N-Point Finite Temperature Expectation Values at Real Times

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Abstract

In both the real- and imaginary-time formalisms of finite temperature field theory it is shown exactly what types of real-time expectation value are being calculated. The relationship between these functions is then considered. The conclusion is that neither the well known results for two-point functions nor the recently obtained results for three-point functions have a simple generalisation to higher order functions. All the results are valid whatever approximation to the full fields is being used and whatever types of field are involved.

1 Introduction

In this paper it will be shown that there is at least one aspect of FTFT (Finite Temperature Field Theory) in which it differs significantly from zero-temperature QFT (Quantum Field Theory). This is the question of what types of expectation values are being calculated and the relationships between them. If one uses the standard approach to FTFT, the ITF (Imaginary-Time Formalism), also known as the Matsubara method [1, 2, 5], one usually calculates retarded and advanced functions functions, though this is something that has not previously been proved for the general case. This is to be contrasted with the time-ordered expectation values normally encountered in zero-temperature QFT. The time-ordered functions are however obtained directly from the alternative FTFT formalisms, which are known as RTF (Real-Time Formalisms) [2, 3, 4, 5]. Thus comparing

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the values of the different types of expectation value, as is done in this paper, is equivalent to comparing the results most easily obtained from the two different FTFT formalisms.

For the two-point functions, the relationships between the various expectation values and the results of FTFT calculations have long been known [1, 5, 6]. The purpose of this paper is generalise the recent work of [7, 8, 9] on three-point functions to N-point functions. In doing so we shall find that not all the results can be generalised, at least not in a straightforward way. We shall see that two-point functions are a very special case, three-point functions are still quite special, and we only see the general behaviour of N-point functions for $N \geq 4$. This work is complementary to that of Kobes [11, 12, 13, 14, 15].

In section 2 we shall set up the two formalisms of FTFT using a path integral approach, as in this approach it is clear that both formalisms contain the same physical information and *cannot* give different answers for the same object. In section 3 we study the ITF and derive a precise form for the result of ITF calculations. The relation of the ITF result to N-point retarded functions is then the subject of section 4. The RTF results are considered in section 5 and the usual results of the two formalisms are compared in section 6. The conclusions are given in section 7.

2 The Finite Temperature Formalisms

We start from the generating functional of the path ordered approach to finite temperature field theory [2, 5],

$$Z[j] = Tr\{e^{-\beta H} T_C \, exp\{\imath \int_C d\tau \, j^\dagger \phi + \phi^\dagger j\}\} \tag{2.1}$$

where the fields $\phi(\tau)$, $\phi^\dagger(\tau)$ are Heisenberg fields with sources $j^\dagger(\tau)$, $j(\tau)$, analytically continued to take values in the complex-time plane. All the times τ lie on the directed path C in the complex-time plane. To use the path integral method, the end of the curve must be $-\imath\beta$ below the starting point and the imaginary time component must always be decreasing (even if only infinitesimally so) as we move along the curve. While different choices of C correspond to 'different' FTFT formalisms, whatever C is chosen the same physical information is encoded in the path integral. However each formalism differs in the way that information about the system (i.e. that we have an equilibrium many-body problem) is encoded in the results. For instance depending on the curve C chosen, each formalism generates different types of connected Green functions, Γ_C, namely

$$\begin{aligned} \Gamma_C(\tau_1, \tau_2, ..., \tau_N) = \\ Tr\{e^{-\beta H} T_C \phi_a(\tau_a)\phi_b(\tau_b)...\phi_N(\tau_N)\}/Tr\{e^{-\beta H}\}. \end{aligned} \tag{2.2}$$

Here the T_C indicates that the operators are path ordered with respect to the relative position of their time arguments on the directed path C. We will suppress any spin or other indices on the fields as they do not effect our arguments. We will likewise only write explicitly the time and energy variables.

The Imaginary-Time Formalism (ITF) [1, 2, 5] can be obtained by letting $C = C_I$ where C_I runs from 0 to $-\imath\beta$, as shown in Figure 1. It is clear that the Green functions of ITF are thermal expectation values of fields at Euclidean times which are time ordered with respect to these Euclidean times.

The alternative finite temperature formalisms are RTF (Real-Time Formalisms). There are two distinct ways of getting the same equilibrium Feynman rules in this case, the approach of Umezawa and co-workers called Thermo Field Dynamics [3], which is closely linked to C^*-algebra approaches [5], and the path integral approach of Niemi and Semenoff [2, 4, 5]. We will use the path integral approach here as it is clear in this approach that ITF and RTF encode the same physical information.

In a path integral approach the curve in the complex time plane used is $C = C_R$ where $C_R = C_1 \oplus C_2 \oplus C_3 \oplus C_4$ runs from $-\infty$ to $+\infty$ (C_1), $+\infty$ to $+\infty - \imath\bar{\alpha}\beta$ (C_2), $+\infty - \imath\bar{\alpha}\beta$ to $-\infty - \imath\bar{\alpha}\beta$ (C_3), and $-\infty - \imath\bar{\alpha}\beta$ to $-\infty - \imath\beta$ (C_4) [2, 4, 5]. This is shown in Figure 1. The parameter $\alpha = 1 - \bar{\alpha}$ is arbitrary and it reflects some of the freedom of choice in the path C. Physical results should not therefore depend on α. It turns out that one can usually ignore the vertical sections C_3, C_4 when, as here, one looks at Green functions [2, 4, 5]. We then write everything in terms of doublet fields, $\phi^\mu(t)$, of real time arguments which vary between $-\infty$ and $+\infty$. Though there are many conventions for signs we choose

$$\phi^\mu(t) = \begin{cases} \phi(t), & t \in C_1 & \text{if } \mu = 1 \\ \phi(t - \imath\bar{\alpha}\beta), & t - \imath\bar{\alpha}\beta \in C_2 & \text{if } \mu = 2 \end{cases} \tag{2.3}$$

and likewise for $\phi^\dagger$, where $\mu = 1, 2$ is the thermal index. To account for the sign discussed in [16], we define

$$j^\dagger(t)^\mu = \begin{cases} j^\dagger(t), & t \in C_1 & \text{if } \mu = 1 \\ -j^\dagger(t - \imath\bar{\alpha}\beta), & t - \imath\bar{\alpha}\beta \in C_2 & \text{if } \mu = 2 \end{cases} \tag{2.4}$$

and likewise for $j(t)$. These definitions, after the usual path integral manipulations [2, 4, 5], lead to the same scalar propagator and Feynman rules as used in [11], i.e. an overall factor of -1 for type two vertices over the type one vertex values. The connected RTF Green functions, as generated by functionally differentiating Z of (2.1), when written in terms of the thermal doublets, (2.3), are

$$\begin{aligned} \Gamma^{\mu_1\mu_2\ldots\mu_N} &:= \left(\frac{1}{Z} \prod_{a=1}^N \frac{-\imath\partial}{\partial j_a^\dagger(t_a)^{\mu_a}} \right) Z |_{j=0} \\ &= Tr\{ e^{-\beta H} T_C \phi_1^{\mu_1}(t_1) \phi_2^{\mu_2}(t_2) \ldots \phi_N^{\mu_N}(t_N) \} / Tr\{ e^{-\beta H} \}. \end{aligned} \tag{2.5}$$

In relating the various types of function that we shall encounter, it is useful to express all expectation values in terms of the thermal Wightman functions where the order of the fields is fixed. The N-point thermal Wightman functions will be denoted by Γ with N subscripts and are defined by

$$\begin{aligned} \Gamma_{a_1 a_2 \ldots a_N}(\tau_1, \tau_2, \ldots, \tau_N) = \\ (-1)^p Tr\{ e^{-\beta H} \phi_{a_1}(\tau_{a_1}) \phi_{a_2}(\tau_{a_2}) \ldots \phi_{a_N}(\tau_{a_N}) \} / Tr\{ e^{-\beta H} \}, \end{aligned} \tag{2.6}$$

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where the $\{\tau\}$ are complex times, the fields ϕ_a having been continued to complex times in the usual spirit of the path integral approach to finite temperature field theory. The subscripts $a_1, a_2, ..., a_N$ are equal to any permutation of $1, 2, ..., N$. In (2.6) p is the number of times we have to swap fermionic fields in going from fields in the order $\phi_1 \phi_2 ... \phi_N$ to $\phi_{a_1} \phi_{a_2} ... \phi_{a_N}$.

From the definition of thermal Wightman functions (2.6) we use the fact that the trace is cyclic and that we can regard the $e^{-\beta H}$ equilibrium density matrix as a time evolution operator to give

$$\Gamma_{a_1 a_2 ... a_N}(\{\tau\}) = \sigma_{a_1} \Gamma_{a_2 ... a_N a_2}(\{\tau'\}) \tag{2.7}$$

where $\tau_j = \tau'_j$ except for $\tau'_{a_1} = \tau_{a_1} + \imath \beta$. To account for the factor of $(-1)^p$ in (2.6), σ_{a_1} is $+1$ (-1) if the field being moved from the front to the back, ϕ_{a_1}, is bosonic (fermionic)

The N-point functions we are considering are expectation values in which each member of a set of N fields $\{\phi_j\}$ appears just once. It does not matter whether some or all of the members of this set are related to each other or not. Lower case Roman subscripts run between 1 and N, while we use Greek indices taking values 1 or 2 only for the thermal labels used by RTF such as in (2.3). Since the order of the fields is completely specified by the subscripts we do not use the time arguments to indicate any order and we just write the time argument of the a-th field, ϕ_a, as the a-th time argument of the thermal Wightman function from the left. For example, with N=4, $\phi_1 = \phi$ a real scalar field, $\phi_2 = \psi$ and $\phi_3 = \bar{\psi}$ a fermion field and its conjugate, and $\phi_4 = A$ a vector field, we have

$$\Gamma_{3142}(r, s, t, u) = -Tr\{e^{-\beta H} \bar{\psi}(t) \phi(r) A(u) \psi(s)\}/Tr\{e^{-\beta H}\}. \tag{2.8}$$

3 Imaginary Time Formalism

We will first look at ITF and try to find out exactly what sort of expectation values are being calculated after one has made the analytic continuation to real energies and times. It is simplest to work in terms of the constituent thermal Wightman functions (2.6). First we must first consider the behaviour of the thermal Wightman functions at complex times. Inserting N complete sets of energy eigenstates we have

$$\begin{aligned}
\Gamma_{a_1 ... a_N}&(\tau_1, ..., \tau_N) = \\
&(-1)^p \sum_{m_1, m_2, ..., m_N} [exp\{-\imath E_{m_1}(b - \tau_{a_1} + \tau_{a_N})\}. \\
&\left(\prod_{j=2}^{N} exp\{-\imath E_{m_j}(-\tau_{a_j} + \tau_{a_{j-1}})\} \right) \cdot \left(\prod_{j=1}^{N} h_j \right)]
\end{aligned} \tag{3.9}$$

where

$$h_j(\{m\}) = \langle m_j | \phi_{a_j}(0) | m_{j+1} \rangle \tag{3.10}$$

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and $\{|m_j\rangle\}$ are a complete set of energy eigenstates for temperature β^{-1} ($m_{1+N} = m_1$). We have replaced the density matrix $e^{-\beta H}$ by $e^{-\imath b H}$ so that the explicit temperature dependence appears through the complex parameter b. To get the physical Green functions we just take the limit of $b \to -\imath\beta$. Note that we still work in the energy eigenstate basis for a temperature β^{-1}. Thus the implicit dependence on β remains e.g. in the energy eigenvalues, $\{E_j\}$.

What we wish to do is to consider the behaviour of $\Gamma_{a_1 a_2 \dots a_N}$ at complex times $\{\tau_j\}$. We shall assume that if all the exponentials in (3.9) are of the form $exp\{z_j E_j\}$, where E_j is an energy eigenvalue for temperature β^{-1} and so real and positive, then the trace sum in (3.9) is uniformly convergent (rather than the absolute convergence suggested in [6]) if all the z_j have negative real parts. Given this assumption, the thermal Wightman function, $\Gamma_{a_1 a_2 \dots a_3}(\{\tau\})$, is bounded for all values of the time variables provided they satisfy $\{\tau\} \in A_{a_1 a_2 \dots a_3}$ where

$$A_{a_1 a_2 \dots a_N} := \{\{\tau_{a_1}, \tau_{a_2}, \dots, \tau_{a_N}\} \,|\, Im(\tau_{a_N}) \geq Im(\tau_{a_{N-1}}) \geq \dots$$
$$\dots \geq Im(\tau_{a_1}) \geq Im(\tau_{a_N} + b)\}. \tag{3.11}$$

Being bounded in this region implies that the thermal Wightman function is also analytic in this region [17].

Now we can express the ITF Green functions in terms of these thermal Wightman functions at complex times. We can do this by extending the definition of the time-ordered function, $\Gamma^{111\dots1}$ of (2.5), from real-times to complex times. We can define a bounded expectation value of path ordered fields, Γ_b, by

$$\Gamma_b(\{\tau\}) = \Gamma_{a_1 a_2 \dots a_n}(\{\tau\})$$
$$\text{if} \quad \{\tau\} \in C_b$$
$$\text{and} \quad |b| > |\tau_{a_1}| > |\tau_{a_2}| > \dots > |\tau_{a_N}| \tag{3.12}$$

where C_b is a straight line running from 0 to b in the lower half of the complex time plane. The condition that $\{\tau\} \in A$ of (3.11) is satisfied given the restriction of b to the lower half plane, so that Γ_b is bounded. This is a very useful function because it is bounded as $b \to -\imath\beta$ i.e. as $C_b \to C_I$, the path used for ITF, and thus we will use it to obtain the ITF Green functions. In the limit of $b \to \infty - \imath\beta$ we obtain the time-ordered Green functions. The cyclicity of the trace has a simple form, namely

$$\Gamma_b(\{\tau\}) = \sigma_l \Gamma_b(\{\tau'\}) \tag{3.13}$$

where the label l is associated with the field whose time argument is the largest, i.e. $|\tau_l| > |\tau_{\text{rest}}|$, and $\tau_l' = \tau_l - b$, $\tau_j' = \tau_j$ ($j \neq l$). This (anti-)periodicity of Γ_b of (3.12) then allows us to look at the Fourier series of the Γ_b function and, given that the function is bounded, the Fourier coefficients, $\gamma_{\nu_1, \dots, \nu_N}$, are well defined and given by

$$\gamma_{\nu_1, \nu_2, \dots, \nu_N} = \lim_{b \to -\imath\beta} \delta_{\{\nu\}} \left(\prod_{j=1}^{N-1} \int_{C_b} \tau_j \right)$$
$$e^{2\pi\imath(\nu_1\tau_1 + \nu_2\tau_2 + \dots + \nu_{N-1}\tau_{N-1})/b}.$$
$$\Gamma^b(\tau_1, \tau_2, \dots, \tau_{N-1}, 0). \tag{3.14}$$
$$\delta_{\{\nu\}} = \delta_{\nu_1, \nu_2, \dots, \nu_N} \tag{3.15}$$

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where the ν_j is (half) integer if the j-th field is bosonic (fermionic) and we have shifted $\tau_j \rightarrow \tau_j - \tau_N$, i.e. we use translational invariance to extract the delta function.

We now look at the inverse Fourier transform of the thermal Wightman functions so that we can write the ITF result (3.14) in terms of the thermal Wightman functions at real energies. Again, because the thermal Wightman functions are assumed well behaved when $\{\tau\} \in A_{a_1...a_N}$ we have

$$
\gamma_{a_1 a_2 ... a_N}(p_1, p_2, ..., p_N) = \left(\prod_{j=1}^{N} \int_{C_1} dt_j e^{+\imath p_j t_j} \right)
$$
$$
\Gamma_{a_1 a_2 ... a_N}(t_1, t_2, ..., t_N)
$$
$$
\Gamma_{a_1 a_2 ... a_N}(\tau_1, \tau_2, ..., \tau_N) = \left(\prod_{j=1}^{N} (2\pi)^{-1} \int_{-\infty}^{\infty} dp_j e^{-\imath p_j \tau_j} \right) \cdot
$$
$$
\gamma_{a_1 a_2 ... a_N}(p_1, p_2, ..., p_N)
$$

$$(3.16)$$

where we take $\{t\} \in C_1$ of RTF that runs along the real time axis with an infinitesimal slope to ensure that $\{t\} \in A_{a_1...a_N}$. We will often use the energy space version of (2.7), the N-point version of the well known KMS (Kubo-Martin-Schwinger) condition [1, 2, 3, 4, 5, 6]. In our notation it is

$$
f_{a_1} \gamma_{a_1 a_2 ... a_N}(p_1, p_2, ..., p_N) = \gamma_{a_2 ... a_N a_1}(p_1, p_2, ..., p_N) \tag{3.17}
$$

where

$$
f_j = \sigma_j e^{-\imath b p_j}, \tag{3.18}
$$

and $\sigma_j = +1$ (-1) if the j-th field, ϕ_j, is bosonic (fermionic). We substitute (3.16) into (3.14) using (3.12) so that we can get $\gamma_{\nu_1,...,\nu_N}$, the ITF function, in terms of the thermal Wightman functions at real energies. Thus we have

$$
\gamma_{\nu_1, \nu_2, ..., \nu_N} =
$$
$$
\delta_{\{\nu\}} \lim_{b \rightarrow -\imath \beta} \left(\prod_{j=1}^{N-1} \int_{C_b} d\tau_j exp\{ \sum_j 2\pi\imath(\nu_j \tau_j)/b \} \right) \cdot
$$
$$
\Gamma_b(\tau_1, \tau_2, ..., \tau_{N-1}, 0) \tag{3.19}
$$

We can now use (3.12) and we find that if we further limit b such that $Re(b) > 0$ then

$$
\gamma_{\nu_1, \nu_2, ..., \nu_N} = \delta_{\{\nu\}} \sum_{perm.\{a\}|a_N=N} \left(\prod_{j=1}^{N-1} \int_0^{\tau_{a_j}} d\tau_{a_j} \right)
$$
$$
(\prod_{j=1}^{N-1} (2\pi)^{-1} \int dk_{a_j} exp\{ \imath \sum_j (2\pi\imath/b)(\nu_j - k_{a_j})\tau_{a_j} \}) \cdot \gamma_{\tau_1 \tau_2 ... \tau_N} \tag{3.20}
$$

where $a_0 = 0$, $\tau_{a_0} = \tau_0 = b$. The first sum takes the sequence $\{a_j\}$ over all permutations of $1, ..., N$ subject to the limitation that $a_N = N$. The further restriction on b was

imposed merely for convenience in defining the order in which the τ integrals are done for a given thermal Wightman function. Doing the time integrals is we find that in ITF one is calculating

$$\gamma_{\nu_1 \ldots \nu_N} = \delta(\{\nu\}).\Phi(\{z_j = 2\pi \imath \nu_j / \beta\}), \tag{3.21}$$

where the result of the ITF calculation is a function of the imaginary Matsubara energies, $\{z_j = 2\pi \imath \nu_j / \beta\}$. At complex energies $\{z\}$ the function Φ is chosen to be

$$
\begin{aligned}
\Phi(\{z\}) \\
&= (-1)^{N-1} (\prod_{j=1}^{N} (2\pi)^{-1} \int dk_j) \, 2\pi \delta(\sum_{j=1}^{N} k_j). \\
&\quad \sum_{perm.\{b\}} \gamma_{b_1 b_2 \ldots b_N}(\{k\}). \prod_{j=2}^{N} \imath (B_j^N)^{-1} \tag{3.22} \\
&= (-1)^{N-1} (\prod_{j=1}^{N} (2\pi)^{-1} \int dk_j) \, 2\pi \delta(\sum_{j=1}^{N} k_j). \\
&\quad \sum_{perm.\{b\}} \rho_{b_1 b_2 \ldots b_N}(\{k\}). \prod_{j=2}^{N} \imath (B_j^N)^{-1}, \tag{3.23}
\end{aligned}
$$

where the first sum takes $\{b_j\}$ through all permutations of the numbers $(1, 2, \ldots, N)$ and

$$B_j^m = \sum_{l=j}^{m} (z_{b_l} - k_{b_l}), \tag{3.24}$$

$$\rho_{b_1 b_2 \ldots b_N}(\{k\}) = \frac{1}{2} (\gamma_{b_1 b_2 \ldots b_N}(\{k\}) + (-1)^{N-1} \gamma_{b_N \ldots b_2 b_1}(\{k\})). \tag{3.25}$$

Note that to get a nice form for Φ we have introduced an N-th redundant complex energy variable defined through the constraint

$$\sum_{j=1}^{N} z_j = 0. \tag{3.26}$$

There is an implicit analytic continuation involved here as Φ of (3.22) is a function of complex energies but it is defined in terms of the ITF result in (3.21). To go from the definition in terms of discrete imaginary energy values, $2\pi \imath \nu_j / \beta$, to complex energies is not a uniquely defined analytic continuation. Other possible expressions for Φ also satisfy (3.21), such as those that are similar to (3.22) but with extra factors of $\sigma_a exp\{2\pi \imath \nu_a / \beta\}$ inserted anywhere.

The choice we have made corresponds to the simplest one where all factors of $exp\{2\pi \imath \nu_a / \beta\}$ are put equal to σ_a. To be more precise about the form for Φ in (3.22) which we have chosen, we look at the analytic and asymptotic behaviour of this form as is done with the two-point functions [6]. We see this form for Φ satisfies two conditions

(A) $\phi(\{z\})$ is analytic for $\{z\} \in \Omega$ where
$\Omega = \{\{z\} | \sum_{a=1}^{N-1} c_a z_a \notin \Re e \ \forall \{c_a\}$ where $c_a = 0, 1\}$,

(B) $\Phi(\{z\}) \to 0$ as we take any number of the z_a to infinity while remaining in Ω.

Note that Ω is an disconnected open set with the hyper-surfaces of real energies splitting it into a number of connected open subsets. The number of these subsets will be noted later.

It remains to be shown that these conditions are sufficient to make the choice of the form of Φ unique. This can be done by building on the proof given for functions of one variable in [6]. There it was shown that conditions (A) and (B) were sufficient to fix the continuation of a function of one complex variable from an infinite set of points in the upper half complex plane, a constant distance apart and lieing on a straight line, to the whole of the upper half complex plane.

In the present case we start by considering Φ as a function of only one variable, say z_a, holding the others constant. Then we know from [6] that (A) and (B) ensure that the continuation in z_a is unique in any connected subset of Ω which contains the points $z_a = 2\pi \imath \nu_a / \beta$ as $\nu_a \to \infty$. However, this does not fix the continuation of Φ in the remaining connected subsets of Ω as in such regions there are not an infinite number of points where we know the function Φ and thus we can not use the result of [6].

The trick is to look at the behaviour of Φ along a different set of lines in each of the different connected subsets of Ω. The simplest choice is pick lines each of which lie parallel to a different $\sum_{a=1}^{N-1} c_a z_a \in \Re e$ boundary hypersurface (where the c_a are those used in the definition of Ω). It is always possible to find a set of lines of this kind as none of the hypersurfaces are parallel and they have the origin as a common point. Further we choose these lines such that they pass through Matsubara energy values at regular and finite intervals. The lines chosen in each region form a basis though not in general an orthogonal one. We can use the proof of [6] that conditions (A) and (B) are sufficient to fix the continuation along each of these lines in the connected subset of Ω being considered. Having shown that the analytic continuation is unique in each of these directions separately it is straightforward to see that the continuation as a function of all $N - 1$ variables is therefore unique [17] in each one of the connected subsets of Ω. Therefore (A) and (B) are sufficient to ensure that the analytic continuation from the Matsubara energies is unique in the several variable case.

For example, consider the case of three-point functions in ITF. Then we are looking at a function of two independent variables, $\Phi(z_1, z_2)$ which is not analytic when $z_1 \in \Re e$, $z_2 \in \Re e$, or $z_1 + z_2 \in \Re e$. Thus it is defined in a region Ω made up of six disconnected regions, I to VI, which are shown in fig.(2). The real parts are not important so they have been projected out. Take region V and consider a pure bosonic function for instance. We can consider Φ along the lines (i) $z_1 = 2\pi \imath / \beta$, $z_2 = -2\pi \imath \eta_1 / \beta$ and (ii) $(z_1 + z_2) = -2\pi \imath / \beta$, $(z_1 - z_2) = 2\pi \imath (2\eta_2 - 1)/\beta$ where the η_a are real parameters. The value of Φ is known at the Matsubara energies where η_a is a (half-)integer for bosons (fermions). We first look at Φ along the straight line (i). One can then apply the proof of [6] directly to show (A) and (B) fix the continuation of Φ uniquely in region V along this line as a function of z_1. This is repeated for the second line to fix Φ as a function of $z_1 - z_2$. Thus the behaviour

of Φ as a function of the $\{z\}$ near the intersection of these lines, the point $\eta_1 = 2$, $\eta_2 = 2$, is now completely fixed. Thus the analytic continuation for the whole region is fixed [17].

Lastly, we started to consider the conditions (A) and (B) as a way of fixing the analytic continuation only after we used the ad-hoc prescription of choosing the simplest form for Φ in (3.22), namely dropping all possible exponential terms. An alternative way to motivate this stage is to look at the definition of various types of real-time Green functions, as we shall do in the next section, and to study their analytic and asymptotic behaviour. Then one could try to choose the form of Φ that matched the behaviour of the function desired at real-times. This presupposes that one knows what can be extracted from ITF which as we shall see is not obvious in general.

Now we have given some additional conditions and fixed the ITF result at complex energies, we can proceed to examine what we have obtained at real physical energies. We look at Φ of (3.22) as $z_a \to p_a \in \Re e$. However we must specify how we approach the real energy axes because there the B terms in (3.22) are singular. There are many different ways of doing this. We are going to start by looking at the simplest approach. The usual and simplest analytic continuation from the imaginary, discrete energies of ITF in (3.21) to real physical energies is

$$ z_a \to p_a + \imath\epsilon_a, \tag{3.27} $$

where the ϵ_a are infinitessimal real constants which may be positive or negative and need not be all the same size.

Note that at this stage we have a unique function, $\Phi(\{z\})$ of (3.23), as the ITF result at complex energies $\{z\}$. The arbitraryness in the continuation from energy values $2\pi\imath\nu_i/\beta$ to general complex energies has been dealt with. The choice left in the continuation from imaginary energies, which we are looking at in (3.27) and in the rest of this paper, is how to approach the real energy axes. This choice exists because in general Φ has cuts along the whole length of every real energy axis and in general the value of both the real and imaginary parts of Φ are discontinuous there.

To see what the analytic continuation of the ITF calculation to real energies has produced, we do the inverse Fourier transform through

$$ \Phi(\{t\}; \{\epsilon\}) \;=\; (\prod_{j=1}^{N}(2\pi)^{-1} \int dp_j e^{-\imath p_j t_j}) \; \Phi(\{z_j = p_j + \epsilon_j\}). \tag{3.28} $$

Using

$$ \theta(t) = (2\pi)^{-1} \int dp \, e^{-\imath pt} \frac{\imath}{p + \imath\epsilon} \tag{3.29} $$

we find that

$$ \Phi(\{t\}; \{\epsilon\}) \;= (-1)^{N-1} \sum_{perm.\{b\}} \Gamma_{b_1 b_2 \dots b_N}(\{t\}). $$
$$ \left(\prod_{j=2}^{N} \psi_j^N \theta(\psi_j^N(t_{b_j} - t_{b_{j-1}})) \right) $$

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$$\tag{3.30}$$

$$= (-1)^{N-1} \sum_{perm.\{b\}} \rho_{b_1 b_2 \ldots b_N}(\{t\})$$

$$\left(\prod_{j=2}^{N} \psi_j^N \theta(\psi_j^N (t_{b_j} - t_{b_{j-1}})) \right)$$

$$\tag{3.31}$$

where

$$\psi_j^m = \theta(\sum_{l=0}^{m-j} \epsilon_{b_{j+l}}) - \theta(-\sum_{l=0}^{m-j} \epsilon_{b_{j+l}}),$$

$$\rho_{b_1 b_2 \ldots b_N}(\{t\}) = \frac{1}{2} \left(\Gamma_{b_1 b_2 \ldots b_N}(\{t\}) + (-1)^{N-1} \Gamma_{b_N \ldots b_2 b_1}(\{t\}) \right).$$

$$\tag{3.32}$$

This is the clearest form for the ITF result in that it tells us that *every* thermal Wightman function appears once in (3.30), each multiplied by a single known product of theta functions of the times (rather than sums of products of theta functions that we started with). Studying the terms of the first form (3.30), we see that the coefficient of any thermal Wightman function is the same, up to a possible minus sign, as the coefficient in front of the thermal Wightman function with indices in the reverse order. This fact is reflected in the second form, (3.31). These pairs of thermal Wightman functions are the spectral functions for general N-point ITF functions.

It is now immediately obvious that there are many results coming out of an ITF calculation where the usual simple type of analytic continuation, (3.27), has been used. One can choose the set of infinitesimal ϵ's to have many different values and signs, always subject to the constraint $\sum_{a=1}^{N} \epsilon_a = 0$, and there are distinct choices for these ϵ's. These correspond to different results which have different sets of theta functions in front of each thermal Wightman function. However, precisely what type of functions have been calculated in ITF is not clear.

4   *N*-Point Retarded Functions and
other ITF results

When one uses the simple prescription (3.27) to extract real-time results from an ITF calculation of a two-point function that one obtains the real-time retarded and advanced two-point functions [1, 5, 6]. More recently [7, 9] it was shown that the same is true for three-point functions. We therefore try to rewrite the N-point retarded function formula in a form similar to (3.30). The N-point retarded functions R_a are given as multiple commutators [18]. For pure bosonic fields they are defined to be

$$R_a(t_1, t_2, \ldots, t_N) =$$

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$$\sum_{perm.\{a\}|a_N=a} \prod_{j=1}^{N-1} \theta(t_{a_{j+1}} - t_{a_j})[[...[[\phi_a, \phi_{a_{N-1}}], \phi_{a_{N-2}}], ...], \phi_{a_1}]$$

$$(4.33)$$

where $\phi_a = \phi_a(t_a)$. The sum is over $\{a_j\}$, $j = 1$ to N, running through all permutations of the numbers 1 to N subject to the constraint $a_N = a$. The advanced functions $\bar{R}_a$ are defined in a similar manner except we replace all the $\theta(t)$ by $\theta(-t)$ and add an overall factor of $(-1)^{N-1}$, c.f. the two-point case in [1, 5].

The general formula for the N-point retarded functions, $R_a(\{t\})$, is

$$R_a(t_1, t_2, ..., t_N) =$$
$$\sum_{perm.\{a\}|a_N=a} \prod_{j=1}^{N-1} \theta(t_{a_{j+1}} - t_{a_j}) \sum_{s=0}^{N-1} \sum_{\{q\}} (-1)^{N-s-1} \Gamma_{b_1 b_2 .... b_N}$$

$$(4.34)$$

The sum over $\{q\}$ is a sum over all possible values of a set of the s integer variables, $\{q\}$, which satisfy $0 < q_1 < q_2 < ... < q_s < N$. The indices, $\{b_j\}$, of the thermal Wightman functions, $\Gamma_{b_1...b_N}$, are then defined through

$$q_l = j \quad \leftrightarrow \quad b_{N-l+1} = a_j$$
$$q_{l-1} < j < q_l \quad \leftrightarrow \quad b_{j-l+1} = a_j$$
$$\forall l \ \text{ s.t. } \ 1 \le l \le s+1 \qquad (4.35)$$

with $q_0 = 0$ and $q_{s+1} = N$. What this definition means is that each term in (4.34) has come from a term in R_a where we have taken the first term of s of the commutators in (4.33), i.e. replaced $[A, B]$ by AB s times, while we take the second term, $[A, B] \rightarrow -\sigma BA$ (where $\sigma = +1$ unless A and B are fermionic, see below), of the other $N - 1 - s$ commutators. The sum over the q's is then all ways of choosing s of the $N - 1$ commutators for which we will take the first term of the commutator. We will take (4.34) as the definition of the retarded functions of fields of mixed statistics. We then see that the sign in the definition of the thermal Wightman functions (2.6) then absorbs the change of sign in replacing the relevant commutators by anti-commutators so that the definition (4.33) holds for fields of mixed statistics.

From the definitions (4.33) and (4.34), we see that the retarded function is (anti-)symmetric in $N - 1$ fields. We can thus give the N distinct retarded functions a single label a ($= 1, ..., N$), the label of the one field, ϕ_a, that plays a special role in the definitions (4.33) and (4.34).

We can now swap the order of the sums in (4.34) to give

$$R_a(t_1, t_2, ..., t_N) = \sum_{s=0}^{N-1} (-1)^{N-s-1} \sum_{\{q\}} \sum_{perm.\{a\}|a_N=a}$$
$$\prod_{j=1}^{n-1} \theta(t_{a_{j+1}} - t_{a_j}) \Gamma_{b_1 b_2 .... b_N} \qquad (4.36)$$

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Since for given s and $\{q\}$, (4.35) is a $1:1$ map for $\{a\} \leftrightarrow \{b\}$, we can replace the sum over permutations of $\{a\}$ by a sum over $\{b\}$ being all permutations of the numbers 1 to N, so that

$$
\begin{aligned}
R_a(t_1, t_2, ..., t_N) &= \sum_{s=0}^{N-1}(-1)^{N-s-1}\sum_{\{q\}}\sum_{perm.\{b\}} \\
&\quad \prod_{j=1}^{n-1}\theta(t_{a_{j+1}} - t_{a_j})\Gamma_{b_1 b_2 .... b_N} \quad\quad (4.37)\\
&= \sum_{s=0}^{N-1}(-1)^{N-s}\sum_{perm.\{b\}}\Gamma_{b_1 b_2 .... b_N} \\
&\quad \sum_{\{q\}}\prod_{j=1}^{n-1}\theta(t_{a_{j+1}} - t_{a_j}) \quad\quad (4.38)
\end{aligned}
$$

The task now is to try to do the sum over the q's and reduce the coefficients to a single product of theta functions so that we can compare N-point retarded function with (3.30). We find that

$$
\begin{aligned}
R_a(t_1, t_2, ..., t_N) &= (-1)^{N-1}\sum_{perm.\{b\}}\Gamma_{b_1 b_2 .... b_N} \\
&\quad \prod_{j=N-s}^{N-1}(-\theta(t_{b_j} - t_{b_{j+1}}))\prod_{j=1}^{N-s-1}\theta(t_{b_{j+1}} - t_{b_j})
\end{aligned}
$$
$$(4.39)$$

where s is now defined via the relation $b_{N-s} = a$.

It is now easy to compare with the ITF result (3.30). Suppose in (3.30) we pick out one epsilon, and set it positive, and then set all the other epsilons negative, say $\epsilon_a = (N-1)\epsilon$, $\epsilon_{other} = -\epsilon$ where ϵ is an infinitesimal positive quantity. Thus we have continued from imaginary energies to just below the real energy axes except for the energy of the a'th field which we approach from above. The ITF result in this case is, from (3.30), just (4.39), and we have

$$
\Phi^N(\{t\}; \epsilon_a > 0, \epsilon_{other} < 0) = R_a(\{t\}). \quad\quad (4.40)
$$

So we have shown that at least a subset of the possible ITF results, the ones with only one epsilon positive in (3.21), always lead to the N retarded functions.

The advanced functions, $\bar{R}_a$, are obtained in an identical manner except that the theta functions are reversed in time and an overall factor of $(-1)^{N-1}$ is added. This corresponds to switching the signs of all the epsilon in (3.22) terms so we have

$$
\Phi^N(\{t\}; \epsilon_a < 0, \epsilon_{other} > 0) = \bar{R}_a(\{t\}). \quad\quad (4.41)
$$

We shall now examine whether or not ITF *only* calculates the N retarded and the N advanced functions. For the two-point functions, $N = 2$, the formula (3.30) only depends

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on the sign of the one independent ϵ since $\sum_{j=1}^{N} \epsilon_j = 0$. In this case then there are only two distinct choices for the ϵ's and so only two independent results [1, 5, 6]. They are also equivalent to all the independent choices of the epsilon's which satisfy $\epsilon_a > 0, \epsilon_{rest} < 0$ or $\epsilon_a < 0, \epsilon_{rest} > 0$. Thus the retarded and advanced functions exhaust the possible results for $N = 2$.

In the case of the three-point functions, $N = 3$, while the sum of two epsilons appears in the formula (3.30) for the result of an ITF calculation, because the sum of epsilons is always zero, we can again always express the answer (3.30) in terms of just single epsilons. The result then depends only on the signs of the individual epsilons and one epsilon will always have a different sign from the other two. Thus from (4.40) and (4.41) the retarded and advanced functions again exhaust all possible results of a three-point ITF calculation [7, 8, 9]. We give all possible results of an ITF calculation of a three-point function using (3.27) in table 1, each distinct result being characterised by a unique set of values for the set of variables $s_j = \theta(\epsilon_j) - \theta(-\epsilon_j)$. In each case one example set of values for the ϵ_j's is given, in terms of an infinitesimal positive quantity, ϵ.

Beyond this things are not so simple. For $N \geq 4$ the constraint $\sum_{j=1}^{N} \epsilon_j = 0$ (3.26) does not ensure that one of the epsilons has a different sign from all the others. There are of course $2^N - 2$ ways of choosing the signs of individual epsilons, once the constraint has been accounted for, whereas there are only $2N$ retarded and advanced functions (though for $N = 2$ only two of these are independent). It is therefore clear from these simple arguments that $N = 2, 3$ are special cases and *only* for these cases is the result of an ITF calculation using the simple analytic continuation (3.27) always one of the retarded and advanced functions.

The situation for $N \geq 4$ is actually more complicated still. As can be seen from (3.30), the result of an ITF calculation always depends not only on the sign of the individual epsilons but also on the signs of all possible sums of subsets of the epsilons. The constraint $\sum_{j=1}^{N} \epsilon_j = 0$ does not allow us to remove all the dependence in (3.30) on the sign of sums of epsilons when $N \geq 4$. Therefore using the simple analytic continuation with ITF leads to more than 2^N different results depending on how the ϵ's are chosen.

For instance for $N = 4$ each distinct result can be characterised by a unique set of values for the six variables $s_1, s_2, s_3, s_{12}, s_{13}, s_{23}$ as can be seen from (3.22). The s_j are the signs of a single epsilon variable as defined above while $s_{ij} = \theta(\epsilon_i + \epsilon_j) - \theta(-\epsilon_i - \epsilon_j)$. Some examples are given in table 2 where the Q_{ij} and $\bar{Q}_{ij}$ functions are the new types of function that we shall discuss in a moment. There are in fact 32 distinct results from a four-point ITF calculation with the usual analytic continuation, of which only 8 are retarded or advanced functions compared, with the 14 ways of choosing different signs s_j for the single epsilons. This is to be compared with the fact that there are 16 distinct RTF four-point functions with two identities between them, as we shall see later.

It is possible to find and count distinct ITF results, and the results of a computer search are shown in table 3. The number of distinct ITF results is also the same as the number of distinct connected subsets of Ω discussed earlier in the context of fixing the analytic continuation. From table 3, we see that the number of different ITF results is vast, larger and growing faster than any other type of relevant function. Studying these new ITF functions a little more confirms that they are not very simple. Equation (3.30)

shows that the results from an ITF calculation using the simple analytic continuation (3.27) satisfy the following rule:

> Let S_y^a be the set of all sums of y different ϵ's, where one epsilon, ϵ_a, is always included in the sum. Then if all the elements of S_y^a are positive (negative) for $y = 1$ to x, the function being studied (the ITF result (3.30) with this $\{\epsilon\}$) is zero unless the a-th time variable (that of the a'th field ϕ_a) is bigger (smaller) than at least x of the other time variables, though it is not restricted with respect to comparisons with the remaining $(N - 1 - x)$ time arguments.

The case where $x = N - 1$ is just the retarded or advanced function case. This rule emphasises the fact that the new functions are nothing very obvious. In table 2 we have used this rule to label the new four-point functions. We define $Q_{ij}^{(4)}$ ($\bar{Q}_{ij}^{(4)}$) to be the result of an ITF calculation using the simple analytic continuation (3.27) where we have chosen $s_i = s_j = +1 \ (-1)$, $s_k = -1 \ (+1)$ and $s_{ik} = +1 \ (-1)$, $k \neq i, j$ (i, j, $k = 1, 2, 3, 4$).

Thus, from this rule, we see that ITF using the usual analytic continuation does not only produce the $2N$ retarded and advanced functions. All of the retarded and advanced functions can always be calculated in ITF, they are the results found when we use the simple analytic continuation (3.27) and pick one epsilon to have a different sign from all the others as (4.40) and (4.41) show. However there are many other choices for the epsilons that lead to distinct results when $N \geq 4$. This is important when we try to compare ITF and RTF calculations as the latter calculates 2^N functions and there are always two relations between the RTF functions.

5 RTF Functions

From (2.5), we have that the Green functions of RTF are of the form

$$
\Gamma^{\mu_1 \mu_2 \cdots \mu_N}(\{t\}) \;=\; \sum_* \Gamma_{a_1 \ldots a_{N_2} a_{N_2+1} \ldots a_N}(\{\tau\}) \prod_{j=2}^{N_2} \theta(t_{a_j} - t_{a_{j-1}})
$$
$$
\prod_{j=N_2+1}^{N-1} \theta(t_{a_j} - t_{a_{j+1}})
$$

(5.42)

where $\tau_j = t$ if $\mu_j = 1$ and $\tau_j = t - \imath\beta\bar{\alpha}$ if $\mu_a = j$ and N_2 is the number of thermal indices μ_j that equal two. The sum $\sum_*$ is over certain permutations of the $\{a\}$ indices, namely

$$
* \;\equiv\; \begin{pmatrix} a_1 \ldots a_{N_2} \\ \{j | \mu_j = 2\} \end{pmatrix} \otimes \begin{pmatrix} a_{N_2+1} \ldots a_N \\ \{j | \mu_j = 1\} \end{pmatrix}.
$$

(5.43)

i.e. the subscripts $a_1, ..., a_{N_2}$ ($a_{N_2+1}, ..., a_N$) run through all permuations of the numbers that are the indices of the type two (one) fields, those with a thermal label $\mu = 2$ ($\mu = 1$) . In this way the fields whose time arguments lie on C_1 (C_2) are the ones labelled $a_1...a_{N_2}$

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$(a_{N_2+1}...a_N)$ and they are all on the left (right) in the thermal Wightman functions in (5.42) as demanded by the path ordering in the definition of RTF Green functions (2.5).

Taking a Fourier transform, we have

$$\Gamma^{\{\mu\}}(\{k\}) = \sum_* \left(\prod_{j=1}^N (2\pi)^{-1} \int dp_{a_j} \right)$$
$$\Delta(\{k-p\}).X^{\bar{\alpha}}(\{\mu\},\{p\})\gamma_{a_1 a_2 ... a_N}(\{p\}).P(\{k-p\}) \qquad (5.44)$$

where

$$\Delta(\{k-p\}) = \begin{cases} 2\pi\delta(D_1^{N_2})2\pi\delta(D_1^N) & \text{if} \quad 0 < N_2 < N \\ 2\pi\delta(D_1^N) & \text{if} \quad N_2 = 0 \text{ or } N \end{cases}$$
$$\qquad (5.45)$$

$$X(\{\mu\},\{p\}) = \prod_{\{j|\mu_j=2\}} e^{-\beta p_j} \qquad (5.46)$$

$$P = \begin{cases} \left(\prod_{j=2}^{N_2} \imath(D_j^{N_2} + \imath\epsilon)^{-1} \right). \\ \quad \left(\prod_{j=N_2+1}^{N-1} \imath(D_{N_2+1}^j + \imath\epsilon)^{-1} \right) & \text{if} \quad 1 < N_2 < N-1 \\ \left(\prod_{j=2}^{N_2} \imath(D_j^{N_2} + \imath\epsilon)^{-1} \right) & \text{if} \quad N_2 = N-1 \text{ or } N \\ \left(\prod_{j=N_2+1}^{N-1} \imath(D_{N_2+1}^j + \imath\epsilon)^{-1} \right) & \text{if} \quad N_2 = 1 \text{ or } 0 \end{cases}$$
$$\qquad (5.47)$$

$$D_j^m(\{k-p\}) = \sum_{l=0}^{|m-j|_N} (k_{a_{j+l}} - p_{a_{j+l}}). \qquad (5.48)$$

The factor X takes care of the fact that the fact that the time arguments lying on C_2 have a constant imaginary part.

Now we can use time translation invariance by assuming that the Fourier transform of the thermal Wightman functions, the $\gamma_{a_1...a_N}(\{p\})$ of (3.16), are proportional to $\delta(\sum_{j=1}^N p_j)$. The $\delta(D_1^N)$ in Δ then gives us an overall $\delta(\sum_{j=1}^N k_j)$ as expected. We also find that we can pull the factor X outside the integral and this gives

$$\Gamma^{\{\mu\}}(\{k\}).X^{-\bar{\alpha}}(\{\mu\},\{k\})$$
$$= \Gamma_{\alpha=1}^{(\{\mu\})}(\{k\})$$
$$= \sum_* \left(\prod_{j=1}^N (2\pi)^{-1} \int dp_{a_j} \right)$$
$$\Delta(\{k-p\})\gamma_{a_1 a_2 ... a_N}(\{p\}).P(\{k-p\}). \qquad (5.49)$$

Note that the right hand sides of (5.49) are independent of α. Also note that if one sets $\alpha = 1$ at the start of the derivation, in the definition of the RTF Green functions (5.42), we find that in deriving the equality between the two right hand sides of (5.49),

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no assumption of equilibrium or time-translational invariance is made. Thus the $\alpha = 1$ limit of the right hand sides of (5.49), when written in space-time coordinates rather than momentum, is true out of equilibrium . While this is not immediately obvious, the fact that there is no factor X and so no explicit temperature factor from the density matrix in the $\alpha = 1$ limit of (5.49) is indicative of this. For the $\alpha = 1$ case both type one and type two fields are the physical field at real times. We shall return to this point in a moment.

Now at this point, we can manipulate the above formula (5.49) into a similar one for $X^\alpha \Gamma^{\{\mu\}}$ just by using the cyclicity of the trace and we find that

$$
\begin{aligned}
\Gamma^{\{\mu\}}&(\{k\}).\bar{\sigma}(\{\mu\}).X^\alpha(\{\mu\},\{k\}) \\
&= \Gamma^{\{\mu\}}_{\alpha=0}(\{k\}).\bar{\sigma}(\{\mu\}) \\
&= \sum_* \left(\prod_{j=1}^{N}(2\pi)^{-1}\int dp_{a_j}\right)\Delta(\{k-p\}) \\
&\quad \gamma_{a_1 a_2 \ldots a_N}(\{p\}).P^*(\{k-p\}),
\end{aligned}
\tag{5.50}
$$

where P^* is the complex conjugate of (5.47) and

$$
\bar{\sigma} = \prod_{\{j|\mu_j=2\}} \sigma_j.
\tag{5.51}
$$

However, this approach hides the fact that the $\alpha = 0$ limit of the right hand sides of (5.50) can actually be derived without using equilibrium or time-translational invariance in a similar way to the $\alpha = 1$ case of (5.49). This is clear if we realise that the RTF can be derived by using the curve $C' = C'_1 \oplus C'_2 \oplus C'_3 \oplus C'_4$ from $+\infty$ to $-\infty$ (C'_1), $-\infty$ to $-\infty - \imath\alpha\beta$ (C'_2), $-\infty - \imath\alpha\beta$ to $+\infty - \imath\alpha\beta$ (C'_3), and $+\infty - \imath\alpha\beta$ to $+\infty - \imath\beta$ (C'_4). This is equivalent to the usual curve used in RTF when one uses the periodicity of the fields (up to a minus sign for each fermionic field whose time argument lies on C_2). The $\alpha = 0$ limit of C' then involves the physical fields at real-times.

There are some useful formulae involving all the RTF functions known for two-point functions [5, 19] and only more recently for three-point functions [8, 9, 11] that tell us that certain sums of all the RTF functions are zero and other such sums are equal to one of the retarded or advanced functions, i.e. the usual results of an ITF calculation. We will now consider whether these results can be generalised for N-point functions.

The quickest way to get at these results is to look at RTF in the rather special $\alpha = 1$ limit. In this limit the choice of curve coincides with that used in the CTP (Closed-Time Path) approach to non-equilibrium problems [5, 20]. As we have already noted above, in this $\alpha = 1$ limit of RTF, we are merely manipulating Green function definitions and have no need to call upon the cyclicity of the trace or time translation invariance in order to deal with fields whose time argument lies on C_2. In particular, we can draw on the results of the report by Chou et al. [20] as their $G_{++-+-++\ldots}$ functions correspond to the RTF Green functions, $\Gamma^{\{\mu\}}$, of (2.5) in the $\alpha = 1$ limit with a $+$ $(-)$ index in [20] corresponding to a type one (two) index in the RTF functions here. From [20], we see that in our notation

$$
\sum_{\{\mu\}=1,2}(-1)^{N_2}\Gamma^{\{\mu\}}_{\alpha=1} = 0
\tag{5.52}
$$

even out of equilibrium. N_2 is the number of type two thermal labels in each term. Thus we have for the equilibrium case, by using (5.49), that

$$\sum_{\{\mu\}=1,2} (-1)^{N_2} X^{-\bar{\alpha}} \Gamma^{\{\mu\}} = 0 \tag{5.53}$$

This is in fact equivalent to the identity noted by Kobes and Semenoff [19] which stated that the sum over all possible ways of circling the vertices of graphs within their formalism was zero, a result derived from their largest time equation (see also [11, 12, 13, 14]).

We can then go beyond the results of [20] by looking at the time reversed version of the closed time path, our $\alpha = 1$ RTF path. The equivalent of the simple non-equilibrium $\alpha = 1$ result is found then for the case of $\alpha = 0$ and is best thought of using the alternative RTF time curve mentioned above. As before all that changes is $P \to P^*$ and a minus sign for every type two fermionic field. What we find is a version of the equation of Kobes and Semenoff derived from their smallest time equation [11, 19] but now directly in terms of the RTF functions (for any α) rather than in terms of the circled and uncircled graphs used in [11, 12, 13, 14, 19] or the $\alpha = 1$ graphs of CTP as used in [20]. In our notation this equation is

$$\sum_{\{\mu\}=1,2} (-1)^{N_2} \bar{\sigma} X^{\alpha} \Gamma^{\{\mu\}} = 0, \tag{5.54}$$

where $\bar{\sigma}$ was defined in (5.51).

Moving on, another key result in the CTP approach is one giving the retarded functions in terms of the RTF functions. Again this can be found in the the paper by Chou et al. [20] for the non-equilibrium $\alpha = 1$ case. We can then use our general α equilibrium relation (5.49) and this gives

$$R_a = \sum_{\mu_a=1, \{\mu_{rest}\}=1,2} (-1)^{N_2} X^{-\bar{\alpha}} \Gamma^{\{\mu\}} \tag{5.55}$$

$$= \sum_{\mu_a=1, \{\mu_{rest}\}=1,2} (-1)^{N_2} \Gamma^{\{\mu\}}_{\alpha=1} \tag{5.56}$$

One can then go beyond the results of [20] as before by finding the equivalent relation for the $\alpha = 0$ limit. This is found to be

$$\bar{R}_a = \sum_{\mu_a=1, \{\mu_{rest}\}=1,2} (-1)^{N_2} \bar{\sigma} X^{\alpha} \Gamma^{\{\mu\}} \tag{5.57}$$

$$= \sum_{\mu_a=1, \{\mu_{rest}\}=1,2} (-1)^{N_2} \bar{\sigma} \Gamma^{\{\mu\}}_{\alpha=0} \tag{5.58}$$

Again this can be derived from (5.56) by using the cyclicity of the trace (2.7) though this hides the fact that the $\alpha = 0$ limit can be derived without assuming equilibrium or time-translation invariance.

We should note that relations similar to (5.53), (5.54), (5.56) and (5.58) can be found within the circled/uncircled graph formalism of Kobes and Semenoff [11, 12, 13, 14, 19]. The precise connection between those results and those given above was pointed out recently by Kobes [14].

Finally, one should note that the approach used here does not make use of any hermitian properties to derive (5.50), (5.54) and (5.57) from (5.49), (5.53) and (5.55).

6 Comparing ITF and RTF

The derivation of the FTFT formalisms using the path integral method [2, 5] shows that ITF and RTF contain the same physical information. It is also clear from the work above that ITF, using the usual analytic continuation (3.27), gives retarded and advanced functions (and other functions for $N \geq 4$) while the $\Gamma^{11\ldots 1}$ functions of RTF are the time-ordered expectation values. Thus the formalisms as *usually* used are calculating different quantities. The question asking how are ITF and RTF results related becomes how are the retarded and time-ordered functions related?

What we would like to do is invert the expressions (5.56), (5.58) to find an expression for the time-ordered function in terms of the retarded and advanced functions.

Unless otherwise specified, all the functions here are functions of energies $\{p\}$. The functions are defined in terms of time (2.5) and (4.34) and the Fourier transforms are defined as in (3.28).

The relationship between ITF and RTF two-point functions is summarised in what is called the spectral representation of RTF [5, 22]. However it is best to consider them in a new and unusual way if we wish to generalise to higher point functions. There is also one small point that is worth clarifying regarding the relationship of these RTF spectral representations to retarded and advanced functions. Using the definitions of retarded and advanced and of the RTF functions, together with equilibrium and time translation invariance (so $p_1 + p_2 = 0$), gives us the relations

$$\Gamma^{11} - \sigma(\sigma f_2)^\alpha \Gamma^{12} - \sigma(\sigma f_1)^\alpha \Gamma^{21} + \Gamma^{22} \;=\; 0 \tag{6.59}$$

$$\Gamma^{11} - (\sigma f_2)^{-\bar{\alpha}} \Gamma^{12} - (\sigma f_1)^{-\bar{\alpha}} \Gamma^{21} + \Gamma^{22} \;=\; 0 \tag{6.60}$$

$$\Gamma^{11} - (\sigma f_2)^{-\bar{\alpha}} \Gamma^{12} \;=\; R_1^{(2)} \tag{6.61}$$

$$\Gamma^{11} - \sigma(\sigma_2 f_2)^\alpha \Gamma^{12} \;=\; \bar{R}_1^{(2)} \tag{6.62}$$

where f_j was defined in (3.18). These are the $N = 2$ versions of (5.53), (5.54), (5.56) and (5.58) where we have $\sigma_1 = \sigma_2 = \sigma$. The two-point retarded and advanced functions, $R_j^{(2)}$ and $\bar{R}_j^{(2)}$, are defined as functions of time by (4.34)

$$R_a^{(2)}(t_a, t_b) \;=\; \theta(t_a - t_b) Tr\{e^{-\beta H}[\phi_a(t_a), \phi_b(t_b)]_\sigma\}/Tr\{e^{-\beta H}\},$$
$$\tag{6.63}$$

$$\bar{R}_a^{(2)}(t_a, t_b) \;=\; -\theta(t_b - t_a) Tr\{e^{-\beta H}[\phi_a(t_a), \phi_b(t_b)]_\sigma\}/Tr\{e^{-\beta H}\},$$
$$\tag{6.64}$$

where $[A, B]_\sigma = AB - \sigma BA$. It is straightforward to invert these relations to give

$$\Gamma^{11} \;=\; (1 - f_1)^{-1}(R_1^{(2)} - f_1 \bar{R}_1^{(2)})$$
$$\Gamma^{12} \;=\; \sigma(\sigma f_1)^\alpha (1 - f_1)^{-1}(R_1^{(2)} - \bar{R}_1^{(2)})$$
$$\Gamma^{21} \;=\; (\sigma f_1)^{\bar{\alpha}} (1 - f_1)^{-1}(R_1^{(2)} - \bar{R}_1^{(2)})$$
$$\Gamma^{22} \;=\; (1 - f_1)^{-1}(f_1 R_1^{(2)} - \bar{R}^{(2)}{}_1) \tag{6.65}$$

Note that we only have factors of $e^{\beta p_j}$ as this is what comes directly from the cyclicity of the statistical trace (2.7) and the $e^{-\beta H}$ density matrix, as seen in (3.17). However, we now depart from the usual approach and summarise these relations directly in a 2×2 matrix relation using $e^{\beta p_j}$ *not* $e^{-\beta|p_j|}$. We find then

$$\Gamma^{\mu_1 \mu_2} = U_1^{\mu_1 \nu_1} U_2^{\mu_2 \nu_2} \bar{\Gamma}^{\nu_1 \nu_2} \tag{6.66}$$

where

$$\bar{\Gamma}^{\nu_1 \nu_2} = \begin{pmatrix} 0 & -R_1^{(2)} \frac{b_{L1}}{b_{L2}} s^{-1/2} \\ -\bar{R}_1^{(2)} \frac{b_{L2}}{b_{L1}} s^{1/2} & 0 \end{pmatrix} \tag{6.67}$$

$$U_j^{\mu\nu} = \begin{pmatrix} (1 + \sigma_j n_j)^{1/2}/b_{Lj} b_{Rj} & \sigma_j n_j^{1/2} b_{Lj}/b_{Rj} \\ n_j^{1/2} b_{Rj}/b_{Lj} & (1 + \sigma_j n_j)^{1/2} b_{Lj} b_{Rj} \end{pmatrix} \tag{6.68}$$

$$n_j = \frac{\sigma_j f_j}{1 - f_j} = (e^{\beta p_j} - \sigma_j)^{-1} \tag{6.69}$$

$$b_{Rj}^2 = exp\{-\beta p_j (\frac{1}{2} - \alpha)\} \tag{6.70}$$

$$s = \frac{n_1}{1 + \sigma_2 n_2} \tag{6.71}$$

The b_L parameters do not effect physical results and it has been included as the usual form for U quoted for the $\alpha \neq 1/2$ case involves a particular choice for b_L. The s parameter in $\bar{\Gamma}$ takes account (remembering that $\sigma_1 = \sigma_2$ in this two-point case) of the fact that there can be factors of $\imath$ from the square roots.

We can also rewrite these equations in a more familiar way, which is easier to use when we are considering two-point functions. Assuming that the two fields involved have the same statistics, $\sigma_1 = \sigma_2$, as they must for a non-trivial two-point function, we find that

$$\Gamma^{\mu_1 \mu_2} = U_1^{\mu_1 \nu_1} \bar{\Pi}^{\nu_1 \nu_2} (\bar{U}_1^{-1})^{\nu_2 \mu_2} \tag{6.72}$$

where

$$\bar{\Pi} = \begin{pmatrix} R_1^{(2)} & 0 \\ 0 & -\bar{R}_1^{(2)} \end{pmatrix} \tag{6.73}$$

$$\bar{U}_j^{-1} = \tau_3 U_j^{-1} \tau_3 \tag{6.74}$$

with τ_3 the usual Pauli spin matrix. All we used was

$$U_2^{\mu_2 \nu_2} = (\bar{U}_1^{-1})^{\lambda_2 \mu_2} \begin{pmatrix} 0 & s^{-\frac{1}{2}} b_{L1}/b_{L2} \\ -s^{\frac{1}{2}} b_{L2}/b_{L1} & 0 \end{pmatrix}^{\nu_2 \lambda_2}. \tag{6.75}$$

While (6.66) is similar to the form usually seen [2, 3, 4, 5, 22], it still involves $e^{\beta p}$ *not* $e^{-\beta|p|}$. We note that in this form the non-zero entries in the diagonal matrix $\bar{\Pi}$ are just the retarded and minus the advanced propagators. A form similar to (6.72) was also

noted in [21] though without the connection between $\bar{\Pi}$ and the retarded and advanced two-point functions being made.

To get usual form [2, 3, 4, 5, 22], which involves involves $e^{-\beta|p|}$ factors, we manipulate the negative energy part of (6.72) to give

$$\Gamma^{\mu_1\mu_2} = \hat{U}_1^{\mu_1\nu_1}\hat{\Pi}^{\nu_1\nu_2}(\bar{\hat{U}}_2^{-1})^{\nu_2\mu_2} \tag{6.76}$$

where

$$\hat{\Pi} = \frac{1}{2}\begin{pmatrix} R_1^{(2)}\theta(p) + \bar{R}_1^{(2)}\theta(-p) & 0 \\ 0 & -R_1^{(2)}\theta(-p) - \theta(p)\bar{R}_1^{(2)} \end{pmatrix}. \tag{6.77}$$

$$\hat{n} = (e^{\beta|p|} - \sigma_j)^{1/2} \tag{6.78}$$

Here $\hat{U}$ $(\bar{\hat{U}}^{-1})$ is defined precisely as U $(\bar{U}^{-1})$ except we replace n by $\hat{n}$ in (6.68) (so the b_R parameter remains as defined in (6.70)) and as a result we have the familiar result that $\lim_{\beta\to\infty}\hat{U} \to 1$. Note that in this form involving $e^{\beta|p|}$ we do *not* get the retarded and advanced functions, $R_1^{(2)}$, $\bar{R}_1^{(2)}$, appearing as the elements of the diagonal matrix $\hat{\Pi}$ in the middle of the representation. However the only difference is a change of sign in the imaginary parts for negative energies only, i.e.

$$\hat{\Pi} = Re\{\bar{\Pi}\} + \theta(p)Im\{\bar{\Pi}\} - \theta(-p)Im\{\bar{\Pi}\} \tag{6.79}$$

The direct connection between the retarded and advanced functions (the usual results obtained from ITF two-point calculations) and the entries of the core matrix in such RTF representations of propagators is possible *only* when one works in terms of n of (6.69) and U of (6.68), functions of the energy, not the modulus of energy. This accounts precisely for the sign difference at negative energies in the imaginary part of self-energies found by Kobes and Semenoff [19] when they compared the result of a two-point ITF calculation, i.e. the true retarded and advanced functions, with the matrix $\hat{\Pi}$ calculated from RTF results.

The essential elements of (6.66) are that for each field we have a factor U with parameters appropriate for that leg and then a core matrix, $\bar{\Gamma}^{\{\nu\}}$, that has zeros for the $\{\nu\} = 1$ entry and another zero for the $\{\nu\} = 2$ entry. These zeros correspond to the two relations that we always have between the RTF functions, equations (5.53) and (5.54). The other entries are then related to the retarded and advanced functions, i.e. represent the equations (5.56) and (5.58). We can try to find a similar matrix structure for the three-point functions. From these equations we have

$$\begin{aligned}
R_a^{(3)} = (1 - f_a)^{-1}\Bigg\{ & \Gamma^{111} - \sum_{\{b,c\}}\left[(\sigma_b f_b)^{-\bar{\alpha}}\Gamma\big|_{\substack{\mu_b=2 \\ \mu_a=\mu_c=1}}\right] \\
& -f_a(\sigma_a f_a)^{-\bar{\alpha}}\Gamma\big|_{\substack{\mu_a=2 \\ \mu_{rest}=1}} + (\sigma_a f_a)^{\bar{\alpha}}\Gamma\big|_{\substack{\mu_a=1 \\ \mu_{rest}=2}} \\
& + \sum_{\{b,c\}}\left[f_a(\sigma_b f_b)^{\bar{\alpha}}\Gamma\big|_{\substack{\mu_b=1 \\ \mu_a=\mu_c=2}}\right] - f_a\Gamma^{222}\Bigg\}
\end{aligned} \tag{6.80}$$

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$$\bar{R}_a^{(3)} = (1 - f_a^{-1})^{-1} \left\{ \Gamma^{111} - \sum_{\{b,c\}} \left[\sigma_b (\sigma_b f_b)^\alpha \Gamma \big|_{\substack{\mu_b=2 \\ \mu_a=\mu_c=1}} \right] \right.$$

$$- (\sigma_a f_a)^{-\bar{\alpha}} \Gamma \big|_{\mu_a=2 \mu_{rest}=1} + \sigma_a (\sigma_a f_a)^{-\alpha} \Gamma \big|_{\substack{\mu_a=1 \\ \mu_{rest}=2}}$$

$$\left. + \sum_{\{b,c\}} \left[\sigma_b f_a^{-1} (\sigma_b f_b)^{-\alpha} \Gamma \big|_{\substack{\mu_b=1 \\ \mu_a=\mu_c=2}} \right] - f_a^{-1} \Gamma^{222} \right\}$$

$$(6.81)$$

$$0 = \Gamma^{111} - \sum_a^3 \left[(\sigma_a f_a)^{-\bar{\alpha}} \Gamma \big|_{\substack{\mu_a=2 \\ \mu_{rest}=1}} \right]$$

$$+ \sum_{a=1}^3 \left[(\sigma_a f_a)^{\bar{\alpha}} \Gamma \big|_{\substack{\mu_a=1 \\ \mu_{rest}=2}} \right] - \Gamma^{222} \qquad (6.82)$$

$$0 = \Gamma^{111} - \sum_{a=1}^3 \left[\sigma_a (\sigma_a f_a)^\alpha \Gamma \big|_{\substack{\mu_a=2 \\ \mu_{rest}=1}} \right]$$

$$+ \sum_{a=1}^3 \left[\sigma_a (\sigma_a f_a)^{-\alpha} \Gamma \big|_{\substack{\mu_a=1 \\ \mu_{rest}=2}} \right] - \Gamma^{222}.$$

$$(6.83)$$

In the above the labels $a, b, c = 1, 2$, or 3, a is fixed and the sum over $\{b, c\}$ means taking the subscripts b, c over all values subject to $a \neq b \neq c$. In going from (5.56) and (5.58) to (6.80) and (6.81) where the latter now involve all the RTF functions, we have used (6.82) and (6.83). We have eight equations involving the eight RTF three-point functions and we can invert these expressions to find

$$\Gamma^{111}(k_1, k_2, k_3) = \sum_{\substack{a=1 \\ a \neq b,c; b<c}}^3 \sigma_a n_b n_c (R_a^{(3)} + \bar{R}_a^{(3)} f_a)$$

$$\Gamma^{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) \big|_{\substack{\mu_a=2 \\ \mu_{b,c}=1}} = (\sigma_a f_a)^{\bar{\alpha}} \left\{ R_a^{(3)} n_b n_c \sigma_a + R_b^{(3)} n_a n_c \sigma_b f_a^{-1} \right.$$

$$+ R_c^{(3)} n_a n_b \sigma_c f_a^{-1} + \bar{R}_a^{(3)} n_b n_c \sigma_a$$

$$\left. + \bar{R}_b^{(3)} n_a n_c \sigma_b f_b + \bar{R}_c^{(3)} n_a n_b \sigma_c f_c \right\}$$

$$\Gamma^{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) \big|_{\substack{\mu_a=1 \\ \mu_{b,c}=2}} = (\sigma_a f_a)^{-\bar{\alpha}} f_a \left\{ R_a^{(3)} n_b n_c \sigma_a + R_b^{(3)} n_a n_c \sigma_b f_b \right.$$

$$+ R_c^{(3)} n_a n_b \sigma_c f_c + \bar{R}_a^{(3)} n_b n_c \sigma_a$$

$$\left. + \bar{R}_b^{(3)} n_a n_c \sigma_b f_a^{-1} + \bar{R}_c^{(3)} n_a n_b \sigma_c f_a^{-1} \right\}$$

$$\Gamma^{222}(k_1, k_2, k_3) = \sum_{\substack{a=1 \\ a \neq b,c; b<c}}^3 \sigma_a n_b n_c (R_a^{(3)} f_a + \bar{R}_a^{(3)})$$

$$(6.84)$$

where n_a etc. were defined in (6.69) and here we always choose $a, b, c = 1, 2,$ or 3, $a \neq b, c$ and $b < c$. The $R_a^{(3)}$ and $\bar{R}_a^{(3)}$ are the six different expectation values of retarded and advanced three-point functions in Fourier space. The retarded functions in space-time are

$$
\begin{aligned}
R^{(3)}(t_1, t_2, t_3) = \\
\theta(t_a - t_b)\theta(t_b - t_c) \quad &Tr\{e^{-\beta H}[[\phi_a(t_a), \phi_b(t_b)]_\sigma, \phi_c(t_c)]_\sigma\}/Tr\{e^{-\beta H}\} \\
+\theta(t_a - t_c)\theta(t_b - t_b) \quad &Tr\{e^{-\beta H}[[\phi_a(t_a), \phi_c(t_c)]_\sigma, \phi_b(t_b)_\sigma]\}/Tr\{e^{-\beta H}\}
\end{aligned}
$$
(6.85)

where (abc) is any permutation of (123) and we are not summing over repeated indices. The commutators are given by $[A, B]_\sigma = AB \pm BA$ where we take the $+$ sign unless A and B are fermionic. The advanced function is the same, except we switch the arguments of all the theta functions, $\theta(t-t') \to \theta(t'-t)$. There is no overall sign change for three-point functions.

Motivated by (6.66), we can try to write the three-point function as

$$
\Gamma^{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = (\prod_{a=1}^{3} U_a(k_a)^{\mu_a\nu_a})\bar{\Gamma}^{\nu_1\nu_2\nu_3}(k_1, k_2, k_3),
$$
(6.86)

and it is straightforward to show that this is possible with

$$
\begin{aligned}
\bar{\Gamma}^{111}(k_1, k_2, k_3) &= 0, \\
\bar{\Gamma}^{\nu_1\nu_2\nu_3}(k_1, k_2, k_3)\Big|_{\substack{\nu_a=2 \\ \nu_{b,c}=1}} &= \bar{R}_a^{(3)}\frac{n_b^{1/2}n_c^{1/2}}{(1+\sigma_a n_a)^{1/2}}\sigma_a.\frac{b_{Lb}b_{Lc}}{b_{La}} \\
\bar{\Gamma}^{\nu_1\nu_2\nu_3}(k_1, k_2, k_3)\Big|_{\substack{\nu_a=1 \\ \nu_{b,c}=2}} &= R_a^{(3)}\frac{n_b^{1/2}n_c^{1/2}}{(1+\sigma_a n_a)^{1/2}}.\frac{b_{La}}{b_{Lb}b_{Lc}} \\
\bar{\Gamma}^{222}(k_1, k_2, k_3) &= 0.
\end{aligned}
$$
(6.87)

Thus we have a three-point spectral function representation in RTF (6.86) as well as a two-point representation (6.66).

For $N \geq 4$-point functions we see that the form (6.86) can always be written down. To do this we *define* the core matrix $\bar{\Gamma}$ through the relation

$$
\Gamma^{\mu_1\mu_2...\mu_N}(k_1, k_2, ..., k_N) = \\
(\prod_{a=1}^{N} U_a(k_a)^{\mu_a\nu_a})\bar{\Gamma}^{\nu_1\nu_2...\nu_N}(k_1, k_2, ..., k_N)
$$
(6.88)

It is simple to invert this relation and we then find that

$$
\bar{\Gamma}^{111..1} = \bar{\Gamma}^{222..2} = 0
$$
(6.89)

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because they are proportional to (5.53) or (5.54) respectively. The entries with only one index different from the others are known in terms of the expressions (5.56) and (5.58) for retarded and advanced functions. For instance we have

$$\bar{\Gamma}^{\nu_a=2,\nu_{rest}=1} = \frac{(\prod_{c=1}^{N}(1+\sigma_c n_c)^{1/2}b_{Lc})}{(1+\sigma_a n_a)^{1/2}(n_a)^{1/2}b_{la}^2}\sigma_a \bar{R}_a \tag{6.90}$$

While it is straightforward to find expressions for the other entries, simple counting arguments show they can not be in one-to-one correspondence with the other ITF functions. There are 2^N entries in $\bar{\Gamma}$. However there are only $2N$ retarded and advanced functions and just two identities, (5.53),(5.54), so for $N \geq 4$ so there is no simple one-to-one relation with these functions. In total there are far more than 2^N different results that can be calculated using ITF so again there is no obvious one-to-one correspondence.

We note that so far everything has been done in terms of expectation values of Heisenberg fields not 1PI functions. It is simple derive the 1PI relations and it allows us to find the generalisation of the results of Kobes [11]. We merely remove the full propagators from each leg using the form (6.66) for the propagators and we find

$$\Gamma_{1PI}^{\{\mu\}}(\{k\}) = (\prod_{a=1}^{N}\bar{U}_a(k_a)^{\mu_a \nu_a})\bar{\Gamma}_{1PI}^{\{\nu\}}(\{k\}) \tag{6.91}$$

where

$$\bar{\Gamma}_{1PI}^{\{\nu\}}(\{k\}) = \left(\prod_{a=1}^{N}\bar{\Pi}_a^{-1}(k_a)^{\nu_a \nu_a}\right)\bar{\Gamma}^{\{\nu\}}(\{k\}), \tag{6.92}$$

and $\bar{\Pi}$ was the diagonal matrix defined in (6.72). The $\bar{\Gamma}_{1PI}^{\{\nu\}}$ matrix has zeros for the entries when all the thermal indices, $\{\nu\}$, are 1 or when all are equal to 2, and these correspond to the 1PI version of the equations (5.53) and (5.54) when we invert (6.91). The entries in $\bar{\Gamma}_{1PI}^{\{\nu\}}$ with just one index different from the others gives retarded and advanced 1PI functions (with an overall factor similar to the connected function case (6.90)).

Finally, one can consider analytic continuations in ITF other than the simplest one of (3.28) that we have also referred to as the usual analytic continuation. In particular one might consider trying a Feynman prescription

$$z_a = p_a + \epsilon_a(\theta(p_a) - \theta(-p_a)) \tag{6.93}$$

in (3.22). The epsilon terms in the denominators $\{B\}$ of (3.22) would then look like the ones seen at zero temperature in the Feynman propagator which there we associate with time-ordered functions. However, by comparing (5.44) with (3.22), we see that this does *not* give the time-ordered functions, $\Gamma^{111\cdots 1}$, directly in ITF. There is *no* analytic continuation in ITF that gives the time-ordered expectation value directly. It also shows that the time-ordered functions (5.44) can *not* be written in terms of the spectral functions, $\rho_{a_1 \ldots a_N}$ of (3.25), as easily as the retarded, advanced and other functions found using ITF (3.23) can be. That the time-ordered function is not simply related to the ITF result

found using the analytic continuation (6.93) can be seen explicitly elsewhere for the case of the two-point functions, for instance in equation (3.1.19) of [5].

Since the analytic continuations in ITF do not lead directly to the time-ordered function, and it is the simplest one (3.27) that always allows one to calculate the retarded and advanced functions, the functions calculated with other analytic continuations in ITF are not as interesting (although we note the work of Kobes [15]). It seems best therefore to stick to the simplest analytic continuation (3.27) when using ITF.

7   Conclusions

There are two major results in this paper. First we have shown the precise connection between the different type of real-time Green functions and the usual results obtained in ITF and RTF, the former being a non-trivial problem. Secondly, the relationship between the different real-time Green functions themselves and thus the relationship between the usual results of ITF and RTF calculations was found. These results are important because the situation turns out to be rather more complicated for general N-point functions compared with the special cases of the two- and three-point functions discussed elsewhere, [1, 5, 6] and [7, 8, 9, 11, 12, 13, 14, 15] respectively.

The most important practical application of these results is the fact that there are large differences in the real and imaginary parts of the different Green functions in momentum space [7, 9] whereas at zero temperature there are only sign differences. Recent calculations of QCD three-point functions in ITF [23] and in RTF [24] are a good illustration of this. This means one must first know precisely which type of real-time Green function is required for a given problem. Only then can one decide how to use the calculational schemes of ITF and RTF in order to obtain the relevant function.

All that was used in the derivation of the results given here was the definition of the various types of full Green functions that appear in equilibrium finite temperature field theory. In particular, we only use their basic properties, such as the cyclicity of the trace and the form of the density matrix, that almost define what we mean by being in thermal equilibrium. Provided that we work in an approximation that gives results that have these same basic properties as the full Green functions, e.g. the general KMS condition (3.17), then these approximations to the full functions will also satisfy the same relations. Thus each individual diagram in a Feynman diagram expansion satisfies these relations. In a similar way, Green functions in non-path integral approaches such as TFD [3] also obey the same relations, even though the derivation of the results such as (3.17) is rather different. For instance, for the two-point case see [25].

We also note that all the calculations performed here make no assumptions about the particle nature of the system. This is just as well since it is clear that the usual role of particles in QFT that is the very basis of QFT is *not* applicable at finite temperature [26, 27], except perhaps as an approximation. The results derived here are valid even if we consider approaches that are more honest about the particle picture used as a basis for finite temperature field theory, such as that of Landsman [27].

The information that we are dealing with is purely thermal and this almost completely

decouples from the details of the theory we are dealing with. The so called 'spectral' representations of RTF are a good example of where the interesting matrix structure is entirely due to the thermal nature of the situations being considered. All the details of the theory, the particle nature, spin indices etc. are in the spectral functions ρ about which we have not made any assumptions. This is also why all the results derived are valid for all types of fields though we only had to account for a few minus signs associated with the different statistics of fermionic and bosonic fields.

Many of the relationships derived here have been checked in actual calculations. For instance in the rather tricky zero energy limit, general N-point one-loop RTF diagrams in scalar field theories, as discussed in [28], can be seen to satisfy (5.56). The method of Kobes and Semenoff [19] provides a useful calculational scheme and the recent application of these methods by Kobes [11, 12, 13, 15], where some of the relationships discussed here were also noted, has given many concrete examples of the relationships discussed in action. Recent one-loop calculations of Baier et al. [29] have also made use of these relations.

The approach used here is complementary to that used by Kobes [11, 12, 13, 15] in several ways. As noted above, the results obtained here are valid independent of the approximation being used while Kobes obtained many of his results in a perturbative approximation. Further, the approach used here is able to identify the precise function that certain sums of RTF Green functions give e.g. (5.56),(5.58) whereas Kobes' approach only identifies for what times such combinations are zero. However, in practice many calculations are perturbative so Kobes' approach is much closer to actual calculations. It also provides a practical way to test the results and shows how one could exploit the relationship between retarded functions and RTF in practice. The study of the properties and relations of the Green functions at finite temperature benefits greatly from having both methods available.

After this work was completed, I recieved the preprint by Aurenche and Becherrawy [30] which is relevant to some of the points discussed here.

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ϵ_1/ϵ	ϵ_2/ϵ	ϵ_3/ϵ	s_1	s_2	s_3	Result
2	-1	-1	+	-	-	$R_1^{(3)}$
-1	2	-1	-	+	-	$R_2^{(3)}$
-1	-1	2	-	-	+	$R_3^{(3)}$
-2	1	1	-	+	+	$\bar{R}_1^{(3)}$
1	-2	1	+	-	+	$\bar{R}_2^{(3)}$
1	1	-2	+	+	-	$\bar{R}_3^{(3)}$

Table 1: Some choices for $\{\epsilon\}$ illustrating all possible results for a three-point ITF real-time function when using the simple analytic continuation.

ϵ_1/ϵ	ϵ_2/ϵ	ϵ_3/ϵ	ϵ_4/ϵ	s_1	s_2	s_3	s_{12}	s_{13}	s_{23}	Result
-1	-1	-1	3	-	-	-	-	-	-	$R_4^{(4)}$
1	1	1	-3	+	+	+	+	+	+	$\bar{R}_4^{(4)}$
3	1	-2	-2	+	+	-	+	+	-	$Q_{12}^{(4)}$
1	3	-2	-2	+	+	-	+	-	+	$Q_{21}^{(4)}$
2	2	-3	-1	+	+	-	+	-	-	$\bar{Q}_{34}^{(4)}$
2	2	-1	-3	+	+	-	+	+	+	$\bar{Q}_{43}^{(4)}$

Table 2: Some distinct results for a four-point ITF real-time function with a set of example ϵ values.

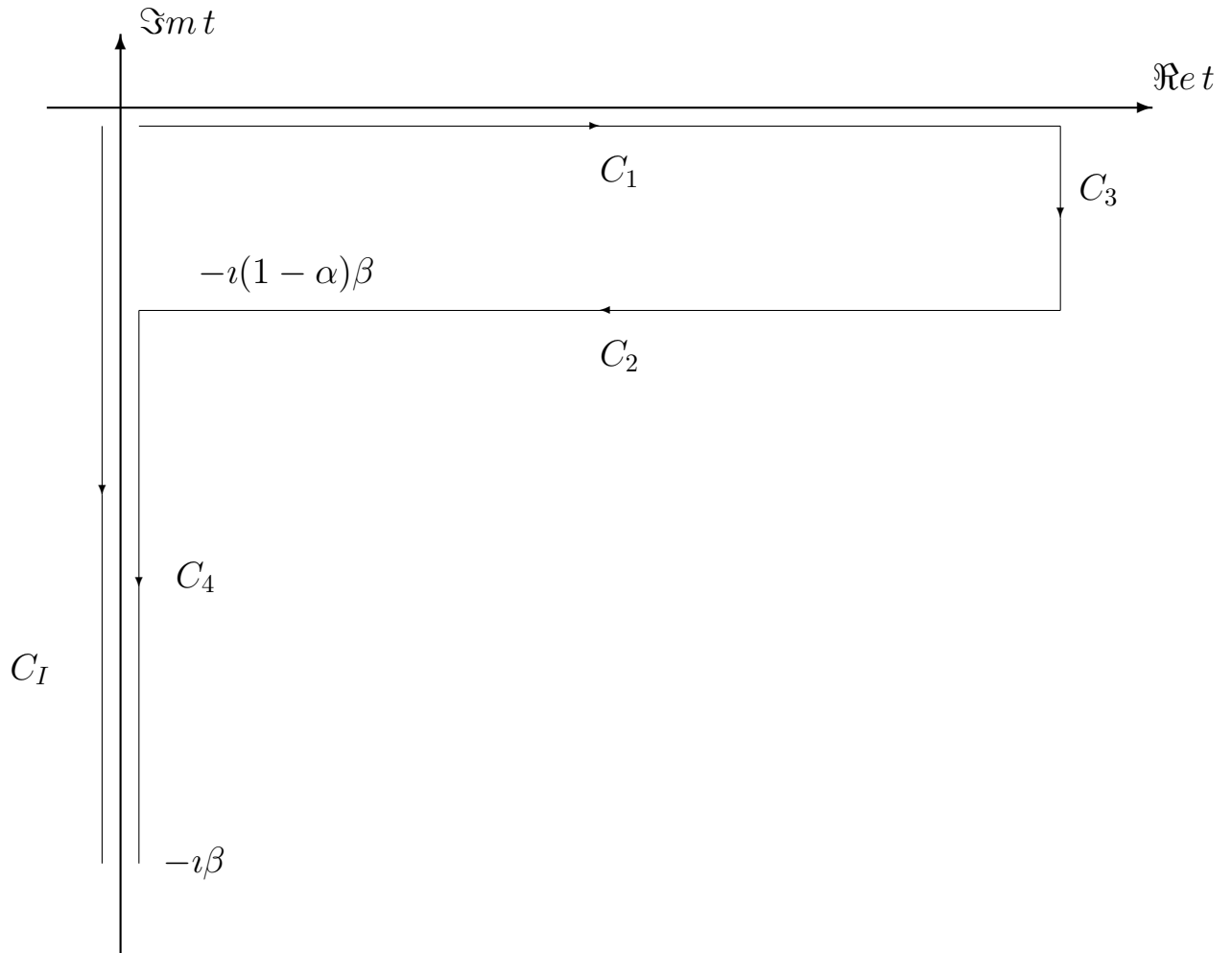| N	$R,\ \bar{R}$	$\Gamma^{\{\mu\}}$	$\Gamma_{\{a_i\}}$	ITF
2	2	4	2	2
3	6	8	6	6
4	8	16	24	32
5	10	32	120	370
6	12	64	720	11292
7	14	128	5040	1066044
8	16	256	40320	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$2N$	2^N	$N!$	?

Table 3: Numbers of N-point functions.
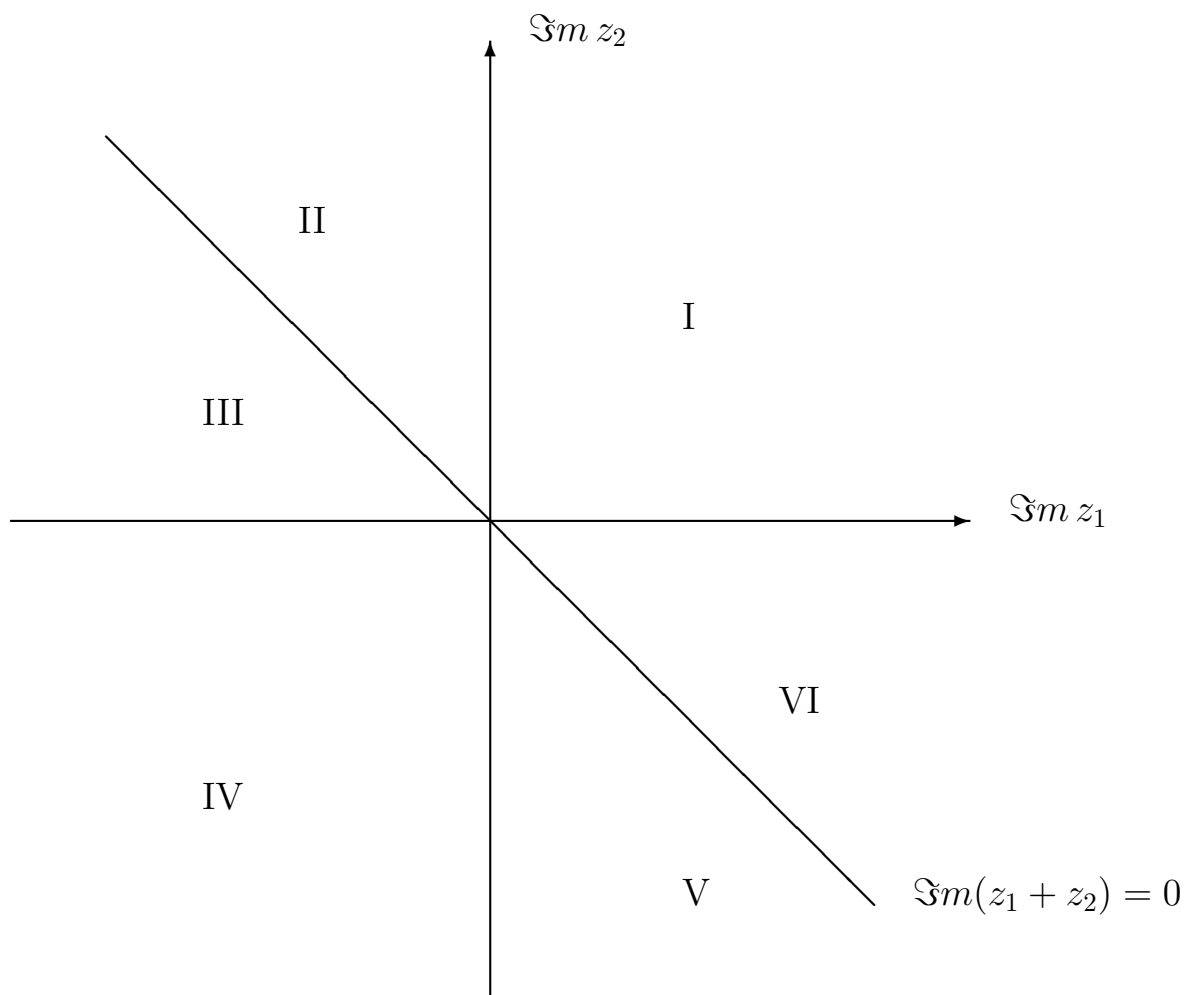
Figure 1: The paths used for RTF and ITF.

Figure 2: Regions of analyticity for the three-point ITF function Φ.