Primes as sums of irrational numbers

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In the Ramanujan notebooks, one formula is particularly remarquable is

$$24\sum_{n=1}^{\infty}\frac{n^{13}}{e^{2\pi n}-1}=1$$

This formula suggests that there could be other identities with rationals or primes, after having searched many forms, one in particular seems to be able to generate the prime numbers. The Ramanujan formula works for n^{4k+1} exponents, if we use exponents of the form n^{4k+3} then,

for example :

$$691=16\sum_{n=1}^{\infty}\frac{n^{11}}{e^{\pi n}-1}-65536\sum_{n=1}^{\infty}\frac{n^{11}}{e^{4\pi n}-1}$$

Here 691 is the numerator of the 12'th Bernoulli number : -691/2730. The Ramanujan formula is closely related to numerators of the Bernoulli numbers, one example is

$$24\sum_{n=1}^{\infty}\frac{n^{673}}{e^{2\pi n}-1}=1563446\ldots\ 036059151, a\ 1077\ digit\ prime$$

That number is $\frac{12\,B(674)}{674}$, see A112548 of the OEIS for references. The biggest known prime of that form being $240\sum_{n=1}^{\infty}\frac{n^{22807}}{e^{2\pi n}-1}$, a 71299 digits prime.

By exploring the form n^{4k+3}, I have found other primes not related to Bernoulli numbers.

$$17=32\sum_{n=1}^{\infty}\frac{n^{7}}{e^{\pi n}-1}-8192\sum_{n=1}^{\infty}\frac{n^{7}}{e^{4\pi n}-1}$$

$$37=-2662\sum_{n=1}^{\infty}\frac{n^{7}}{e^{2\pi n}-1}+5632\sum_{n=1}^{\infty}\frac{n^{7}}{e^{4\pi n}-1}+\sum_{n=1}^{\infty}\frac{n^{11}}{e^{\pi n}-1}-4096\sum_{n=1}^{\infty}\frac{n^{11}}{e^{4\pi n}-1}$$

$$251=6087\sum_{n=1}^{\infty}\frac{n^{3}}{e^{\pi n}-1}+68343\sum_{n=1}^{\infty}\frac{n^{3}}{e^{2\pi n}-1}+2016\sum_{n=1}^{\infty}\frac{n^{11}}{e^{4\pi n}-1}$$

If we pose

$$S_{k,m} = \sum_{n=1}^{\infty} \frac{n^k}{e^{m\pi n} - 1}$$

Then we can write each prime as a sum of the $S(k,m)$ function, like

$$691 \equiv [S_{11,1}, S_{11,4}]$$

Disregarding the coefficients of the integer relation.

We have the following list of primes.

Prime	Linear combination of
2	$S_{3,1}, S_{3,2}, S_{3,4}$
3	$S_{7,1}, S_{7,2}, S_{7,4}$
5	$S_{7,1}, S_{7,2}, S_{7,4}$
7	$S_{3,1}, S_{3,2}, S_{3,4}$
11	$S_{3,1}, S_{3,2}, S_{7,1}, S_{7,2}$
13	$S_{3,1}, S_{3,4}, S_{7,2}, S_{7,4}$
17	$S_{7,1}, S_{7,4}$
19	$S_{7,1}, S_{7,2}, S_{7,4}$
23	$S_{3,2}, S_{3,4}, S_{7,1}, S_{7,2}$
83	$S_{7,2}, S_{7,4}, S_{11,1}, S_{11,2}$
167	$S_{3,1}, S_{3,3}, S_{3,6}, S_{3,12}$
251	$S_{3,1}, S_{3,2}, S_{3,4}$
691	$S_{11,1}, S_{11,4}$
71840783209099	$S_{27,1}, S_{27,4}, S_{31,2}, S_{31,4}$

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