Complete Metric Spaces

Definition 1. Let (X, d) be a metric space. A sequence (x_n) in X is called a *Cauchy sequence* if for any $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for any $m \geq n_\varepsilon$, $n \geq n_\varepsilon$.

Theorem 2. Any convergent sequence in a metric space is a Cauchy sequence.

Proof. Assume that (x_n) is a sequence which converges to x. Let $\varepsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$. Let m, $n \in N$ be such that $m \geq N$, $n \geq N$. Then

$$d(x_m, x_n) \;\leq\; d(x_m, x) + d(x_n, x) \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (x_n) is a Cauchy sequence. □

Then converse of this theorem is not true. For example, let $X = (0, 1]$. Then $(\frac{1}{n})$ is a Cauchy sequence which is not convergent in X.

Definition 3. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges (to a point in X).

Theorem 4. A closed subset of a complete metric space is a complete subspace.

Proof. Let S be a closed subspace of a complete metric space X. Let (x_n) be a Cauchy sequence in S. Then (x_n) is a Cauchy sequence in X and hence it must converge to a point x in X. But then $x \in \overline{S} = S$. Thus S is complete. □

Theorem 5. A complete subspace of a metric space is a closed subset.

Proof. Let S be a complete subspace of a metric space X. Let $x \in \overline{S}$. Then there is a sequence (x_n) in S which converges to x (in X). Hence (x_n) is a Cauchy sequence in S. Since S is complete, (x_n) must converge to some point, say, y in S. By the uniqueness of limit, we must have $x = y \in S$. Hence $\overline{S} = S$, i.e. S is closed. □

Definition 6. Let A be a nonempty subset of a metric space (X, d). The *diameter* of A is defined to be

$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

We say that A is *bounded* if $\operatorname{diam}(A)$ is finite.

Theorem 7. Let (X, d) be a complete metric space. If (F_n) is a sequence of nonempty closed subsets of X such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and $(\operatorname{diam}(F_n))$ converges to 0, then $\bigcap_{n=1}^{\infty} F_n$ is a singleton.

Proof. Assume that (X, d) is a complete metric space. Let (F_n) be a sequence of nonempty closed subsets of X such that $F_{n+1} \subseteq F_n$ for every $n \in \mathbb{N}$ and $(\operatorname{diam}(F_n))$ converges to 0. First, we show that $\bigcap_{n=1}^{\infty} F_n$ is nonempty. For each $n \in \mathbb{N}$, choose $x_n \in F_n$. To show that (x_n) is a Cauchy sequence, let $\varepsilon > 0$. Since $(\operatorname{diam}(F_n))$ converges to 0, there is an $N \in \mathbb{N}$ such that $\operatorname{diam}(F_N) < \varepsilon$. Let m, $n \in \mathbb{N}$ with $n \geq N$, $m \geq N$. Then $x_m \in F_m \subseteq F_N$, $x_n \in F_n \subseteq F_N$. Hence

$$d(x_m, x_n) \leq \operatorname{diam}(F_N) < \varepsilon.$$

Thus (x_n) is a Cauchy sequence in X. Since X is complete, it follows that (x_n) is a convergent sequence in X. Let

$$x = \lim_{n \to \infty} x_n.$$

We now show that $x \in \bigcap_{n=1}^{\infty} F_n$. Let $n \in \mathbb{N}$. Note that $x_m \in F_m \subseteq F_n$ for any $m \geq n$. Thus the sequence $(x_n, x_{n+1}, \dots)$ is a sequence in F_n and is a subsequence of (x_n), so it converges to x. This implies that $x \in \bar{F}_n = F_n$. Thus $x \in \bigcap_{n=1}^{\infty} F_n$. Next, we show that $\bigcap_{n=1}^{\infty} F_n$ is a singleton. To see this, let x, $y \in \bigcap_{n=1}^{\infty} F_n$ and $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $\operatorname{diam}(F_N) < \varepsilon$. Since x, $y \in F_N$, it follows that $d(x, y) \leq \operatorname{diam}(F_N) < \varepsilon$. This is true for any $\varepsilon > 0$. Hence $d(x, y) = 0$, which means $x = y$. □

Definition 8. Let f be a function from a metric space (X, d) into a metric space (Y, ρ). We say that f is *uniformly continuous* if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any x, $y \in X$, $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$.

Theorem 9. A uniformly continuous function maps Cauchy sequences into Cauchy sequences.

Proof. Let $f\colon (X, d) \to (Y, \rho)$ be a uniformly continuous function. Let (x_n) be a Cauchy sequence in X. To see that $(f(x_n))$ is a Cauchy sequence, let $\varepsilon > 0$. Then there is a $\delta > 0$ such that

$$\forall\, x, y \in X,\ d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon.$$

Thus there exists an $N \in \mathbb{N}$ such that $d(x_m, x_n) < \delta$ for any m, $n \geq N$. It follows that $\rho(f(x_m), f(x_n)) < \varepsilon$ for any m, $n \geq N$. Hence $(f(x_n))$ is a Cauchy sequence in Y. □

Remark. If f is not uniformly continuous, then the theorem may not be true. For example, $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$ and $(x_n) = (\frac{1}{n})$ is a Cauchy sequence in $(0, \infty)$ but $(f(x_n)) = (n)$ is not a Cauchy sequence.

Definition 10. Let f be a function from a metric space (X, d) into a metric space (Y, ρ). We say that f is an *isometry* if $d(a, b) = \rho(f(a), f(b))$ for any a, $b \in X$.

Theorem 11. Let $f: (X, d) \to (Y, \rho)$ be an isometry. Then it is injective and uniformly continuous. Moreover, its inverse $f^{-1}: (f[X], \rho) \to (X, d)$ is also an isometry.

Proof. Let $f: (X, d) \to (Y, \rho)$ be an isometry. Let $\varepsilon > 0$. Choose $\delta = \varepsilon > 0$. Let a, $b \in X$ be such that $d(a, b) < \delta$. Then

$$\rho(f(a), f(b)) = d(a, b) < \delta = \varepsilon.$$

Hence f is uniformly continuous. Next, let a, $b \in X$ be such that $f(a) = f(b)$. Thus $d(a, b) = \rho(f(a), f(b)) = 0$. This shows that $a = b$. Hence f is injective. To see that f^{-1} is an isometry, let y, $z \in f[X]$ and let a, $b \in X$ be such that $f(a) = y$ and $f(b) = z$. Thus

$$\rho(y, z) = \rho(f(a), f(b)) = d(a, b) = d(f^{-1}(y), f^{-1}(z)).$$

Hence f^{-1} is an isometry. □

Theorem 12. Let A be a dense subset of a metric space (X, d). Let f be a uniformly continuous function (isometry) from A into a complete metric space (Y, ρ). Then there is a unique uniformly continuous function (isometry) g from X into Y which extends f.

Proof. We will give a proof only for a uniformly continuous function. The proof for an isometry is similar and somewhat easier.

Let (X, d) be a metric space and (Y, ρ) a complete metric space. Let A be a dense subset of X and let f be a uniformly continuous from A into Y.

Step 1: define a function $g: X \to Y$.

For each $x \in X = \overline{A}$, there is a sequence (x_n) in A which converges to x. Then (x_n) is a Cauchy sequence in X. Thus $(f(x_n))$ is a Cauchy sequence in Y. Since Y is complete, $(f(x_n))$ is a convergent sequence. Define

$$g(x) = \lim_{n\to\infty} f(x_n)$$

for any $x \in X$, where (x_n) is a sequence in A which converges to x.

Step 2: g is well-defined, i.e., independent of the choice of (x_n).

Let (x_n) and (y_n) be any sequence in A which converges to $x \in \overline{A} = X$. Then the sequence $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$ must converge to x. Hence, the sequence $(f(x_1), f(y_1), f(x_2), f(y_2), \ldots, f(x_n), f(y_n), \ldots)$ converges to some point $z \in Y$. Since $(f(x_1), f(x_2), \dots)$ and $(f(y_1), f(y_2), \dots)$ are its subsequences, they must also converge to z. Hence $z = g(x)$ does not depend on the choice of the sequences.

Step 3: g is an extension of f.

Let $a \in A$ and let $a_n = a$ for each $n \in \mathbb{N}$. Then (a_n) is a sequence in A which converges to a. Hence $g(a) = \lim_{n\to\infty} f(a_n) = f(a)$. This shows that g is an extension of f.

Step 4: g is uniformly continuous on X.

Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that

$$\forall\, a, b \in A,\ d(a, b) < \delta \Rightarrow \rho(f(a), f(b)) < \frac{\varepsilon}{3}.$$

Let x, $y \in X$ be such that $d(x, y) < \delta$. Then there are sequence (x_n) and (y_n) in A such that $x_n \to x$ and $y_n \to y$. Hence $f(x_n) \to g(x)$ and $f(y_n) \to g(y)$. Choose $N \in \mathbb{N}$ such that

$$d(x_N, x) < \frac{\delta - d(x, y)}{2} \quad \text{and} \quad d(y_N, y) < \frac{\delta - d(x, y)}{2} \tag{0.1}$$

$$\text{and} \quad \rho(f(x_N), g(x)) < \frac{\varepsilon}{3} \quad \text{and} \quad \rho(f(y_N), g(y)) < \frac{\varepsilon}{3}. \tag{0.2}$$

Then

$$d(x_N, y_N) \le d(x_N, x) + d(x, y) + d(y, y_N) < \delta,$$

which implies that $\rho(f(x_N), f(y_N)) < \varepsilon$, by the uniform continuity of f on A. Hence,

$$\begin{aligned} \rho(g(x), g(y)) &\le \rho(g(x), f(x_N)) + \rho(f(x_N), f(y_N)) + \rho(f(y_N), g(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This shows that g is uniformly continuous on X.

Step 5: g is unique.

Let g and h be (uniformly) continuous functions on X which extends f on a dense subset A. To see that $g = h$, let $x \in X$. Then there is a sequence (x_n) in A which converges to x. By continuity of g and h,

$$g(x) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = h(x).$$

Hence $g = h$. □

Definition 13. A *completion* of a metric space (X, d) is a pair consisting of a complete metric space (X^*, d^*) and an isometry φ of X into X^* such that $\varphi[X]$ is dense in X^*.

Theorem 14. Every metric space has a completion.

Proof. Let (X, d) be a metric space. Denote by $\mathcal{C}[X]$ the collection of all Cauchy sequences in X. Define a relation $\sim$ on $\mathcal{C}[X]$ by

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

It is easy to see that this is an equivalence relation on $\mathcal{C}[X]$. Let X^* be the set of all equivalence classes for $\sim$:

$$X^* = \{ [(x_n)] : (x_n) \in \mathcal{C}[X] \}.$$

Define $d^*\colon X^* \times X^* \to [0, \infty)$ by

$$d^*([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n),$$

where $[(x_n)]$, $[(y_n)] \in X^*$. To show that d^* is well-defined, let (x_n') and (y_n') be two Cauchy sequences in X such that $(x_n) \sim (x_n')$ and $(y_n) \sim (y_n')$. Then

$$\lim_{n\to\infty} d(x_n, x_n') = \lim_{n\to\infty} d(y_n, y_n') = 0.$$

By the triangle inequality,

$$\begin{aligned} d(x_n, y_n) &\le d(x_n, x_n') + d(x_n', y_n') + d(y_n', y_n) \quad \text{and} \\ d(x_n', y_n') &\le d(x_n', x_n) + d(x_n, y_n) + d(y_n, y_n'). \end{aligned}$$

Hence,

$$|d(x_n, y_n) - d(x_n', y_n')| \le d(x_n, x_n') + d(y_n, y_n') \longrightarrow 0.$$

Since both $(d(x_n, y_n))$ and $(d(x_n', y_n'))$ are convergent, this shows that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n').$$

Thus d^* is well-defined.

Next, we show that d^* is a metric on X^*. Let $[(x_n)]$, $[(y_n)]$, $[(z_n)] \in X^*$. Then

$$d^*([(x_n)], [(y_n)]) = 0 \Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0 \Leftrightarrow (x_n) \sim (y_n) \Leftrightarrow [(x_n)] = [(y_n)].$$

Also,

$$d^*([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = d^*([(y_n)], [(x_n)]).$$

Since $d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$,

$$\lim_{n\to\infty} d(x_n, z_n) \le \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n).$$

Thus

$$d^*([(x_n)],[(z_n)]) \;\le\; d^*([(x_n)],[(y_n)]) + d^*([(y_n)],[(z_n)]).$$

Hence d^* is a metric on X^*.

For each $x \in X$, let $\hat{x} = [(x, x, \dots)] \in X^*$, the equivalence classes of the constant sequence $(x, x, \dots)$. Define $\varphi\colon X \to X^*$ by $\varphi(x) = \hat{x}$. Then for any x, $y \in X$,

$$d^*(\varphi(x), \varphi(y)) = d^*(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Hence φ is an isometry from X into X^*. To show that $\varphi[X]$ is dense in X^*, let $x^* = [(x_n)] \in X^*$ and let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for any m, $n \ge N$, $d(x_m, x_n) < \frac{\varepsilon}{2}$. Let $z = x_N$. Then $\hat{z} \in \varphi[X]$ and

$$d^*(x^*, \hat{z}) \;=\; \lim_{n\to\infty} d(x_n, z) \;=\; \lim_{n\to\infty} d(x_n, x_N) \;\le\; \frac{\varepsilon}{2} \;<\; \varepsilon.$$

Thus $\hat{z} \in B_{d^*}(x^*, \varepsilon) \cap \varphi[X]$. Hence, $\varphi[X]$ is dense in X^*.

Finally we show that (X^*, d^*) is complete. To establish this, we apply the following lemma of which proof is left as an exercise:

Lemma 15. Let (X, d) be a metric space and A a dense subset such that every Cauchy sequence in A converges in X. Prove that X is complete.

Hence, it suffices to show that every Cauchy sequence in the dense subspace $\varphi[X]$ converges in X^*. Let $(\widehat{z}_k)$ be a Cauchy sequence in $\varphi[X]$, where each $\widehat{z}_k$ is represented by the Cauchy sequence $(z_k, z_k, \dots)$. Since φ is an isometry,

$$d(z_n, z_m) \;=\; d^*(\widehat{z}_n, \widehat{z}_m) \qquad \text{for each } m,\ n.$$

Hence, $(z_1, z_2, z_3, \dots)$ is a Cauchy sequence in X. Let $z^* = [(z_1, z_2, z_3, \dots)] \in X^*$. To show that $(\widehat{z}_k)$ converges to z^*, let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $d(z_k, z_n) < \frac{\varepsilon}{2}$ for any k, $n \ge N$. Hence, for each $k \ge N$,

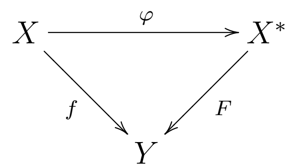
$$d^*(\widehat{z}_k, z^*) \;=\; \lim_{n\to\infty} d(z_k, z_n) \;\le\; \frac{\varepsilon}{2} \;<\; \varepsilon.$$

This shows that $(\widehat{z}_k)$ converges to a point z^* in X^* and that X^* is complete. $\square$

Before proving that a completion of a metric space is unique up to isometry, we will give an alternative definition of a completion in terms of a universal mapping property. We state this fact in the following definition:

Theorem 16 (Universal Mapping Property). Let (X, d) be a metric space, (X^*, d^*) a complete metric space and $\varphi\colon (X, d) \to (X^*, d^*)$ an isometry. Then $\varphi[X]$ is dense in X^* if and only if it satisfies the following universal mapping property:

Given any complete metric space (Y, ρ) and an isomtry $f: X \to Y$, there exists a unique isometry $F: X^* \to Y$ such that $F \circ \varphi = f$.



Proof. Assume that $\varphi[X]$ is dense in X^*. We will show that it satisfies the universal mapping property. Let (Y, ρ) be a complete metric space and $f: X \to Y$ an isometry from X into Y. Since φ is an isometry, φ is 1-1. Thus $\varphi^{-1}: \varphi[X] \to X$ is an isometry from $\varphi[X]$ onto X. Since f is an isometry from X onto Y, it follows that $h := f \circ \varphi^{-1}$ is an isometry from a dense subset $\varphi[X]$ of X^* into a complete metric space Y. Hence it can be extended uniquely into an isometry F from X^* into Y. Then for any $x \in X$

$$F \circ \varphi(x) = h(\varphi(x)) = f \circ \varphi^{-1}(\varphi(x)) = f(x).$$

Thus $F \circ \varphi = f$.

If $G: X^* \to Y$ is another isometry such that $G \circ \varphi = f$, then $F = G$ on the dense subset $\varphi[X]$ of X^*; hence they must be equal on X^*, i.e. $F = G$.

Conversely, assume that it satisfies universal mapping property and show that $\varphi[X]$ is dense in X^*. Since $\overline{\varphi[X]}$ is closed in a complete metric space X^*, $\overline{\varphi[X]}$ is also complete. Let $Y = \overline{\varphi[X]}$ and $f = \varphi$. By the universal mapping property, there is a unique isometry $F: X^* \to Y$ such that $F \circ \varphi = f = \varphi$. This shows that F is the identity on the subspace $\varphi[X]$. It implies that F is the identity on $\overline{\varphi[X]}$. From this, we must have $X^* = \overline{\varphi[X]}$. □

Theorem 17. A completion of a metric space is unique up to isometry. More precisely, if $\{\varphi_1, (X_1^*, d_1^*)\}$ and $\{\varphi_2, (X_2^*, d_2^*)\}$ are two completions of (X, d), then there is a unique isometry f from X_1^* onto X_2^* such that $f \circ \varphi_1 = \varphi_2$.

Proof. First, letting $Y = X_2^*$ and $f = \varphi_2$, by the universal mapping property of (X_1^*, d_1^*) there is a unique isometry $F: X_1^* \to X_2^*$ such that $F \circ \varphi_1 = \varphi_2$. Similarly, letting $Y = X_1^*$ and $f = \varphi_1$, by the universal mapping property of (X_2^*, d_2^*) there is a unique isometry $G: X_2^* \to X_1^*$ such that $G \circ \varphi_2 = \varphi_1$.

Hence, $G \circ F$ is an isometry on X_1^* such that $G \circ F \circ \varphi_1 = \varphi_1$. But then the identity map $I_{X_1^*}$ is an isometry on X_1^* such that $I_{X_1^*} \circ \varphi_1 = \varphi_1$. By the uniqueness property, we have $G \circ F = I_{X_1^*}$. By the same argument, we can show that $F \circ G = I_{X_2^*}$. This shows that F and G are inverses of each other. Thus F is an isometry from (X_1^*, d_1^*) onto (X_2^*, d_2^*). □

Definition 18. A function $f : (X, d) \to (X, d)$ is said to be a *contraction map* if there is a real number $k < 1$ such that $d(f(x), f(y)) \leq k\, d(x, y)$ for all x, $y \in X$.

Theorem 19. Let f be a contraction map of a complete metric space (X, d) into itself. Then f has a unique fixed point.

Proof. Let (X, d) be a complete metric space and $f \colon X \to X$ a d-contractive map. Then there is a real number $k < 1$ and $d(f(x), f(y)) \le kd(x, y)$ for all x, y in X. Fix $x_0 \in X$ and let $x_n = f(x_{n-1})$ for each $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x_1, x_2) &= d(f(x_0), f(x_1)) \le k\, d(x_0, x_1), \\ d(x_2, x_3) &= d(f(x_1), f(x_2)) \le k\, d(x_1, x_2) \le k^2 d(x_0, x_1). \end{aligned}$$

Assume that $d(x_{m-1}, x_m) \le k^{m-1} d(x_0, x_1)$. Then

$$d(x_m, x_{m+1}) = d(f(x_{m-1}), f(x_m)) \le k\, d(x_{m-1}, x_m) \le k^m d(x_0, x_1).$$

By induction, we have $d(x_m, x_{m+1}) \le k^m d(x_0, x_1)$ for each $m \in \mathbb{N}$. Hence, for any m, $n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\le k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1) \\ &\le (k^n + k^{n+1} + k^{n+2} + \cdots)\, d(x_0, x_1) \\ &= \frac{k^n}{1-k}\, d(x_0, x_1). \end{aligned}$$

Since $0 \le k < 1$, the sequence (k^n) converges to 0. This implies that (x_n) is a Cauchy sequence. Since X is complete, the sequence (x_n) is convergent to, say, x in X. To show that $f(x) = x$, let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for any $n \ge N$. Hence $d(f(x_N), x) = d(x_{N+1}, x) < \frac{\varepsilon}{2}$. Therefore

$$\begin{aligned} d(f(x), x) &\le d(f(x), f(x_N)) + d(f(x_N), x) \\ &\le k\, d(x, x_N) + d(x_{N+1}, x) < k\frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $f(x) = x$. Now we show that the fixed point is unique. Let x and y be fixed points of f. Then $d(x, y) = d(f(x), f(y)) \le k\, d(x, y)$. If $d(x, y) > 0$, then $1 \le k$, a contradiction. Thus $d(x, y) = 0$. This shows that $x = y$. Hence, f has a unique fixed point. □

Theorem 20 (Baire's Theorem). If X is a complete metric space, the intersection of a countable number of dense open subsets is dense in X.

Proof. In the proof of this theorem, we will denote the closed ball centered at x with radius r by $B[x, r]$:

$$B[x, r] = \{y \in X \mid d(y, x) \le r\}.$$

Note that any open set in a metric space contains a closed ball. Indeed, if we shrink the radius of an open ball slightly, we obtain a closed ball contained in that open ball.

Suppose that $V_1, V_2, \dots$ are dense and open in X and let W be a nonempty open set in X. We will show that

$$\Big(\bigcap_{n=1}^{\infty} V_n\Big) \cap W \neq \varnothing.$$

Since V_1 is dense in X, $W \cap V_1$ is a nonempty open set. Hence, we can find $x_1 \in X$ and $0 < r_1 < 1$ such that

$$B[x_1, r_1] \subseteq W \cap V_1. \tag{0.3}$$

If $n \geq 2$ and x_{n-1} and r_{n-1} are chosen, the denseness of V_n shows that $V_n \cap B(x_{n-1}, r_{n-1})$ is a nonempty open set, and therefore we can find $x_n \in X$ and $0 < r_n < \frac{1}{n}$ such that

$$B[x_n, r_n] \subseteq V_n \cap B(x_{n-1}, r_{n-1}). \tag{0.4}$$

By induction, this process produces the sequence $\{x_n\}$ in X. If $m, n \geq N$, then x_m and x_n are in $B(x_N, r_N)$, and thus

$$d(x_m, x_n) \leq d(x_m, x_N) + d(x_N, x_n) < 2\, r_N < \frac{2}{N}.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, x_n converges to some $x \in X$. If $k \geq n$, then x_k lies in a closed set $B[x_n, r_n]$. Thus, $x \in B[x_n, r_n]$ for all $n \geq 1$. By (0.3), $x \in W \cap V_1$, and by (0.4), we have $x \in V_n$ for all $n \geq 2$. Hence,

$$x \in \Big(\bigcap_{n=1}^{\infty} V_n\Big) \cap W.$$

We can conclude that the intersection of all V_n is dense in X. □

Remark. The completeness assumption is necessary in this theorem as the following example illustrates. Let $X = \mathbb{Q}$. Write $\mathbb{Q} = \{\, r_n \mid n \in \mathbb{N}\,\}$ and let $G_n = \mathbb{Q} - \{r_n\}$ for each $n \in \mathbb{N}$. Then G_n is open and dense in $\mathbb{Q}$ for each n, but $\bigcap_{n=1}^{\infty} G_n = \varnothing$.

Corollary 21. If a complete metric space is a union of countably many closed sets, then at least one of the closed sets has nonempty interior.

Proof. Let X be a complete metric space. Assume that $X = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed. For each $n \in \mathbb{N}$, let $G_n = F_n^c$. Then $\bigcap_{n=1}^{\infty} G_n = \varnothing$. By Baire's theorem, there exists an open set G_n which is not dense in X. Thus, $\overline{G}_n \neq X$. But then $\operatorname{Int} F_n = X - \overline{G}_n$, and hence F_n has nonempty interior. □