^k(k+1)$. Using B.3 from Lemma 40 in (Sun et al., 2017) we have

$$\mathbb{E}\left[\sum_{k=0}^{\infty} b_k Z^{k+1}\mathbb{1}_{Y>0}\right] = \sum_{k=0}^{\infty} b_k\mathbb{E}\left[e^{-2(k+1)Y/\mu}\mathbb{1}_{Y>0}\right]$$

$$= \sum_{k=0}^{\infty} b_k \exp\left(\frac{1}{2}\left(\frac{2(k+1)}{\mu}\right)^2\sigma_Y^2\right)\Phi^c\left(\frac{2(k+1)}{\mu}\sigma_Y\right)$$

Where $\Phi^c(x)$ is the complementary Gaussian CDF (The exchange of summation and expectation is justified since $Y > 0$ implies $Z \in [0,1]$, see proof of Lemma 18 in (Sun et al., 2017) for details). Using the following bounds $\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2} \leq \Phi^c(x) \leq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}+\frac{3}{x^5}\right)e^{-x^2/2}$ by applying the upper (lower) bound to the even (odd) terms in the sum, and then adding a non-negative quantity, we obtain

$$\leq \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}(-\beta)^k(k+1)\left(\frac{1}{\frac{2(k+1)}{\mu}\sigma_Y} - \frac{1}{\left(\frac{2(k+1)}{\mu}\sigma_Y\right)^3}\right)$$

$$+\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\beta^k(k+1)\left(\frac{3}{\left(\frac{2(k+1)}{\mu}\sigma_Y\right)^5}\right)$$

and using $\sum_{k=0}^{\infty}(-\beta)^k = \frac{1}{1+\beta}, \sum_{k=0}^{\infty}\frac{b_k}{(k+1)^3} \geq 0, \sum_{k=0}^{\infty}\frac{|b_k|}{(k+1)^5} \leq 2$ (from Lemma 17 in (Sun et al., 2017)) and taking $T \to \infty$ so that $\beta \to 1$ we have

$$\sum_{k=0}^{\infty} b_k\mathbb{E}\left[Z^{k+1}\mathbb{1}_{Y>0}\right] \leq \frac{1}{2\sqrt{2\pi}}\frac{1}{\frac{2}{\mu}\sigma_Y} + \frac{1}{\sqrt{2\pi}}\frac{6}{\left(\frac{2}{\mu}\sigma_Y\right)^5}$$

giving the upper bound

$$\mathbb{E}\left[\mathrm{sech}^2(Y/\mu)\right] = \mathbb{E}\left[1-\tanh^2(Y/\mu)\right] \le 8\sum_{k=0}^{\infty} b_k \mathbb{E}\left[Z^{k+1}\mathbb{1}_{Y>0}\right]$$

$$\le \sqrt{\frac{2}{\pi}}\frac{\mu}{\sigma_Y} + \frac{3\mu^5}{2\sqrt{2\pi}\sigma_Y^5}$$

while the lower bound (Lemma 20 in (Sun et al., 2017)) is

$$\sqrt{\frac{2}{\pi}}\frac{\mu}{\sigma_Y} - \frac{2\mu^3}{\sqrt{2\pi}\sigma_Y^3} - \frac{3\mu^5}{2\sqrt{2\pi}\sigma_Y^5} \le \mathbb{E}\left[\mathrm{sech}^2(Y)\right]$$

### B.2. Gradient bounds

After conditioning on $\mathcal{J}\backslash\{n,i\}$ the variables $X+q_n v_n, X+q_i v_i$ are Gaussian. We can thus plug the bounds into 16 to obtain

$$\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w})) \ge \sqrt{\frac{2}{\pi}} w_i\theta(1-\theta)$$

$$*\mathbb{E}_{\mathcal{J}\backslash\{n,i\}}\left[\begin{array}{c}\frac{1}{\sqrt{\sigma_X^2+w_i^2}} - \frac{\mu^2}{\left(\sigma_X^2+w_i^2\right)^{3/2}} - \frac{3\mu^4}{4\left(\sigma_X^2+w_i^2\right)^{5/2}} \\ -\frac{1}{\sqrt{\sigma_X^2+q_n^2}} - \frac{3\mu^4}{4\left(\sigma_X^2+q_n^2\right)^{5/2}}\end{array}\right]$$

$$\ge \sqrt{\frac{2}{\pi}} w_i\theta(1-\theta)\left(\begin{array}{c}\mathbb{E}_{\mathcal{J}\backslash\{n,i\}}\left[\frac{\sqrt{\sigma_X^2+q_n^2}-\sqrt{\sigma_X^2+w_i^2}}{\sqrt{\sigma_X^2+q_n^2}\sqrt{\sigma_X^2+w_i^2}}\right] \\ -\frac{\mu^2}{w_i^3} - \frac{3\mu^4}{2w_i^5}\end{array}\right)$$

the term in the expectation is positive since $q_n > ||w||_\infty(1+\zeta) > w_i$ giving

$$\ge \sqrt{\frac{2}{\pi}} w_i\theta(1-\theta)\left(\begin{array}{c}\mathbb{E}_{\mathcal{J}\backslash\{n,i\}}\left[\begin{array}{c}\sqrt{\sigma_X^2+q_n^2} \\ -\sqrt{\sigma_X^2+w_i^2}\end{array}\right] \\ -\frac{\mu^2}{w_i^3} - \frac{3\mu^4}{2w_i^5}\end{array}\right)$$

. To extract the $\zeta$ dependence we plug in $q_n > w_i(1+\zeta)$ and develop to first order in $\zeta$ (since the resulting function of $\zeta$ is convex) giving

$$\ge \sqrt{\frac{2}{\pi}} w_i\theta(1-\theta)\left(\begin{array}{c}\mathbb{E}_{\mathcal{J}\backslash\{n,i\}}\left[\frac{w_i^2\zeta}{\sqrt{\sigma_X^2+w_i^2}}\right] \\ -\frac{\mu^2}{w_i^3} - \frac{3\mu^4}{2w_i^5}\end{array}\right)$$

$$\ge \sqrt{\frac{2}{\pi}}\theta(1-\theta)\left(w_i^3\zeta - \frac{\mu^2}{w_i^2} - \frac{3\mu^4}{2w_i^4}\right)$$

Given some $\zeta$ and $r$ such that $w_i > r$, if we now choose $\mu$ such that $\mu < \sqrt{\frac{\sqrt{1+\frac{3}{4}r^3\zeta}-1}{3}}r$ we have the desired result. This can be achieved by requiring $\mu < c_1 r^{5/2}\sqrt{\zeta}$ for a suitably chosen $c_1 > 0$.

□

**Lemma 8** (Point-wise concentration of projected gradient)**.** *For $\boldsymbol{u}^{(i)}$ defined in 7, the gradient of the objective 1 obeys*

$$\mathbb{P}\left[\left|\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}) - \mathbb{E}\left[\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q})\right]\right| \ge t\right]$$

$$\le 2\exp\left(-\frac{pt^2}{4+2\sqrt{2}t}\right)$$

**Proof of Lemma 8: (Point-wise concentration of projected gradient)** .
If we denote by $\boldsymbol{x}^i$ a column of the data matrix with entries $x_j^i \sim BG(\theta)$, we have

$$\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}))$$

$$= \frac{1}{p}\sum_{k=1}^{p}\tanh\left(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}^k}{\mu}\right)\left(x_i^k - x_n^k\frac{w_i}{q_n}\right) \equiv \frac{1}{p}\sum_{k=1}^{p} Z_k$$

. Since $\tanh(x)$ is bounded by 1,

$$|Z_k| \le \left|\left(x_i^k - x_n^k\frac{w_i}{q_n}\right)\right| \equiv \left|u^T x^k\right|$$

. Invoking Lemma 21 from (Sun et al., 2017) and $\|u\|^2 = 1 + \frac{w_i^2}{q_n^2} \le 2$ we obtain

$$\mathbb{E}\left[|Z_k|^m\right] \le \mathbb{E}_{Z\sim\mathcal{N}(0,2)}\left[|Z|^m\right] \le \sqrt{2}^m(m-1)!!$$

$$\le 2\sqrt{2}^{m-2}\frac{m!}{2}$$

and using Lemma 36 in (Sun et al., 2017) with $R = \sqrt{2}, \sigma = \sqrt{2}$ we have

$$\mathbb{P}\left[\left|\nabla g_{DL}(\boldsymbol{w})_i - \mathbb{E}\left[\nabla g_{DL}(\boldsymbol{w})_i\right]\right| \ge t\right]$$

$$\le 2\exp\left(-\frac{pt^2}{4+2\sqrt{2}t}\right)$$

□

**Lemma 9** (Projection Lipschitz Constant)**.** *The Lipschitz constant for $\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}))$ is*

$$L = 2\sqrt{n}\,\|\boldsymbol{X}\|_\infty\left(\frac{\|\boldsymbol{X}\|_\infty}{\mu}+1\right)$$

**Proof of Lemma 9: (Projection Lipschitz Constant).**
We have

$$|\boldsymbol{u}^{(j)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w})) - \boldsymbol{u}^{(j)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}'))|$$

$$= \left| \frac{1}{p}\sum_{i=1}^{p} \left[ \begin{array}{c} \tanh(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}^i}{\mu}) \left( x_j^i - \frac{x_n^i}{q_n(\boldsymbol{w})} w_j \right) \\ -\tanh(\frac{\boldsymbol{q}^*(\boldsymbol{w}')\boldsymbol{x}^i}{\mu}) \left( x_j^i - \frac{x_n^i}{q_n(\boldsymbol{w}')} w_j' \right) \end{array} \right] \right|$$

$$\equiv \left| \frac{1}{p}\sum_{i=1}^{p} \left[ \tanh(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}^i}{\mu}) s(\boldsymbol{w}) - \tanh(\frac{\boldsymbol{q}^*(\boldsymbol{w}')\boldsymbol{x}^i}{\mu}) s(\boldsymbol{w}') \right] \right|$$

where we have defined $s(\boldsymbol{w}) = x_j^i - \frac{x_n}{q_n(\boldsymbol{w})} w_j$. Using $\boldsymbol{q}(\boldsymbol{w}), \boldsymbol{q}(\boldsymbol{w}') \in C \Rightarrow q_n(\boldsymbol{w}), q_n(\boldsymbol{w}') \geq \frac{1}{2\sqrt{n}}$ we have

$$\begin{aligned} |s(\boldsymbol{w}) - s(\boldsymbol{w}')| &= |x_n^i| \left| \frac{w_j}{q_n(\boldsymbol{w})} - \frac{w_j'}{q_n(\boldsymbol{w}')} \right| \\ &\leq |x_n| \, 2\sqrt{n} \, \|\boldsymbol{w} - \boldsymbol{w}'\| \end{aligned}$$

Lemma 25 in (Sun et al., 2017) gives

$$\left| \tanh(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}}{\mu}) - \tanh(\frac{\boldsymbol{q}^*(\boldsymbol{w}')\boldsymbol{x}}{\mu}) \right| \leq \frac{2\sqrt{n}}{\mu} \|x\| \, \|\boldsymbol{w} - \boldsymbol{w}'\|$$

We also use the fact that tanh is bounded by 1 and $s(\boldsymbol{w})$ is bounded by $\|\boldsymbol{X}\|_\infty$. We can then use Lemma 23 in (Sun et al., 2017) to obtain

$$\begin{aligned} &|\boldsymbol{u}^{(j)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w})) - \boldsymbol{u}^{(j)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}'))| \\ &\leq \frac{2\sqrt{n}}{p}\sum_{i=1}^{p}(\frac{1}{\mu}\left\|x^i\right\|_\infty^2 + \left\|x^i\right\|_\infty) \|\boldsymbol{w} - \boldsymbol{w}'\| \\ &\leq 2\sqrt{n}\, \|\boldsymbol{X}\|_\infty \left( \frac{\|\boldsymbol{X}\|_\infty}{\mu} + 1 \right) \|\boldsymbol{w} - \boldsymbol{w}'\| \end{aligned}$$

we thus have $L = 2\sqrt{n}\, \|\boldsymbol{X}\|_\infty \left( \frac{\|\boldsymbol{X}\|_\infty}{\mu} + 1 \right)$. □

**Lemma 10** (Uniformized gradient fluctuations)**.** *For all $\boldsymbol{w} \in \mathcal{C}_\zeta, i \in [n]$, with probability $\mathbb{P} > \mathbb{P}_y$*

*we have*

$$\left| \begin{array}{c} \boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w})) \\ -\mathbb{E}\left[\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}))\right] \end{array} \right| \leq y(\theta,\zeta)$$

*where*

$$\mathbb{P}_y \equiv 2\exp\left( \begin{array}{c} -\frac{1}{4}\frac{py(\theta,\zeta)^2}{4+\sqrt{2}y(\theta,\zeta)} + \log(n) \\ +n\log\left( \frac{48\sqrt{n}\left(\frac{4\log(np)}{\mu} + \sqrt{\log(np)}\right)}{y(\theta,\zeta)} \right) \end{array} \right)$$

Proof: B

**Proof of Lemma 10:(Uniformized gradient fluctuations).** For $\boldsymbol{X} \in \mathbb{R}^{n\times p}$ with i.i.d. $BG(\theta)$ entries, we define the event $\mathcal{E}_\infty \equiv \{1 \leq \|\boldsymbol{X}\|_\infty \leq 4\sqrt{\log(np)}\}$. We have

$$\mathbb{P}[\mathcal{E}_\infty^c] \leq \theta(np)^{-7} + e^{-0.3\theta np}$$

For any $\varepsilon \in (0,1)$ we can construct an $\varepsilon$-net $N$ for $\mathcal{C}_\zeta \backslash B^2_{1/20\sqrt{5(n-1)}}(0)$ with at most $(3/\varepsilon)^n$ points. Using Lemma 9, on $\mathcal{E}_\infty$, $\mathrm{grad}[f_{DL}](\boldsymbol{q})_i$ is $L$-Lipschitz with

$$L = 8\sqrt{n}\left( \frac{4\log(np)}{\mu} + \sqrt{\log(np)} \right)$$

. If we choose $\varepsilon = \frac{y(\theta,\zeta)}{2L}$ we have

$$|N| \leq (\frac{6L}{y(\theta,\zeta)})^n$$

. We then denote by $\mathcal{E}_g$ the event

$$\max_{\boldsymbol{w}\in N, i\in[n]} \left| \begin{array}{c} \boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w})) \\ -\mathbb{E}\left[\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}))\right] \end{array} \right| \leq \frac{y(\theta,\zeta)}{2}$$

and obtain that on $\mathcal{E}_g \cap \mathcal{E}_\infty$

$$\sup_{\boldsymbol{w}\in\mathcal{C}_\zeta, i\in[n]} |\nabla g_{DL}(\boldsymbol{w})_i - \mathbb{E}\left[\nabla g_{DL}(\boldsymbol{w})_i\right]| \leq y(\theta,\zeta)$$

. Setting $t = \frac{b(\theta)}{2}$ in the result of Lemma 8 gives that for all $\boldsymbol{w} \in \mathcal{C}_\zeta, i \in [n]$,

$$\begin{aligned} &\mathbb{P}\left[ \left| \begin{array}{c} \boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w})) \\ -\mathbb{E}\left[\boldsymbol{u}^{(i)*}\mathrm{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}))\right] \end{array} \right| \geq \frac{y(\theta,\zeta)}{2} \right] \\ &\leq 2\exp\left( -\frac{1}{4}\frac{py(\theta,\zeta)^2}{4+2\sqrt{2}y(\theta,\zeta)} \right) \end{aligned}$$

and thus

$$\mathbb{P}\left[\mathcal{E}_g^c\right] \leq 2\exp\left( \begin{array}{c} -\frac{1}{4}\frac{py(\theta,\zeta)^2}{4+\sqrt{2}y(\theta,\zeta)^2} \\ +n\log\left(\frac{6L}{b(\theta)}\right) + \log(n) \end{array} \right)$$

□

**Lemma 11** (Gradient descent convergence rate for dictionary learning - population)**.** *For any $1 > \zeta_0 > 0$ and $s > \frac{\mu}{4\sqrt{2}}$, Riemannian gradient descent with step size $\eta < \frac{c_2 s}{n}$ on the dictionary learning population objective 8 with $\mu < \frac{c_4\sqrt{\zeta_0}}{n^{5/4}}, \theta \in (0, \frac{1}{2})$, enters a ball of radius $c_3 s$ from a target solution in*

$$T < \frac{C_1}{\eta\theta}\left( \frac{1}{s} + n\log\frac{1}{\zeta_0} \right)$$

*iterations with probability*

$$\mathbb{P} \geq 1 - 2\log(n)\zeta_0$$

*where the $c_i, C_i$ are positive constants.*

**Proof of Lemma 11: (Gradient descent convergence rate for dictionary learning - population)**

The rate will be obtained by splitting $\mathcal{C}_{\zeta_0}$ into three regions. We consider convergence to $B_s^2(0)$ since this set contains a global minimizer. Note that the balls in the proof are defined with respect to $\boldsymbol{w}$.

### B.3. $\mathcal{C}_{\zeta_0} \backslash B^2_{1/20\sqrt{5}}(0)$

The analysis in this region is completely analogous to that in the first part of the proof of Lemma 1. For every point in this set we have

$$\|\boldsymbol{w}\|_\infty > \frac{1}{20\sqrt{5(n-1)}}$$

. From Lemma 16 we know that $\sqrt{\frac{n-1}{(2+\zeta^{(t)})\zeta^{(t)}+n}} < \frac{1}{20\sqrt{5}} \Rightarrow \boldsymbol{w}^{(t)} \in B^2_{1/20\sqrt{5}}(0)$ hence in this set $\zeta < 8$. If we choose $r = \frac{1}{40\sqrt{5(n-1)}}$, since for every point in this region $r^3\zeta < 1$, we have $\frac{r^{5/2}\sqrt{\zeta}}{2\sqrt{3}} < \sqrt{\frac{\sqrt{1+\frac{3}{4}r^3\zeta}-1}{3}}r = z(r,\zeta)$ and we thus demand $\mu < \frac{\sqrt{\zeta_0}}{\left(40\sqrt{5(n-1)}\right)^{5/2}2\sqrt{3}} \leq \frac{r^{5/2}\sqrt{\zeta}}{2\sqrt{3}}$ and obtain from Lemma 4 that for $|w_i| > r$

$$\boldsymbol{u}^{(i)*}\text{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w})) \geq \frac{c_{DL}}{(8000(n-1))^{3/2}}$$

. We now require $\eta < \frac{1}{360\sqrt{5\theta n(n-1)}} = \frac{b-r}{3M}$ we can apply Lemma 17 with $b = \frac{1}{20\sqrt{5(n-1)}}, r = \frac{1}{40\sqrt{5(n-1)}}, M = \sqrt{\theta n}$ (since the maximal norm of the Riemannian gradient is $\sqrt{\theta n}$ from Lemma 12), obtaining that at every iteration in this region

$$\zeta' \geq \zeta\left(1 + \frac{\sqrt{n}c_{DL}}{2(8000(n-1))^{3/2}}\eta\right)$$

and the maximal number of iterations required to obtain $\zeta > 8$ and exit this region is given by

$$t_1 = \frac{\log(8/\zeta_0)}{\log\left(1 + \frac{\sqrt{n}c_{DL}}{2(8000(n-1))^{3/2}}\eta\right)} \tag{17}$$

### B.4. $B^2_{1/20\sqrt{5}}(0) \backslash B_s^2(0)$

According to Proposition 7 in (Sun et al., 2017), which we can apply since $s \geq \frac{\mu}{4\sqrt{2}}, \mu < \frac{9}{50}$, in this region we have

$$\frac{\boldsymbol{w}^*\nabla_{\boldsymbol{w}} g_{DL}^{pop}(\boldsymbol{w})}{\|\boldsymbol{w}\|} \geq c\theta$$

A simple calculation shows that $\nabla_{\boldsymbol{w}} g_{DL}^{pop}(\boldsymbol{w}) = \left(\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{w}}\right)^* \text{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w}))$ where $\boldsymbol{\varphi}$ is the map defined in 3, and thus

$$\frac{\boldsymbol{w}^*\left(\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{w}}\right)^* \text{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w}))}{\|\boldsymbol{w}\|} = \frac{\begin{pmatrix} \boldsymbol{w}^* \\ -\frac{\|\boldsymbol{w}\|^2}{q_n}\end{pmatrix} \text{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w}))}{\|\boldsymbol{w}\|} > \theta c \tag{18}$$

. Defining $h(\boldsymbol{q}) = \frac{\|\boldsymbol{w}\|^2}{2}$, and denoting by $\boldsymbol{q}'$ an update of Riemannian gradient descent with step size $\eta$, we have (using a Lagrange remainder term)

$$h(\boldsymbol{q}') = h(\boldsymbol{q}) + \frac{\partial h(\boldsymbol{q}')}{\partial\eta}\eta + \underbrace{\int_0^\eta dt \frac{\partial^2 h(\boldsymbol{q}')}{\partial\eta^2}_{\eta=t}(\eta - t)}_{\equiv R}$$

$$= \frac{\|\boldsymbol{w}\|^2}{2} - \left\langle \text{grad}[f_{DL}^{pop}](\boldsymbol{q}), \frac{\partial h(\boldsymbol{q})}{\partial\boldsymbol{q}}\right\rangle + R$$

where in the last line we used $\boldsymbol{q}' = \cos(g\eta)\boldsymbol{q} - \sin(g\eta)\frac{\text{grad}[f_{DL}^{pop}](\boldsymbol{q})}{g}$ where $g \equiv \|\text{grad}[f_{DL}^{pop}](\boldsymbol{q})\|$. Since $\left\langle \text{grad}[f_{DL}^{pop}](\boldsymbol{q}), \frac{\partial h(\boldsymbol{q})}{\partial\boldsymbol{q}}\right\rangle = \left\langle \text{grad}[f_{DL}^{pop}](\boldsymbol{q}), (\boldsymbol{I} - \boldsymbol{q}\boldsymbol{q}^*)\frac{\partial h(\boldsymbol{q})}{\partial\boldsymbol{q}}\right\rangle$ and

$$(\boldsymbol{I} - \boldsymbol{q}\boldsymbol{q}^*)\frac{\partial h(\boldsymbol{q})}{\partial\boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{q}\boldsymbol{q}^*)\begin{pmatrix}\boldsymbol{w} \\ -q_n\end{pmatrix}$$

$$= \begin{pmatrix}\boldsymbol{w} \\ -q_n\end{pmatrix} - (\|\boldsymbol{w}\|^2 - q_n^2)\boldsymbol{q} = 2(1 - \|\boldsymbol{w}\|^2)\begin{pmatrix}\boldsymbol{w} \\ -\frac{\|\boldsymbol{w}\|^2}{q_n}\end{pmatrix}$$

we obtain (using 18)

$$\frac{\|\boldsymbol{w}'\|^2}{2} = \frac{\|\boldsymbol{w}\|^2}{2} + 2(1 - \|\boldsymbol{w}\|^2)\eta\left\langle \text{grad}[f_{DL}^{pop}](\boldsymbol{q}), \begin{pmatrix}\boldsymbol{w} \\ -\frac{\|\boldsymbol{w}\|^2}{q_n}\end{pmatrix}\right\rangle + R$$

$$< \frac{\|\boldsymbol{w}\|^2}{2} - 2(1 - \|\boldsymbol{w}\|^2)\|\boldsymbol{w}\|\theta c\eta + R$$

It remains to bound $R$. Denoting $r = \begin{pmatrix}\boldsymbol{w} \\ -q_n\end{pmatrix}^* \text{grad}[f](\boldsymbol{q})$ we have

$$\frac{\partial^2 h(\boldsymbol{q}')}{\partial\eta^2}_{\eta=t} = \left(\frac{\partial\boldsymbol{q}'}{\partial\eta}\right)^* \frac{\partial^2 h(\boldsymbol{q})}{\partial\boldsymbol{q}\partial\boldsymbol{q}}\frac{\partial\boldsymbol{q}'}{\partial\eta}_{\eta=t} + \frac{\partial h(\boldsymbol{q})}{\partial\boldsymbol{q}}^* \frac{\partial^2\boldsymbol{q}'}{\partial\eta^2}_{\eta=t}$$

$$= \begin{array}{c}\cos^2(gt)\left(\overline{\text{grad}}[f_{DL}^{pop}](\boldsymbol{q})^2 - \text{grad}[f_{DL}^{pop}](\boldsymbol{q})_n^2\right) \\ +g^2\left(\sin^2(gt) - \cos(gt)\right)\left(\|\boldsymbol{w}\|^2 - q_n^2\right) \\ +g\sin(gt)r(1 + 2\cos(gt))\end{array}$$

hence for some $C > 0$, if $\|\text{grad}[f_{DL}^{pop}](\boldsymbol{q})\| < M$ we have

$$R < CM^2\eta^2$$

and thus choosing $\eta < \frac{(1-\|\boldsymbol{w}\|^2)\|\boldsymbol{w}\|\theta c}{CM^2}$ we find

$$\|\boldsymbol{w}'\|^2 < \|\boldsymbol{w}\|^2 - 2(1 - \|\boldsymbol{w}\|^2)\|\boldsymbol{w}\| c\theta\eta$$

and in our region of interest $\|\boldsymbol{w}'\|^2 < \|\boldsymbol{w}\|^2 - \tilde{c}s\theta\eta$ for some $\tilde{c} > 0$ and thus summing over iterations, we obtain for some $\tilde{C}_2 > 0$

$$t_2 = \frac{\tilde{C}_2}{s\theta\eta}. \tag{19}$$

From Lemma 12, $M = \sqrt{\theta n}$ and thus with a suitably chosen $c_2 > 0$, $\eta < \frac{c_2 s}{n}$ satisfies the above requirement on $\eta$ as well as the previous requirements, since $\theta < 1$.

### B.5. Final rate and distance to minimizer

Combining these results gives, we find that when initializing in $\mathcal{C}_{\zeta_0}$, the maximal number of iterations required for Riemannian gradient descent to enter $B_s^2(0)$ is

$$T \le t_1 + t_2 < \frac{C_1}{\eta\theta}\left(n\log\frac{1}{\zeta_0} + \frac{1}{s}\right)$$

for some suitably chosen $C_1$, where $t_1, t_2$ are given in 17,19. The probability of such an initialization is given by the probability of initializing in one of the $2n$ possible choices of $\mathcal{C}_\zeta$, which is bounded in Lemma 3.

Once $\boldsymbol{w} \in B_s^2(0)$, the distance in $\mathbb{R}^{n-1}$ between $\boldsymbol{w}$ and a solution to the problem (which is a signed basis vector, given by the point $\boldsymbol{w} = \boldsymbol{0}$ or an analog on a different symmetric section of the sphere) is no larger than $s$, which in turn implies that the Riemannian distance between $\boldsymbol{\varphi}(\boldsymbol{w})$ and a solution is no larger than $c_3 s$ for some $c_3 > 0$. We note that the conditions on $\mu$ can be satisfied by requiring $\mu < \frac{c_4\sqrt{\zeta_0}}{n^{5/4}}$. $\square$

**Lemma 12** (Dictionary learning gradient upper bound)**.** *The dictionary learning population gradient obeys*

$$\|\nabla_{\boldsymbol{w}} g_{DL}^{pop}(\boldsymbol{w})\| \le \sqrt{2\theta n}$$
$$\|\text{grad}[f_{DL}^{pop}](\boldsymbol{q})\| \le \sqrt{\theta n}$$

*while in the finite sample case*

$$\|\nabla_{\boldsymbol{w}} g_{DL}(\boldsymbol{w})\|^2 \le \sqrt{2n}\,\|\boldsymbol{X}\|_\infty$$
$$\|\text{grad}[f_{DL}](\boldsymbol{q})\| \le \sqrt{n}\,\|\boldsymbol{X}\|_\infty$$

*where $\boldsymbol{X}$ is the data matrix with i.i.d. $BG(\theta)$ entries.*

*Proof.* Denoting $\boldsymbol{x} \equiv (\overline{\boldsymbol{x}}, x_n)$ we have

$$\|\nabla_{\boldsymbol{w}} g_{DL}^{pop}(\boldsymbol{w})\|^2 = \left\|\mathbb{E}\left[\tanh\left(\frac{\boldsymbol{q}^*\boldsymbol{x}}{\mu}\right)\left(\overline{\boldsymbol{x}} - x_n\frac{\boldsymbol{w}}{q_n}\right)\right]\right\|^2$$

and using Jensen's inequality, convexity of the $L^2$ norm and the triangle inequality to obtain

$$\le \mathbb{E}\left[\left\|\tanh\left(\frac{\boldsymbol{q}^*\boldsymbol{x}}{\mu}\right)\overline{\boldsymbol{x}}\right\|^2 + \left\|\tanh\left(\frac{\boldsymbol{q}^*\boldsymbol{x}}{\mu}\right)\left(x_n\frac{\boldsymbol{w}}{q_n}\right)\right\|^2\right]$$
$$\le \mathbb{E}\left[\|\overline{\boldsymbol{x}}\|^2 + \left\|x_n\frac{\boldsymbol{w}}{q_n}\right\|^2\right] \le 2\theta n$$

while

$$\|\text{grad}[f_{DL}^{pop}](\boldsymbol{q})\| \le \|\nabla f_{DL}^{pop}(\boldsymbol{q})\|$$
$$= \left\|\mathbb{E}\left[\tanh\left(\frac{\boldsymbol{q}^*\boldsymbol{x}}{\mu}\right)\boldsymbol{x}\right]\right\| \le \sqrt{\theta n}$$

Similarly, in the finite sample size case one obtains

$$\|\nabla_{\boldsymbol{w}} g_{DL}(\boldsymbol{w})\|^2 \le \frac{1}{p}\sum_{i=1}^{p}\|\overline{\boldsymbol{x}}^i\|^2 + \left\|x_n^i\frac{\boldsymbol{w}}{q_n}\right\|^2 \le 2n\,\|\boldsymbol{X}\|_\infty^2$$

$$\|\text{grad}[f_{DL}](\boldsymbol{q})\| \le \frac{1}{p}\sum_{i=1}^{p}\left\|\tanh\left(\frac{\boldsymbol{q}^*\boldsymbol{x}^i}{\mu}\right)\boldsymbol{x}^i\right\|$$
$$\le \sqrt{n}\,\|\boldsymbol{X}\|_\infty$$

$\square$

**Proof of Theorem 2: (Gradient descent convergence rate for dictionary learning)** .
The proof will follow exactly that of Lemma 11, with the finite sample size fluctuations decreasing the guaranteed change in $\zeta$ or $||\boldsymbol{w}||$ at every iteration (for the initial and final stages respectively) which will adversely affect the bounds.

### B.6. $\mathcal{C}_{\zeta_0}\backslash B^2_{1/20\sqrt{5}}(0)$

To control the fluctuations in the gradient projection, we choose

$$y(\theta, \zeta_0) = \frac{\zeta_0 c_{DL}}{2(8000(n-1))^{3/2}}$$

which can be satisfied by choosing $y(\theta, \zeta_0) = \frac{c_7\theta(1-\theta)\zeta_0}{n^{3/2}}$ for an appropriate $c_7 > 0$ . According to Lemma 10, with probability greater than $\mathbb{P}_y$ we then have

$$\left|\begin{array}{c}\boldsymbol{u}^{(i)*}\text{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w})) \\ -\mathbb{E}\left[\boldsymbol{u}^{(i)*}\text{grad}[f_{DL}](\boldsymbol{q}(\boldsymbol{w}))\right]\end{array}\right| \le y(\theta, \zeta)$$

With the same condition on $\mu$ as in Lemma 11, combined with the uniformized bound on finite sample fluctuations, we have that at every point in this set

$$\boldsymbol{u}^{(i)*}\text{grad}[f_{DL}^{pop}](\boldsymbol{q}(\boldsymbol{w})) \ge \frac{c_{DL}}{2(8000(n-1))^{3/2}}$$

. According to Lemma 12 the Riemannian gradient norm is bounded by $M = \sqrt{n}\,\|\boldsymbol{X}\|_\infty$. Choosing $r, b$ as in Lemma 11, we require $\eta < \frac{1}{360\|\boldsymbol{X}\|_\infty\sqrt{5n(n-1)}} = \frac{b-r}{3M}$ and obtain from Lemma 17

$$\zeta' \geq \zeta\left(1 + \frac{\sqrt{n}c_{DL}}{4(8000(n-1))^{3/2}}\eta\right)$$

$$t_1 = \frac{\log(8/\zeta_0)}{\log\left(1 + \frac{\sqrt{n}c_{DL}}{4(8000(n-1))^{3/2}}\eta\right)} \tag{20}$$

**B.7.** $B^2_{1/20\sqrt{5}}(0)\backslash B^2_s(0)$

From Theorem 2 in (Sun et al., 2017) there are numerical constants $c_b, c_\star$ such that in this region

$$\frac{\boldsymbol{w}^*\nabla_{\boldsymbol{w}} g_{DL}(\boldsymbol{w})}{\|\boldsymbol{w}\|} = \frac{\boldsymbol{w}^*\left(\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{w}}\right)^* \mathrm{grad}[f](\boldsymbol{q}(\boldsymbol{w}))}{\|\boldsymbol{w}\|} \geq c_\star\theta$$

with probability $\mathbb{P} > 1 - c_b p^{-6}$. Following the same analysis as in Lemma 11, since from Lemma 12 the norm of the gradient gradient is bounded by $\sqrt{n}||\boldsymbol{X}||_\infty$ we require $\eta < \frac{(1-\|\boldsymbol{w}\|^2)\|\boldsymbol{w}\|\theta c_\star}{Cn||\boldsymbol{X}||^2_\infty}$ which is satisfied by requiring $\eta < \frac{\tilde{c}\theta s}{n||\boldsymbol{X}||^2_\infty}$ for some chosen $\tilde{c} > 0$. We then obtain

$$t_3 = \frac{C_2}{s\theta\eta} \tag{21}$$

for a suitably chosen $C_2 > 0$.

### B.8. Final rate and distance to minimizer

The final bound on the rate is obtained by summing over the terms for the three regions as in the population case, and convergence is again to a distance of less than $c_3 s$ from a local minimizer. The probability of achieving this rate is obtained by taking a union bound over the probability of initialization in $\mathcal{C}_{\zeta_0}$ (given in Lemma 3) and the probabilities of the bounds on the gradient fluctuations holding (from Lemma 10 and (Sun et al., 2017)). Note that the fluctuation bound events imply by construction the event $\mathcal{E}_\infty = \{1 \leq \|\boldsymbol{X}\|_\infty \leq 4\sqrt{\log(np)}\}$ hence we can replace $\|\boldsymbol{X}\|_\infty$ in the conditions on $\eta$ above by $4\sqrt{\log(np)}$. The conditions on $\eta, \mu$ can be satisfied by requiring $\eta < \frac{c_5\theta s}{n\log np}, \mu < \frac{c_6\sqrt{\zeta_0}}{n^{5/4}}$ for suitably chosen $c_5, c_6 > 0$. The bound on the number of iterations can be simplified to the form in the theorem statement as in the population case. □

## C. Generalized Phase Retrieval

We show below that negative curvature normal to stable manifolds of saddle points in strict saddle functions is a feature that is found not only in dictionary learning, and can be used to obtain efficient convergence rates for other nonconvex problems as well, by presenting an analysis of generalized phase retrieval that is along similar lines to the dictionary learning analysis. We stress that this contribution is not novel since a more thorough analysis was carried out by (Chen et al., 2018). The resulting rates are also suboptimal, and pertain only to the population objective.

Generalized phase retrieval is the problem of recovering a vector $\boldsymbol{x} \in \mathbb{C}^n$ given a set of magnitudes of projections $y_k = |\boldsymbol{x}^*\boldsymbol{a}_k|$ onto a known set of vectors $\boldsymbol{a}_k \in \mathbb{C}^n$. It arises in numerous domains including microscopy (Miao et al., 2002), acoustics (Balan et al., 2006), and quantum mechanics (Corbett, 2006) (see (Shechtman et al., 2015) for a review). Clearly $\boldsymbol{x}$ can only be recovered up to a global phase. We consider the setting where the elements of every $\boldsymbol{a}_k$ are i.i.d. complex Gaussian, (meaning $(a_k)_j = u + iv$ for $u, v \sim \mathcal{N}(0, 1/\sqrt{2})$). We analyze the least squares formulation of the problem (Candes et al., 2015) given by

$$\min_{\boldsymbol{z}\in\mathbb{C}^n} f(\boldsymbol{z}) = \frac{1}{2p}\sum_{k=1}^{p}\left(y_k^2 - |\boldsymbol{z}^*\boldsymbol{a}_k|^2\right)^2.$$

Taking the expectation (large $p$ limit) of the above objective and organizing its derivatives using Wirtinger calculus (Kreutz-Delgado, 2009), we obtain

$$\mathbb{E}[f] = \|\boldsymbol{x}\|^4 + \|\boldsymbol{z}\|^4 - \|\boldsymbol{x}\|^2\|\boldsymbol{z}\|^2 - |\boldsymbol{x}^*\boldsymbol{z}|^2 \tag{22}$$

$$\nabla\mathbb{E}[f] = \begin{bmatrix} \nabla_{\boldsymbol{z}}\mathbb{E}[f] \\ \nabla_{\overline{\boldsymbol{z}}}\mathbb{E}[f] \end{bmatrix} = \begin{bmatrix} \left((2\|\boldsymbol{z}\|^2 - \|\boldsymbol{x}\|^2)\boldsymbol{I} - \boldsymbol{x}\boldsymbol{x}^*\right)\boldsymbol{z} \\ \left((2\|\boldsymbol{z}\|^2 - \|\boldsymbol{x}\|^2)\boldsymbol{I} - \overline{\boldsymbol{x}}\boldsymbol{x}^T\right)\overline{\boldsymbol{z}} \end{bmatrix}.$$

For the remainder of this section, we analyze this objective, leaving the consideration of finite sample size effects to future work.

### C.1. The geometry of the objective

In (Sun et al., 2016) it was shown that aside from the manifold of minima

$$\breve{A} \equiv \boldsymbol{x}e^{i\theta},$$

the only critical points of $\mathbb{E}[f]$ are a maximum at $\boldsymbol{z} = \boldsymbol{0}$ and a manifold of saddle points given by

$$\hat{A}\setminus\{\boldsymbol{0}\} \equiv \left\{\boldsymbol{z}\;\middle|\;\boldsymbol{z}\in W, \|\boldsymbol{z}\| = \frac{\|\boldsymbol{x}\|}{\sqrt{2}}\right\}$$

where $W \equiv \{\boldsymbol{z}|\boldsymbol{z}^*\boldsymbol{x} = 0\}$. We decompose $\boldsymbol{z}$ as

$$\boldsymbol{z} = \boldsymbol{w} + \zeta e^{i\phi}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \tag{23}$$

where $\zeta > 0, \boldsymbol{w} \in W$. This gives $\|\boldsymbol{z}\|^2 = \|\boldsymbol{w}\|^2 + \zeta^2$. The choice of $\boldsymbol{w}, \zeta, \phi$ is unique up to factors of $2\pi$ in $\phi$, as can be seen by taking an inner product with $\boldsymbol{x}$. Since the gradient decomposes as follows:

$$\nabla_{\boldsymbol{z}}\mathbb{E}[f] = \left(2\|\boldsymbol{z}\|^2 I - \|\boldsymbol{x}\|^2 I - \boldsymbol{x}\boldsymbol{x}^*\right)(\boldsymbol{w} + \zeta e^{i\phi}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|})$$

$$= \left(2\|\boldsymbol{z}\|^2 - \|\boldsymbol{x}\|^2\right)\boldsymbol{w} + 2\zeta e^{i\phi}\left(\|\boldsymbol{z}\|^2 - \|\boldsymbol{x}\|^2\right)\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \tag{24}$$

the directions $e^{i\phi}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}$ are unaffected by gradient descent and thus the problem reduces to a two-dimensional one in the space $(\zeta, \|\boldsymbol{w}\|)$. Note also that the objective for this two-dimensional problem is a Morse function, despite the fact that in the original space there was a manifold of saddle points. It is also clear from this decomposition of the gradient that the stable manifolds of the saddles are precisely the set $W$.

It is evident from 24 that the dispersive property does not hold globally in this case. For $\boldsymbol{z} \notin B_{||\boldsymbol{x}||}$ we see that gradient descent will cause $\zeta$ to decrease, implying positive curvature normal to the stable manifolds of the saddles. This is a consequence of the global geometry of the objective. Despite this, in the region of the space that is more "interesting", namely $B_{||\boldsymbol{x}||}$, we do observe the dispersive property, and can use it to obtain a convergence rate for gradient descent.

We define a set that contains the regions that feeds into small gradient regions around saddle points within $B_{||\boldsymbol{x}||}$ by

$$\overline{Q}_{\zeta_0} \equiv \{\boldsymbol{z}(\zeta, \|\boldsymbol{w}\|)|\zeta \leq \zeta_0\}.$$

We will show that, as in the case of orthogonal dictionary learning, we can both bound the probability of initializing in (a subset of) the complement of $\overline{Q}_{\zeta_0}$ and obtain a rate for convergence of gradient descent in the case of such an initialization. [9]

We now define four regions of the space which will be used in the analysis of gradient descent:

$$S_1 \equiv \left\{\boldsymbol{z} \,\middle|\, \|\boldsymbol{z}\|^2 \leq \tfrac{1}{2}\|\boldsymbol{x}\|^2\right\}$$
$$S_2 \equiv \left\{\boldsymbol{z} \,\middle|\, \tfrac{1}{2}\|\boldsymbol{x}\|^2 < \|\boldsymbol{z}\|^2 \leq (1-c)\|\boldsymbol{x}\|^2\right\}$$
$$S_3 \equiv \left\{\boldsymbol{z} \,\middle|\, (1-c)\|\boldsymbol{x}\|^2 < \|\boldsymbol{z}\|^2 \leq \|\boldsymbol{x}\|^2\right\}$$
$$S_4 \equiv \left\{\boldsymbol{z} \,\middle|\, \|\boldsymbol{x}\|^2 < \|\boldsymbol{z}\|^2 \leq (1+c)\|\boldsymbol{x}\|^2\right\}$$

defined for some $c < \frac{1}{4}$. These are shown in Figure 4.

We now define

$$\boldsymbol{z'} \equiv \boldsymbol{z} - \eta\nabla_{\boldsymbol{z}}\mathbb{E}[f] \equiv \boldsymbol{w}' + \zeta' e^{i\phi}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \tag{25}$$

[9] $\overline{Q}_{\zeta_0}$ is equivalent to the complement of the set $C_\zeta$ used in the analysis of the separable objective and dictionary learning.

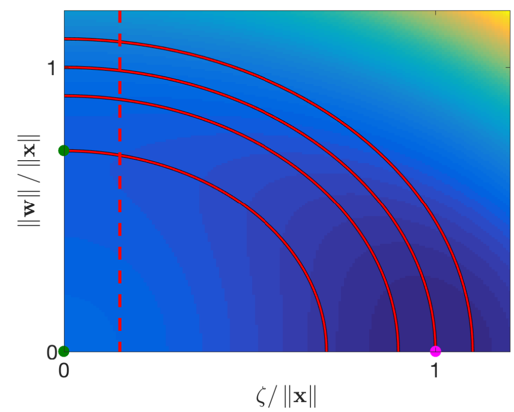


*Figure 4.* The projection of the objective of generalized phase retrieval on the $(\frac{\zeta}{\|\boldsymbol{x}\|}, \frac{\|\boldsymbol{w}\|}{\|\boldsymbol{x}\|})$ plane. The full red curves are the boundaries between the sets $S_1, S_2, S_3, S_4$ used in the analysis. The dashed red line is the boundary of the set $\overline{Q}_{\zeta_0}$ that contains small gradient regions around critical points that are not minima. The maximizer and saddle point are shown in dark green, while the minimizer is in pink.

and using 24 obtain

$$\zeta' = \left(1 - 2\eta(\|\boldsymbol{z}\|^2 - \|\boldsymbol{x}\|^2)\right)\zeta \tag{26a}$$
$$\|\boldsymbol{w}'\| = \left(1 - \eta\left(2\|\boldsymbol{z}\|^2 - \|\boldsymbol{x}\|^2\right)\right)\|\boldsymbol{w}\|. \tag{26b}$$

These are used to find the change in $\zeta, \|\boldsymbol{w}\|$ at every iteration in each region:

**On $S_1$:**
$$\zeta' \geq (1 + \eta\|\boldsymbol{x}\|^2)\zeta \tag{27a}$$
$$\|\boldsymbol{w}'\| \geq \|\boldsymbol{w}\| \tag{27b}$$

**On $S_2$:**
$$\zeta' \geq (1 + 2c\eta\|\boldsymbol{x}\|^2)\zeta \tag{27c}$$
$$\|\boldsymbol{w}'\| \leq \|\boldsymbol{w}\| \tag{27d}$$

**On $S_3$:**
$$\left(1 - \eta\|\boldsymbol{x}\|^2\right)\|\boldsymbol{w}\| \leq \|\boldsymbol{w}'\| \leq \left(1 - (1-2c)\eta\|\boldsymbol{x}\|^2\right)\|\boldsymbol{w}\| \tag{27e}$$
$$\zeta \leq \zeta' \leq (1 + 2c\eta\|\boldsymbol{x}\|^2)\zeta \tag{27f}$$

**On $S_4$:**
$$\left(1 - (1+2c)\eta\|\boldsymbol{x}\|^2\right)\|\boldsymbol{w}\| \leq \|\boldsymbol{w}'\| \leq \left(1 - \eta\|\boldsymbol{x}\|^2\right)\|\boldsymbol{w}\| \tag{27g}$$
$$(1 - 2c\eta\|\boldsymbol{x}\|^2)\zeta \leq \zeta' \leq \zeta \tag{27h}$$

### C.2. Behavior of gradient descent in $\cup_{i=1}^4 S_i$

We now show that gradient descent initialized in $S_1 \backslash \overline{Q}_{\zeta_0}$ cannot exit $\cup_{i=1}^4 S_i$ or enter $\overline{Q}_{\zeta_0}$. Lemma 14 guarantees that gradient descent initialized in $\cup_{i=1}^4 S_i$ remains in this set. From equation 27 we see that a gradient descent step can only decrease $\zeta$ if $\boldsymbol{z} \in S_4$. Under the mild assumption

$\zeta_0^2 < \frac{7}{16}\|\boldsymbol{x}\|^2$ we are guaranteed from Lemma 13 that at every iteration $\zeta \geq \zeta_0$. Thus the region with $\zeta < \zeta_0$ can only be entered if gradient descent is initialized in it. It follows that initialization in $S_1 \backslash \overline{Q}_{\zeta_0}$ rules out entering $\overline{Q}_{\zeta_0}$ at any future iteration of gradient descent. Since this guarantees that regions that feed into small gradient regions are avoided, an efficient convergence rate can again be obtained.

### C.3. Convergence rate

**Theorem 3** (Gradient descent convergence rate for generalized phase retrieval)**.** *Gradient descent on 22 with step size $\eta < \frac{\sqrt{c}}{4\|\boldsymbol{x}\|^2}, c < \frac{1}{4}$, initialized uniformly in $S_1$ converges to a point $\boldsymbol{z}$ such that $\operatorname{dist}(\boldsymbol{z}, \breve{A}) < \sqrt{5c}\|\boldsymbol{x}\|$ in*

$$T < \frac{\log\left(\frac{\|\boldsymbol{x}\|}{\zeta\sqrt{2}}\right)}{\log(1+\eta\|\boldsymbol{x}\|^2)} + \frac{\log(2)}{2\log(1+2c\eta\|\boldsymbol{x}\|^2)} + \frac{\log(2c)\log(\frac{4}{\sqrt{7}})}{\log\left(1-(1-2c)\eta\|\boldsymbol{x}\|^2\right)\log(1+2c\eta\|\boldsymbol{x}\|^2)}$$

*iterations with probability*

$$\mathbb{P} \geq 1 - \sqrt{\frac{8}{\pi}}\operatorname{erf}\left(\frac{\sqrt{2n}}{\|\boldsymbol{x}\|}\zeta\right),$$

*Proof.* Please see Appendix C.3. ☐

We find that in order to prevent the failure probability from approaching 1 in a high dimensional setting, if we assume that $\|\boldsymbol{x}\|$ does not depend on $n$ we require that $\zeta$ scale like $\frac{1}{\sqrt{n}}$. This is simply the consequence of the well-known concentration of volume of a hypersphere around the equator. Even with this dependence the convergence rate itself depends only logarithmically on dimension, and this again is a consequence of the logarithmic dependence of $\zeta$ due to the curvature properties of the objective.

**Lemma 13.** *For any iterate $\boldsymbol{z}$ of gradient descent on 22, assuming $\eta < \frac{\sqrt{c}}{4\|\boldsymbol{x}\|^2}, c < \frac{1}{4}$ and defining $\zeta'$ as in 25, we have i)*

$$\boldsymbol{z} \in \bigcup_{i=1}^{4} S_i \Rightarrow \|\boldsymbol{w}\|^2 \leq \frac{\|\boldsymbol{x}\|^2}{2}$$

*ii)*

$$\boldsymbol{z} \in S_4 \Rightarrow \zeta'^2 \geq \frac{7}{16}\|\boldsymbol{x}\|^2$$

**Proof of Lemma 13.** i) From 27 we see that in $\bigcup_{i=2}^{4} S_i$ the quantity $\|\boldsymbol{w}\|^2$ cannot increase, hence this can only happen in $S_1$. We show that for some $\boldsymbol{z} \in S_1$, a point with $\|\boldsymbol{w}\| = (1-\varepsilon)\frac{\|\boldsymbol{x}\|}{\sqrt{2}}, \varepsilon < 1$ cannot reach a point with $\|\boldsymbol{w}\|' = \frac{\|\boldsymbol{x}\|}{\sqrt{2}}$ by a gradient descent step. This would mean

$$\left(1 - \eta\left(2\|\boldsymbol{w}\|^2 + 2\zeta^2 - \|\boldsymbol{x}\|^2\right)\right)\|\boldsymbol{w}\|$$

$$= \left(1 - \eta\left((1-\varepsilon)^2\|\boldsymbol{x}\|^2 + 2\zeta^2 - \|\boldsymbol{x}\|^2\right)\right)(1-\varepsilon)\frac{\|\boldsymbol{x}\|}{\sqrt{2}}$$

$$= \frac{\|\boldsymbol{x}\|}{\sqrt{2}}$$

and since $\zeta^2 \geq 0$ this implies

$$\left(1 + \varepsilon\eta\|\boldsymbol{x}\|^2(2-\varepsilon)\right)(1-\varepsilon) \geq 1$$

by considering the product of these two factors, this in turn implies

$$\frac{1}{2b}(2-\varepsilon) \geq \eta\|\boldsymbol{x}\|^2(2-\varepsilon) \geq 1$$

where we have used $\eta < \frac{\sqrt{c}}{b\|\boldsymbol{x}\|^2}, c < \frac{1}{4}$. Thus if we choose $b = 4$ this inequality cannot be satisfied.

Additionally, if we initialize in $S_1 \cap \overline{Q}_{\zeta_0}$ then we cannot initialize at a point where $\|\boldsymbol{w}\|' = \frac{\|\boldsymbol{x}\|}{\sqrt{2}}$ and hence the inequality is strict.

ii) Since only a step from $S_4$ can decrease $\zeta$, we have that for the initial point $\|\boldsymbol{z}\|^2 > \|\boldsymbol{x}\|^2$. Combined with $\|\boldsymbol{w}\|^2 \leq \frac{\|\boldsymbol{x}\|^2}{2}$ this gives

$$\zeta^2 \geq \frac{\|\boldsymbol{x}\|^2}{2}$$

and using the lower bound $(1 - 2\eta\|\boldsymbol{x}\|^2 c)\zeta \leq \zeta'$ we obtain

$$\zeta'^2 \geq \frac{\|\boldsymbol{x}\|^2}{2}(1 - 2\eta\|\boldsymbol{x}\|^2 c)^2 \geq \frac{\|\boldsymbol{x}\|^2}{2}(1 - 4\eta\|\boldsymbol{x}\|^2 c)$$

$$\geq (1 - \frac{1}{2b})\frac{\|\boldsymbol{x}\|^2}{2}$$

where in the last inequality we used $c < \frac{1}{4}, \eta < \frac{\sqrt{c}}{b\|\boldsymbol{x}\|^2}$. Choosing $b = 4$ gives

$$\zeta'^2 \geq \frac{7}{16}\|\boldsymbol{x}\|^2$$

If we require $\zeta_0^2 < \frac{7}{16}\|\boldsymbol{x}\|^2$ this also ensures that the next iterate cannot lie in the small gradient regions around the stable manifolds of the saddles. ☐

**Lemma 14.** *Defining $\boldsymbol{z}'$ as in 25, under the conditions of Lemma 13 and we have*

*i)*

$$\boldsymbol{z} \in \bigcup_{i=2}^{4} S_i \Rightarrow \boldsymbol{z}' \in \bigcup_{i=2}^{4} S_i$$

*ii)*

$$z \in S_1 \Rightarrow z' \in S_1 \cup S_2$$

**Proof of Lemma 14.** We use the fact that for the next iterate we have

$$\begin{aligned} \|z'\|^2 &= \left(1 - \eta(2\|z\|^2 - \|x\|^2)\right)^2 \|w\|^2 \\ &+ \left(1 - 2\eta(\|z\|^2 - \|x\|^2)\right)^2 \zeta^2 \end{aligned} \tag{28}$$

We will also repeatedly use $\eta < \frac{\sqrt{c}}{b\|x\|^2}, c < \frac{1}{4}$ and $z \in \bigcup_{i=1}^{4} S_i \Rightarrow \|w\|^2 \leq \frac{\|x\|^2}{2}$ which is a shown in Lemma 13.

### C.4. $z \in S_3 \Rightarrow z' \in \bigcup_{i=2}^{4} S_i$

We want to show $\frac{\|x\|^2}{2} \underset{(1)}{<} \|z'\|^2 \underset{(2)}{\leq} (1+c)\|x\|^2$.

1) We have $z \in S_3 \Rightarrow \|z\|^2 = (1-\varepsilon)\|x\|^2$ for some $\varepsilon \leq c$ and using 28 we must show

$$\frac{\|x\|^2}{2} \leq \begin{array}{c} \left(1 - \eta\|x\|^2(1-2\varepsilon)\right)^2 \|w\|^2 \\ + \left(1 + 2\eta\|x\|^2 \varepsilon\right)^2 \zeta^2 \end{array}$$

or equivalently

$$A \equiv \varepsilon - \frac{\|x\|^2}{2}$$

$$\leq \eta\|x\|^2 \left[ \begin{array}{c} \left(-2(1-2\varepsilon) + (1-2\varepsilon)^2\eta\|x\|^2\right)\|w\|^2 \\ +4\left(\varepsilon + \varepsilon^2\eta\|x\|^2\right)\zeta^2 \end{array} \right] \equiv B$$

and using $\eta < \frac{\sqrt{c}}{b\|x\|^2}, c < \frac{1}{4}$

$$\frac{-\|x\|^2}{b} < \frac{-2\|x\|\sqrt{c}}{b} < -2\eta\|x\|^4 \leq B$$

while on the other hand

$$A \leq c - \frac{\|x\|^2}{2} < -\frac{\|x\|^2}{4}$$

thus picking $b = 4$ guarantees the desired result.

2) By a similar argument, $\|z'\|^2 \leq (1+c)\|x\|^2$ is equivalent to

$$A \equiv \eta\|x\|^2 \left[ \begin{array}{c} \left(-2(1-2\varepsilon) + \eta\|x\|^2(1-2\varepsilon)^2\right)\|w\|^2 \\ +4\left(\varepsilon + \eta\|x\|^2\varepsilon^2\right)\zeta^2 \end{array} \right]$$

$$\leq \|x\|^2 (c+\varepsilon) \equiv B$$

. Since $\|w\|^2 \leq \frac{\|x\|^2}{2}$ and $\|z\|^2 \leq \|x\|^2 \Rightarrow \zeta^2 \leq \frac{\|x\|^2}{2}$ we obtain

$$A \leq \eta\left[\eta\|x\|^4 + 4\left(\|x\|^2\varepsilon + \eta\|x\|^4\varepsilon^2\right)\right]\frac{\|x\|^2}{2}$$

$$< \frac{1}{2b}\left[\frac{1}{b} + 2\left(1 + \frac{1}{8b}\right)\right] c\|x\|^2$$

. If we choose $b = 4$ we thus have $A < B$ which implies

$$\|z'\|^2 < (1+c)\|x\|^2$$

### C.5. $z \in S_4 \Rightarrow z' \in \bigcup_{i=2}^{4} S_i$

We have $z \in S_4 \Rightarrow \|z\|^2 = \|w\|^2 + \zeta^2 = (1+\varepsilon)\|x\|^2$ for some $\varepsilon \leq c$ .

1) $\frac{\|x\|^2}{2} < \|z'\|^2$ is equivalent to

$$A \equiv -(\varepsilon + \frac{1}{2})\|x\|^2$$

$$\leq \eta\|x\|^2 \left[ \begin{array}{c} \left(-4(1+2\varepsilon) + \eta\|x\|^2(1+2\varepsilon)^2\right)\|w\|^2 \\ +4\left(-\varepsilon + \eta\|x\|^2\varepsilon^2\right)\zeta^2 \end{array} \right] \equiv B$$

. We have

$$B \geq -4\eta\|x\|^2\left((1+2\varepsilon)\|w\|^2 + \varepsilon\zeta^2\right) \geq -\frac{15}{8b}\|x\|^2$$

where the last inequality used $\|w\|^2 \leq \frac{\|x\|^2}{2}$ and $\|z\|^2 \leq \|x\|^2(1+c) \Rightarrow \zeta^2 \leq \|x\|^2(\frac{1}{2}+c)$. The choice $b = 4$ gaurantees $A \leq B$ which ensures the desired result.

2) This is trivial since $\|z\|^2 \leq (1+c)\|x\|^2$ and in $S_4$ both $\zeta$ and $\|w\|$decay at every iteration (ref eq).

### C.6. $z \in S_2 \Rightarrow z' \in \bigcup_{i=2}^{4} S_i$

1) We use $z \in S_2 \Rightarrow \|z\|^2 = \|w\|^2 + \zeta^2 = (\frac{1}{2}+\varepsilon)\|x\|^2$ for some $\varepsilon \leq \frac{1}{2} - c$ . Using a similar argument as in the previous section, we are required to show

$$-\varepsilon\|x\|^2 < \eta\|x\|^2 \left[ \begin{array}{c} 4\left(-\varepsilon + \varepsilon^2\eta\|x\|^2\right)\|w\|^2 \\ +\left(2(1-2\varepsilon) + (1-2\varepsilon)^2\eta\|x\|^2\right)\zeta^2 \end{array} \right]$$

$$\equiv B$$

where $B \geq -\varepsilon\frac{\|x\|^2}{b}$ implies that $b = 4$ gives the desired result.

2) The condition is equivalent to

$$A \equiv \eta \|\boldsymbol{x}\|^2 \left[ \begin{array}{c} 4\left(-\varepsilon + \varepsilon^2\eta \|\boldsymbol{x}\|^2\right)\|\boldsymbol{w}\|^2 \\ +\left(2(1-2\varepsilon) + (1-2\varepsilon)^2\eta\|\boldsymbol{x}\|^2\right)\zeta^2 \end{array} \right] + \varepsilon \|\boldsymbol{x}\|^2$$

$$\leq (\frac{1}{2} + c)\|\boldsymbol{x}\|^2 \equiv B$$

One can show by looking for critical points of $A(\varepsilon)$ in the range $0 \leq \varepsilon \leq \frac{1}{2}$ that $A$ is maximized at $\varepsilon = 0$, since there is only one critical point at $\varepsilon^* = \frac{4 - \frac{b}{\sqrt{c}} + 2\frac{\sqrt{c}}{b}}{8\frac{\sqrt{c}}{b}}$ and $A(\varepsilon^*) < 0$, while

$$A(\frac{1}{2}) \leq \left[\left(-2\frac{\sqrt{c}}{b} + \frac{c}{b^2}\right)\|\boldsymbol{w}\|^2\right] + \frac{1}{2}\|\boldsymbol{x}\|^2$$

$$A(0) \leq \frac{1}{2b}\left(2 + \frac{1}{2b}\right)\frac{\|\boldsymbol{x}\|^2}{2}$$

and in both cases $b = 4$ ensures $A \leq B$.

**C.7. $\boldsymbol{z} \in S_1 \Rightarrow \boldsymbol{z}' \in S_1 \cup S_2$**

We must show $\|\boldsymbol{z}'\| \leq (1-c)\|\boldsymbol{x}\|^2$ using $\|\boldsymbol{z}\|^2 = (1-\varepsilon)\frac{\|\boldsymbol{x}\|^2}{2}$ for $0 \leq \varepsilon \leq 1$.

$$\|\boldsymbol{z}'\|^2 = \left(1 + \varepsilon\eta\|\boldsymbol{x}\|^2\right)^2\|\boldsymbol{w}\|^2 + \left(1 + 2(\varepsilon+1)\eta\frac{\|\boldsymbol{x}\|^2}{2}\right)^2\zeta^2$$

$$A \equiv \eta\|\boldsymbol{x}\|^2 \left[ \begin{array}{c} \left(2\varepsilon + \varepsilon^2\eta\|\boldsymbol{x}\|^2\right)\|\boldsymbol{w}\|^2 \\ +\left(2(\varepsilon+1) + (\varepsilon+1)^2\eta\frac{\|\boldsymbol{x}\|^2}{4}\right)\zeta^2 \end{array} \right] - \varepsilon\|\boldsymbol{x}\|^2$$

$$\leq (\frac{1}{2} - c)\|\boldsymbol{x}\|^2 \equiv B$$

and since $A \leq \frac{1}{2b}\left[2 + \frac{1}{b}\right]\frac{\|\boldsymbol{x}\|^2}{2}$ and $B \geq \frac{\|\boldsymbol{x}\|^2}{4}$ once again $b = 4$ suffices to obtain the desired result. □

**Lemma 15.** *For $\boldsymbol{z}$ parametrized as in 23,*

$$\|\boldsymbol{w}\|^2 < c\|\boldsymbol{x}\|^2 \vee \zeta^2 > (1-c)\|\boldsymbol{x}\|^2$$

$$\Rightarrow \operatorname{dist}(\boldsymbol{z}, \breve{A}) < \sqrt{5c}\|\boldsymbol{x}\|$$

**Proof of Lemma 15.** Once $\|\boldsymbol{w}\|^2 < c\|\boldsymbol{x}\|^2$ for some $\boldsymbol{z} \in S_3 \cup S_4$ we have

$$\|\boldsymbol{z}\|^2 = \zeta^2 + \|\boldsymbol{w}\|^2 \geq (1-c)\|\boldsymbol{x}\|^2$$

$$\zeta^2 \geq (1-c)\|\boldsymbol{x}\|^2 - \|\boldsymbol{w}\|^2 > (1-2c)\|\boldsymbol{x}\|^2 \quad (29)$$

For some $\boldsymbol{z} = \boldsymbol{w} + \zeta e^{i\phi}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ we have

$$\operatorname{dist}^2(\boldsymbol{z}, \breve{A}) = \min_\theta \left\|e^{i\theta}\boldsymbol{x} - \boldsymbol{w} - \zeta e^{i\phi}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\|^2$$
$$= \|\boldsymbol{w}\|^2 + \min_\theta \left\|e^{i\theta}\boldsymbol{x} - \zeta e^{i\phi}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\|^2$$

$$= \|\boldsymbol{w}\|^2 + (1 - \frac{\zeta}{\|\boldsymbol{x}\|})^2\|\boldsymbol{x}\|^2 = \|\boldsymbol{z}\|^2 + \|\boldsymbol{x}\|^2 - 2\zeta\|\boldsymbol{x}\|$$

if we assume $\|\boldsymbol{z}\|^2 \leq (1+c)\|\boldsymbol{x}\|^2$

$$\operatorname{dist}^2(\boldsymbol{z}, \breve{A}) \leq (c+2)\|\boldsymbol{x}\|^2 - 2\zeta\|\boldsymbol{x}\| \quad (30)$$

plugging in the value of $\zeta$ from 29 and using fact that $-\sqrt{1-x} \leq -1 + x$ for $x < 1$ we have

$$\operatorname{dist}^2(\boldsymbol{z}, \breve{A}) < (c+2)\|\boldsymbol{x}\|^2 - 2\sqrt{1-2c}\|\boldsymbol{x}\|^2 \leq 5c\|\boldsymbol{x}\|^2$$

Alternatively, if $\zeta^2 > (1-c)\|\boldsymbol{x}\|^2$ we have from 30

$$\operatorname{dist}^2(\boldsymbol{z}, \breve{A}) \leq (c+2)\|\boldsymbol{x}\|^2 - 2\zeta\|\boldsymbol{x}\|$$
$$< (c+2)\|\boldsymbol{x}\|^2 - 2\sqrt{1-c}\|\boldsymbol{x}\|^2 \leq 3c\|\boldsymbol{x}\|^2$$

which gives the desired result. In particular, if we choose $c = \frac{1}{35}$ we converge to $\operatorname{dist}^2(\boldsymbol{z}, \breve{A}) < \frac{\|\boldsymbol{x}\|^2}{7}$, a region which is strongly convex according to (Sun et al., 2017). □

**Proof of Theorem 3: (Gradient descent convergence rate for generalized phase retrieval)** .
We now bound the number of iterations that gradient descent, after random initialization in $S_1$, requires to reach a point where one of the convergence criteria detailed in Lemma 15 is fulfilled. From Lemma 14, we know that after initialization in $S_1$ we need to consider only the set $\bigcup_{i=1}^{4} S_i$. The number of iterations in each set will be determined by the bounds on the change in $\zeta, ||\boldsymbol{w}||$ detailed in 27.

### C.7.1. ITERATIONS IN $S_1$

Assuming we initialize with some $\zeta = \zeta_0$. Then the maximal number of iterations in this region is

$$\zeta_0(1 + \eta\|\boldsymbol{x}\|^2)^{t_1} = \frac{\|\boldsymbol{x}\|}{\sqrt{2}}$$

$$t_1 = \frac{\log\left(\frac{\|\boldsymbol{x}\|}{\zeta_0\sqrt{2}}\right)}{\log(1+\eta\|\boldsymbol{x}\|^2)}$$

since after this many iterations $\|\boldsymbol{z}\|^2 \geq \zeta^2 \geq \frac{\|\boldsymbol{x}\|^2}{2}$.

#### C.7.2. ITERATIONS IN $\bigcup_{i=2}^{4} S_i$

The convergence criteria are $\|\boldsymbol{w}\|^2 < c\|\boldsymbol{x}\|^2$ or $\zeta^2 > (1-c)\|\boldsymbol{x}\|^2$.

After exiting $S_1$ and assuming the next iteration is in $S_2$, the maximal number of iterations required to reach $S_3 \cup S_4$ is obtained using

$$\zeta' \geq (1+2\eta\|\boldsymbol{x}\|^2 c)\zeta$$

and is given by

$$\frac{\|\boldsymbol{x}\|}{\sqrt{2}}(1+2\eta\|\boldsymbol{x}\|^2 c)^{t_2} = (1-c)\|\boldsymbol{x}\|^2$$

$$t_2 = \frac{\log\left(\sqrt{2(1-c)}\right)}{\log(1+2\eta\|\boldsymbol{x}\|^2 c)} \leq \frac{\log(2)}{2\log(1+2\eta\|\boldsymbol{x}\|^2 c)}$$

since after this many iterations $\|\boldsymbol{z}\|^2 \geq \zeta^2 \geq (1-c)\|\boldsymbol{x}\|^2$.

For every iteration in $S_3 \cup S_4$ we are guaranteed

$$\|\boldsymbol{w}'\| \leq \left(1-(1-2c)\eta\|\boldsymbol{x}\|^2\right)\|\boldsymbol{w}\|$$

thus using Lemmas 13.i and 15 the number of iterations in $S_3 \cup S_4$ required for convergence is given by

$$\frac{\|\boldsymbol{x}\|^2}{2}\left(1-(1-2c)\eta\|\boldsymbol{x}\|^2\right)^{t_{3+4}} = c\|\boldsymbol{x}\|^2$$

$$t_{3+4} = \frac{\log(2c)}{\log\left(1-(1-2c)\eta\|\boldsymbol{x}\|^2\right)}$$

The only concern is that after an iteration in $S_3 \cup S_4$ the next iteration might be in $S_2$. To account for this situation, we find the maximal number of iterations required to reach $S_3 \cup S_4$ again. This is obtained from the bound on $\zeta$ in Lemma 13.

Using this result, and the fact that for every iteration in $S_2$ we are guaranteed $\zeta' \geq (1+2\eta\|\boldsymbol{x}\|^2 c)\zeta$ the number of iterations required to reach $S_3 \cup S_4$ again is given by

$$\frac{\sqrt{7}}{4}\|\boldsymbol{x}\|(1+2\eta\|\boldsymbol{x}\|^2 c)^{t_r} = \sqrt{1-c}\|\boldsymbol{x}\|$$

$$t_r = \frac{\log\left(\frac{4\sqrt{1-c}}{\sqrt{7}}\right)}{\log(1+2\eta\|\boldsymbol{x}\|^2 c)} \leq \frac{\log(\frac{4}{\sqrt{7}})}{\log(1+2\eta\|\boldsymbol{x}\|^2 c)}$$

### C.8. Final rate

The final rate to convergence is

$$\begin{aligned} T &< t_1 + t_2 + t_{3+4}t_r \\ &= \frac{\log\left(\frac{\|\boldsymbol{x}\|}{\zeta\sqrt{2}}\right)}{\log(1+\eta\|\boldsymbol{x}\|^2)} + \frac{\log(2)}{2\log(1+2c\eta\|\boldsymbol{x}\|^2)} \\ &+ \frac{\log(2c)\log(\frac{4}{\sqrt{7}})}{\log\left(1-(1-2c)\eta\|\boldsymbol{x}\|^2\right)\log(1+2c\eta\|\boldsymbol{x}\|^2)} \end{aligned}$$

### C.9. Probability of the bound holding

The bound applies to an initialization with $\zeta \geq \zeta_0$, hence in $S_1 \backslash \overline{Q}_{\zeta_0}$. Assuming uniform initialization in $S_1$, the set $\overline{Q}_{\zeta_0}$ is simply a band of width $2\zeta_0$ around the equator of the ball $B_{\|\boldsymbol{x}\|/\sqrt{2}}$ (in $\mathbb{R}^{2n}$, using the natural identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$). This volume can be calculated by integrating over $2n-1$ dimensional balls of varying radius.

Denoting $r = \frac{\zeta_0\sqrt{2}}{\|\boldsymbol{x}\|}$ and by $V(n) = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})}$ the hypersphere volume, the probability of initializing in $S_1 \cap \overline{Q}_{\zeta_0}$ (and thus in a region that feeds into small gradient regions around saddle points) is

$$\begin{aligned} \mathbb{P}(\text{fail}) &= \frac{\text{Vol}(\overline{Q}_{\zeta_0})}{\text{Vol}(B_{\|\boldsymbol{x}\|/\sqrt{2}})} \\ &= \frac{V(2n-1)\int_{-r}^{r}(1-x^2)^{\frac{2n-1}{2}}dx}{V(2n)} \\ &\leq \frac{V(2n-1)\int_{-r}^{r}e^{-\frac{2n-1}{2}x^2}dx}{V(2n)} \\ &= \frac{1}{\sqrt{n-\frac{1}{2}}}\frac{n}{n-\frac{1}{2}}\frac{\Gamma(n)}{\Gamma(\frac{2n-1}{2})}\text{erf}(\sqrt{\frac{2n-1}{2}}r) \\ &\leq \sqrt{\frac{8}{\pi}}\text{erf}(\sqrt{n}r) \end{aligned}$$

. For small $\zeta$ we again find that $\mathbb{P}(\text{fail})$ scales linearly with $\zeta$, as was the case for the previous problems considered. □

## D. Auxiliary Lemmas

### D.1. Separable objective

$$\frac{\partial g_s(\boldsymbol{w})}{\partial w_i} = \tanh\left(\frac{w_i}{\mu}\right) - \tanh\left(\frac{q_n}{\mu}\right)\frac{w_i}{q_n}$$

$$\frac{\partial^2 g_s(\boldsymbol{w})}{\partial w_i \partial w_j} = \left[\frac{1}{\mu}\text{sech}^2\left(\frac{w_i}{\mu}\right) - \tanh\left(\frac{q_n}{\mu}\right)\frac{1}{q_n}\right]\delta_{ij} + \left[\frac{1}{\mu}\text{sech}^2\left(\frac{q_n}{\mu}\right)\frac{1}{q_n^2} - \tanh\left(\frac{q_n}{\mu}\right)\frac{1}{q_n^3}\right]w_i w_j$$

### D.2. Dictionary Learning

$$\nabla_{\boldsymbol{w}} g_{DL}^{pop}(\boldsymbol{w}) = \mathbb{E}\left[\tanh\left(\frac{\boldsymbol{q}^*(\boldsymbol{w})\boldsymbol{x}}{\mu}\right)\left(\overline{x} - \frac{x_n}{q_n(\boldsymbol{w})}w\right)\right]$$

### D.3. Properties of $\mathcal{C}_\zeta$

**Proof of Lemma 3: (Volume of $\mathcal{C}_\zeta$).** We are interested in the relative volume $\frac{\mathrm{Vol}(\mathcal{C}_\zeta)}{\mathrm{Vol}(\mathbb{S}^{n-1})} \equiv V_\zeta$. Using the standard solid angle formula, it is given by

$$V_\zeta = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{n/2}}\int_0^\infty e^{-\frac{\pi}{\varepsilon}x_1^2}\prod_{i=2}^{n}\int_{-x_1/(1+\zeta)}^{x_1/(1+\zeta)} e^{-\frac{\pi}{\varepsilon}x_i^2}dx_i dx_1$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\int_0^\infty e^{-\frac{\pi}{\varepsilon}x^2}\left[\mathrm{erf}\left(\frac{x}{(1+\zeta)}\sqrt{\frac{\pi}{\varepsilon}}\right)\right]^{n-1}dx$$

changing variables to $\tilde{x} = \sqrt{\frac{\pi}{\varepsilon}}\frac{x}{(1+\zeta)}$

$$V_\zeta = \frac{(1+\zeta)}{\sqrt{\pi}}\int_0^\infty e^{-(1+\zeta)^2x^2}\mathrm{erf}^{n-1}(x)dx$$

This integral admits no closed form solution but one can construct a linear approximation around small $\zeta$ and show that it is convex. Thus the approximation provides a lower bound for $V_\zeta$ and an upper bound on the failure probability.

From symmetry considerations the zero-order term is $V_0 = \frac{1}{2n}$. The first-order term is given by

$$\left.\frac{\partial V_\zeta}{\partial\zeta}\right|_{\zeta=0} = \frac{1}{n} - \frac{2}{\sqrt{\pi}}\int_0^\infty x^2e^{-x^2}\mathrm{erf}^{n-1}(x)dx$$

We now require an upper bound for the second integral since we are interested in a lower bound for $V_\zeta$. We can express it in terms of the second moment of the $L^\infty$ norm of a Gaussian vector as follows:

$$\frac{1}{\sqrt{\pi}}\int_0^\infty x^2e^{-x^2}\mathrm{erf}^{n-1}(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty x^2e^{-x^2}\prod_i\frac{1}{\sqrt{\pi}}\int_{-x}^{x}e^{-t_i^2}dt_i dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{x^2}{2}e^{-x^2/2}\prod_i\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-t_i^2/2}dt_i dx$$

$$= \frac{1}{4n}\int\left\|\boldsymbol{X}\right\|_\infty^2 d\mu(\boldsymbol{X})$$

$$= \frac{1}{4n}\left(\mathrm{Var}\left[\left\|\boldsymbol{X}\right\|_\infty\right] + \left(\mathbb{E}\left[\left\|\boldsymbol{X}\right\|_\infty\right]\right)^2\right)$$

where $\mu(\boldsymbol{X})$ is the Gaussian measure on the vector $\boldsymbol{X} \in \mathbb{R}^n$. We can bound the first term using

$$\mathrm{Var}\left[\left\|\boldsymbol{X}\right\|_\infty\right] \leq \max_i \mathrm{Var}\left[|X_i|\right] = \mathrm{Var}\left[|X_i|\right] < \mathrm{Var}\left[X_i\right] = 1$$

To bound the second term, we use the fact that for a standard Gaussian vector $\boldsymbol{X}$ ($X_i \sim \mathcal{N}(0,1)$) and any $\lambda > 0$ we have

$$\exp\left(\lambda\mathbb{E}\left[\left\|\boldsymbol{X}\right\|_\infty\right]\right) \leq \mathbb{E}\left[\exp\left(\lambda\max_i|X_i|\right)\right]$$

$$\leq \mathbb{E}\left[\sum_i \exp\left(\lambda|X_i|\right)\right] = n\mathbb{E}\left[\exp\left(\lambda|X_i|\right)\right]$$

(using convexity and non-negativity of the exponent respectively)

$$n\mathbb{E}\left[\exp\left(\lambda|X_i|\right)\right] = 2n\int_0^\infty \exp\left(\lambda X_i\right)d\mu(X_i)$$

$$\leq 2n\mathbb{E}\left[\exp\left(\lambda X_i\right)\right] = 2n\exp\left(\frac{\lambda^2}{2}\right)$$

taking the log of both sides gives

$$\mathbb{E}\left[\max_i|X_i|\right] \leq \frac{\log(2n)}{\lambda} + \frac{\lambda}{2}$$

and the bound is minimized for $\lambda = \sqrt{2\log(2n)}$ giving

$$\mathbb{E}\left[\max_i|X_i|\right] \leq \sqrt{2\log(2n)} \sim \sqrt{2\log(n)}$$

Combining these bounds, the leading order behavior of the gradient is

$$\left.\frac{\partial V_\zeta}{\partial\zeta}\right|_{\zeta=0} \geq \frac{3-4\log(2n)}{4n} \geq -\frac{\log(n)}{n}.$$

This linear approximation is indeed a lower bound, since using integration by parts twice we have

$$\frac{\partial^2 V_\zeta}{\partial\zeta^2} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-(1+\zeta)^2x^2}\begin{pmatrix} -6(1+\zeta)x^2 \\ +4(1+\zeta)^3x^4\end{pmatrix}\mathrm{erf}^{n-1}(x)dx$$

$$= -\frac{2(n-1)}{\pi}\int_0^\infty e^{-(1+\zeta)^2x^2}\left(1-2(1+\zeta)^2x^2\right)e^{-x^2}\text{erf}^{n-2}(x)dx$$

$$= \frac{4(n-1)(n-2)(1+\zeta)}{\pi^{3/2}}\int_0^\infty e^{-((1+\zeta)^2+2)x^2}\text{erf}^{n-3}(x)dx > 0$$

where the last inequality holds for any $n > 2$ since the integrand is non-negative everywhere. This gives

$$V_\zeta \geq \frac{1}{2n} - \frac{\log(n)}{n}\zeta$$

□

**Lemma 16.** *$B^\infty_{s(\zeta)}(0) \subseteq \mathcal{C}_\zeta \subseteq B^2_{\sqrt{n-1}s(\zeta)}(0)$ where $s(\zeta) = \frac{1}{\sqrt{(2+\zeta)\zeta+n}}$. $B^\infty_{s(\zeta)}(0)$ is the largest $L^\infty$ ball contained in $\mathcal{C}_\zeta$, and $B^2_{\sqrt{n-1}s(\zeta)}(0)$ is the smallest $L^2$ ball containing $\mathcal{C}_\zeta$ (where these balls are defined in terms of the $\boldsymbol{w}$ vector). All three intersect only at the points where all the coordinates of $\boldsymbol{w}$ have equal magnitude. Additionally, $\mathcal{C}_\zeta \subseteq B^\infty_{1/\sqrt{2+\zeta}}(0)$ and this is the smallest $L^\infty$ ball containing $\mathcal{C}_\zeta$.*

*Proof.* Given the surface of some $L^\infty$ ball for $\boldsymbol{w}$ , we can ask what is the minimal $\zeta$ such that $\partial C_{\zeta_m}$ intersects this surface. This amounts to finding the minimal $q_n$ given some $\|\boldsymbol{w}\|_\infty$. Yet this is clearly obtained by setting all the coordinates of $w$ to be equal to $\|\boldsymbol{w}\|_\infty$ (this is possible since we are guaranteed $q_n \geq \|\boldsymbol{w}\|_\infty \Rightarrow \|\boldsymbol{w}\|_\infty \leq \frac{1}{\sqrt{n}}$), giving

$$\frac{\sqrt{1-(n-1)\|\boldsymbol{w}\|_\infty^2}}{\|\boldsymbol{w}\|_\infty} = 1+\zeta_m$$

$$\|\boldsymbol{w}\|_\infty = \frac{1}{\sqrt{(1+\zeta_m)^2+n-1}}$$

thus, given some $\zeta$, the maximal $L^\infty$ ball that is contained in $\mathcal{C}_\zeta$ has radius $\frac{1}{\sqrt{(2+\zeta)\zeta+n}}$. The minimal $L^\infty$ norm containing $\mathcal{C}_\zeta$ can be shown by a similar argument to be $B^\infty_{1/\sqrt{1+(1+\zeta)^2}}(0)$, where one instead maximizes $q_n$ with some fixed $\|\boldsymbol{w}\|_\infty$.

Given some surface of an $L^2$ ball, we can ask what is the maximal $\mathcal{C}_\zeta$ such that $\mathcal{C}_\zeta \subseteq B^2_r(0)$. This is equivalent to finding the maximal $\zeta_M$ such that $\partial C_{\zeta_M}$ intersects the surface of the $L^2$ ball. Since $q_n$ is fixed, maximizing $\zeta$ is equivalent to minimizing $\|\boldsymbol{w}\|_\infty$. This is done by setting $\|\boldsymbol{w}\|_\infty = \frac{\|\boldsymbol{w}\|}{\sqrt{n-1}}$, which gives

$$\frac{\sqrt{1-\|\boldsymbol{w}\|^2}}{\|\boldsymbol{w}\|}\sqrt{n-1} = 1+\zeta_M$$

$$\sqrt{\frac{n-1}{(2+\zeta_M)\zeta_M+n}} = \|\boldsymbol{w}\|$$

The statement in the lemma follows from combining these results. □

**Lemma 17** (Geometric Increase in $\zeta$)**.** *For $\boldsymbol{w} \in C_{\zeta_0}\backslash B^\infty_b$ (where $\zeta \equiv \frac{q_n}{\|\boldsymbol{w}\|_\infty} - 1$), assume $|w_i| > r \Rightarrow \boldsymbol{u}^{(i)*}\text{grad}[f](\boldsymbol{q}(\boldsymbol{w})) \geq c(\boldsymbol{w})\zeta$ where $\boldsymbol{u}^{(i)}$ is defined in 7 and $1 > b > r$. Then if $\|\text{grad}[f](\boldsymbol{q}(\boldsymbol{w}))\| < M$ and we define*

$$\boldsymbol{q}' \equiv \exp_{\boldsymbol{q}}(-\eta\text{grad}[f](\boldsymbol{q}))$$

*for $\eta < \frac{b-r}{3M}$, defining $\zeta'$ in an analogous way to $\zeta$ we have*

$$\zeta' \geq \zeta\left(1+\frac{\sqrt{n}}{2}\eta c(\boldsymbol{w})\right)$$

**Proof of Lemma 17:(Geometric Increase in $\zeta$).** Denoting $g \equiv \|\text{grad}[f](\boldsymbol{q})\|$, we have

$$\boldsymbol{q}' = \cos(g\eta)\boldsymbol{q} - \sin(g\eta)\frac{\text{grad}[f](\boldsymbol{q})}{g}$$

hence, using Lagrange remainder terms,

$$\frac{q'_n}{w'_i} = \frac{q_n - \eta\text{grad}[f](\boldsymbol{q})_n - \int_0^{g\eta}\cos(t)(g\eta-t)dt q_n + \int_0^{g\eta}\sin(t)(g\eta-t)dt\frac{\text{grad}[f](\boldsymbol{q})_n}{g}}{w_i - \eta\text{grad}[f](\boldsymbol{q})_i - \int_0^{g\eta}\cos(t)(g\eta-t)dt w_i + \int_0^{g\eta}\sin(t)(g\eta-t)dt\frac{\text{grad}[f](\boldsymbol{q})_i}{g}}$$

. We assume $w_i > 0$, and the converse case is analogous. From convexity of $\frac{1}{1+x}$

$$\frac{q'_n}{w'_i} \geq \frac{q_n}{w_i} + \left[\frac{\eta}{w_i} - \frac{\int_0^{g\eta}\sin(t)(g\eta-t)dt}{w_i g}\right] * \left(\text{grad}[f](\boldsymbol{q})_i - \frac{w_i}{q_n}\text{grad}[f](\boldsymbol{q})_n\right)$$

$$= \frac{q_n}{w_i} + \frac{\sin(g\eta)}{w_i g}\left(\text{grad}[f](\boldsymbol{q})_i - \frac{w_i}{q_n}\text{grad}[f](\boldsymbol{q})_n\right)$$

$$= \frac{q_n}{w_i} + \frac{\sin(g\eta)}{w_i g} \boldsymbol{u}^{(i)*} \mathrm{grad}[f](\boldsymbol{q}(\boldsymbol{w}))$$

We now use $\eta < \frac{b-r}{3M} < \frac{\pi}{2M} \Rightarrow g\eta < \frac{\pi}{2} \Rightarrow \sin(g\eta) \geq \frac{g\eta}{2}$ and consider two cases. If $|w_i| > r$ we use the bound on the gradient projection in the lemma statement to obtain

$$\frac{q'_n}{w'_i} \geq \frac{q_n}{w_i} + \frac{\eta}{2w_i} c(\boldsymbol{w})\zeta \geq \frac{q_n}{w_i} + \frac{\sqrt{n}}{2} \eta c(\boldsymbol{w})\zeta$$

hence

$$\frac{q'_n}{w'_i} - 1 \geq \frac{q_n}{\|\boldsymbol{w}\|_\infty} - 1 + \frac{\sqrt{n}}{2} \eta c(\boldsymbol{w})\zeta = \zeta \left(1 + \frac{\sqrt{n}}{2} \eta c(\boldsymbol{w})\right) \tag{31}$$

If $|w_i| < r$ we rule out the possibility that $|w'_i| = \|\boldsymbol{w}'\|_\infty$ by demanding $\eta < \frac{b-r}{3M}$. Since $b(b-r) < 1$ we have $1 + \frac{1}{3} b(b-r) < \sqrt{1 + b(b-r)}$ hence the requirement on $\eta$ implies

$$\eta < \frac{\sqrt{1 + b(b-r)} - 1}{gb} = \frac{-2g + \sqrt{4g^2 + 4g^2 b(b-r)}}{2g^2 b}$$

. If we now combine this with the fact that after a Riemannian gradient step $\cos(g\eta)q_i - \sin(g\eta) \leq q'_i \leq \cos(g\eta)q_i + \sin(g\eta)$, the above condition on $\eta$ implies the inequality $(*)$, which in turn ensures that $|w_i| < r \Rightarrow |w'_i| < \|\boldsymbol{w}'\|_\infty$:

$$|w'_i| < |w_i| + \sin(g\eta) < r + g\eta \underset{(*)}{<} (1 - g^2\eta^2)b - g\eta$$

$$< \cos(g\eta) \|\boldsymbol{w}\|_\infty - \sin(g\eta) \leq \|\boldsymbol{w}'\|_\infty$$

Due to the above analysis, it is evident that any $w'_i$ such that $|w'_i| = \|\boldsymbol{w}'\|_\infty$ obeys $|w_i| > r$, from which it follows that we can use 31 to obtain

$$\frac{q'_n}{\|\boldsymbol{w}'\|_\infty} - 1 = \zeta' \geq \zeta \left(1 + \frac{\sqrt{n}}{2} \eta c(\boldsymbol{w})\right)$$

□