Supplementary Material for Weak Detection of Signal in the Spiked Wigner Model

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In this appendix, we prove the theorems and technical results in the main article, "Weak Detection of Signal in the Spiked Wigner Model."

NOTATIONAL REMARKS

We use the standard big-O and little-o notation: $a_N = O(b_N)$ implies that there exists N_0 such that $a_N \le C b_N$ for some constant $C > 0$ independent of N for all $N \ge N_0$; $a_N = o(b_N)$ implies that for any positive constant ϵ there exists N_0 such that $a_N \le \epsilon b_N$ for all $N \ge N_0$.

For X and Y, which can be deterministic numbers and/or random variables depending on N, we use the notation $X = \mathcal{O}(Y)$ if for any (small) $\epsilon > 0$ and (large) $D > 0$ there exists $N_0 \equiv N_0(\epsilon, D)$ such that $\mathbb{P}(|X| > N^{\epsilon}|Y|) < N^{-D}$ whenever $N > N_0$.

For an event Ω, we say that Ω holds with high probability if for any (large) $D > 0$ there exists $N_0 \equiv N_0(D)$ such that $\mathbb{P}(\Omega^c) < N^{-D}$ whenever $N > N_0$.

A. Proof of Theorem 5

We adapt the strategy of Bai and Silverstein (Bai & Silverstein, 2004), and Bai and Yao (Bai & Yao, 2005). In this method, we first express the left-hand side of (4) by using a contour integral via Cauchy's integration formula. The integral is then written in terms of the Stieltjes transforms of the empirical spectral measure and the semicircle measure. Since the Stieltjes transform of the empirical spectral measure converges weakly to a Gaussian process, we find that the linear eigenvalue statistic also converges to a Gaussian random variable. Precise control of error terms requires estimates on the resolvents from random matrix theory, which are known as the local laws.

Denote by ρ_N the empirical spectral distribution of M, i.e.,

$$\rho_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\mu_i}. \tag{A.1}$$

As $N \to \infty$, ρ_N converges to the Wigner semicircle measure ρ, defined by

$$\rho(\mathrm{d}x) = \frac{\sqrt{(4-x^2)_+}}{2\pi} \mathrm{d}x. \tag{A.2}$$

Choose (N-independent) constants $a_- \in (-3, -2)$, $a_+ \in (2, 3)$, and $v_0 \in (0, 1)$ so that the function f is analytic on the rectangular contour Γ whose vertices are $(a_- \pm \mathrm{i}v_0)$ and $(a_+ \pm \mathrm{i}v_0)$. Since $\|M\| \to 2$ almost surely, we assume that all

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eigenvalues of M are contained in Γ. Thus, from Cauchy's integral formula, we find that

$$\begin{aligned}\sum_{i=1}^N f(\mu_i) &= \sum_{i=1}^N \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(z)}{z-\mu_i} \mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma f(z) \left(\sum_{i=1}^N \frac{1}{z-\mu_i} \right) \mathrm{d}z \\ &= -\frac{N}{2\pi\mathrm{i}} \oint_\Gamma f(z) \left(\int_{-\infty}^{\infty} \frac{\rho_N(\mathrm{d}x)}{x-z} \right) \mathrm{d}z.\end{aligned} \tag{A.3}$$

The procedure decouples the randomness of μ_i and the function f, and we can solely focus on the randomness of μ_i via the integral of the function $(x-z)^{-1}$ with respect to the random measure $\rho_N(\mathrm{d}x)$.

Let us recall the Stieltjes transform to handle the random integral of $(x-z)^{-1}$. For a measure ν and a variable $z \in \mathbb{C}^+$, the Stieltjes transform $s_\nu(z)$ of ν is defined by

$$s_\nu(z) = \int_{-\infty}^{\infty} \frac{\nu(\mathrm{d}x)}{x-z}. \tag{A.4}$$

We abbreviate $s_{\rho_N}(z) \equiv s_N(z)$. Then, (A.3) can be rewritten as

$$\sum_{i=1}^N f(\mu_i) = -\frac{N}{2\pi\mathrm{i}} \oint_\Gamma f(z) s_N(z) \mathrm{d}z. \tag{A.5}$$

Similarly, we also find that

$$N \int_{-2}^{2} \frac{\sqrt{4-x^2}}{2\pi} f(x)\, \mathrm{d}x = \frac{N}{2\pi\mathrm{i}} \oint_\Gamma f(z) s(z) \mathrm{d}z, \tag{A.6}$$

where we let $s(z) = s_\rho(z)$, the Stieltjes transform of the Wigner semicircle measure. Thus, we obtain that

$$\sum_{i=1}^N f(\mu_i) - N \int_{-2}^{2} \frac{\sqrt{4-x^2}}{2\pi} f(x)\, \mathrm{d}x = -\frac{N}{2\pi\mathrm{i}} \oint_\Gamma f(z) \big(s_N(z) - s(z) \big) \mathrm{d}z. \tag{A.7}$$

We remark that $s(z)$ satisfies

$$s(z) = \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4-x^2}}{x-z} \mathrm{d}x = \frac{-z + \sqrt{z^2-4}}{2}. \tag{A.8}$$

We use the results from the random matrix theory to analyze the right-hand side of (A.7). For $z \in \mathbb{C}^+$, define the resolvent $R(z)$ of M by

$$R(z) = (M - zI)^{-1}. \tag{A.9}$$

Note that the normalized trace of the resolvent satisfies

$$\frac{1}{N} \operatorname{Tr} R(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\mu_i - z} = s_N(z). \tag{A.10}$$

Let

$$\xi_N(z) = N(s_N(z) - s(z)) = \sum_{i=1}^N [R_{ii}(z) - s(z)]. \tag{A.11}$$

As discussed in Section 1, Theorem 5 was proved in (Baik & Lee, 2017) for

$$\boldsymbol{x} = \mathbf{1} = \frac{1}{\sqrt{N}} (1, 1, \dots, 1)^T.$$

We introduce an interpolation between $\boldsymbol{x}$ and $\mathbf{1}$ as follows: Since $\boldsymbol{x}, \mathbf{1} \in \mathbb{S}^{N-1}$, the $(N-1)$-dimensional unit sphere, we can consider a parametrized curve $\boldsymbol{y} : [0,1] \to \mathbb{S}^{N-1}$, a segment of the geodesic on $\mathbb{S}^{N-1}$ joining $\boldsymbol{x}$ and $\mathbf{1}$ such that $\boldsymbol{y}(0) = \boldsymbol{x}$ and $\boldsymbol{y}(1) = \mathbf{1}$. We write

$$\boldsymbol{y}(\theta) = (y_1(\theta), y_2(\theta), \dots, y_N(\theta))^T \tag{A.12}$$

and also define

$$M_{ij}(\theta) = \sqrt{\lambda} y_i(\theta) y_j(\theta) + H_{ij}, \quad R(\theta, z) = (M(\theta) - zI)^{-1}, \quad \xi_N(\theta, z) = \sum_{i=1}^{N} [R_{ii}(\theta, z) - s(z)]. \tag{A.13}$$

Our strategy of the proof is to show that the limiting distribution of $\xi_N(\theta, z)$ does not change with θ. More precisely, we claim that

$$\frac{\partial}{\partial \theta} \xi_N(\theta, z) = \mathcal{O}(N^{-\frac{1}{2}}) \tag{A.14}$$

for any $z \in \Gamma$. Once we prove the claim, we can use the lattice argument to prove Theorem 5 as follows: Choose points $z_1, z_2, \dots, z_{16N} \in \Gamma$ so that $|z_i - z_{i+1}| \le N^{-1}$ for $i = 1, 2, \dots, 16N$ (with the convention $z_{16N+1} = z_1$). For each z_i, the claim (A.14) shows that

$$\xi_N(1, z_i) - \xi_N(0, z_i) = \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.15}$$

For any $z \in \Gamma$, if z_i is the nearest lattice point from z, then $|z - z_i| \le N^{-1}$. From the Lipschitz continuity of ξ_N, we then find $|\xi_N(\theta, z) - \xi_N(\theta, z_i)| = \mathcal{O}(N^{-1})$ uniformly on z and z_i. Hence,

$$|\xi_N(1, z) - \xi_N(0, z)| \le |\xi_N(1, z) - \xi_N(1, z_i)| + |\xi_N(1, z_i) - \xi_N(0, z_i)| + |\xi_N(0, z_i) - \xi_N(0, z)| = \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.16}$$

Now, integrating over Γ, we get

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} f(z) \xi_N(1, z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} f(z) \xi_N(0, z) \mathrm{d}z = \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.17}$$

This shows that the limiting distribution of the right-hand side of (A.7) does not change even if we change $\boldsymbol{x}$ into $\mathbf{1}$. Therefore, we get the desired theorem from Theorem 1.6 and Remark 1.7 of (Baik & Lee, 2017).

We now prove the claim (A.14). For the ease of notation, we omit the z-dependence in some occasions. Using the formula

$$\frac{\partial R_{jj}(\theta)}{\partial M_{ab}(\theta)} = \begin{cases} -R_{ja}(\theta) R_{bj}(\theta) - R_{jb}(\theta) R_{aj}(\theta) & \text{if } a \neq b, \\ -R_{ja}(\theta) R_{aj}(\theta) & \text{if } a = b, \end{cases} \tag{A.18}$$

and the fact that M and $R(\theta)$ are symmetric, it is straightforward to check that

$$\frac{\partial}{\partial \theta} \xi_N(\theta) = \sum_{a,b=1}^{N} \frac{\partial M_{ab}(\theta)}{\partial \theta} \frac{\partial \xi_N(\theta)}{\partial M_{ab}(\theta)} = -\sqrt{\lambda} \sum_{a,b=1}^{N} \dot{y}_a(\theta) y_b(\theta) \sum_{j=1}^{N} R_{ja}(\theta) R_{bj}(\theta) \,, \tag{A.19}$$

where we use the notation $\dot{y}_a \equiv \dot{y}_a(\theta) = \frac{\mathrm{d} y_a(\theta)}{\mathrm{d}\theta}$.

To estimate the right-hand side of (A.19), we first note that

$$\sum_{a,b=1}^{N} \dot{y}_a(\theta) y_b(\theta) \sum_{j=1}^{N} R_{ja}(\theta) R_{bj}(\theta) = \langle \dot{\boldsymbol{y}}(\theta), R(\theta)^2 \boldsymbol{y}(\theta) \rangle \tag{A.20}$$

For the resolvents of the Wigner matrices, we have the following lemma from (Knowles & Yin, 2013).

Lemma A.1 (Isotropic local law)**.** *For an N-independent constant $\epsilon > 0$, let Γ^ϵ be the ϵ-neighborhood of Γ, i.e.,*

$$\Gamma^\epsilon = \{ z \in \mathbb{C} : \min_{w \in \Gamma} |z - w| \le \epsilon \}.$$

Choose ϵ small so that the distance between Γ^ϵ and $[-2, 2]$ is larger than 2ϵ, i.e.,

$$\min_{w \in \Gamma^\epsilon, x \in [-2,2]} |x - w| > 2\epsilon. \tag{A.21}$$

Then, for any deterministic $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^N$ with $\|\boldsymbol{v}\| = \|\boldsymbol{w}\| = 1$, the following estimate holds uniformly on $z \in \Gamma^\epsilon$:

$$\big| \langle \boldsymbol{v}, (H - zI)^{-1} \boldsymbol{w} \rangle - s(z) \langle \boldsymbol{v}, \boldsymbol{w} \rangle \big| = \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.22}$$

Proof of Lemma A.1. We prove the lemma by using the results in (Knowles & Yin, 2013). If $z = E + \mathrm{i}\eta \in \Gamma^\epsilon$ for some $E \in [a_- - \epsilon, a_+ + \epsilon]$ and $\eta \in [v_0 - \epsilon, v_0 + \epsilon]$, we get the estimate from Theorem 2.2 of (Knowles & Yin, 2013) since the control parameter $\Psi(z)$ in Equation (2.7) of (Knowles & Yin, 2013) is bounded by

$$\Psi(E + \mathrm{i}v_0) \equiv \sqrt{\frac{\operatorname{Im} s(E + \mathrm{i}\eta)}{N\eta}} + \frac{1}{N\eta} = O(N^{-\frac{1}{2}}).$$

A similar estimate holds for $z = E - \mathrm{i}\eta \in \Gamma^\epsilon$ with $E \in [a_- - \epsilon, a_+ + \epsilon]$ and $\eta \in [-v_0 - \epsilon, -v_0 + \epsilon]$. On the other hand, if $z = E + \mathrm{i}\eta \in \Gamma$ for $E \in [a_- - \epsilon, a_- + \epsilon] \cup [a_+ - \epsilon, a_+ + \epsilon]$ and $\eta \in (0, v_0 + \epsilon]$, we can check from an elementary calculation that $|\operatorname{Im} s(E + \mathrm{i}\eta)| \leq C\eta$ for some constant C independent of N. Thus, the upper bound in Equation (2.10) of (Knowles & Yin, 2013) becomes

$$\sqrt{\frac{\operatorname{Im} s(E + \mathrm{i}\eta)}{N\eta}} = O(N^{-\frac{1}{2}}).$$

A similar estimate holds for $z = E - \mathrm{i}\eta \in \Gamma^\epsilon$ with $E \in [a_- - \epsilon, a_- + \epsilon] \cup [a_+ - \epsilon, a_+ + \epsilon]$ and $\eta \in (0, v_0 + \epsilon]$. This completes the proof of the lemma. □

To show that the right-hand side of (A.20) is negligible, we want to use Lemma A.1. The main difference between the right-hand side of (A.20) and the left-hand side of (A.22) is that the former contains the square of the resolvent, and it is not the resolvent of H but of $M(\theta)$. We can overcome the first difficulty by rewriting $R(\theta, z)$ as

$$R(\theta, z)^2 = (M(\theta) - zI)^{-2} = \frac{\partial}{\partial z}(M(\theta) - zI)^{-1} = \frac{\partial}{\partial z} R(\theta, z), \tag{A.23}$$

which can be checked from the definition of the resolvent. Hence we find that

$$\langle \dot{\boldsymbol{y}}(\theta), R(\theta, z)^2 \boldsymbol{y}(\theta) \rangle = \frac{\partial}{\partial z} \langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle. \tag{A.24}$$

Later, we will apply Cauchy's integral formula to estimate the derivative in (A.20) by an integral of the inner product $\langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle$.

Next, we obtain an analogue of Lemma A.1 by using the resolvent expansion. Set $S(z) = (H - zI)^{-1}$. We have from the definition of the resolvents that

$$R(\theta, z)^{-1} - S(z)^{-1} = \sqrt{\lambda} \boldsymbol{y}(\theta) \boldsymbol{y}(\theta)^T, \tag{A.25}$$

and after multiplying $S(z)$ from the right and $R(\theta, z)$ from the left, we find that

$$S(z) - R(\theta, z) = \sqrt{\lambda} R(\theta, z) \boldsymbol{y}(\theta) \boldsymbol{y}(\theta)^T S(z). \tag{A.26}$$

Thus,

$$\begin{aligned} \langle \dot{\boldsymbol{y}}(\theta), S(z) \boldsymbol{y}(\theta) \rangle &= \langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle + \sqrt{\lambda} \langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \boldsymbol{y}(\theta)^T S(z) \boldsymbol{y}(\theta) \rangle \\ &= \langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle + \sqrt{\lambda} \langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle \langle \boldsymbol{y}(\theta), S(z) \boldsymbol{y}(\theta) \rangle \\ &= \langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle \left(1 + \sqrt{\lambda} \langle \boldsymbol{y}(\theta), S(z) \boldsymbol{y}(\theta) \rangle \right). \end{aligned} \tag{A.27}$$

From the isotropic local law, Lemma A.1, we find that

$$\langle \boldsymbol{y}(\theta), S(z) \boldsymbol{y}(\theta) \rangle = s(z) + \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.28}$$

Recall that $\|\boldsymbol{y}(\theta)\| = 1$. Then, it is obvious that $\langle \dot{\boldsymbol{y}}(\theta), \boldsymbol{y}(\theta) \rangle = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\theta} \|\boldsymbol{y}(\theta)\|^2 = 0$. Hence, again from Lemma A.1, we also find that

$$\langle \dot{\boldsymbol{y}}(\theta), S(z) \boldsymbol{y}(\theta) \rangle = s(z) \langle \dot{\boldsymbol{y}}(\theta), \boldsymbol{y}(\theta) \rangle + \mathcal{O}(N^{-\frac{1}{2}}) = \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.29}$$

We then have from (A.27) that

$$\langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle = \frac{\langle \dot{\boldsymbol{y}}(\theta), S(z) \boldsymbol{y}(\theta) \rangle}{1 + \sqrt{\lambda} \langle \boldsymbol{y}(\theta), S(z) \boldsymbol{y}(\theta) \rangle} = \mathcal{O}(N^{-\frac{1}{2}}), \tag{A.30}$$

where we used that $|s| \le 1$ and $\lambda < 1$, hence $1 + \sqrt{s} > c > 0$ for some (N-independent) constant c.

Consider the boundary of the ϵ-neighborhood of z, $\partial B_\epsilon(z) = \{w \in \mathbb{C} : |w - z| = \epsilon\}$. If we choose ϵ as in the assumption of Lemma A.1, $\partial B_\epsilon(z)$ does not intersect $[-2, 2]$. Applying Cauchy's integral formula, we get

$$\frac{\partial}{\partial z} \langle \dot{\boldsymbol{y}}(\theta), R(\theta, z) \boldsymbol{y}(\theta) \rangle = \frac{1}{2\pi \mathrm{i}} \oint_{\partial B_\epsilon(z)} \frac{\langle \dot{\boldsymbol{y}}(\theta), R(\theta, w) \boldsymbol{y}(\theta) \rangle}{(w - z)^2} \mathrm{d}w = \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.31}$$

Thus, we get from (A.20) and (A.31) that

$$\langle \dot{\boldsymbol{y}}(\theta), R(\theta)^2 \boldsymbol{y}(\theta) \rangle = \mathcal{O}(N^{-\frac{1}{2}}). \tag{A.32}$$

Plugging the estimate into the right-hand side of (A.19), we get the claim (A.14).

B. Proof of Theorem 6 and Theorem 8

In this section, we prove Theorem 6 by applying Theorem 5. (The proof of Theorem 8 is exactly same as the proof of Theorem 6 except that we use Theorem 7 instead of Theorem 5.) First, we notice that

$$m_M(f) - m_H(f) = \sum_{\ell=1}^{\infty} \sqrt{\lambda^\ell} \tau_\ell(f). \tag{B.1}$$

Recall that

$$\begin{aligned} V_M(f) &= (w_2 - 2)\tau_1(f)^2 + 2(w_4 - 3)\tau_2(f)^2 + 2\sum_{\ell=1}^{\infty} \ell \tau_\ell(f)^2 \\ &= w_2 \tau_1(f)^2 + 2(w_4 - 1)\tau_2(f)^2 + 2\sum_{\ell=3}^{\infty} \ell \tau_\ell(f)^2. \end{aligned} \tag{B.2}$$

Assuming $w_2 > 0$ and $w_4 > 1$, by Cauchy's inequality, we obtain that

$$|m_M(f) - m_H(f)|^2 \le \left(\frac{\lambda}{w_2} + \frac{\lambda^2}{2(w_4 - 1)} + \sum_{\ell=3}^{\infty} \frac{\lambda^\ell}{2\ell} \right) V_M(f). \tag{B.3}$$

From the identity $\log(1 - \lambda) = -\sum_{\ell=1}^{\infty} \lambda^\ell / \ell$, we get

$$\frac{|m_M(f) - m_H(f)|^2}{V_M(f)} \le \frac{\lambda}{w_2} + \frac{\lambda^2}{2(w_4 - 1)} + \sum_{\ell=3}^{\infty} \frac{\lambda^\ell}{2\ell} = \left(\frac{1}{w_2} - \frac{1}{2} \right) \lambda + \left(\frac{1}{2(w_4 - 1)} - \frac{1}{4} \right) \lambda^2 - \frac{1}{2} \log(1 - \lambda), \tag{B.4}$$

which proves the first part of the theorem.

Since we only used Cauchy's inequality, the equality in (B.3) holds if and only if

$$\frac{w_2 \tau_1(f)}{\sqrt{\lambda}} = \frac{2(w_4 - 1)\tau_2(f)}{\lambda} = \frac{2\ell \tau_\ell(f)}{\sqrt{\lambda^\ell}} \qquad (\ell = 3, 4, \dots). \tag{B.5}$$

We now find all functions f that satisfy (B.5). Letting $2C$ be the common value in (B.5), we rewrite (B.5) as

$$\tau_1(f) = \frac{2C\sqrt{\lambda}}{w_2}, \quad \tau_2(f) = \frac{C\lambda}{w_4 - 1}, \quad \tau_\ell(f) = \frac{C\sqrt{\lambda^\ell}}{\ell} \qquad (\ell = 3, 4, \dots). \tag{B.6}$$

Since f is analytic, we can consider the Taylor expansion of it. Using the Chebyshev polynomials, we can expand f as

$$f(x) = \sum_{\ell=0}^{\infty} C_\ell T_\ell \left(\frac{x}{2} \right). \tag{B.7}$$

Then, from the orthogonality relation of the Chebyshev polynomials, we get for $\ell \geq 1$ that

$$\tau_\ell(f) = \frac{C_\ell}{\pi} \int_{-2}^{2} T_\ell\left(\frac{x}{2}\right) T_\ell\left(\frac{x}{2}\right) \frac{\mathrm{d}x}{\sqrt{4-x^2}} = \frac{C_\ell}{\pi} \int_{-1}^{1} T_\ell(y)\, T_\ell(y) \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \frac{C_\ell}{2}. \tag{B.8}$$

Thus, (B.6) holds if and only if

$$\begin{aligned} f(x) &= c_0 + 2C\left(\frac{2\sqrt{\lambda}}{w_2} T_1\left(\frac{x}{2}\right) + \frac{\lambda}{w_4 - 1} T_2\left(\frac{x}{2}\right) + \sum_{\ell=3}^{\infty} \frac{\sqrt{\lambda}^\ell}{\ell} T_\ell\left(\frac{x}{2}\right)\right) \\ &= c_0 + 2C\sqrt{\lambda}\left(\frac{2}{w_2} - 1\right) T_1\left(\frac{x}{2}\right) + 2C\lambda\left(\frac{1}{w_4 - 1} - \frac{1}{2}\right) T_2\left(\frac{x}{2}\right) + 2C \sum_{\ell=1}^{\infty} \frac{\sqrt{\lambda}^\ell}{\ell} T_\ell\left(\frac{x}{2}\right) \end{aligned} \tag{B.9}$$

for some constant c_0. It is well-known from the generating function of the Chebyshev polynomials that

$$\sum_{\ell=1}^{\infty} \frac{t^\ell}{\ell} T_\ell(x) = \log\left(\frac{1}{\sqrt{1 - 2tx + t^2}}\right). \tag{B.10}$$

(See, e.g., (18.12.9) of (Olver et al., 2010).) Since $T_1(x) = x$ and $T_2(x) = 2x^2 - 1$, we find that (B.9) is equivalent to

$$f(x) = c_0 + C\sqrt{\lambda}\left(\frac{2}{w_2} - 1\right) x + C\lambda\left(\frac{1}{w_4 - 1} - \frac{1}{2}\right)(x^2 - 2) + C \log\left(\frac{1}{1 - \sqrt{\lambda}x + \lambda}\right). \tag{B.11}$$

This concludes the proof of Theorem 6.

C. Computation of the test statistic

Lemma C.1. *Let*

$$L_\lambda = \sum_{i=1}^{N} \phi_\lambda(\mu_i) - N \int_{-2}^{2} \frac{\sqrt{4-y^2}}{2\pi} \phi_\lambda(y)\, \mathrm{d}y \tag{C.1}$$

where ϕ_λ is defined as in (7). Then,

$$L_\lambda = -\log\det\left((1+\lambda)I - \sqrt{\lambda}M\right) + \frac{\lambda N}{2} + \sqrt{\lambda}\left(\frac{2}{w_2} - 1\right) \operatorname{Tr} M + \lambda\left(\frac{1}{w_4 - 1} - \frac{1}{2}\right)(\operatorname{Tr} M^2 - N). \tag{C.2}$$

Proof. It is straightforward to see that

$$\sum_{i=1}^{N} \phi_\lambda(\mu_i) = -\log\det\left((1+\lambda)I - \sqrt{\lambda}M\right) + \sqrt{\lambda}\left(\frac{2}{w_2} - 1\right) \operatorname{Tr} M + \lambda\left(\frac{1}{w_4 - 1} - \frac{1}{2}\right) \operatorname{Tr} M^2. \tag{C.3}$$

To compute the integral in the definition of L_λ, we use the formula

$$\int_{-2}^{2} \log(z - y) \frac{\sqrt{4-y^2}}{2\pi}\, \mathrm{d}y = \frac{z}{4}\left(z - \sqrt{z^2 - 4}\right) + \log\left(z + \sqrt{z^2 - 4}\right) - \log 2 - \frac{1}{2} \tag{C.4}$$

for $z > 2$. See, e.g., Equation (8.5) of (Baik & Lee, 2016). Putting $z = (1+\lambda)/\sqrt{\lambda}$, we get

$$\int_{-2}^{2} \log\left(\frac{1}{1 - \sqrt{\lambda}y + \lambda}\right) \frac{\sqrt{4-y^2}}{2\pi}\, \mathrm{d}y = -\int_{-2}^{2} \left(\log\sqrt{\lambda} + \log\left(\frac{1+\lambda}{\sqrt{\lambda}} - y\right)\right) \frac{\sqrt{4-y^2}}{2\pi}\, \mathrm{d}y = -\frac{\lambda}{2}. \tag{C.5}$$

Finally, it is elementary to check that

$$\int_{-2}^{2} \frac{y\sqrt{4-y^2}}{2\pi}\, \mathrm{d}y = 0, \qquad \int_{-2}^{2} \frac{y^2\sqrt{4-y^2}}{2\pi}\, \mathrm{d}y = 1. \tag{C.6}$$

This proves Equation (C.2). □

Lemma C.2. *Let*

$$m_H(\phi_\lambda) = \frac{1}{4}\left(\phi_\lambda(2) + \phi_\lambda(-2)\right) - \frac{1}{2}\tau_0(\phi_\lambda) + (w_2 - 2)\tau_2(\phi_\lambda) + (w_4 - 3)\tau_4(\phi_\lambda) \tag{C.7}$$

and

$$m_M(\phi_\lambda) = \frac{1}{4}\left(\phi_\lambda(2) + \phi_\lambda(-2)\right) - \frac{1}{2}\tau_0(\phi_\lambda) + (w_2 - 2)\tau_2(\phi_\lambda) + (w_4 - 3)\tau_4(\phi_\lambda) + \sum_{\ell=1}^{\infty} \sqrt{\lambda^\ell}\tau_\ell(\phi_\lambda) \tag{C.8}$$

where ϕ_λ is defined as in (7). Then,

$$m_H(\phi_\lambda) = -\frac{1}{2}\log(1-\lambda) + \left(\frac{w_2 - 1}{w_4 - 1} - \frac{1}{2}\right)\lambda + \frac{(w_4 - 3)\lambda^2}{4} \tag{C.9}$$

and

$$m_M(\phi_\lambda) = m_H(\phi_\lambda) - \log(1 - \sqrt{\lambda^2}) + \left(\frac{2}{w_2} - 1\right)\sqrt{\lambda^2} + \left(\frac{1}{w_4 - 1} - \frac{1}{2}\right)\lambda^2. \tag{C.10}$$

In particular, $m_H(\phi_\lambda) < m_M(\phi_\lambda)$ if $\lambda \in (0, 1)$.

Proof. Recall that ϕ_λ is the function f in (B.11) with $C = 1$ and $c_0 = (\frac{2}{w_4 - 1} - 1)t$. Thus, from (B.6),

$$\tau_1(\phi_\lambda) = \frac{2\sqrt{\lambda}}{w_2}, \quad \tau_2(\phi_\lambda) = \frac{\lambda}{w_4 - 1}, \quad \tau_\ell(\phi_\lambda) = \frac{\sqrt{\lambda^\ell}}{\ell} \qquad (\ell = 3, 4, \dots). \tag{C.11}$$

Moreover,

$$\tau_0(\phi_\lambda) = c_0 = \left(\frac{2}{w_4 - 1} - 1\right)\lambda. \tag{C.12}$$

Since

$$\begin{aligned} \phi_\lambda(2) + \phi_\lambda(-2) &= \log\left(\frac{1}{1 - 2\sqrt{\lambda} + \lambda}\right) + \log\left(\frac{1}{1 + 2\sqrt{\lambda} + \lambda}\right) + 8\lambda\left(\frac{1}{w_4 - 1} - \frac{1}{2}\right) \\ &= -2\log(1-\lambda) + 8\lambda\left(\frac{1}{w_4 - 1} - \frac{1}{2}\right), \end{aligned} \tag{C.13}$$

we find that

$$\begin{aligned} m_H(\phi_\lambda) &= -\frac{1}{2}\log(1-\lambda) + 2\lambda\left(\frac{1}{w_4 - 1} - \frac{1}{2}\right) - \frac{\lambda}{2}\left(\frac{2}{w_4 - 1} - 1\right) + \frac{(w_2 - 2)\lambda}{w_4 - 1} + \frac{(w_4 - 3)\lambda^2}{4} \\ &= -\frac{1}{2}\log(1-\lambda) + \left(\frac{w_2 - 1}{w_4 - 1} - \frac{1}{2}\right)\lambda + \frac{(w_4 - 3)\lambda^2}{4}. \end{aligned} \tag{C.14}$$

Moreover, we also find that

$$\begin{aligned} m_M(\phi_\lambda) &= m_H(\phi_\lambda) + \frac{2\lambda}{w_2} + \frac{\lambda^2}{w_4 - 1} + \sum_{\ell=3}^{\infty} \frac{\lambda^\ell}{\ell} \\ &= m_H(\phi_\lambda) + \left(\frac{2}{w_2} - 1\right)\lambda + \left(\frac{1}{w_4 - 1} - \frac{1}{2}\right)\lambda^2 + \sum_{\ell=1}^{\infty} \frac{\lambda^\ell}{\ell} \\ &= m_H(\phi_\lambda) - \log(1-\lambda) + \left(\frac{2}{w_2} - 1\right)\lambda + \left(\frac{1}{w_4 - 1} - \frac{1}{2}\right)\lambda^2. \end{aligned} \tag{C.15}$$

Finally, it is obvious $m_M(\phi_\lambda) > m_H(\phi_\lambda)$ if $\lambda \in (0, 1)$ since $\tau_\ell(\phi_\lambda) > 0$ for all $\ell = 1, 2, \dots$. □

Remark C.3. *For any λ,*

$$V_M(\phi_\lambda) = V_H(\phi_\lambda) = -2\log(1-\lambda) + \left(\frac{4}{w_2} - 2\right)\lambda + \left(\frac{2}{w_4 - 1} - 1\right)\lambda^2, \tag{C.16}$$

which can be easily checked from (C.11).

D. Proof of Theorem 7

Recall that the normalized off-diagonal entries $\sqrt{N}H_{ij}$ are identically distributed with density g and the normalized diagonal entries $\sqrt{N/w_2}H_{ii}$ are identically distributed with density g_d. In Assumption 1, we further assumed that the densities g and g_d are smooth, positive everywhere, with subexponential tails, and symmetric (about 0). We also assumed that

$$\|\boldsymbol{x}\|_\infty = O(N^{-\phi})$$

for some $\frac{3}{8} < \phi \le \frac{1}{2}$.

As discussed in Section 4, we consider the entrywise transformation defined by a function

$$h(w) := -\frac{g'(w)}{g(w)}. \tag{D.1}$$

If $\lambda = 0$, it is immediate to see that for $i \neq j$

$$\mathbb{E}[h(\sqrt{N}M_{ij})] = \int_{-\infty}^{\infty} h(w)g(w)\mathrm{d}w = -\int_{-\infty}^{\infty} g'(w)\mathrm{d}w = 0.$$

Further, with $\lambda = 0$, as shown in Proposition 4.2 of (Perry et al., 2018),

$$F^H := \mathbb{E}[h(\sqrt{N}M_{ij})^2] = \int_{-\infty}^{\infty} h(w)^2 g(w)\mathrm{d}w = \int_{-\infty}^{\infty} \frac{g'(w)^2}{g(w)}\mathrm{d}w \ge 1, \tag{D.2}$$

where the equality holds if and only if $\sqrt{N}H_{ij}$ is a standard Gaussian (hence $h(w) = w$). For the diagonal entries, we similarly define

$$h_d(w) := -\frac{g_d'(w)}{g_d(w)}. \tag{D.3}$$

Then, if $\lambda = 0$, $\mathbb{E}[h_d(\sqrt{N/w_2}M_{ii})] = 0$ and

$$F_d^H := \mathbb{E}[h_d(\sqrt{N/w_2}M_{ii})^2] = \int_{-\infty}^{\infty} \frac{g_d'(w)^2}{g_d(w)}\mathrm{d}w \ge 1, \tag{D.4}$$

We define a transformed matrix $\widetilde{M}$ as follows: the off-diagonal terms of $\widetilde{M}$ are defined by

$$\widetilde{M}_{ij} = \frac{1}{\sqrt{F^H N}} h(\sqrt{N}M_{ij}) \quad (i \neq j), \qquad \widetilde{M}_{ii} = \sqrt{\frac{w_2}{F_d^H N}} h_d\Big(\sqrt{\frac{N}{w_2}}M_{ii}\Big). \tag{D.5}$$

Note that the entries of $\widetilde{M}$ are independent up to symmetry. Since g is smooth, h is also smooth and all moments of $\sqrt{N}\widetilde{M}_{ij}$ are $O(1)$. Thus, applying a high-order Markov inequality, it is immediate to find that $\widetilde{M}_{ij} = \mathcal{O}(N^{-\frac{1}{2}})$.

D.1. Decomposition of the transformed matrix

We first evaluate the mean and the variance of each off-diagonal entry by using the comparison method with the pre-transformed entries. For $i \neq j$, we find that

$$\begin{aligned}\mathbb{E}[\widetilde{M}_{ij}] &= \frac{1}{\sqrt{F^H N}} \int_{-\infty}^{\infty} h(w) g(w - \sqrt{\lambda N}x_i x_j)\mathrm{d}w \\ &= -\frac{1}{\sqrt{F^H N}} \int_{-\infty}^{\infty} \frac{g'(w)}{g(w)} \Big(g(w - \sqrt{\lambda N}x_i x_j) - g(w) \Big)\mathrm{d}w.\end{aligned} \tag{D.6}$$

In the Taylor expansion

$$\begin{aligned}&g(w - \sqrt{\lambda N}x_i x_j) - g(w) \\ &= \sum_{\ell=1}^{4} \frac{g^{(\ell)}(w)}{\ell!} \Big(-\sqrt{\lambda N}x_i x_j\Big)^\ell + \frac{g^{(5)}(w - \theta\sqrt{\lambda N}x_i x_j)}{5!} \Big(-\sqrt{\lambda N}x_i x_j\Big)^5\end{aligned} \tag{D.7}$$

for some $\theta \in (0,1)$. Note that the second term and the fourth term in the summation are even functions. Since g'/g is an odd function, from the symmetry we find that

$$\begin{aligned}\mathbb{E}[\widetilde{M}_{ij}] &= \frac{\sqrt{\lambda}x_i x_j}{\sqrt{F^H}} \int_{-\infty}^{\infty} \frac{g'(w)^2}{g(w)} \mathrm{d}w + C_3 N x_i^3 x_j^3 + O(N^3 x_i^5 x_j^5) \\ &= \sqrt{\lambda F^H} x_i x_j + C_3 N x_i^3 x_j^3 + O(N^3 x_i^5 x_j^5)\end{aligned} \tag{D.8}$$

for some (N-independent) constants C_3 and C_5. Similarly,

$$\begin{aligned}\mathbb{E}[\widetilde{M}_{ij}^2] &= \frac{1}{F^H N} \int_{-\infty}^{\infty} \left(\frac{g'(w)}{g(w)}\right)^2 g(w - \sqrt{\lambda N} x_i x_j) \mathrm{d}w \\ &= \frac{1}{N} + \frac{1}{F^H N} \int_{-\infty}^{\infty} \left(\frac{g'(w)}{g(w)}\right)^2 \Big(g(w - \sqrt{\lambda N} x_i x_j) - g(w)\Big) \mathrm{d}w \\ &= \frac{1}{N} + \frac{\lambda x_i^2 x_j^2}{2F^H} \int_{-\infty}^{\infty} \frac{g'(w)^2 g''(w)}{g(w)^2} \mathrm{d}w + O(N x_i^4 x_j^4) = \frac{1}{N} + \lambda G^H x_i^2 x_j^2 + O(N x_i^4 x_j^4).\end{aligned} \tag{D.9}$$

For the diagonal entries, we similarly get

$$\mathbb{E}[\widetilde{M}_{ii}] = \sqrt{\lambda F_d^H} x_i^2 + O(N x_i^6) \tag{D.10}$$

and

$$\mathbb{E}[\widetilde{M}_{ii}^2] = \frac{w_2}{N} + \frac{\lambda x_i^4}{2F^H} \int_{-\infty}^{\infty} \frac{g_d'(w)^2 g_d''(w)}{g_d(w)^2} \mathrm{d}w + O(N x_i^8) =: \frac{w_2}{N} + \lambda G_d^H x_i^4 + O(N x_i^8). \tag{D.11}$$

We omit the detail.

The evaluation of the mean and the variance shows that the transformed matrix $\widetilde{M}$ is not a spiked Wigner matrix when $\lambda > 0$, since the variances of the off-diagonal entries are not identical. Our strategy is to approximate $\widetilde{M}$ as a spiked generalized Wigner matrix for which the sum of the variances of the entries in each row is equal to 1. Let S be the variance matrix of $\widetilde{M}$ defined as

$$S_{ij} = \mathbb{E}[\widetilde{M}_{ij}^2] - (\mathbb{E}[\widetilde{M}_{ij}])^2. \tag{D.12}$$

From (D.8), (D.9), (D.10), and (D.11),

$$S_{ij} = \frac{1}{N} + \lambda(G^H - F^H) x_i^2 x_j^2 + O(N\|\boldsymbol{x}\|_\infty^8) \quad (i \neq j), \qquad S_{ii} = \frac{w_2}{N} + \lambda(G_d^H - F_d^H) x_i^4 + O(N\|\boldsymbol{x}\|_\infty^8) \quad (i \neq j), \tag{D.13}$$

hence

$$\begin{aligned}\sum_{j=1}^N S_{ij} &= \frac{w_2}{N} + \lambda(G_d^H - F_d^H) x_i^4 + \sum_{j: j \neq i} \left(\frac{1}{N} + \lambda(G^H - F^H) x_i^2 x_j^2\right) + O(N^2 \|\boldsymbol{x}\|_\infty^8) \\ &= 1 + \frac{w_2 - 1}{N} + \lambda(G^H - F^H) x_i^2 + O(N^2 \|\boldsymbol{x}\|_\infty^8),\end{aligned} \tag{D.14}$$

which shows that $\widetilde{M}$ is indeed approximately a spiked generalized Wigner matrix.

D.2. CLT for a general Wigner-type matrix

To adapt the strategy of Section A, we use the local law for general Wigner-type matrices in (Ajanki et al., 2017). Consider a matrix $W = (W_{ij})_{1 \le i,j \le N}$ defined by

$$W_{ij} = \frac{1}{\sqrt{N S_{ij}}} (\widetilde{M}_{ij} - \mathbb{E}[\widetilde{M}_{ij}]) \quad (i \neq j), \qquad W_{ii} = \sqrt{\frac{w_2}{N S_{ii}}} (\widetilde{M}_{ii} - \mathbb{E}[\widetilde{M}_{ii}]) \tag{D.15}$$

Note that $\mathbb{E}[W_{ij}] = 0$, $\mathbb{E}[W_{ij}^2] = \frac{1}{N}$ $(i \neq j)$, and $\mathbb{E}[W_{ii}^2] = \frac{w_2}{N}$. Then, the matrix W is a Wigner matrix. We set

$$R^W(z) = (W - zI)^{-1} \quad (z \in \mathbb{C}^+). \tag{D.16}$$

Next, we introduce an interpolation for W. For $0 \le \theta \le 1$, we define a matrix $W(\theta)$ by

$$\begin{aligned} W_{ij}(\theta) &= (1-\theta)W_{ij} + \theta(\widetilde{M}_{ij} - \mathbb{E}[\widetilde{M}_{ij}]) = \left(1 - \theta + \theta\sqrt{NS_{ij}}\right) W_{ij} \\ &= \left(1 + \frac{\theta N\lambda(G^H - F^H)x_i^2 x_j^2}{2} + O(N^2 x_i^4 x_j^4)\right) W_{ij} \quad (i \neq j) \end{aligned} \tag{D.17}$$

and

$$\begin{aligned} W_{ii}(\theta) &= (1-\theta)W_{ii} + \theta(\widetilde{M}_{ii} - \mathbb{E}[\widetilde{M}_{ii}]) = \left(1 - \theta + \theta\sqrt{\frac{NS_{ii}}{w_2}}\right) W_{ii} \\ &= \left(1 + \frac{\theta N\lambda(G_d^H - F_d^H)x_i^4}{2w_2} + O(N^2 x_i^8)\right) W_{ii}. \end{aligned} \tag{D.18}$$

Note that $W(0) = W$ and $W(1) = \widetilde{M} - \mathbb{E}[\widetilde{M}]$. For $0 \le \theta \le 1$, $W(\theta)$ is a general Wigner-type matrix considered in (Ajanki et al., 2017) satisfying the conditions (A)-(D) therein. Moreover, if we let

$$R^W(\theta, z) = (W(\theta) - zI)^{-1} \quad (z \in \mathbb{C}^+), \tag{D.19}$$

then Theorem 1.7 of (Ajanki et al., 2017) asserts that the limiting distribution of $R_{ij}^W(z)$ is $m_i(z)\delta_{ij}$, where $m_i(\theta, z)$ is the unique solution to the quadratic vector equation

$$-\frac{1}{m_i(\theta, z)} = z + \sum_{j=1}^N \mathbb{E}[W_{ij}(\theta)^2] m_j(\theta, z). \tag{D.20}$$

Recall that $s(z) = (-z + \sqrt{z^2 - 4})/2$ is the Stieltjes transform of the Wigner semicircle measure. It is direct to check that $1 + zs(z) + s(z)^2 = 0$. With an ansatz $m_i(\theta, z) = s(z) + C_1 x_i^2 + C_2 N^{-1}$, we can then find $m_i(\theta, z) = s(z) + O(\|\boldsymbol{x}\|_\infty^2)$; see also Theorem 4.2 of (Ajanki et al., 2017).

For the resolvent $R^W(\theta, z)$, we have the following lemma from (Ajanki et al., 2017).

Lemma D.1 (Anisotropic local law)**.** *Let Γ^ϵ be the ϵ-neighborhood of Γ as in Lemma A.1. Then, for any deterministic $\boldsymbol{v} = (v_1, \ldots, v_N), \boldsymbol{w} = (w_1, \ldots, w_N) \in \mathbb{C}^N$ with $\|\boldsymbol{v}\| = \|\boldsymbol{w}\| = 1$, the following estimate holds uniformly on $z \in \Gamma^\epsilon \cap \{z \in \mathbb{C}^+ : \operatorname{Im} z > N^{-\frac{1}{2}}\}$:*

$$\left| \sum_{i,j=1}^N \overline{v_i} R_{ij}^W(\theta, z) w_j - \sum_{i=1}^N m_i(\theta, z) \overline{v_i} w_i \right| = \mathcal{O}(N^{-\frac{1}{2}}). \tag{D.21}$$

Proof. See Theorem 1.13 of (Ajanki et al., 2017). Note that $\rho(z), \kappa(z) = O(\operatorname{Im} z)$ in Theorem 1.13 of (Ajanki et al., 2017), which can be checked from Equations (1.25), (4.5a), (4.5f), and (1.17) of (Ajanki et al., 2017). □

Let $\Gamma_{1/2}^\epsilon := \Gamma^\epsilon \cap \{z \in \mathbb{C}^+ : |\operatorname{Im} z| > N^{-\frac{1}{2}}\}$. On $\Gamma_{1/2}^\epsilon$, as a simple corollary to Lemma D.1, we obtain

$$\left| \langle \boldsymbol{v}, R^W(\theta, z), \boldsymbol{w} \rangle - s(z) \langle \boldsymbol{v}, \boldsymbol{w} \rangle \right| = \mathcal{O}(N^{-\frac{1}{2}}), \tag{D.22}$$

which is analogous to Lemma A.1.

We have the following lemma for the difference between $\operatorname{Tr} R^W(0, z)$ and $\operatorname{Tr} R^W(1, z)$ on $\Gamma_{1/2}^\epsilon$.

Lemma D.2. *Let $R^W(\theta, z)$ be defined as in Equations (D.17) and (D.19). Then, the following holds uniformly for $z \in \Gamma_{1/2}^\epsilon$:*

$$\operatorname{Tr} R^W(1, z) - \operatorname{Tr} R^W(0, z) = \lambda(G^H - F^H) s'(z) s(z) + \mathcal{O}(N^{\frac{3}{2}} \|\boldsymbol{x}\|_\infty^4). \tag{D.23}$$

We will prove Lemma D.2 later in this section.

On $\Gamma \backslash \Gamma^{\epsilon}_{1/2}$, we use the following results on the rigidity of eigenvalues.

Lemma D.3. *Denote by $\mu_1^W(\theta) \geq \mu_2^W(\theta) \geq \cdots \geq \mu_N^W(\theta)$ the eigenvalues of $W(\theta)$. Let γ_i be the classical location of the eigenvalues with respect to the semicircle measure defined by*

$$\int_{\gamma_i}^{2} \frac{\sqrt{4-x^2}}{2\pi} \mathrm{d}x = \frac{1}{N}\left(i - \frac{1}{2}\right) \tag{D.24}$$

for $i = 1, 2, \dots, N$. Then,

$$|\mu_i^W(\theta) - \gamma_i| = \mathcal{O}(N^{-\frac{2}{3}}). \tag{D.25}$$

Proof. See Corollary 1.11 of (Ajanki et al., 2017). Note that the limiting measure ρ is the semicircle measure in Corollary 1.11 of (Ajanki et al., 2017) since $N\mathbb{E}[W_{ij}(\theta)^2] = 1 + O(N\|\boldsymbol{x}\|_\infty^4) = 1 + o(1)$ for $i \neq j$. □

From Lemma D.3, we find that

$$\begin{aligned} |\operatorname{Tr} R^W(1,z) - \operatorname{Tr} R^W(0,z)| &= \left| \sum_{i=1}^N \left(\frac{1}{\mu_i^W(1) - z} - \frac{1}{\mu_i^W(0) - z} \right) \right| = \left| \sum_{i=1}^N \frac{\mu_i^W(0) - \mu_i^W(1)}{(\mu_i^W(1) - z)(\mu_i^W(0) - z)} \right| \\ &\leq \left| \sum_{i=1}^N \frac{|\mu_i^W(0) - \gamma_i| + |\gamma_i - \mu_i^W(1)|}{(\mu_i^W(1) - z)(\mu_i^W(0) - z)} \right| = \mathcal{O}(N^{\frac{1}{3}}). \end{aligned} \tag{D.26}$$

Thus, from (D.23) and (D.26),

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^W(1,z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^W(0,z) \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma^\epsilon_{1/2}} f(z) \left(\operatorname{Tr} R^W(1,z) - \operatorname{Tr} R^W(0,z) \right) \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma \backslash \Gamma^\epsilon_{1/2}} f(z) \left(\operatorname{Tr} R^W(1,z) - \operatorname{Tr} R^W(0,z) \right) \mathrm{d}z \\ &= \frac{\lambda(G^H - F^H)}{2\pi \mathrm{i}} \int_{\Gamma^\epsilon_{1/2}} f(z) s'(z) s(z) \mathrm{d}z + \mathcal{O}(N^{\frac{3}{2}} \|\boldsymbol{x}\|_\infty^4) + \mathcal{O}(N^{-\frac{1}{6}}) \\ &= \frac{\lambda(G^H - F^H)}{2\pi \mathrm{i}} \oint_\Gamma f(z) s'(z) s(z) \mathrm{d}z + \mathcal{O}(N^{\frac{3}{2}} \|\boldsymbol{x}\|_\infty^4) + \mathcal{O}(N^{-\frac{1}{6}}) \end{aligned} \tag{D.27}$$

D.3. CLT for a general Wigner-type matrix with a spike

Recall that $W(1) = \widetilde{M} - \mathbb{E}[\widetilde{M}]$. Our next step in the approximation is to consider $\widetilde{M} = W(1) + \mathbb{E}[\widetilde{M}]$. Since $\mathbb{E}[\widetilde{M}]$ is not a rank-1 matrix, we instead consider

$$A(\theta) = W(1) + \theta \sqrt{\lambda F^H} \boldsymbol{x}\boldsymbol{x}^T, \qquad R^A(\theta, z) = (A(\theta) - zI)^{-1} \tag{D.28}$$

for $\theta \in [0, 1]$. Note that $A(0) = W(1)$.

We follow the same strategy as in Section A. For $z \in \Gamma^\epsilon_{1/2}$, we use

$$\begin{aligned} \frac{\partial}{\partial \theta} \operatorname{Tr} R^A(\theta, z) &= -\sum_{i=1}^N \sum_{a,b=1}^N \frac{\partial A_{ab}(\theta)}{\partial \theta} R^A_{ia}(\theta, z) R^A_{bi}(\theta, z) \\ &= -\sqrt{\lambda F^H} \frac{\partial}{\partial z} \sum_{a,b=1}^N x_a x_b R^A_{ba}(\theta, z) = -\sqrt{\lambda F^H} \frac{\partial}{\partial z} \langle \boldsymbol{x}, R^A(\theta, z) \boldsymbol{x} \rangle. \end{aligned} \tag{D.29}$$

Recall that $R^A(0,z) = R^W(1,z)$ satisfies the isotropic local law in (D.22),

$$\left| \langle \boldsymbol{v}, R^A(0,z), \boldsymbol{w} \rangle - s(z) \langle \boldsymbol{v}, \boldsymbol{w} \rangle \right| = \left| \langle \boldsymbol{v}, R^W(1,z), \boldsymbol{w} \rangle - s(z) \langle \boldsymbol{v}, \boldsymbol{w} \rangle \right| = \mathcal{O}(N^{-\frac{1}{2}}). \tag{D.30}$$

As in (A.25) and (A.26), we can easily check that

$$R^A(0,z) - R^A(\theta,z) = \theta\sqrt{\lambda F^H} R^A(\theta,z)\boldsymbol{x}\boldsymbol{x}^T R^A(0,z), \tag{D.31}$$

hence

$$\langle \boldsymbol{x}, R^A(0,z)\boldsymbol{x}\rangle = \langle \boldsymbol{x}, R^A(\theta,z)\boldsymbol{x}\rangle + \theta\sqrt{\lambda F^H}\langle \boldsymbol{x}, R^A(\theta,z)\boldsymbol{x}\rangle\langle \boldsymbol{x}, R^A(0,z)\boldsymbol{x}\rangle. \tag{D.32}$$

We thus find that

$$\langle \boldsymbol{x}, R^A(\theta,z)\boldsymbol{x}\rangle = \frac{\langle \boldsymbol{x}, R^A(0,z)\boldsymbol{x}\rangle}{1+\theta\sqrt{\lambda F^H}\langle \boldsymbol{x}, R^A(0,z)\boldsymbol{x}\rangle} = \frac{s(z)}{1+\theta\sqrt{\lambda F^H}s(z)} + \mathcal{O}(N^{-\frac{1}{2}}). \tag{D.33}$$

Plugging it back to (D.29) and applying Cauchy's integral formula again, we find that

$$\frac{\partial}{\partial\theta}\operatorname{Tr} R^A(\theta,z) = -\frac{\sqrt{\lambda F^H}s'(z)}{(1+\theta\sqrt{\lambda F^H}s(z))^2} + \mathcal{O}(N^{-\frac{1}{2}}). \tag{D.34}$$

Now, integrating over θ, we get

$$\begin{aligned}\operatorname{Tr} R^A(1,z) - \operatorname{Tr} R^A(0,z) &= \frac{s'(z)}{s(z)}\left(\frac{1}{1+\theta\sqrt{\lambda F^H}s(z)}\right)\Bigg|_{\theta=0}^{\theta=1} + \mathcal{O}(N^{-\frac{1}{2}}) \\ &= -\frac{\sqrt{\lambda F^H}s'(z)}{1+\sqrt{\lambda F^H}s(z)} + \mathcal{O}(N^{-\frac{1}{2}}).\end{aligned} \tag{D.35}$$

On $\Gamma\backslash\Gamma^{\epsilon}_{1/2}$, we use the interlacing property of the eigenvalues. Let E_0^A and E_1^A be the cumulative distribution functions for the eigenvalue counting measures of $A(0)$ and $A(1)$, respectively, i.e., if we let $\mu_i^A(\theta)$ be the i-th eigenvalue of $A(\theta)$ and denote by $\mu_1^A(\theta) \geq \mu_2^A(\theta) \geq \cdots \geq \mu_N^A(\theta)$ the eigenvalues of $A(\theta)$, then

$$E_0^A(w) = \frac{1}{N}|\{\mu_i^A(0) : \mu_i^A(0) < w\}|, \qquad E_1^A(w) = \frac{1}{N}|\{\mu_i^A(1) : \mu_i^A(1) < w\}|. \tag{D.36}$$

The interlacing property is that

$$N|E_0^A(w) - E_1^A(w)| \leq 1. \tag{D.37}$$

In terms of E_0^A, we can represent the trace of the resolvent $R^A(0,z)$ by

$$\operatorname{Tr} R^A(0,z) = \sum_{i=1}^N \frac{1}{\mu_i^A(0) - z} = N\int_{-\infty}^{\infty}\frac{E_0^A(\mathrm{d}x)}{(x-z)^2}, \tag{D.38}$$

where we used integration by parts with empirical spectral measure of $A(0)$. Similarly,

$$\operatorname{Tr} R^A(1,z) = N\int_{-\infty}^{\infty}\frac{E_1^A(\mathrm{d}x)}{(x-z)^2},$$

and we get

$$\operatorname{Tr} R^A(1,z) - \operatorname{Tr} R^A(0,z) = N\int_{-\infty}^{\infty}\frac{E_1^A(\mathrm{d}x) - E_0^A(\mathrm{d}x)}{(x-z)^2}. \tag{D.39}$$

From the rigidity, Lemma D.3, we have that $\|A(0)\| - 2 = o(1)$. Moreover, since $A(0) = W(1)$ is a general Wigner-type matrix and $A(\theta)$ is a rank-1 perturbation of $A(0)$ with $\|A(0) - A(\theta)\| < 1$, it is not hard to see that $\|A(\theta)\| - 2 = o(1)$ with high probability as well. Thus,

$$\operatorname{Tr} R^A(1,z) - \operatorname{Tr} R^A(0,z) = N\int_{-\infty}^{\infty}\frac{E_1^A(\mathrm{d}x) - E_0^A(\mathrm{d}x)}{(x-z)^2} = N\int_{-2-\epsilon}^{2+\epsilon}\frac{E_1^A(\mathrm{d}x) - E_0^A(\mathrm{d}x)}{(x-z)^2} = \mathcal{O}(1). \tag{D.40}$$

Following the idea in (D.27), we obtain from (D.35) and (D.40) that

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\oint_\Gamma f(z)\operatorname{Tr} R^A(1,z)\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\oint_\Gamma f(z)\operatorname{Tr} R^A(0,z)\mathrm{d}z \\ &= -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma f(z)\frac{\sqrt{\lambda F^H}s'(z)}{1+\sqrt{\lambda F^H}s(z)}\mathrm{d}z + \mathcal{O}(N^{-\frac{1}{2}}).\end{aligned} \tag{D.41}$$

D.4. CLT for a general Wigner-type matrix with a spike and small perturbation

While the rank-1 spike in A is $\sqrt{\lambda F^H}\boldsymbol{x}\boldsymbol{x}^T$, the mean of the diagonal entry

$$\mathbb{E}[\widetilde{M}_{ii}] = \sqrt{\lambda F_d^H} x_i^2 + O(N\|\boldsymbol{x}\|_\infty^6), \tag{D.42}$$

which is different from $\sqrt{\lambda F^H} x_i^2$ in general. We thus define a matrix $B(\theta)$ for $0 \le \theta \le 1$ by

$$B_{ij}(\theta) = A_{ij}(1) \quad (i \neq j), \qquad B_{ii}(\theta) = A_{ii}(1) + \theta(\mathbb{E}[\widetilde{M}_{ii}] - \sqrt{\lambda F^H} x_i^2 - C_3 N x_i^6) \tag{D.43}$$

for the constant C_3 in (D.8). By definition, $B(0) = A(1)$ and

$$\widetilde{M}_{ii} = B_{ii}(1) + C_3 N x_i^6. \tag{D.44}$$

We also set

$$R^B(\theta, z) = (B(\theta) - zI)^{-1}.$$

For $z \in \Gamma_{1/2}^{\epsilon}$,

$$\begin{aligned}\frac{\partial}{\partial\theta} \operatorname{Tr} R^B(\theta, z) &= -\sum_{i,a=1}^{N} \left(\mathbb{E}[\widetilde{M}_{aa}] - \sqrt{\lambda F^H} x_a^2 - C_3 N x_a^6\right) R_{ia}^B(\theta, z) R_{ai}^B(\theta, z) \\ &= -\frac{\partial}{\partial z} \sum_{a=1}^{N} \left(\mathbb{E}[\widetilde{M}_{aa}] - \sqrt{\lambda F^H} x_a^2 - C_3 N x_a^6\right) R_{aa}^B(\theta, z).\end{aligned} \tag{D.45}$$

Since $\|B(\theta) - A(1)\| = O(\|\boldsymbol{x}\|_\infty^2)$, we find that

$$R_{aa}^B(\theta, z) - R_{aa}^B(0, z) = R_{aa}^B(\theta, z) - R_{aa}^A(1, z) = O(\|\boldsymbol{x}\|_\infty^2)$$

for $a = 1, 2, \dots, N$. Denote by $\boldsymbol{e}_a$ a standard basis vector whose a-th coordinate is 1 and all other coordinates are zero. From (D.31), we find that

$$\langle \boldsymbol{e}_a, R^A(0, z)\boldsymbol{x}\rangle = \langle \boldsymbol{e}_a, R^A(1, z)\boldsymbol{x}\rangle + \sqrt{\lambda F^H} \langle \boldsymbol{e}_a, R^A(1, z)\boldsymbol{x}\rangle \langle \boldsymbol{x}, R^A(0, z)\boldsymbol{x}\rangle, \tag{D.46}$$

hence

$$\langle \boldsymbol{e}_a, R^A(1, z)\boldsymbol{x}\rangle = \frac{\langle \boldsymbol{e}_a, \boldsymbol{x}\rangle s(z)}{1 + \sqrt{\lambda F^H} s(z)} + \mathcal{O}(N^{-\frac{1}{2}}). \tag{D.47}$$

Using the same argument again, we obtain that

$$R_{aa}^A(1, z) = \langle \boldsymbol{e}_a, R^A(1, z)\boldsymbol{e}_a\rangle = s(z) - \frac{\sqrt{\lambda F^H} s(z)^2}{1 + \sqrt{\lambda F^H} s(z)} |\langle \boldsymbol{x}, \boldsymbol{e}_a\rangle|^2 + \mathcal{O}(N^{-\frac{1}{2}}) = s(z) + \mathcal{O}(N^{-\frac{1}{2}}), \tag{D.48}$$

hence

$$R_{aa}^B(\theta, z) = R_{aa}^A(1, z) + \mathcal{O}(N^{-\frac{1}{2}}) = s(z) + \mathcal{O}(N^{-\frac{1}{2}}) \tag{D.49}$$

as well. Thus,

$$\begin{aligned}&\sum_{a=1}^{N} \left(\mathbb{E}[\widetilde{M}_{aa}] - \sqrt{\lambda F^H} x_a^2 - C_3 N x_a^6\right) R_{aa}^B(\theta, z) \\ &= \sum_{a=1}^{N} \left(\mathbb{E}[\widetilde{M}_{aa}] - \sqrt{\lambda F^H} x_a^2\right) s(z) + O(N\|\boldsymbol{x}\|_\infty^4) + \mathcal{O}(N^{-\frac{1}{2}}) \\ &= \sqrt{\lambda}(\sqrt{F^H} - \sqrt{F_d^H}) s(z) + \mathcal{O}(N\|\boldsymbol{x}\|_\infty^4) + \mathcal{O}(N^{-\frac{1}{2}})\end{aligned} \tag{D.50}$$

and

$$\frac{\partial}{\partial\theta} \operatorname{Tr} R^B(\theta, z) = -\sqrt{\lambda}(\sqrt{F_d^H} - \sqrt{F^H}) s'(z) + \mathcal{O}(N\|\boldsymbol{x}\|_\infty^4) + \mathcal{O}(N^{-\frac{1}{2}}). \tag{D.51}$$

Applying the estimate $R^B_{aa}(\theta, z) - R^A_{aa}(1, z) = O(\|\boldsymbol{x}\|_\infty^2)$ on $\Gamma \backslash \Gamma^\epsilon_{1/2}$, we obtain that

$$\begin{aligned}&\frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^B(1, z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^B(0, z) \mathrm{d}z \\ &= -\frac{\sqrt{\lambda}(\sqrt{F^H_d} - \sqrt{F^H})}{2\pi \mathrm{i}} \oint_\Gamma f(z) s'(z) \mathrm{d}z + \mathcal{O}(\sqrt{N}\|\boldsymbol{x}\|_\infty^2) + \mathcal{O}(N^{-\frac{1}{2}}).\end{aligned} \tag{D.52}$$

By construction, for all i, j,

$$\widetilde{M}_{ij} = B_{ij}(1) + C_3 N x_i^3 x_j^3 + O(N^2 x_i^5 x_j^5). \tag{D.53}$$

Set $\boldsymbol{x}^3 = (x_1^3, x_2^3, \dots, x_N^3)^T$, $B' = B(1) + C_3 N \boldsymbol{x}^3 (\boldsymbol{x}^3)^T$, and $R^{B'}(z) = (B' - zI)^{-1}$. Then, $z \in \Gamma^\epsilon_{1/2}$,

$$\langle \boldsymbol{e}_a, R^B(z) \boldsymbol{e}_a \rangle - \langle \boldsymbol{e}_a, R^{B'}(z) \boldsymbol{e}_a \rangle = C_3 N \langle \boldsymbol{e}_a, R^{B'} \boldsymbol{x}^3 \rangle \langle \boldsymbol{x}^3, R^B \boldsymbol{e}_a \rangle = \mathcal{O}(N \|\boldsymbol{x}\|_\infty^6). \tag{D.54}$$

On $\Gamma \backslash \Gamma^\epsilon_{1/2}$, we use the estimate

$$R^B_{aa}(z) - R^{B'}_{aa}(z) = O(N \|\boldsymbol{x}\|_\infty^6). \tag{D.55}$$

Then,

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^{B'}(z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^B(1, z) \mathrm{d}z = \mathcal{O}(N^2 \|\boldsymbol{x}\|_\infty^6) + \mathcal{O}(N\sqrt{N} \|\boldsymbol{x}\|_\infty^6). \tag{D.56}$$

Finally, if we set $E = \widetilde{M} - B''$, then $E_{ij} = O(N^2 x_i^5 x_j^5)$. Then, since $\|\boldsymbol{x}\|_\infty = N^{-\phi}$ for some $\phi > \frac{3}{8}$,

$$\|E\| \le \|E\|_{HS} = \left(\sum_{i,j=1}^N |E_{ij}|^2 \right)^{\frac{1}{2}} = O\left(N^2 \|\boldsymbol{x}\|_\infty^8 \left(\sum_{i,j=1}^N x_i^2 x_j^2 \right)^{\frac{1}{2}} \right) = O\left(N^2 \|\boldsymbol{x}\|_\infty^8 \right) = o(N^{-1}). \tag{D.57}$$

Thus, if we let $R^{\widetilde{M}}(z) = (\widetilde{M} - z)^{-1}$, for any $z \in \Gamma_\epsilon$,

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^{\widetilde{M}}(z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^{B'}(z) \mathrm{d}z = o(1) \tag{D.58}$$

with high probability.

D.5. Proof of Theorem 7 and Theorem 8

We are now ready to prove Theorem 7.

Denote by $\widetilde{\mu}_1 \ge \widetilde{\mu}_2 \ge \dots \ge \widetilde{\mu}_N$ the eigenvalues of $\widetilde{M}$. Recall that we denoted by $\mu_1^W(0) \ge \mu_2^W(0) \ge \dots \ge \mu_N^W(0)$ the eigenvalues of $W(0)$. From Cauchy's integral formula, as in (A.3), we have

$$\begin{aligned}&\sum_{i=1}^N f(\widetilde{\mu}_i) - N \int_{-2}^2 \frac{\sqrt{4 - x^2}}{2\pi} f(x) \, \mathrm{d}x \\ &= \left(\sum_{i=1}^N f(\mu_i^W(0)) - N \int_{-2}^2 \frac{\sqrt{4 - x^2}}{2\pi} f(x) \, \mathrm{d}x \right) + \left(\sum_{i=1}^N f(\widetilde{\mu}_i) - \sum_{i=1}^N f(\mu_i^W(0)) \right) \\ &= \left(\sum_{i=1}^N f(\mu_i^W(0)) - N \int_{-2}^2 \frac{\sqrt{4 - x^2}}{2\pi} f(x) \, \mathrm{d}x \right) - \left(\frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^{\widetilde{M}}(z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \operatorname{Tr} R^W(0, z) \mathrm{d}z \right).\end{aligned} \tag{D.59}$$

Since W is a Wigner matrix, the first term in the right-hand side converges to a Gaussian random variable, and the mean and the variance of the limiting Gaussian distribution are given by

$$m_W(f) = \frac{1}{4} \left(f(2) + f(-2) \right) - \frac{1}{2} \tau_0(f) + (w_2 - 2) \tau_2(f) + (\widetilde{w_4} - 3) \tau_4(f) \tag{D.60}$$

and

$$V_W(f) = (w_2 - 2)\tau_1(f)^2 + 2(\widetilde{w_4} - 3)\tau_2(f)^2 + 2\sum_{\ell=1}^{\infty} \ell \tau_\ell(f)^2, \tag{D.61}$$

respectively, where

$$\widetilde{w_4} = \frac{1}{(F^H)^2}\int_{-\infty}^{\infty} (h(w))^4 g(w)\, \mathrm{d}w = \frac{1}{(F^H)^2}\int_{-\infty}^{\infty} \frac{(g'(w))^4}{(g(w))^3} \mathrm{d}w, \tag{D.62}$$

corresponding to the leading order term in the fourth moment of W_{ij}. (Note that the fourth moments of W_{ij} are not equal, but the difference between $N^2\mathbb{E}[(W_{ij})^4]$ and $\widetilde{w_4}$ is negligible.)

For the second term in the right-hand side of (D.59), combining (D.27), (D.41), (D.52), (D.56), and (D.58), we obtain that

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\oint_\Gamma f(z) \operatorname{Tr} R^{\widetilde{M}}(z)\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\oint_\Gamma f(z) \operatorname{Tr} R^{W}(0,z)\mathrm{d}z \\
&= \frac{\lambda(G^H - F^H)}{2\pi \mathrm{i}} \oint_\Gamma f(z) \frac{s(z)^3}{1-s(z)^2}\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\oint_\Gamma f(z) \frac{\sqrt{\lambda F^H} s'(z)}{1+\sqrt{\lambda F^H} s(z)}\mathrm{d}z \\
&\quad - \frac{\sqrt{\lambda}(\sqrt{F_d^H} - \sqrt{F^H})}{2\pi \mathrm{i}} \oint_\Gamma f(z) s'(z)\mathrm{d}z + o(1)
\end{aligned} \tag{D.63}$$

with high probability. From (D.59), we thus find that the CLT for the LSS holds, i.e.,

$$\left(\sum_{i=1}^{N} f(\mu_i^{\widetilde{M}}) - N\int_{-2}^{2} \frac{\sqrt{4-x^2}}{2\pi} f(x)\, \mathrm{d}x \right) \to \mathcal{N}(m_{\widetilde{M}}(f), V_{\widetilde{M}}(f)), \tag{D.64}$$

and the variance $V_{\widetilde{M}}(f) = V_W(f)$ since the second term in (D.59) converges to a deterministic number as $N \to \infty$, which corresponds to the change of the mean. In particular,

$$\begin{aligned}
m_{\widetilde{M}}(f) - m_W(f) &= -\frac{\lambda(G^H - F^H)}{2\pi \mathrm{i}} \oint_\Gamma f(z) s'(z) s(z)\mathrm{d}z + \frac{1}{2\pi \mathrm{i}}\oint_\Gamma f(z) \frac{\sqrt{\lambda F^H} s'(z)}{1+\sqrt{\lambda F^H} s(z)}\mathrm{d}z \\
&\quad + \frac{\sqrt{\lambda}(\sqrt{F_d^H} - \sqrt{F^H})}{2\pi \mathrm{i}} \oint_\Gamma f(z) s'(z)\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}}\oint_\Gamma f(z) s'(z) \left[-\lambda(G^H - F^H)s(z) + \frac{\sqrt{\lambda F^H}}{1+\sqrt{\lambda F^H} s(z)} + \sqrt{\lambda}(\sqrt{F_d^H} - \sqrt{F^H}) \right] \mathrm{d}z.
\end{aligned} \tag{D.65}$$

Following the computation in the proof of Lemma 4.4 in (Baik & Lee, 2017) with the identity $s'(z) = \frac{s(z)^2}{1-s(z)^2}$, we find that the right-hand side of (D.65) is given by

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\oint_\Gamma f(z) s'(z) \left[-\lambda(G^H - F^H)s(z) + \frac{\sqrt{\lambda F^H}}{1+\sqrt{\lambda F^H} s(z)} + \sqrt{\lambda}(\sqrt{F_d^H} - \sqrt{F^H}) \right] \mathrm{d}z \\
&= (\sqrt{\lambda F_d^H} - \sqrt{\lambda F^H})\tau_1(f) + (\lambda G^H - \lambda F^H)\tau_2(f) + \sum_{\ell=1}^{\infty} \sqrt{(\lambda F^H)^\ell}\tau_\ell(f).
\end{aligned} \tag{D.66}$$

(See also Remark 1.7 of (Baik & Lee, 2017).) This proves Theorem 7.

D.6. Proof of Lemma D.2

In this subsection, we prove Lemma D.2.

NOTATIONAL REMARKS

In the rest of the section, we use C order to denote a constant that is independent of N. Even if the constant is different from one place to another, we may use the same notation C as long as it does not depend on N for the convenience of the presentation.

Proof of Lemma D.2. To prove the lemma, we consider

$$\begin{aligned}\frac{\partial}{\partial\theta}\operatorname{Tr} R^W(\theta,z) &= -\sum_{i=1}^N\sum_{a,b=1}^N \frac{\partial W_{ab}(\theta)}{\partial\theta} R^W_{ia}(\theta,z)R^W_{bi}(\theta,z)\\ &= -\frac{\partial}{\partial z}\sum_{a,b=1}^N \frac{\partial W_{ab}(\theta)}{\partial\theta} R^W_{ba}(\theta,z),\end{aligned} \tag{D.67}$$

where we again used that $\frac{\partial}{\partial z}R^W(\theta,z) = R^W(\theta,z)^2$. We expand the right-hand side by using the definition of $W(\theta)$,

$$W_{ab}(\theta) = \left(1-\theta+\theta\sqrt{NS_{ab}}\right)W_{ab}, \tag{D.68}$$

and get

$$\begin{aligned}\sum_{a,b=1}^N \frac{\partial W_{ab}(\theta)}{\partial\theta} R^W_{ba}(\theta,z) &= \sum_{a,b=1}^N\left(-1+\sqrt{NS_{ab}}\right)W_{ab}R^W_{ba}(\theta,z) = \sum_{a,b=1}^N \frac{-1+\sqrt{NS_{ab}}}{1-\theta+\theta\sqrt{NS_{ab}}}W_{ab}(\theta)R^W_{ba}(\theta,z)\\ &= \frac{N\lambda(G^H-F^H)}{2}\sum_{a,b=1}^N x_a^2x_b^2W_{ab}(\theta)R^W_{ba}(\theta,z) + \mathcal{O}(\sqrt{N}\|\boldsymbol{x}\|_\infty^2).\end{aligned} \tag{D.69}$$

Here, we used the properties that $W_{ab}(\theta) = \mathcal{O}(N^{-\frac12})$, $R^W_{ba}(\theta,z) = \mathcal{O}(N^{-\frac12})$ for $b\neq a$, $R^W_{aa}(\theta,z) = \mathcal{O}(1)$, and $\sum_a x_a^2 = \sum_b x_b^2 = 1$, which imply

$$\left|N^2\sum_{a,b=1}^N x_a^4x_b^4W_{ab}(\theta)R^W_{ba}(\theta,z)\right| \le N^2\|\boldsymbol{x}\|_\infty^4\sum_{a,b=1}^N x_a^2x_b^2|W_{ab}(\theta)R^W_{ba}(\theta,z)| = \mathcal{O}(N\|\boldsymbol{x}\|_\infty^4) \tag{D.70}$$

and

$$\left|N\sum_{a=1}^N x_a^4x_b^4W_{aa}(\theta)R^W_{aa}(\theta,z)\right| \le N\|\boldsymbol{x}\|_\infty^2\sum_{a=1}^N x_a^2|W_{aa}(\theta)R^W_{aa}(\theta,z)| = \mathcal{O}(\sqrt{N}\|\boldsymbol{x}\|_\infty^2). \tag{D.71}$$

Since $W(\theta)R^W(\theta,z) = I + zR^W(\theta,z)$,

$$\begin{aligned}\sum_{a,b=1}^N x_b^2W_{ab}(\theta)R^W_{ba}(\theta,z) &= \sum_{b=1}^N x_b^2(W(\theta)R^W(\theta,z))_{bb} = 1 + z\sum_{b=1}^N x_b^2R^W_{bb}(\theta,z)\\ &= 1 + zs(z) + \mathcal{O}(N^{-\frac12}).\end{aligned} \tag{D.72}$$

Plugging it into (D.69), we get

$$\begin{aligned}&\sum_{a,b=1}^N \frac{\partial W_{ab}(\theta)}{\partial\theta}R^W_{ba}(\theta,z)\\ &= \frac{\lambda(G^H-F^H)}{2}(1+zs(z)) + \frac{N\lambda(G^H-F^H)}{2}\sum_{a,b=1}^N\left(x_a^2-\frac1N\right)x_b^2W_{ab}(\theta)R^W_{ba}(\theta,z) + \mathcal{O}(\sqrt{N}\|\boldsymbol{x}\|_\infty^2).\end{aligned} \tag{D.73}$$

It remains to estimate the second term in the right-hand side of (D.73). Set

$$X \equiv X(\theta,z) := \sum_{a,b=1}^N\left(x_a^2-\frac1N\right)x_b^2W_{ab}(\theta)R^W_{ba}(\theta,z). \tag{D.74}$$

We notice that $|X| = \mathcal{O}(N^{-1})$ on $\Gamma^\epsilon_{1/2}$ by a naive power counting as in (D.69). To obtain a better bound for X, we use a method based on a recursive moment estimate, introduced in (Lee & Schnelli, 2018). We need the following lemma:

Lemma D.4. *Let X be as in (D.74). Define an event Ω_ϵ by*

$$\Omega_\epsilon = \bigcap_{i,j=1}^{N} \left(\{|W_{ij}(\theta)| \le N^{-\frac{1}{2}+\epsilon}\} \cap \{|R^W_{ij}(\theta,z) - \delta_{ij}s(z)| \le N^{-\frac{1}{2}+\epsilon}\} \right).$$

Then, for any fixed (large) D and (small) ϵ, which may depend on D,

$$\begin{aligned}\mathbb{E}[|X|^{2D}|\Omega_\epsilon] &\le CN^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4 \mathbb{E}[|X|^{2D-1}|\Omega_\epsilon] + CN^{1+4\epsilon}\|\boldsymbol{x}\|_\infty^8 \mathbb{E}[|X|^{2D-2}|\Omega_\epsilon] \\ &\quad + CN^{1+5\epsilon}\|\boldsymbol{x}\|_\infty^{12}\mathbb{E}[|X|^{2D-3}|\Omega_\epsilon] + N^{1+9\epsilon}\|\boldsymbol{x}\|_\infty^{16}\mathbb{E}[|X|^{2D-4}|\Omega_\epsilon].\end{aligned} \tag{D.75}$$

We will prove Lemma D.4 at the end of this section. With Lemma D.4, we are ready to obtain an improved bound for X. First, note that $\mathbb{P}(\Omega_\epsilon^c) < N^{-D^2}$, which can be checked by applying a high-order Markov inequality with the moment condition on $\widetilde{M}$ (Assumption 1(iii)). We decompose $\mathbb{E}[|X|^{2D}]$ by

$$\mathbb{E}[|X|^{2D}] = \mathbb{E}[|X|^{2D}\cdot \mathbf{1}(\Omega_\epsilon)] + \mathbb{E}[|X|^{2D}\cdot \mathbf{1}(\Omega_\epsilon^c)] = \mathbb{E}[|X|^{2D}|\Omega_\epsilon]\cdot\mathbb{P}(\Omega_\epsilon) + \mathbb{E}[|X|^{2D}\cdot \mathbf{1}(\Omega_\epsilon^c)]. \tag{D.76}$$

The second term in the right-hand side of (D.76), the contribution from the exceptional event Ω_ϵ^c is negligible, since $\mathbb{P}(\Omega_\epsilon^c) < N^{-D^2}$,

$$\mathbb{E}[|X|^{2D}\cdot \mathbf{1}(\Omega_\epsilon^c)] \le \left(\mathbb{E}[|X|^{4D}]\right)^{\frac{1}{2}} \left(\mathbb{P}(\Omega_\epsilon^c)\right)^{\frac{1}{2}} \le N^{-\frac{D^2}{2}} \left(\mathbb{E}[|X|^{4D}]\right)^{\frac{1}{2}} \tag{D.77}$$

and

$$\mathbb{E}[|X|^{4D}] \le \left(\sum_{a,b=1}^{N} |W_{ab}R^W_{ba}|\right)^{4D} \le \frac{N^{8D}}{(\operatorname{Im} z)^{4D}} \max_{a,b}\mathbb{E}[|W_{ab}|^{4D}] \le N^{10D}, \tag{D.78}$$

where we used a trivial bound $|R^W_{ba}| \le \|R^W\| \le \frac{1}{\operatorname{Im} z}$.

From Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

which holds for any $a, b > 0$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, we find that

$$\begin{aligned}N^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4 |X|^{2D-1} &= N^{\frac{(2D-1)\epsilon}{2D}} N^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4 \cdot N^{-\frac{(2D-1)\epsilon}{2D}}|X|^{2D-1} \\ &\le \frac{1}{2D}N^{(2D-1)\epsilon}(N^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4)^{2D} + \frac{2D-1}{2D}N^{-\epsilon}|X|^{2D}.\end{aligned} \tag{D.79}$$

Applying Young's inequality for other terms in (D.75), we get

$$\begin{aligned}\mathbb{E}[|X|^{2D}|\Omega_\epsilon] &\le CN^{(2D-1)\epsilon}(N^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4)^{2D} + CN^{(D-1)\epsilon}(N^{1+4\epsilon}\|\boldsymbol{x}\|_\infty^8)^{D} \\ &\quad + CN^{(\frac{2D}{3}-1)\epsilon}(N^{1+5\epsilon}\|\boldsymbol{x}\|_\infty^{12})^{\frac{2D}{3}} + CN^{(\frac{D}{2}-1)\epsilon}(N^{1+9\epsilon}\|\boldsymbol{x}\|_\infty^{16})^{\frac{D}{2}} + CN^{-\epsilon}\mathbb{E}[|X|^{2D}|\Omega_\epsilon].\end{aligned} \tag{D.80}$$

Absorbing the last term in the right-hand side to the left-hand side and plugging the estimates (D.77) and (D.78) into (D.76), we now get

$$\begin{aligned}\mathbb{E}[|X|^{2D}] &\le CN^{(2D-1)\epsilon}(N^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4)^{2D} + CN^{(D-1)\epsilon}(N^{1+4\epsilon}\|\boldsymbol{x}\|_\infty^8)^{D} \\ &\quad + CN^{(\frac{2D}{3}-1)\epsilon}(N^{1+5\epsilon}\|\boldsymbol{x}\|_\infty^{12})^{\frac{2D}{3}} + CN^{(\frac{D}{2}-1)\epsilon}(N^{1+9\epsilon}\|\boldsymbol{x}\|_\infty^{16})^{\frac{D}{2}} + N^{-\frac{D^2}{2}+5D}.\end{aligned} \tag{D.81}$$

For any fixed $\epsilon' > 0$ independent of D, from the $(2D)$-th order Markov inequality,

$$\mathbb{P}\big(|X| \ge N^{\epsilon'}\sqrt{N}\|\boldsymbol{x}\|_\infty^4\big) \le N^{-2D\epsilon'}\frac{\mathbb{E}[|X|^{2D}]}{(\sqrt{N}\|\boldsymbol{x}\|_\infty^4)^{2D}} \le N^{-2D\epsilon'}N^{5D\epsilon}. \tag{D.82}$$

Thus, by choosing D sufficiently large and $\epsilon = 1/D$, we find that

$$|X| = \mathcal{O}(\sqrt{N}\|\boldsymbol{x}\|_\infty^4).$$

We now go back to (D.67) and use (D.73) with the bound $|X| = \mathcal{O}(\sqrt{N}\|\boldsymbol{x}\|_\infty^4)$. Since $\|\boldsymbol{x}\|_\infty = O(N^{-\phi})$ for some $\frac{3}{8} < \phi \le \frac{1}{2}$,

$$\sum_{a,b=1}^{N} \frac{\partial W_{ab}(\theta)}{\partial \theta} R_{ba}^W(\theta, z) = \frac{\lambda(G^H - F^H)}{2}(1 + zs(z)) + \mathcal{O}(N^{\frac{3}{2}}\|\boldsymbol{x}\|_\infty^4). \tag{D.83}$$

To handle the derivative of the right-hand side, we use Cauchy's integral formula as in (A.31) with a rectangular contour, contained in $\Gamma_{1/2}^{\epsilon}$, whose perimeter is larger than ϵ. Then, we get from (D.67) that

$$\frac{\partial}{\partial \theta} \operatorname{Tr} R^W(\theta, z) = -\frac{\lambda(G^H - F^H)}{2}\frac{\partial}{\partial z}(1 + zs(z)) + \mathcal{O}(N^{\frac{3}{2}}\|\boldsymbol{x}\|_\infty^4). \tag{D.84}$$

Since $1 + zs(z) + s(z)^2 = 0$,

$$\frac{\partial}{\partial z}(1 + zs(z)) = \frac{\partial}{\partial z}(-s(z)^2) = -2s(z)s'(z). \tag{D.85}$$

After integrating over θ from 0 to 1, we conclude that (D.23) holds for a fixed $z \in \Gamma_{1/2}^{\epsilon}$. To prove the uniform bound in the lemma, we can use the lattice argument in Section A; see Equations (A.14)-(A.17). $\square$

Finally, we prove the recursive moment estimate in Lemma D.4.

Proof of Lemma D.4. We consider

$$\mathbb{E}[|X|^{2D}] = \mathbb{E}\left[\sum_{a,b=1}^{N} \big(x_a^2 - \frac{1}{N}\big)x_b^2 W_{ab}(\theta) R_{ba}^W(\theta, z) X^{D-1}\overline{X}^D\right].$$

For simplicity, we omit the θ-dependence and z-dependence of $W \equiv W(\theta)$ and $R^W \equiv R^W(\theta, z)$.

We use the following inequality that generalizes Stein's lemma (see Proposition 5.2 of (Baik et al., 2018)): Let Φ be a C^2 function. Fix a (small) $\epsilon > 0$, which may depend on D. Recall that Ω_ϵ is the complement of the exceptional event on which $|W_{ab}|$ or $|R_{ba}^W|$ is exceptionally large for some a, b, defined by

$$\Omega_\epsilon = \bigcap_{i,j=1}^{N} \Big(\{|W_{ij}| \le N^{-\frac{1}{2}+\epsilon}\} \cap \{|R_{ij}^W - \delta_{ij}s| \le N^{-\frac{1}{2}+\epsilon}\}\Big).$$

Then,

$$\mathbb{E}[W_{ab}\Phi(W_{ab})|\Omega_\epsilon] = \mathbb{E}[W_{ab}^2]\mathbb{E}[\Phi'(W_{ab})|\Omega_\epsilon] + \epsilon_1, \tag{D.86}$$

where the error term ϵ_1 admits the bound

$$|\epsilon_1| \le C_1 \mathbb{E}\Big[|W_{ab}|^3 \sup_{|t|\le 1} \Phi''(tW_{ab})\Big|\Omega_\epsilon\Big] \tag{D.87}$$

for some constant C_1. The estimate (D.86) follows from the proof of Proposition 5.2 of (Baik et al., 2018) with $p = 1$, where we use the inequality (5.38) therein only up to second to the last line.

In the estimate (D.86), we let

$$\Phi(W_{ab}) = R_{ba}^W X^{D-1}\overline{X}^D \tag{D.88}$$

so that

$$\mathbb{E}[|X|^{2D}|\Omega_\epsilon] = \sum_{a,b=1}^{N} \big(x_a^2 - \frac{1}{N}\big)x_b^2 \mathbb{E}\left[W_{ab}\Phi(W_{ab})|\Omega_\epsilon\right]. \tag{D.89}$$

We now consider the term $\mathbb{E}\left[W_{ab}\Phi(W_{ab})|\Omega_\epsilon\right]$ in (D.89). Applying the equation (D.86),

$$\begin{aligned}
&\mathbb{E}[W_{ab}\Phi(W_{ab})|\Omega_\epsilon] = \mathbb{E}[W_{ab}^2]\mathbb{E}[\Phi'(W_{ab})|\Omega_\epsilon] + \epsilon_1 \\
&= \mathbb{E}[W_{ab}^2]\Big(-\mathbb{E}\left[R_{bb}^W R_{aa}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right] - \mathbb{E}\left[R_{ba}^W R_{ba}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right] \\
&\quad + (D-1)\mathbb{E}\left[R_{ba}^W \frac{\partial X}{\partial W_{ab}} X^{D-2}\overline{X}^D|\Omega_\epsilon\right] + D\mathbb{E}\left[R_{ba}^W \frac{\partial \overline{X}}{\partial W_{ab}} X^{D-1}\overline{X}^{D-1}|\Omega_\epsilon\right]\Big) + \epsilon_1.
\end{aligned} \tag{D.90}$$

We plug it into (D.89) and estimate each term. We decompose the term originated from the first term in (D.90) as

$$\begin{aligned}&\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{bb}^W R_{aa}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right]\\&=\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{bb}^W(R_{aa}^W-s)X^{D-1}\overline{X}^D|\Omega_\epsilon\right]+s\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{bb}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right].\end{aligned} \tag{D.91}$$

The first term satisfies that

$$\begin{aligned}&\left|\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{bb}^W(R_{aa}^W-s)X^{D-1}\overline{X}^D|\Omega_\epsilon\right]\right|\\&\le CN^2\|\boldsymbol{x}\|_\infty^4 N^{-1}N^{-\frac{1}{2}+\epsilon}\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]=CN^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]\end{aligned} \tag{D.92}$$

for some constant C. For the second term, we recall that $\sum_a(x_a^2-\frac{1}{N})=0$ and $\mathbb{E}[W_{ab}^2]$ are identical except for $a\neq b$. Thus,

$$\begin{aligned}&\left|s\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{bb}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right]\right|\\&\le C\left|\sum_{b=1}^{N}|x_b^2-\frac{1}{N}|x_b^2|w_2-1|N^{-1}\mathbb{E}\left[R_{bb}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right]\right|\\&\le C'N\|\boldsymbol{x}\|_\infty^4 N^{-1}\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]=C'\|\boldsymbol{x}\|_\infty^4\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]\end{aligned} \tag{D.93}$$

for some constants C and C'. We then find that

$$\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{bb}^W R_{aa}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right]\le CN^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon] \tag{D.94}$$

for some constant C. For the second term in (D.90), we also have

$$\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{ba}^W R_{ba}^W X^{D-1}\overline{X}^D|\Omega_\epsilon\right]\le CN^{2\epsilon}\|\boldsymbol{x}\|_\infty^4\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]. \tag{D.95}$$

To estimate the third term and the fourth term in (D.90), we notice that on Ω_ϵ

$$\left|\frac{\partial X}{\partial W_{ab}}\right|=\left|\sum_{i,j=1}^{N}(x_i^2-\frac{1}{N})x_j^2W_{ij}R_{bi}^W R_{ja}^W+(x_a^2-\frac{1}{N})x_b^2R_{ba}^W\right|\le CN^{\frac{1}{2}+3\epsilon}\|\boldsymbol{x}\|_\infty^4. \tag{D.96}$$

for some constant C. Thus, we obtain that

$$\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{ba}^W\frac{\partial X}{\partial W_{ab}}X^{D-2}\overline{X}^D|\Omega_\epsilon\right]\le CN^{1+4\epsilon}\|\boldsymbol{x}\|_\infty^8\mathbb{E}[|X|^{2D-2}|\Omega_\epsilon] \tag{D.97}$$

and

$$\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}[W_{ab}^2]\mathbb{E}\left[R_{ba}^W\frac{\partial \overline{X}}{\partial W_{ab}}X^{D-1}\overline{X}^{D-1}|\Omega_\epsilon\right]\le CN^{1+4\epsilon}\|\boldsymbol{x}\|_\infty^8\mathbb{E}[|X|^{2D-2}|\Omega_\epsilon]. \tag{D.98}$$

Hence, from (D.90), (D.94), (D.95), (D.97), and (D.98),

$$\begin{aligned}\left|\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}\left[W_{ab}\Phi(W_{ab})|\Omega_\epsilon\right]\right|&\le CN^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]\\&\quad+CN^{1+4\epsilon}\|\boldsymbol{x}\|_\infty^8\mathbb{E}[|X|^{2D-2}|\Omega_\epsilon]+\epsilon_1.\end{aligned} \tag{D.99}$$

It remains to estimate $|\epsilon_1|$ in (D.87). Proceeding as before,

$$\begin{aligned}&\sum_{a,b=1}^{N}(x_a^2-\frac{1}{N})x_b^2\mathbb{E}\Big[|W_{ab}|^3\Phi''(W_{ab})\Big|\Omega_\epsilon\Big]\\&\le CN^{\epsilon}\|\boldsymbol{x}\|_\infty^4\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]+CN^{1+2\epsilon}\|\boldsymbol{x}\|_\infty^8\mathbb{E}[|X|^{2D-2}|\Omega_\epsilon]+CN^{1+5\epsilon}\|\boldsymbol{x}\|_\infty^{12}\mathbb{E}[|X|^{2D-3}|\Omega_\epsilon].\end{aligned} \tag{D.100}$$

We want to compare $\Phi''(W_{ab})$ and $\Phi''(tW_{ab})$ for some $|t|<1$. Let $R^{W,t}$ be the resolvent of W where W_{ab} and W_{ba} are replaced by tW_{ab} and tW_{ba}, respectively, and let X^t be defined as X in (D.74) with the same replacement for W_{ab} (and W_{ba}) and also R^W is replaced by $R^{W,t}$. Then,

$$R_{ji}^{W,t}-R_{ji}^{W}=(1-t)R_{ja}^{W}W_{ab}R_{bi}^{W,t}, \tag{D.101}$$

and

$$X^t-X=\sum_{i,j=1}^{N}(x_i^2-\frac{1}{N})x_j^2W_{ij}(R_{ji}^{W,t}-R_{ji}^{W})-(1-t)(x_a^2-\frac{1}{N})x_b^2W_{ab}R_{ba}^{W,t}. \tag{D.102}$$

Thus, on Ω_ϵ,

$$|X^t-X|\le CN^{4\epsilon}\|\boldsymbol{x}\|_\infty^4. \tag{D.103}$$

Using the estimates (D.101) and (D.103), on Ω_ϵ, we obtain that

$$|\Phi''(W_{ab})-\Phi''(tW_{ab})|\le C|\Phi''(W_{ab})|+N^{\frac{1}{2}+5\epsilon}\|\boldsymbol{x}\|_\infty^{12}|X|^{2D-4} \tag{D.104}$$

uniformly on $t\in(-1,1)$.

Combining (D.89) and (D.99) with (D.100), (D.104), and (D.87), we finally get

$$\begin{aligned}\mathbb{E}[|X|^{2D}|\Omega_\epsilon]&\le CN^{\frac{1}{2}+\epsilon}\|\boldsymbol{x}\|_\infty^4\mathbb{E}[|X|^{2D-1}|\Omega_\epsilon]+CN^{1+4\epsilon}\|\boldsymbol{x}\|_\infty^8\mathbb{E}[|X|^{2D-2}|\Omega_\epsilon]\\&\quad+CN^{1+5\epsilon}\|\boldsymbol{x}\|_\infty^{12}\mathbb{E}[|X|^{2D-3}|\Omega_\epsilon]+CN^{1+9\epsilon}\|\boldsymbol{x}\|_\infty^{16}\mathbb{E}[|X|^{2D-4}|\Omega_\epsilon].\end{aligned} \tag{D.105}$$

This proves the desired lemma. □

References

Ajanki, O. H., Erdős, L., and Krüger, T. Universality for general Wigner-type matrices. *Probab. Theory Related Fields*, 169 (3-4):667–727, 2017.

Bai, Z. D. and Silverstein, J. W. CLT for linear spectral statistics of large-dimensional sample covariance matrices. *Ann. Probab.*, 32(1A):553–605, 2004.

Bai, Z. D. and Yao, J. On the convergence of the spectral empirical process of Wigner matrices. *Bernoulli*, 11(6):1059–1092, 2005.

Baik, J. and Lee, J. O. Fluctuations of the free energy of the spherical Sherrington-Kirkpatrick model. *J. Stat. Phys.*, 165(2): 185–224, 2016.

Baik, J. and Lee, J. O. Fluctuations of the free energy of the spherical Sherrington-Kirkpatrick model with ferromagnetic interaction. *Ann. Henri Poincaré*, 18(6):1867–1917, 2017.

Baik, J., Lee, J. O., and Wu, H. Ferromagnetic to Paramagnetic Transition in Spherical Spin Glass. *Journal of Statistical Physics*, 2018.

Knowles, A. and Yin, J. The isotropic semicircle law and deformation of Wigner matrices. *Comm. Pure Appl. Math.*, 66 (11):1663–1750, 2013.

Lee, J. O. and Schnelli, K. Local law and Tracy-Widom limit for sparse random matrices. *Probab. Theory Related Fields*, 171(1-2):543–616, 2018.

Olver, F. W. J., Lozier, D. W., Boisvert, R. F., and Clark, C. W. (eds.). *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

Perry, A., Wein, A. S., Bandeira, A. S., and Moitra, A. Optimality and sub-optimality of PCA I: Spiked random matrix models. *Ann. Statist.*, 46(5):2416–2451, 2018.