Random Walks on Hypergraphs with Edge-Dependent Vertex Weights: Supplementary Material

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A. Incorrect Stationary Distribution in Earlier Work

Li et al. (2018) claim in Equation 4 that the stationary distribution π of a random walk on a hypergraph $H = (V, E, \gamma, \omega)$ with edge-dependent vertex weights is

$$\pi_v = \frac{d(v)}{\sum_{u \in V} d(u)}, \tag{1}$$

where $d(v) = \sum_{e \in E(v)} \omega(e)$ is the sum of edge weights of incident hyperedges. Curiously, the stationary distribution given by this formula does not depend on the vertex weights. A counterexample to this formula is shown in hypergraph H in Figure 1 of the main text, with edge-dependent vertex weights as described in the caption (i.e. $\gamma_{e_1}(b) = 1, \gamma_{e_2}(b) = 2$). Computing the stationary distribution π of a random walk on H yields that $\pi_b = 7/20$, while Equation (1) incorrectly yields $\pi_b = 2/7$.

B. Proof of Theorem 3.1

First we need the following definition and lemma.

Definition B.1. *Let M be a Markov chain with state space X and transition probabilities $p_{x,y}$, for $x, y \in S$. We say M is* reversible *if there exists a probability distribution π over S such that*

$$\pi_x p_{x,y} = \pi_y p_{y,x}. \tag{2}$$

Lemma B.1. *Let M be an irreducible Markov chain with finite state space S and transition probabilities $p_{x,y}$ for $x, y, \in S$. M is reversible if and only if there exists a weighted, undirected graph G with vertex set S such that a random walk on G and M are equivalent.*

Proof of Lemma. First, suppose M is reversible. Since M is irreducible, let π be the stationary distribution of M. Note that, because M is irreducible, $\pi_x \neq 0$ for all states x.

Let G be a graph with vertices S, and edge weights $w_{x,y} = \pi_x p_{x,y}$. By reversibility, G is well-defined. In a random walk on G, the probability of going from x to y in one time-step is

$$\frac{w_{x,y}}{\sum_{z \in S} w_{x,z}} = \frac{\pi_x p_{x,y}}{\sum_{z \in S} \pi_x p_{x,z}} = \frac{p_{x,y}}{\sum_{z \in S} p_{x,z}} = p_{x,y},$$

since $\sum_{z \in S} p_{x,z} = 1$.

Thus, if M is reversible, the stated claim holds. The other direction follows from the fact that a random walk on an undirected graph is always reversible (Aldous & Fill, 2002). □

Theorem 3.1. *Let $H = (V, E, \omega, \gamma)$ be a hypergraph with edge-*independent *vertex weights. Then, there exist weights $w_{u,v}$ on the clique graph G^H such that a random walk on H is equivalent to a random walk on G^H.*

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Proof of Theorem 3.1. Let $\gamma(v) = \gamma_e(v)$ for vertices v and incident hyperedges e. We first show that a random walk on H is reversible. By Kolmogorov's criterion, reversibility is equivalent to

$$p_{v_1,v_2}p_{v_2,v_3}\cdots p_{v_n,v_1} = p_{v_1,v_n}p_{v_n,v_{n-1}}\cdots p_{v_2,v_1}. \tag{3}$$

for any set of vertices $v_1, \ldots, v_n$.

Since the transition probabilities for any two vertices u, v are

$$p_{u,v} = \sum_{e\in E(u,v)} \frac{\omega(e)}{d(u)}\frac{\gamma(u)}{\delta(e)} = \frac{\gamma(u)}{\delta(u)}\sum_{e\in E(u,v)}\frac{\omega(e)}{\delta(e)}, \tag{4}$$

we have

$$\begin{aligned} p_{v_1,v_2}p_{v_2,v_3}\cdots p_{v_n,v_1} &= \left(\frac{\gamma(v_1)}{\delta(v_1)}\sum_{e\in E(v_1,v_2)}\frac{\omega(e)}{\delta(e)}\right)\cdots\left(\frac{\gamma(v_n)}{\delta(v_n)}\sum_{e\in E(v_n,v_1)}\frac{\omega(e)}{\delta(e)}\right) \\ &= \prod_{i=1}^{n}\left(\frac{\gamma(v_i)}{\delta(v_i)}\sum_{e\in E(v_i,v_{i+1})}\frac{\omega(e)}{\delta(e)}\right), \text{where we define } v_{n+1} = v_1 \\ &= \left(\frac{\gamma(v_1)}{\delta(v_1)}\sum_{e\in E(v_n,v_1)},\frac{\omega(e)}{\delta(e)}\right)\cdots\left(\frac{\gamma(v_2)}{\delta(v_2)}\sum_{e\in E(v_2,v_1)}\frac{\omega(e)}{\delta(e)}\right) \\ &= p_{v_1,v_n}p_{v_n,v_{n-1}}\cdots p_{v_2,v_1}. \end{aligned} \tag{5}$$

So by Kolmogorov's criterion, a random walk on H is reversible.

Furthermore, because H is connected, random walks on H are irreducible. Thus, by Lemma B.1, there exists a graph G with vertex set V and edge weights $w_{u,v}$ such that random walks on G and H are equivalent. The equivalence of the random walks implies that $p_{u,v} > 0$ if and only if $w_{u,v} > 0$, so it follows that G is the clique graph of H. □

C. Non-Lazy Random Walks on Hypergraphs

First we generalize the random walk framework of Cooper et al. (2013) to random walks on hypergraphs with edge-dependent vertex weights. Informally, in a non-lazy random walk, a random walker at vertex v will do the following:

1. pick an edge e containing v, with probability $\frac{\omega(e)}{d(v)}$,
2. pick a vertex $w \neq v$ from e, with probability $\frac{\gamma_e(w)}{\delta(e)-\gamma_e(v)}$, and
3. move to vertex w.

Formally, we have the following.

Definition C.1. *A* non-lazy random walk *on a hypergraph with edge-dependent vertex weights $H = (V, E, \omega, \gamma)$ is a Markov chain on V with transition probabilities*

$$p_{v,w} = \sum_{e\in E(v)}\left(\frac{\omega(e)}{d(v)}\right)\left(\frac{\gamma_e(w)}{\delta(e)-\gamma_e(v)}\right). \tag{6}$$

for all states $v \neq w$.

It is also useful to define a modified version of the clique graph without self-loops.

Definition C.2. *Let $H = (V, E, \omega, \gamma)$ be a hypergraph with edge-dependent vertex weights. The* clique graph of H without self-loops*, G^H_{nl}, is a weighted, undirected graph with vertex set V, and edges E' defined by*

$$E' = \{(v, w) \in V \times V : v, w \in e \textit{ for some } e \in E, \textit{ and } v \neq w\}. \tag{7}$$

In contrast to the lazy random walk, a non-lazy random walk on a hypergraph with edge-independent vertex weights is not guaranteed to satisfy reversibility. However, if H has *trivial* vertex weights, then reversibility holds, and we get the following result.

Theorem C.1. *Let $H = (V, E, \omega, \gamma)$ be a hypergraph with trivial vertex weights, i.e. $\gamma_e(v) = 1$ for all vertices v and incident hyperedges e. Then, there exist weights $w_{u,v}$ on the clique graph without self-loops G^H_{nl} such that a non-lazy random walk on H is equivalent to a random walk on G^H_{nl}.*

Proof. Again, we first show that a non-lazy random walk on H is reversible. Define the probability mass function $\pi_v = c \cdot d(v)$ for normalizing constant $c > 0$. Let $p_{u,v}$ be the probability of going from u to v in a non-lazy random walk on H, where $u \neq v$. Then,

$$\begin{aligned}\pi_u P_{u,v} &= c \cdot d(u) \cdot \left(\sum_{e \in E(u,v)} \frac{w(e)}{d(u)} \cdot \frac{1}{|e|-1} \right) \\ &= \sum_{e \in E(u,v)} \left(\omega(e) \cdot \frac{c}{|e|-1} \right).\end{aligned}$$

By symmetry, $\pi_u p_{u,v} = \pi_v p_{v,u}$, so a non-lazy random is reversible. Thus, by Lemma B.1, there exists a graph G with vertex set V and edge weights $w_{u,v}$ such that a random walk on G and a non-lazy random walk on H are equivalent. The equivalence of the random walks implies that $p_{u,v} > 0$ if and only if $w_{u,v} > 0$, so it follows that G is the clique graph of H without self-loops. ☐

D. Relationships between Random Walks on Hypergraphs and Markov Chains on Vertex Set

In the main text, we show that there are hypergraphs with edge-dependent vertex weights whose random walks are not equivalent to a random walk on a graph. A natural follow-up question is to ask whether all Markov chains on a vertex set V can be represented as a random walk on some hypergraph with the same vertex set and edge-dependent vertex weights. Below, we show that the answer is no. Since random walks on hypergraphs with edge-dependent vertex weights are lazy, in the sense that $p_{v,v} > 0$ for all vertices v, we restrict our attention to lazy Markov chains with $p_{v,v} = 0$.

Claim D.1. *There exists a lazy Markov chain M with state space V such that M is not equivalent to a random walk on a hypergraph with vertex set V and edge-dependent vertex weights.*

Proof. Suppose for the sake of contradiction that any lazy Markov chain with V is equivalent to a random walk on some hypergraph with vertex set V. Let M be a lazy Markov chain with states V and transition probabilities p^M, with the following property. For some states $x, y \in V$, let

$$\begin{aligned}p^M_{x,x} &= 0.9 \\ p^M_{x,y} &= 0.01 \\ p^M_{y,x} &= 0.1 \\ p^M_{y,y} &= 0.001.\end{aligned} \tag{8}$$

By assumption, let $H = (V, E, \omega, \gamma)$ be a hypergraph with vertex set V and edge-dependent vertex weights, such that a random walk on H is equivalent to M. Let p^H be the transition probabilities of a random walk on H. We have

$$\begin{aligned}d(x) \cdot p^M_{x,x} &= d(x) \cdot p^H_{x,x} \\ &= \sum_{e \in E(x)} \omega(e) \cdot \left(\frac{\gamma_e(x)}{\delta(x)} \right) \\ &\geq \sum_{e \in E(x,y)} \omega(e) \cdot \left(\frac{\gamma_e(x)}{\delta(x)} \right) \\ &= d(y) \cdot p^H_{y,x} \\ &= d(y) \cdot p^M_{y,x}\end{aligned} \tag{9}$$

Plugging in Equations (8) to the above yields $d(x) \cdot 0.9 \geq d(y) \cdot 0.1$, or $9d(x) \geq d(y)$.

By similar reasoning, we also have $d(y) \cdot p^M_{y,y} \geq d(x) \cdot p^M_{x,y}$, and plugging in Equations (8) gives us $d(y) \cdot 0.001 \geq d(x) \cdot 0.01$, or $d(y) \geq 10d(x)$.

Combining both of these inequalities, we obtain

$$9d(x) \geq d(y) \geq 10d(x). \tag{10}$$

Since the vertex degree $d(x) \geq 0$, we obtain a contradiction. □

Next, for any $k > 1$, define a k-hypergraph to be a hypergraph with edge-dependent vertex weights whose hyperedges have cardinality at most k. We show that, for any k, there exists a k-hypergraph with vertex set V whose random walk is not equivalent to the random walk of any $(k-1)$-hypergraph with vertex set V. We first prove the result for $k = 3$.

Lemma D.1. *There exists a* 3*-hypergraph with vertex set* V*, whose random walk is not equivalent to a random walk on any* 2*-hypergraph with vertex set* V*.*

Proof. Let $H_3 = (V, E_3, \omega, \gamma)$ be a 3-hypergraph with four vertices, $V = \{v_1, v_2, v_3, v_4\}$, and two hyperedges $e_1 = \{v_1, v_2, v_3\}$ and $e_2 = \{v_1, v_3, v_4\}$. Let the hyperedge weights be $\omega(e_1) = \omega(e_2) = 1$ and the vertex weights be $\gamma_{e_1}(v_1) = 2$, and $\gamma_{e_i}(v_j) = 1$ for all other v_j, e_i such that $v_j \in e_i$.

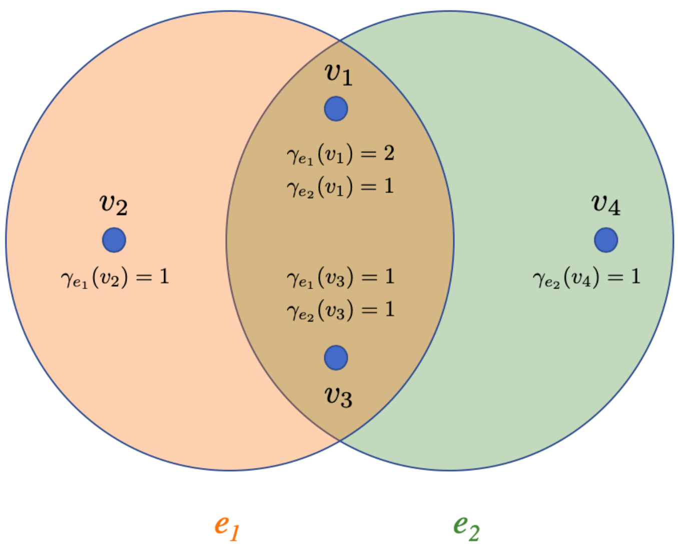

Figure 1. Pictured above is H_3.

For the sake of contradiction, suppose a random walk on H_3 is equivalent to a random walk on $H_2 = (V, E_2, \omega, \gamma)$, where H_2 is a 2-hypergraph with vertex set V. Let p^{H_i} be the transition probabilities of H_i for $i = 2, 3$; by assumption, $p^{H_2} = p^{H_3}$.

H_2 must have the following edges: $e'_{12} = \{v_1, v_2\}$, $e'_{14} = \{v_1, v_4\}$, $e'_{23} = \{v_2, v_3\}$, $e'_{34} = \{v_3, v_4\}$, and $e'_{13} = \{v_1, v_3\}$. WLOG let $\gamma_{e_{ij}}(v_i) + \gamma_{e_{ij}}(v_j) = 1$ for each i, j. Moreover, while we do not depict these edges in the figure below, H_2 also has edges $e'_i = \{v_i\}$ for $i = 1, 2, 3, 4$, though it may be the case that $\omega(e'_i) = 0$.

For shorthand, we write ω_{ij} for $\omega(e'_{ij})$, ω_i for $\omega(e'_i)$, and γ_{ijk} for $\gamma_{e'_{ij}}(v_k)$ where $k \in \{i, j\}$.

By definition, we have

$$\frac{1}{2} = p^{H_3}_{v_2,v_1} = p^{H_2}_{v_2,v_1} = \left(\frac{\omega_{12}}{\omega_{12} + \omega_{23} + \omega_2}\right) \gamma^{121} \tag{11}$$

Thus, $\left(\frac{\omega_{12}}{\omega_{12}+\omega_{23}+\omega_2}\right) = (2 \cdot \gamma_{121})^{-1}$.

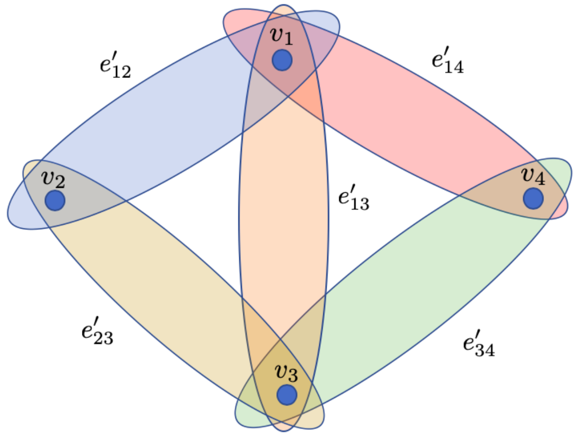

Figure 2. Pictured above is H_2. For illustrative purposes, we do not draw out singleton edges.

By similar analysis of $p^{H_3}_{v_2,v_3}$, and using that $\gamma_{232} + \gamma_{233} = 1$, we also have $\left(\frac{\omega_{23}}{\omega_{12}+\omega_{23}+\omega_2}\right) = \big(4(1-\gamma_{232})\big)^{-1}$. Thus, adding together the bounds on $p^{H_2}_{v_2,v_1}$ and $p^{H_3}_{v_2,v_1}$

$$\frac{1}{2\gamma_{121}} + \frac{1}{4(1-\gamma_{232})} = \left(\frac{\omega_{12}}{\omega_{12}+\omega_{23}+\omega_2}\right) + \left(\frac{\omega_{23}}{\omega_{12}+\omega_{23}+\omega_2}\right) \leq 1. \tag{12}$$

Note that, to get the bound in Equation (12), we summed $p^{H_2}_{v_2,v_i}$ for $i \neq 2$. If we follow the same steps but replace v_2 with v_1, v_3, we get the following bounds, respectively:

$$\frac{1}{8\cdot\gamma_{121}} + \frac{7}{24(1-\gamma_{131})} + \frac{1}{6(1-\gamma_{141})} \leq 1 \tag{13}$$

$$\frac{1}{8\gamma_{232}} + \frac{5}{12\gamma_{131}} + \frac{1}{6\gamma_{344}} \leq 1. \tag{14}$$

Now, solving for γ_{121} in Equation (12) yields

$$\gamma_{121} \geq \frac{2(1-\gamma_{232})}{3-4\gamma_{232}}. \tag{15}$$

Next, using that $\gamma_{ijk} \in [0,1]$, we bound Equation (13):

$$\begin{aligned} 1 &\geq \frac{1}{8\gamma_{121}} + \frac{7}{24(1-\gamma_{131})} + \frac{1}{6(1-\gamma_{141})} \\ &\geq \frac{1}{8\gamma_{121}} + \frac{7}{24} + \frac{1}{6} \\ &= \frac{1}{8\gamma_{121}} + \frac{11}{24}. \end{aligned} \tag{16}$$

Solving for γ_{121} yields $\gamma_{121} \leq \frac{10}{13}$. Combining with Equation (15):

$$\frac{10}{13} \geq \gamma_{121} \geq \frac{2(1-\gamma_{232})}{3-4\gamma_{232}} \implies \gamma_{232} \leq \frac{2}{7}. \tag{17}$$

Bounding Equation (14) in a similar way to Equation (16) gives us:

$$\begin{aligned} 1 &\geq \frac{1}{8\gamma_{232}} + \frac{5}{12\gamma_{131}} + \frac{1}{6\gamma_{344}} \\ &\geq \frac{1}{8\gamma_{232}} + \frac{5}{12} + \frac{1}{6} \\ &= \frac{1}{8\gamma_{232}} + \frac{7}{12}. \end{aligned} \tag{18}$$

Solving for γ_{232} gives us

$$\gamma_{232} \geq \frac{3}{10}. \tag{19}$$

Finally, putting together Equations (17) and (19):

$$\frac{3}{10} \leq \gamma_{232} \leq \frac{2}{7}, \tag{20}$$

which yields a contradiction, as $\frac{3}{10} > \frac{2}{7}$. □

We prove the result for general k by extending the above proof.

Theorem D.1. *Let $k > 1$. Then, there exists a k-hypergraph with vertex set V whose random walk is not equivalent to a random walk on any $(k-1)$-hypergraph with vertex set V.*

Proof. For simplicity, assume k is even (our argument can be adapted to odd k). Write $k = 2(n+1)$. For the sake of contradiction, suppose all k-hypergraphs have random walks equivalent to the random walk of some $(k-1)$-hypergraph.

Let $H_k = (V, E_k, \omega, \gamma)$ be a k-hypergraph with vertices $V = \{v_1, \ldots, v_n, w_1, \ldots, w_n, x, y\}$, and hyperedges $e_1 = \{v_1, \ldots, v_n, b, c\}$ and $e_2 = \{w_1, \ldots, w_n, b, c\}$. The edge weights are $\omega(e_1) = \omega(e_2) = 1$, and the edge-dependent vertex weights are $\omega_{e_1}(b) = 2$, and $\omega_{e_i}(v) = 1$ for all other v, e_i with $v \in e_i$.

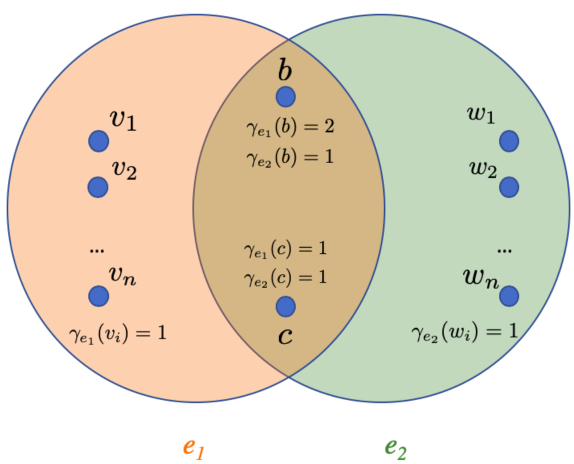

Figure 3. Pictured above is H_k.

By assumption, let $H_{k-1} = (V, E_{k-1}, \omega, \gamma)$ be a $(k-1)$-hypergraph whose random walk is equivalent to a random walk on H_k. Let $p^{H_k}, p^{H_{k-1}}$ be the transition probabilities of H_k, H_{k-1}, respectively.

Then, in H_{k-1}, we have

$$d(v_i) \cdot p^{H_{k-1}}_{v_i,v_j} = \sum_{e \in E(v_i,v_j)} \omega(e) \cdot \left(\frac{\gamma_e(v_j)}{\delta(e)}\right) \leq \sum_{e \in E(v_j)} \omega(e) \cdot \left(\frac{\gamma_e(v_j)}{\delta(e)}\right) = d(v_j) \cdot p^{H_{k-1}}_{v_j,v_j} \tag{21}$$

for all $i, j \in \{1, \ldots, n\}$. Since $p^{H_{k-1}}_{v_i,v_j} = p^{H_{k-1}}_{v_j,v_j}$, the above equation implies $d(v_i) \leq d(v_j)$. So by symmetry, $d(v_i) = d(v_j)$ for all i, j.

This means that Equation (21) is actually a strict equality, so

$$\sum_{e \in E(v_i,v_j)} \omega(e) \cdot \left(\frac{\gamma_e(v_j)}{\delta(e)} \right) = \sum_{e \in E(v_j)} \omega(e) \cdot \left(\frac{\gamma_e(v_j)}{\delta(e)} \right). \tag{22}$$

Since every term in the above sums are positive and equal, it must be the case that every hyperedge in H_{k-1} containing v_j also contains v_i, for all i, j. Because they all are in the same hyperedges in both H_{k-1} and H_k, we can view $\{v_1, \ldots, v_n\}$ as a single "supernode" v. By symmetry, we can also view $\{w_1, \ldots, w_n\}$ as a single supernode w.

Thus, we have reduced our problem to the counterexample in Lemma D.1, and the result follows. □

Putting all of our results together gives us the following (informal) hierarchy of Markov chains

$$\begin{aligned}
\{\text{random walks on hypergraphs with edge-independent vertex weights}\} &= \{\text{random walks on graphs}\} \\
&\subsetneq \{\text{random walks on 2-hypergraphs}\} \\
&\subsetneq \{\text{random walks on 3-hypergraphs}\} \\
&\subsetneq \ldots \\
&\subsetneq \{\text{all lazy Markov chains}\}.
\end{aligned}$$

E. Proof of Theorem 4.1

We first prove the following lemma.

Lemma E.1. *Let $H = (V, E)$ be a hypergraph with edge-dependent vertex weights $\gamma_e(v)$ and hyperedge weights $\omega(e)$. Without loss of generality, assume $\sum_{v \in e} \gamma_e(v) = 1$. There exist $\rho_e > 0$ satisfying*

$$\rho_e = \sum_{v \in e} \sum_{f \in E(v)} d(v)^{-1} \cdot \rho_f \cdot \omega(f) \cdot \gamma_f(v) \tag{23}$$

and

$$\sum_{e \in E} \rho_e \cdot \omega(e) = 1. \tag{24}$$

Proof of Lemma. Our proof outline is as follows. First, we prove the lemma in the case where the hyperedge weights are all equal to each other. Then, we extend that result to the case where the hyperedge weights are rational. Finally, we use the density of $\mathbb{Q}$ in $\mathbb{R}$ to extend our result from rational hyperedge weights to real ones.

First, suppose all of the hyperedge weights are equal to each other. WLOG let $\omega(e) = 1$ for all $e \in E$. Switching the order of summation in Equation 23, we have

$$\begin{aligned}
\sum_{v \in e} \sum_{f \in E(v)} d(v)^{-1} \cdot \rho_f \cdot \omega(f) \cdot \gamma_f(v) &= \sum_{v \in e} \sum_{f \in E(v)} d(v)^{-1} \cdot \rho_f \cdot \gamma_f(v) \\
&= \sum_{f \in E} \sum_{v \in e \cap f} d(v)^{-1} \cdot \rho_f \cdot \gamma_f(v) \\
&= \sum_{f \in E} \rho_f \cdot \left(\sum_{v \in e \cap f} d(v)^{-1} \gamma_f(v) \right).
\end{aligned} \tag{25}$$

Now let A be a square matrix of size $|E| \times |E|$, with entries $A_{e,f} = \sum_{v \in e \cap f} d(v)^{-1} \gamma_f(v)$. Note that the column sums of

A are equal to 1:

$$\begin{aligned}\sum_{e\in E} A_{e,f} &= \sum_{e\in E}\sum_{v\in e\cap f} d(v)^{-1}\gamma_f(v)\\ &= \sum_{v\in f}\sum_{e\in E(v)} d(v)^{-1}\gamma_f(v)\\ &= \sum_{v\in f} d(v)^{-1}\gamma_f(v)\cdot d(v)\\ &= \sum_{v\in f}\gamma_f(v)\\ &= 1.\end{aligned} \tag{26}$$

Thus, by the Perron-Frobenius theorem, A has a positive eigenvector ρ with eigenvalue 1.

So by construction, ρ satisfies Equation 23. Moreover, $t\cdot\rho$ also satisfies Equation 23 for any $t>0$. Thus, $t\cdot\rho$ with $t=\left(\sum_{e\in E}\rho_e\cdot\omega(e)\right)^{-1}$ satisfies both Equation 23 and Equation 24, and so the lemma is proved in the case where the hyperedge weights are all equal.

Next, assume H is a hypergraph with rational hyperedge weights, i.e. $\omega(e)\in\mathbb{Q}$ for all $e\in E$. Multiplying through by denominators, we can assume $\omega(e)\in\mathbb{N}$. Create hypergraph H' with vertices V in the following way. For each hyperedge e, replace e with hyperedges $e_1,...,e_{\omega(e)}$, where each hyperedge e_i:

- contains the same vertices as e,
- has weight $\omega'(e_i)=1$,
- has the same vertex weights as e, so that $\gamma'_{e_i}(v)=\gamma_e(v)$ for all $v\in e$.

Let E' be the hyperedges of H', and let $M(v)$ be the hyperedges incident to vertex v in H'. Since H' has equal hyperedge weights, we can find constants ρ'_{e_i} that satisfy Equations 23 and 24 for H'. Note that $\rho'_{e_i}=\rho'_{e_j}$ by symmetry.

Now, for each hyperedge e of H, let $\rho_e=\rho'_{e_1}$. I claim that ρ_e satisfies Equations 23 and 24 for H. Equation 24 is satisfied since

$$\omega(e)\cdot\rho_e=\omega(e)\cdot\rho'_{e_1}=\rho'_{e_1}+\cdots+\rho'_{e_{\omega(e)}}=\sum_{i=1}^{\omega(e)}\rho'_{e_i}\omega'(e_i), \tag{27}$$

which implies

$$\sum_{e\in E}\rho_e\cdot\omega(e)=\sum_{e\in E}\sum_{i=1}^{\omega(e)}\rho'_{e_i}\omega'(e_i)=\sum_{e\in E'}\rho'_{e_i}\omega(e_i)=1. \tag{28}$$

To show Equation 23 holds for H, note that

$$d(v)^{-1}\cdot\rho_f\cdot\omega(f)\cdot\gamma_f(v)=\sum_{i=1}^{\omega(f)}\left(d(v)^{-1}\cdot\rho'_{f_i}\cdot\omega'(f_i)\cdot\gamma'_{f_i}(v)\right). \tag{29}$$

Summing over both sides yields

$$\begin{aligned}\sum_{v\in e}\sum_{f\in E(v)} d(v)^{-1}\cdot\rho_f\cdot\omega(f)\cdot\gamma_f(v) &= \sum_{v\in e}\sum_{f\in E(v)}\sum_{i=1}^{\omega(f)}\big(d(v)^{-1}\cdot\rho'_{f_i}\cdot\omega'(f_i)\cdot\gamma'_{f_i}(v)\big)\\ &= \sum_{v\in e}\sum_{f\in M(v)} d(v)^{-1}\cdot\rho'_f\cdot\omega'(f)\cdot\gamma'_f(v)\\ &= \sum_{v\in e_1}\sum_{f\in M(v)} d(v)^{-1}\cdot\rho'_f\cdot\omega'(f)\cdot\gamma'_f(v)\\ &= \rho'_{e_1}, \text{ since Equation 23 holds for } H'\\ &= \rho_e.\end{aligned} \tag{30}$$

Thus, Equations 23 and 24 hold for H when H has rational hyperedge weights.

Finally, we consider the general case, where we assume nothing about the hyperedge weights besides that they are positive real numbers. By similar reasoning to our proof of the equal hyperedge weight case, we are done if we can find positive ρ_e satisfying Equation 23.

We have

$$\begin{aligned}\sum_{v\in e}\sum_{f\in E(v)} d(v)^{-1}\cdot\rho_f\cdot\omega(f)\cdot\gamma_f(v) &= \sum_{v\in e}\sum_{f\in E(v)} d(v)^{-1}\cdot\omega(f)\cdot\rho_f\cdot\gamma_f(v)\\ &= \sum_{f\in E}\sum_{v\in e\cap f} d(v)^{-1}\cdot\rho_f\cdot\omega(f)\cdot\gamma_f(v)\\ &= \sum_{f\in E}\rho_f\cdot\left(\sum_{v\in e\cap f} d(v)^{-1}\cdot\omega(f)\cdot\gamma_f(v)\right).\end{aligned} \tag{31}$$

Let A be a matrix of size $|E|\times|E|$ with entries

$$A_{e,f} = \sum_{v\in e\cap f} d(v)^{-1}\cdot\omega(f)\cdot\gamma_f(y). \tag{32}$$

Showing that there exist positive ρ_e that satisfy Equation 23 is equivalent to showing that A has a positive eigenvector with eigenvalue 1. By the Perron-Frobenius theorem, this equivalent to A having spectral radius 1.

For each hyperedge $e\in E$, let $q_1^e, q_2^e, \dots$ be a sequence of rational numbers that converges to $\omega(e)$, i.e. $\lim_{n\to\infty} q_n^e = \omega(e)$. Let H_n be H except we replace all hyperedge weights $\omega(e)$ with q_n^e. By the previous part of the proof, there exist positive constants $\rho^n(e)$ that satisfy Equation 23 for H_n; equivalently, if we let A_n be the matrix from Equation 32 for hypergraph H_n, then A_n has spectral radius 1.

Since A_n has a continuous dependence on the hyperedge weights, and spectral radius is a continuous function, it follows that the spectral radius of A is the limit of the spectral radius of A_n. Thus, the spectral radius of A is 1, and we are done. □

Theorem 4.1 is now a relatively straightforward corollary of Lemma E.1.

Theorem 4.1. *Let $H = (V, E, \omega, \gamma)$ be a hypergraph with edge-independent vertex weights. There exist positive constants ρ_e such that the stationary distribution π of a random walk on H is*

$$\pi_v = \sum_{e\in E(v)} \omega(e)\cdot\big(\rho_e\gamma_e(v)\big). \tag{33}$$

Moreover, ρ_e can be computed in time $O\big(|E|^3 + |E|^2\cdot|V|\big)$.

Proof of Theorem 4.1. Without loss of generality, assume $\delta(e) = \sum_{v \in e} \gamma_e(v) = 1$ for all hyperedges e, i.e. by scaling ρ_e appropriately.

Let $\rho_e > 0$ be from Lemma E.1, and define

$$\pi_v = \sum_{e \in E(v)} \omega(e)\big(\rho_e \gamma_e(v)\big). \tag{34}$$

I claim that π_v is the stationary distribution for a random walk on H.

First, note that

$$\begin{aligned}
\sum_{v \in V} \pi_v &= \sum_{v \in V} \sum_{e \in E(v)} \omega(e)\big(\rho_e \gamma_e(v)\big) \\
&= \sum_{e \in E} \sum_{v \in e} \omega(e)\big(\rho_e \gamma_e(v)\big) \\
&= \sum_{e \in E} \rho_e \omega(e) \sum_{v \in e} \gamma_e(v) \\
&= \sum_{e \in E} \rho_e \omega(e) \\
&= 1, \text{ by Equation 24}
\end{aligned} \tag{35}$$

so π is indeed a probability distribution on V. Now, for any vertex $w \in V$, we have

$$\begin{aligned}
\sum_{v \in V} \pi_v p_{v,w} &= \sum_{v \in V} \pi_v \left(\sum_{e \in E(v)} \frac{\omega(e)}{d(v)} \gamma_e(w) \right) \\
&= \sum_{v \in V} \sum_{e \in E(v,w)} \pi_v \cdot \gamma_e(w) \cdot \omega(e) \cdot d(v)^{-1} \\
&= \sum_{e \in E(w)} \sum_{v \in e} \pi_v \cdot \gamma_e(w) \cdot \omega(e) \cdot d(v)^{-1} \\
&= \sum_{e \in E(w)} \omega(e) \cdot \gamma_e(w) \left(\sum_{v \in e} \frac{\pi_v}{d(v)} \right).
\end{aligned} \tag{36}$$

If we simplify the inner sum, we get

$$\sum_{v \in e} \frac{\pi_v}{d(v)} = \sum_{v \in e} d(v)^{-1} \sum_{f \in E(v)} \rho_f \cdot \omega(f) \cdot \gamma_f(v) = \sum_{v \in e} \sum_{f \in E(v)} d(v)^{-1} \cdot \rho_f \cdot \omega(f) \cdot \gamma_f(v) = \rho_e. \tag{37}$$

Plugging this back in, we get

$$\sum_{e \in E(w)} \omega(e) \cdot \gamma_e(w) \left(\sum_{v \in e} \frac{\pi_v}{d(v)} \right) = \sum_{e \in E(w)} \omega(e) \cdot \gamma_e(w) \cdot \rho_e = \pi_w. \tag{38}$$

Thus, $\sum_{v \in V} \pi_v p_{v,w} = \pi_w$, so π is a stationary distribution for H.

Finally, note that computing A (Equation 32) takes time $O(|E|^2 \cdot |V|)$ when $d(v)$ is precomputed, and solving $A\rho = \rho$ takes time $O(|E|^3)$, so the total runtime to compute ρ_e is $O(|E|^3 + |E|^2 \cdot |V|)$. □

F. Proof of Theorem 4.2

For completeness, we include the definition of the Cheeger constant of a Markov chain (Montenegro & Tetali, 2006).

Definition F.1. *Let M be an ergodic Markov chain with finite state space V, transition probabilities $p_{u,v}$, and stationary distribution π. The* Cheeger constant *of M is*

$$\Phi = \min_{S \subset V, 0 < \pi(S) \leq 1/2} \frac{\sum_{x \in S, y \notin S} \pi_x p_{x,y}}{\pi(S)}, \tag{39}$$

where $\pi(S) = \sum_{v \in S} \pi_v$.

First, we prove the following lemma for the mixing time of any lazy Markov chain.

Lemma F.1. *Let M be a finite, irreducible Markov chain with states S and transition probabilities $p_{x,y}$, satisfying $p_{x,x} \geq \delta$ for all $x \in S$. Let π be the stationary distribution of M, and let π_{min} be the smallest element of π. Then,*

$$t_{mix}(\epsilon) \leq \left\lceil \frac{8\delta}{\Phi^2} \log \left(\frac{1}{2\epsilon\sqrt{\pi_{min}}} \right) \right\rceil \tag{40}$$

Proof of Lemma. We use the notation of Jerison (2013). Let P^* be the time-reversal transition matrix of P. Note that P^*P and $\frac{P+P^*}{2}$ are both reversible Markov chains. Let α be the square-root of the second-largest eigenvalue of P^*P, and let b be the second-largest eigenvalue of $\frac{P+P^*}{2}$. By the Cheeger inequality, we have $1 - b \geq \frac{\Phi^2}{2}$. Combining this with Lemma 1.21 of Montenegro & Tetali (2006) yields

$$\frac{\mathcal{E}_{\frac{P+P^*}{2}}(f,f)}{\mathrm{Var}_\pi(f)} \geq \frac{\Phi^2}{2}, \tag{41}$$

where $f : S \to \mathbb{R}$ is any function, $\mathcal{E}_{\frac{P+P^*}{2}}(f,f)$ is the Dirichlet form of the Markov chain $\frac{P+P^*}{2}$, and $\mathrm{Var}_\pi(f)$ is the variance of f (see Montenegro & Tetali (2006) for more details).

From Jerison (2013),

$$\mathcal{E}_{P^*P}(f,f) \geq 2\delta \mathcal{E}_{\frac{P+P^*}{2}}(f,f). \tag{42}$$

Combining Equations 41 and 42 yields

$$\frac{\mathcal{E}_{P^*P}(f,f)}{\mathrm{Var}_\pi(f)} \geq \frac{\Phi^2}{4\delta}. \tag{43}$$

Now, from Lemma 1.2 of Montenegro & Tetali (2006), $1 - \alpha^2 \geq \frac{\mathcal{E}_{P^*P}(f,f)}{\mathrm{Var}_\pi(f)}$; plugging this into the above equation and rearranging yields $\alpha \leq \left(1 - \frac{\Phi^2}{4\delta}\right)^{1/2} \leq 1 - \frac{\Phi^2}{8\delta}$. Plugging this into Equation 1.6 of Jerison (2013) yields

$$t_{mix}(\epsilon) \leq \left\lceil \frac{1}{1-\alpha} \log \left(\frac{1}{2\epsilon\sqrt{\pi_{min}}} \right) \right\rceil \leq \left\lceil \frac{8\delta}{\Phi^2} \log \left(\frac{1}{2\epsilon\sqrt{\pi_{min}}} \right) \right\rceil. \qquad \square$$

Theorem 4.2. *Let $H = (V, E, \omega, \gamma)$ be a hypergraph with edge-dependent vertex weights. Without loss of generality, assume $\rho_e = 1$ (i.e. by multiplying the vertex weights in hyperedge e by ρ_e). Then,*

$$t^H_{mix}(\epsilon) \leq \left\lceil \frac{8\beta_1}{\Phi^2} \log \left(\frac{1}{2\epsilon\sqrt{d_{min}\beta_2}} \right) \right\rceil, \tag{44}$$

where

- *Φ is the Cheeger constant of a random walk on H (Montenegro & Tetali, 2006; Jerison, 2013)*
- *d_{min} is the minimum degree of a vertex in H, i.e. $d_{min} = \min_v d(v)$,*
- *$\beta_1 = \min_{e \in E, v \in e} \left(\frac{\gamma_e(v)}{\delta(e)} \right)$, and*
- *$\beta_2 = \min_{e \in E, v \in e} \left(\gamma_e(v) \right)$.*

Proof of Theorem 4.2. We have

$$p_{v,v} = \sum_{e \in E(v)} \frac{\omega(e)}{d(v)} \frac{\gamma_e(v)}{\delta(e)} \geq \beta_1 \sum_{e \in E(v)} \frac{\omega(e)}{d(v)} = \beta_1 \tag{45}$$

for all vertices v. Similarly,

$$\pi_v = \sum_{e \in E(v)} \omega(e)\gamma_e(v) \geq \beta_2 d(v). \tag{46}$$

Applying Lemma F.1 to a random walk on H yields the desired bound:

$$t_{mix}(\epsilon) \leq \left\lceil \frac{8\delta}{\Phi^2} \log\left(\frac{1}{2\epsilon\sqrt{\pi_{min}}}\right)\right\rceil \leq \left\lceil \frac{8\beta_1}{\Phi^2} \log\left(\frac{1}{2\epsilon\sqrt{d_{min}\beta_2}}\right)\right\rceil$$ □

G. Proof of Theorem 5.2

Theorem 5.2. *Let $H = (V, E, \omega, \gamma)$ be a hypergraph with edge-dependent vertex weights, with vertex weights normalized so that $\rho_e = 1$ for all hyperedges e. Let G^H be the clique graph of H, with edge weights*

$$w_{u,v} = \sum_{e \in E(u,v)} \frac{\omega(e)\gamma_e(u)\gamma_e(v)}{\delta(e)}. \tag{47}$$

Let L^H, L^G be the Laplacians of H and G^H, respectively, and let λ_1^H, λ_1^G be the second-smallest eigenvalues of L^H, L^G, respectively. Then

$$\frac{1}{c(H)}\lambda_1^H \leq \lambda_1^G \leq c(H)\lambda_1^H, \tag{48}$$

where $c(H) = \max_{v \in V}\left(\frac{\max_{e \in E}\gamma_e(v)}{\min_{e \in E}\gamma_e(v)}\right)$.

Proof of Theorem 5.2. As shorthand, we write $G = G^H$. Let $p^H_{u,v}$ and π^H_vbe the transition probabilities, stationary distribution of a random walk on H. Define $p^G_{u,v}$ and π^G_v similarly for G. Furthermore, let $d^H(v)$ and $d^G(v)$ be the degree in H and G respectively.

We will use Theorem 8 of Chung (2005) to prove our theorem, which requires us to have lower and upper bounds on $\frac{\pi_v^G}{\pi_v^H}$ and $\frac{\pi_v^G p_{v,u}}{\pi_v^G p_{v,u}}$.

First, for an arbitrary vertex v, we have

$$\begin{aligned}
\pi_v^G \propto \sum_{u \in V} w_{u,v} &= \sum_{u \in V}\sum_{e \in E(u,v)} \frac{\omega(e)\gamma_e(u)\gamma_e(v)}{\delta(e)} \\
&= \sum_{e \in E(v)}\sum_{u \in e} \frac{\omega(e)\gamma_e(u)\gamma_e(v)}{\delta(e)} \\
&= \sum_{e \in E(v)} \omega(e)\gamma_e(v)\left(\frac{\sum_{u \in e}\gamma_e(u)}{\delta(e)}\right) \\
&= \sum_{e \in E(v)} \omega(e)\gamma_e(v) \\
&= \pi_v^H,
\end{aligned} \tag{49}$$

so random walks on G^H and H have the same stationary distributions. Next, for any two vertices u, v, we have

$$\frac{\pi^G(v)p^G_{u,v}}{\pi^H(v)p^H_{u,v}} = \frac{p^G_{u,v}}{p^H_{u,v}} = \frac{\dfrac{w_{u,v}}{d^G(u)}}{\displaystyle\sum_{e\in E(u,v)} \frac{\omega(e)}{d^H(u)}\frac{\gamma_e(v)}{\delta(e)}} = \frac{\displaystyle\sum_{e\in E(u,v)} \frac{\omega(e)\gamma_e(u)\gamma_e(v)}{\delta(e)}}{\displaystyle\sum_{e\in E(u,v)} \frac{\omega(e)\gamma_e(v)d^G(u)}{\delta(e)d^H(u)}}. \tag{50}$$

The RHS is upper-bounded by the maximum ratio of each term in the sum, which is

$$\begin{aligned}\max_{u,v} \frac{d^H(u)\gamma_e(u)}{d_G(u)} &= \max_{u,v} \frac{\left(\displaystyle\sum_{f\in E(u)} \omega(f)\right)\gamma_e(u)}{\left(\displaystyle\sum_{f\in E(u)} \omega(f)\gamma_f(u)\right)} \\ &\leq \max_{u,v} \left(\frac{\max_e \gamma_e(u)}{\min_f \gamma_f(u)}\right) \\ &= \max_{u} \left(\frac{\max_e \gamma_e(u)}{\min_e \gamma_f(u)}\right).\end{aligned} \tag{51}$$

Similarly, it is lower bounded by $\min_u \frac{\min_e \gamma_e(u)}{\max_e \gamma_e(u)}$. Applying Theorem 8 of Chung (2005) gives the desired bound. □

H. Rank Aggregation Experiments with Synthetic Data

Data: We use a variant of the TrueSkill model to generate our data. We assume each player has an intrinsic "skill" level (for simplicity, assume skill does not change over time), and a player's performance in match is proportional to their skill plus some added Gaussian noise. Such a model can represent many different kinds of games, including shooting games (e.g. Halo, scores represent kill/death ratios in a timed free-for-all match) and racing games (e.g. Mario Kart, scores are inversely proportional to the time a player takes to finish a course).

The players are $\{1, \ldots, n\}$. Player i has intrinsic skill i, so the true ranking of players, τ^*, is player $1 <$ player $2 < \cdots$ player n. We create k partial rankings, $\tau_1, \ldots, \tau_k$, where each partial ranking τ_i corresponds to a noisy subsampling of τ^*. More specifically, to create each partial ranking, we do the following.

1. Choose a subset of players $A \subset \{1, ..., n\}$, where player i is included in A with probability p.
2. Choose a scale factor c uniformly at random from $[1/3, 3]$.
3. For each player $i \in A$, independently draw a score for player i from a $N(0.2 \cdot i, \sigma)$ distribution, and scale that score by c.
4. Set τ_j to be a ranking of the players in A according to their score.

The tuneable parameters are: n, the number of players to be ranked; σ, the amount of noise in our partial rankings; k, the number of partial rankings; and p, which controls the size of each partial ranking. We set the mean score for player i to be $0.2 \cdot i$, so that the the scale of the simulated scores is similar to the scores from the Halo dataset.

Methods: As with the real data, we create a Markov chain-based rank aggregation algorithm where the Markov chain is a random walk on a hypergraph $H = (V, E, \omega, \gamma)$. The vertices are $V = \{1, ..., n\}$, and the hyperedges E correspond to the partial rankings $\tau_1, \ldots, \tau_k$. We set vertex weights $\gamma_{e_j}(v) = \exp[(\text{score of } v \text{ in partial ranking } \tau_j)]$, and edge weights $\omega(e_j) = (\text{standard deviation of scores in } \tau_j) + 1$.

Our baselines are MC3 and a rank aggregation algorithm using the *clique graph* G^H, both of which are described in the main text.

Results: We fix universe size $n = 100$, and set k to be the smallest number of hyperedges until all n vertices are included at least once. We set $\sigma = 1$ and $p = 0.03, 0.05, 0.07$.

To assess performance, we measure the weighted Kendall τ correlation coefficient (Yilmaz et al., 2008) between the estimated ranking and the true ranking τ^*. Our weighted hypergraph algorithm outperforms both MC3 and the clique graph algorithm in all cases (figure below), with the most significant gains when p is small, i.e. when there is less information in each partial ranking. Moreover, the performance of the clique graph algorithm is much worse than both MC3 and the weighted hypergraph, which suggests that the clique graph is not a good approximation of H.

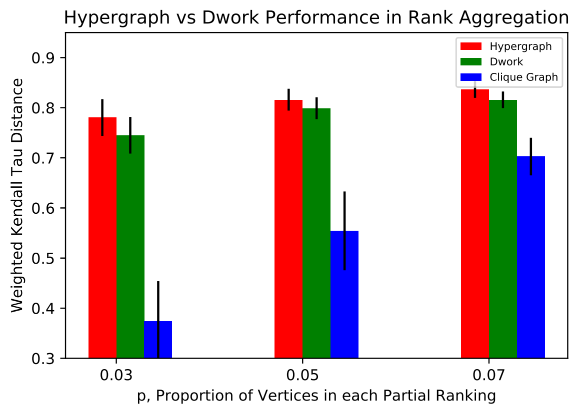

Figure 4. Results of rank aggregation experiment using synthetic data.

References

Aldous, D. and Fill, J. A. *Reversible Markov Chains and Random Walks on Graphs*. 2002.

Chung, F. Laplacians and the cheeger inequality for directed graphs. *Annals of Combinatorics*, 9(1):1–19, Apr 2005. ISSN 0219-3094. doi: 10.1007/s00026-005-0237-z.

Cooper, C., Frieze, A., and Radzik, T. The cover times of random walks on random uniform hypergraphs. *Theoretical Computer Science*, 509:51 – 69, 2013. ISSN 0304-3975. doi: http://dx.doi.org/10.1016/j.tcs.2013.01.020.

Jerison, D. General mixing time bounds for finite markov chains via the absolute spectral gap, October 2013.

Li, J., He, J., and Zhu, Y. E-tail product return prediction via hypergraph-based local graph cut. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '18, pp. 519–527, New York, NY, USA, 2018. ACM. ISBN 978-1-4503-5552-0.

Montenegro, R. and Tetali, P. Mathematical aspects of mixing times in markov chains. *Found. Trends Theor. Comput. Sci.*, 1 (3):237–354, May 2006. ISSN 1551-305X. doi: 10.1561/0400000003.

Yilmaz, E., Aslam, J. A., and Robertson, S. A new rank correlation coefficient for information retrieval. In *Proceedings of the 31st Annual International ACM SIGIR Conference on Research and Development in Information Retrieval*, SIGIR '08, pp. 587–594, New York, NY, USA, 2008. ACM. ISBN 978-1-60558-164-4. doi: 10.1145/1390334.1390435.