Math 7410: Lie Combinatorics and Hyperplane Arrangements

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Cornell University
Fall 2016

Last updated January 9, 2017.
The latest version is online here.

Contents

Contents by Lecture

Administrative

There is a website with an indefinite schedule and some information.

For the first few weeks I will focus on combinatorics, but an algebraic approach to it. This is the topic of species. There won't be any prerequisites to this portion for a few weeks. Later, we will discuss hyperplane arrangements, which involves discrete geometry.

1 Species and Enumeration

Definition 1.1. A **species** is a functor $\mathbf{set}^{\times}$ to **Set**, where $\mathbf{set}^{\times}$ is the category of finite sets and bijections and **Set** is the category of all sets and all maps. A morphism of species is a natural transformation of functors.

Let P be a species. For each finite set I, we have a set $P[I]$. For each bijection $\sigma\colon I \to J$, we have a map $\sigma^*\colon P[I] \to P[J]$. Give $I \xrightarrow{\sigma} J \xrightarrow{\tau} K$, we have

$$\begin{tikzcd}
P[I] \arrow[rr, "\sigma^*"] \arrow[dr, "(\tau\sigma)^*"'] & & P[J] \arrow[dl, "\tau^*"] \\
& P[K] &
\end{tikzcd}$$

Also, $\mathrm{id}_I^* = \mathrm{id}_{P[I]}$. Note that each σ^* is invertible, and $(\sigma^*)^{-1} = (\sigma^{-1})^*$.

It also follows that each set $P[m]$ (see Remark 1.2) is acted upon by the symmetric group S_m. The action is $\sigma \cdot x = \sigma^*(x)$, for $\sigma \in S_m$, $x \in P[m]$. In particular, $\sigma^*(x) \in P[m]$.

Remark 1.2 (Convention). $[m] := \{1, 2, 3, \dots, m\}$. We write $P[m] = P[\{1, 2, \dots, m\}]$.

This action of the symmetric group allows us to reinterpret a species as a collection $\{P[m]\}_{m\geq 0}$ of S_m-sets. This uniquely determines P.

A **morphism of species** $f\colon P \to Q$ consists of maps $f_I\colon P[I] \to Q[I]$, one for each finite set I, such that for any bijection $\sigma\colon I \to J$,

$$\sigma^* f_I(x) = f_J(\sigma^*(x))$$

for all $x \in P[I]$. That is, the diagram below commutes.

$$\begin{tikzcd}
P[I] \arrow[r, "f_I"] \arrow[d, "\sigma^*"] & Q[I] \arrow[d, "\sigma^*"] \\
P[J] \arrow[r, "f_J"] & Q[J]
\end{tikzcd}$$

So in the end, a species is not actually that much. It's just a collection of S_m-sets. We want to use it to do some enumeration, and some algebra as well, just as we use finite groups to encode combinatorial information.

Definition 1.3. The **species** L **of linear orders** is defined on a set I as the set of all linear orders on I.

$$L[I] = \{\text{linear orders on I}\}.$$

Example 1.4. For example,

$$L[a,b,c] = \{abc, bac, acb, bcacab, cba\}.$$

If we have the bijection $\sigma = \begin{pmatrix} a & b & c \\ y & z & x \end{pmatrix}$, then

$$\sigma^* = \begin{pmatrix} abc & bac & acb & bca & cab & cba \\ yzx & zyx & yxz & \dots & & \end{pmatrix}.$$

Definition 1.5. A **partition** X of a set I is a collection of disjoint nonempty subsets of I whose union is I. The notation $X \vdash I$ means X is a partition of I.

Definition 1.6. The **species** Π **of set partitions** is the species determined by

$$\Pi[I] = \{\text{partitions } X \text{ of I}\}$$

Definition 1.7. A **composition** F of a set I is a totally ordered partition of I.

Definition 1.8. The **sepcies** Σ **of set compositions** is the species determined by

$$\Sigma[I] = \{\text{compositions } F \text{ of I}\}.$$

Example 1.9. If $I = \{a,b,c,d\}$, with a partition $X = \{\{a,c\},\{b\},\{d\}\}$, the following two composition aren't the same but have the same underlying partition X.

$$(\{a,c\},\{b\},\{d\}) \neq (\{b\},\{a,c\},\{d\})$$

There are morphisms

$$L \to \Sigma \to \Pi,$$

where the first morphism is viewing a linear order as a composition into singletons and the second just forgets the order.

In fact, we have

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, two heads] & \Sigma \arrow[d, two heads] \\ E \arrow[r, hook] & \Pi \end{tikzcd}$$

where E is the **exponential species**, defined by $E[I] = \{*_I\}$. That is, $E[I]$ always has a single element, denoted $*_I$.

Remark 1.10. In combinatorics, one is interested in the cardinality of a set. When we talk about species, we get a generating function instead.

Definition 1.11. Given a species P, it's **generating function** is

$$P(x) = \sum_{n \geq 0} \#P[m] \frac{x^m}{m!} \quad \in \mathbb{Q}[[x]].$$

This says that species are a categorification of power series where we replace numbers by sets.

Example 1.12.

$$L(x) = \sum_{n \geq 0} \#L[n] \frac{x^n}{n!} = \sum_{n \geq 0} n! \frac{x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

$$E(x) = \sum_{n \geq 0} \#E[n] \frac{x^n}{n!} = \sum_{n \geq 0} \frac{x^n}{n!} = e^x$$

$\Pi(x)$ is the generating function for the number of set partitions and $\Sigma(x)$ is the generating function for the number of ordered set partitions.

Definition 1.13. Given species P and Q, their **Cauchy Product** $P \cdot Q$ is the species defined by

$$(P \cdot Q)[I] = \coprod_{I = S \sqcup T} P[S] \times Q[T]$$

where the disjoint union is taken over all ordered decompositions of I (order of S and T matters) such that S and T partition I.

On bijections (which are arrows in $\mathbf{set}^{\times}$), the Cauchy product acts as follows. Given $\sigma \colon I \to J$,

$$\begin{tikzcd}
(P \cdot Q)[I] \arrow[d, dashed, "\sigma^*"] & \supseteq & P[S] \times Q[T] \arrow[d, "(\sigma_S^*) \times (\sigma_T)^*"] \\
(P \cdot Q)[J] & \supseteq & P[\sigma(S)] \times P[\sigma(T)]
\end{tikzcd}$$

Example 1.14. $E \cdot E$ is the species of subsets.

$$(E \cdot E)[I] = \coprod_{I = S \sqcup T} E[S] \times E[T] \cong \{S \colon S \subseteq I\}$$

Example 1.15. $E^{\cdot k}$ is the k-fold Cauchy product. This is the species of functions to $[k]$.

$$(E^{\cdot k})[I] = \{f \colon I \to [k] \mid f \text{function}\}$$

$$(E^{\cdot k})[I] = \coprod_{I = S_1 \sqcup \ldots \sqcup S_k} E[S_1] \times \ldots \times E[S_k]$$

To see that these two are the same, notice that $S_i = f^{-1}(i)$ for each $i \in [k]$.

Proposition 1.16. *The generating series for the Cauchy product is the product of power series in* $\mathbb{Q}[[x]]$.

$$(P \cdot Q)(x) = P(x)Q(x)$$

Proof. First, notice that

$$\#(P \cdot Q)[n] = \#\left(\coprod_{i+j=n} P[i] \times Q[j]\right) = \sum_{i+j=n} \#P[i]\#Q[j]$$

Therefore,

$$\begin{aligned}(P \cdot Q)(x) &= \sum_{n\geq 0} \#(P \cdot Q)[n]\frac{x^n}{n!} \\ &= \sum_{n\geq 0}\sum_{i+j=n} \#P[i]\#Q[j]\frac{x^n}{n!} \\ &= \left(\sum_{n\geq 0} \#P[n]\frac{x^n}{n!}\right)\left(\sum_{n\geq 0} \#Q[n]\frac{x^n}{n!}\right) = P(x)Q(x)\end{aligned}$$

□

Example 1.17.

$$(E \cdot E)(x) = E(x)E(x) = e^{2x} = \sum_{n\geq 0} 2^n \frac{x^n}{n!}.$$

This is a proof of the fact that a set with n elements has 2^n subsets, since $E \cdot E$ is the species of subsets, so now we know that $\#(E \cdot E)[n] = 2^n$.

Example 1.18.

$$(E^{\cdot k})(x) = e^{kx}$$

This proves that the number of functions $[m] \to [k]$ is k^m.

Definition 1.19. Let B be the species of bijections from a set I to itself, and D the species of **derrangements** of I. (A derrangement is a bijection without fixed points).

Claim 1.20. $B = E \cdot D$

Proof. We have

$$B[I] \cong \coprod_{I=S\sqcup T} E[S] \times D[T],$$

where the map is $\sigma \mapsto (S = \mathrm{Fix}(\sigma), \sigma|_T)$, where $T = I \setminus S$. □

Now using the fact that the Cauchy product corresponds to the product of generating functions, we get that $B(x) = E(x)D(x)$, and therefore

$$\frac{1}{1-x} = e^x D(x) \implies D(x) = \frac{e^{-x}}{1-x}$$

So we have derived the generating function for derrangements.

$$\#D[m] = m! \sum_{i=0}^{m} \frac{(-1)^i}{i!}.$$

Remark 1.21. We can see that

$$\#L[m] = m! = \#B[m].$$

This begs the question: is L isomorphic to B? The answer is no; they are not isomorphic because there is no canonical way to identify bijections on I with orders on I unless the set I comes with a order already. Here is a proof of this fact.

Proof. Let S_n act on $L[I]$ by relabelling. This action has only one orbit.

Let S_n act on $B[I]$ by relabelling. The number of orbits is the same as the number of cycle types of the bijections, which is the number of partitions of #I.

So L and B are not the same species. □

1.1 Substitution

Definition 1.22. Given species P and Q, with $Q[\emptyset] = \emptyset$, their **substitution** $P \circ Q$ is defined by

$$(P \circ Q)[I] = \coprod_{X \vdash I} \left(P[X] \times \prod_{S \in X} Q[S] \right)$$

This looks strange, but it has a nice consequence for generating functions.

Proposition 1.23. *When* $Q(0) = 0$*,*

$$(P \circ Q)(x) = P(Q(x)).$$

Proof. Exercise. □

The point of the next definition is to make species we have amenable to substitutions.

Definition 1.24. Given a species P, let P_+ be the species defined by

$$P_+[I] = \begin{cases} P[I] & I \neq \emptyset \\ \emptyset & I = \emptyset \end{cases}$$

Example 1.25.

$$E \circ E_+ \cong \Pi$$

This gives us the generating function for Π.

$$\Pi(x) = e^{e^x - 1}$$

Example 1.26.

$$L \circ E_+ \cong \Sigma$$

This gives us the generating function for Σ.

$$\Sigma(x) = \frac{1}{2 - e^x}$$

Definition 1.27. Let A be the species of rooted trees,

$$A[I] = \{\text{rooted trees with vertex set } I\}.$$

Recall that a rooted tree is a connected acyclic graph with a chosen vertex.

Let $\vec{A}$ be the species of planar rooted trees (that is, rooted trees with a linear order on the set of children of each node).

Exercise 1.28. Prove that

(a) $A = X \cdot (E \circ A)$ and $\vec{A} = X \cdot (L \circ \vec{A})$

(b) $A(x) = xe^{A(x)}$ and $\vec{A}(x) = \frac{x}{1 - \vec{A}(x)}$

(c) $\vec{A}(x) = \frac{1 - \sqrt{1-4x}}{2}$ and $\vec{A}[m] = m!C_{m-1}$

where X is the species defined by

$$X[I] = \begin{cases} \{*\} & \#I = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

(Note that $X(x) = x$) and C_n is the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

2 Monoids in Species

Remark 2.1 (Idea)**.** A species M is a monoid if it carries an operation which is associative and unital. If M were a set, this would be the definition of a usual monoid.

Definition 2.2. A **monoid in the category of species** is a species M with an associative, unital operation.

By an operation, we mean a morphism of species $\mu\colon M \cdot M \to M$. This is a collection of maps $(M \cdot M)[I] \to M[I]$ for each finite set I. So we have a collection of maps

$$\mu\colon \coprod_{I=S\sqcup T} M[S] \times M[T] \to M[I].$$

The map μ has components $\mu_{ST}\colon M[S] \times M[T] \to M[I]$. For $x \in M[S]$ and $y \in M[T]$, we write $\mu_{ST}(x, y) = x \cdot y$.

Now, associativity simply means that whenever we have a set I with a partition into three pieces $I = R \sqcup S \sqcup T$ and $x \in M[R]$, $y \in M[S]$, $z \in M[T]$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

We can also describe what it means to be unital. There is $1 \in M[\emptyset]$ such that $1 \cdot x = x = x \cdot 1$ for every $x \in M[I]$ for every I.

We need one more condition. For any bijection $\sigma\colon I \to J$, with $I = S \sqcup T$, $x \in M[S]$ and $y \in M[T]$, we must have $\sigma^*(x \cdot y) = \sigma|_S^*(x)\sigma|_T^*(y)$.

Definition 2.3. A monoid is **commutative** if $x \cdot y = y \cdot x$ for all $x \in M[S]$, $y \in M[T]$, $I = S \sqcup T$.

Definition 2.4. A morphism of monoids $f\colon M \to N$ is a morphism of species if

(a) $f_I(x \cot y) = f_S(x) \cdot f_T(y)$

(b) $f_\emptyset(1) = 1$.

Example 2.5. Recall that we had the species

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, two heads] & \Sigma \arrow[d, two heads] \\ E \arrow[r, hook] & \Pi \end{tikzcd} \tag{1}$$

These are all monoids, and all of these morphisms are morphisms of monoids. L, Σ are noncommutative and E, Π are commutative.

How is L a monoid? Well, we define

$$\begin{array}{rccc} \mu_{S,T}\colon & L[S] \times L[T] & \to & L[I] \\ & (\ell_1, \ell_2) & \mapsto & \ell_1\ell_2 \end{array}$$

by concatenating the two orders. For example, if $I = \{a, b, c, d, e\}$, and $\ell_1 = adc$ and $\ell_2 = be$, then $\ell_1 \ell_2 = adcbe$.

We make E into a monoid in the only possibly way

$$\begin{array}{rccc} \mu_{S,T}\colon & E[S] \times E[T] & \to & E[I] \\ & (*_S, *_T) & \mapsto & *_{ST} \end{array}$$

We make Σ into a monoid by

$$\begin{array}{rccc} \mu_{S,T}\colon & \Sigma[S] \times \Sigma[T] & \to & E[I] \\ & (F, G) & \mapsto & F \cdot G \end{array}$$

where F is a composition of S and G is a composition of T, and $F \cdot G$ is the concatenation of compositions. If $F = (S_1, \ldots, S_k)$ and $G = (T_1, \ldots, T_h)$, then $F \cdot G = (S_1, \ldots, S_k, T_1, \ldots, T_h)$.

We make Π into a monoid by

$$\begin{array}{rccc} \mu_{S,T}\colon & \Pi[S] \times \Pi[T] & \to & \Pi[I] \\ & (X, Y) & \mapsto & X \cup Y \end{array}$$

If X is a partition of S and Y a partition of T, then we get a partition of I by taking the union of the two partitions.

The arrows in the diagram (1) are now morphisms of monoids.

2.1 Free Monoids

Definition 2.6. A species is **positive** if $Q[\emptyset] = \emptyset$.

Definition 2.7. Given a positive species Q, let $\mathcal{T}(Q) := L \circ Q$. (This is the substitution operation we defined last time.) This will be the **free monoid on** Q**.**

What does this look like? We have

$$\begin{aligned} \mathcal{T}(Q)[I] &= \coprod_{X \vdash I} \left(L[X] \times \prod_{S \in X} Q[S] \right) \\ &= \left\{ (F, x) \;\middle|\; F \in \Sigma[I], F = (S_1, \ldots, S_k), x = (x_1, \ldots, x_k), x_i \in Q[S_i] \right\}. \end{aligned}$$

We make $\mathcal{T}(Q)$ into a monoid by

$$\begin{array}{ccc} \mathcal{T}(Q)[S] \times \mathcal{T}(Q)[T] & \to & T(Q)[I] \\ ((F, x), (G, y)) & \mapsto & (F \cdot G, (x, y)) \end{array}$$

where we concatenate the two compositions and the two tuples x and y. By (x, y) we mean $(x_1, \ldots, x_k, y_1, \ldots, y_h)$.

The unit is $((), ())$, where the first $()$ is the empty composition of $\emptyset$ and the second $()$ is the empty sequence of Q-structures.

There is a morphism of species $Q \to \mathcal{T}(Q)$ given by

$$\begin{array}{rcl} Q[I] & \to & \mathcal{T}(Q)[I] \\ x & \mapsto & ((I), x) \end{array}$$

where (I) is the composition of I with one block.

Proposition 2.8. *Given a monoid* M *and a morphism of species* $f\colon Q \to M$*, there is a unique morphism of monoids* $\widehat{f}\colon \mathcal{T}(Q) \to M$ *such that*

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, hook] & M \\ \mathcal{T}(Q) \arrow[ur, dashed, "\widehat{f}"'] & \end{tikzcd}$$

Proof. We only need to define $\widehat{f}$. So for a set I, define $\widehat{f}_I\colon \mathcal{T}(Q)[I] \to M[I]$ by

$$(F, x) \mapsto f_{S_1}(x_1) \cdot \cdots \cdot f_{S_k}(x_k).$$

Note that each $f_{S_i}(x_i) \in M[S_i]$, and associativity allows us to omit the parentheses. $\square$

Exercise 2.9. Let Q be as before, let $\mathcal{S}(Q) := E \circ Q$.

$$\mathcal{S}(Q)[I] = \left\{ (X, x) \;\middle|\; X \in \Pi[I], x = (x_S)_{S \in X}, x_S \in Q[S] \right\}.$$

Show that $\mathcal{S}(Q)$ is the free commutative monoid on Q.

Exercise 2.10. What is the generating function for the free monoid $\mathcal{T}(Q)$ on Q? What about $\mathcal{S}(Q)$.

Example 2.11.

- $\Sigma \cong \mathcal{T}(E_+) = L \circ E_+$
- $\Pi \cong \mathcal{S}(E_+)$.
- $L = \mathcal{T}(X)$, where X is the species concentrated on singletons,

$$X[I] = \begin{cases} \{*\} & \text{if I is a singleton} \\ \emptyset & \text{otherwise} \end{cases}$$

- $E = \mathcal{S}(X) = E \circ X$.

Exercise 2.12. If X is the species concentrated on singletons, then $P \circ X \cong P \cong X \circ P$ for any species P.

This means that L is the free monoid on X, and E is the free commutative monoid on X. Compare with the free monoid on one element, which is $\mathbb{N}$. This is also the free commutative monoid on one generator, but this is not the case in the category of species.

2.2 Comonoids in Species

Next time, we will investigate monoids and comonoids in monoidal categories and see what the underlying thread is. We are trying right now to do things by hand, but this may be confusing at the moment. That's okay. It will all make sense soon.

Definition 2.13. A **comonoid** is a species C together with maps

$$\begin{array}{rccc} \Delta_{ST}\colon & C[I] & \to & C[S] \times C[T] \\ & z & \mapsto & (z|_S, z/_S) \end{array}$$

for each I and for each $I = S \sqcup T$.

Terminology: $z|_S$ is the **restriction of** z **to** S and $z/_S$ is the **contraction of** S **from** z.

The maps Δ_{ST} must be **coassociative**: whenever $I = R \sqcup S \sqcup T$, and $z \in C[I]$,

$$(z|_{R\sqcup S})|_R = z|_R \in C[R]$$

$$(z|_{R\sqcup S})/_R = (z/_R)|_S \in C[S]$$

$$(z/_R)/_S = z/_{R\sqcup S} \in C[T]$$

The structure should also be **counital**: for any $z \in C[I]$,

$$z|_I = z = z/_\varphi.$$

Finally, the naturality of the map Δ_{ST} is captures in the following. For any $\sigma\colon I \to J$ a bijection, and any $z \in C[I]$, then

$$\sigma^*(z)|_{\sigma(S)} = \sigma^*_S(z|_S)$$

$$\sigma^*(z)/_{\sigma(S)} = \sigma|^*_T(z/_S)$$

Definition 2.14. A comonoid is **cocommutative** (in the category of species) if $z|_S = z/_T$ whenever $z \in C[I]$ and $I = S \sqcup T$.

Definition 2.15. A **morphism of comonoids** is a morphism of species $f\colon C \to D$ such that

(a) $f_I(z)|_S = f_S(z|_S)$

(b) $f_I(z)/_S = f_T(z/_S)$.

Remark 2.16 (Motivation)**.** Associativity of monoid operations can be expressed in terms of the commutativity of the following diagram.

$$\begin{tikzcd} M[R] \times M[S] \times M[T] \arrow[r, "\mathrm{id}\times\mu_{S,T}"] \arrow[d, "\mu_{R,S}\times\mathrm{id}"] & M[R] \times M[S \sqcup T] \arrow[d, "\mu_{R,S\sqcup T}"] \\ M[R \sqcup S] \times M[T] \arrow[r, "\mu_{R\sqcup S,T}"] & M[I] \end{tikzcd}$$

Coassociativity can be expressed by turning all of the arrows in the above diagram around.

$$\begin{tikzcd} C[I] \arrow[r, "\Delta_{R,S\sqcup T}"] \arrow[d, "\Delta_{R\sqcup S,T}"] & C[R] \times C[S \sqcup T] \arrow[d, "\mathrm{id}\times\Delta_{S,T}"] \\ C[R \sqcup S] \times C[T] \arrow[r, "\Delta_{R,S}\times \mathrm{id}"] & C[R] \times C[S] \times C[T] \end{tikzcd}$$

Similarly, we an write commutativity as

$$\begin{tikzcd} M[S] \times M[T] \arrow[rr, "\text{switch}"] \arrow[dr, "\mu_{S,T}"'] & & M[T] \times M[S] \arrow[dl, "\mu_{T,S}"] \\ & M[I] & \end{tikzcd}$$

and cocommuativity as

$$\begin{tikzcd} C[S] \times C[T] \arrow[rr, "\text{switch}"] & & C[T] \times C[S] \\ & C[I] \arrow[ul, "\Delta_{S,T}"] \arrow[ur, "\Delta_{T,S}"'] & \end{tikzcd}$$

Example 2.17. L, Σ, E, Π are all cocommutative comonoids, with operations

$$\begin{array}{rcl} L[I] & \xrightarrow{\Delta_{S,T}} & L[S] \times L[T] \\ \ell & \mapsto & (\ell|_S, \ell|_T) \end{array}$$

with $\ell|_S$, $\ell|_T$ induced by ℓ on S.

$$\begin{array}{rcl} E[I] & \xrightarrow{\Delta_{S,T}} & E[S] \times E[T] \\ *_I & \mapsto & (*_S, *_T) \end{array}$$

$$\begin{array}{rcl} \Sigma[I] & \xrightarrow{\Delta_{S,T}} & \Sigma[S] \times \Sigma[T] \\ F & \mapsto & (F|_S, F|_T) \end{array}$$

If $F = (S_1, \ldots, S_k)$, then $F|_S = (S_1 \cap S, \ldots, S_k \cap S)$ with empty intersections removed.

Remark 2.18. We have the dual notions of monoids and comonoids in species. Monoids in the category of species are an elaboration on the idea of monoids in sets. But what are comonoids in the category of sets?

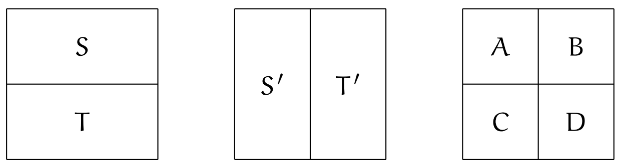
This is a set C with a map $\Delta\colon C \to C \times C$. This must be coassociative and counital. Counital means that $\pi_1\Delta(x) = x$ and $\pi_2\Delta(x) = x$, where π_1 and π_2 are the projections $C \times C \to C$. This means that in **Set**, each object has a unique comonoid structure that is given by the diagonal, so they're not that interesting to study.

Let B be a species that is both a monoid and a comonoid. This means that it has multiplication and comultiplication maps (where $I = S \sqcup T$).

$$\begin{array}{rccc} \mu_{S,T}\colon & B[S] \times B[T] & \to & B[I] \\ & (x, y) & \mapsto & x \cdot y \end{array}$$

$$\begin{array}{rccc} \Delta_{S,T}\colon & B[I] & \to & B[S] \times B[T] \\ & z & \mapsto & (z|_S, z/_S) \end{array}$$

Definition 2.19. We say that B is a **bimonoid** if the following holds for any $I = S \sqcup T = S' \sqcup T'$. Let A, B, C, D denote the resulting intersections $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, $D = T \cap T'$.



Then we should have that for any $x \in B[S]$, $y \in B[T]$,

$$(x \cdot y)|_{S'} = x|_A \cdot y|_C \qquad (x \cdot y)/_{S'} = x/_A \cdot y/_C$$

Example 2.20. Recall that we have

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, two heads] & \Sigma \arrow[d, two heads] \\ E \arrow[r, hook] & \Pi \end{tikzcd}$$

These are all bimonoids, and moreover they are cocommutative. Given $\ell_1 \in L[S]$, $\ell_2 \in L[T]$, we can check that

$$(\ell_1 \cdot \ell_2)|_{S'} = \ell_1|_A \cdot \ell_2|_C$$

3 Monoidal Categories

We're going to set aside species for now and talk about monoidal categories. This framework will make it easier to talk about species and give a general definition of monoids, comonoids, and bimonoids.

Remark 3.1 (Idea)**.** A monoidal category is a category with an operation that is associative and unital up to coherent isomorphism.

Definition 3.2. A monoical category consists of the following data:

- A category **C**,
- A functor $\bullet\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, called the **tensor product** or **monoidal product**,
- an object I of **C** called the **unit object**.
- natural isomorphisms α, λ, ρ

$$\begin{aligned} \alpha_{A,B,C}\colon (A \bullet B) \bullet C &\xrightarrow{\sim} A \bullet (B \bullet C) \\ \lambda_A\colon A &\xrightarrow{\sim} I \bullet A \\ \rho_A\colon A &\xrightarrow{\sim} A \bullet I \end{aligned}$$

that satisfy the axioms

$$\begin{tikzcd} & (A \bullet B) \bullet (C \bullet D) \arrow[dr, "\alpha_{A,B,C\bullet D}"] & \\ ((A \bullet B) \bullet C) \bullet D \arrow[ur, "\alpha_{A\bullet B,C,D}"] \arrow[d, "\alpha_{A\bullet B\bullet C}\bullet \mathrm{id}_D"] & & A \bullet (B \bullet (C \bullet D)) \\ (A \bullet (B \bullet C)) \bullet D \arrow[rr, "\alpha_{A,B\bullet C,D}"] & & A \bullet ((B \bullet C) \bullet D) \arrow[u, "\mathrm{id}_A \bullet \alpha_{B,C,D}"'] \end{tikzcd}$$

$$\begin{tikzcd} (A \bullet I) \bullet B \arrow[rr, "\alpha_{A,I,B}"] & & A \bullet (I \bullet B) \\ & A \bullet B \arrow[ul, "\rho_A \bullet \mathrm{id}_B"] \arrow[ur, "\mathrm{id}_A \bullet \lambda_B"'] & \end{tikzcd}$$

What if we have 5 objects? There are then 24 ways to parenthesize them, so we'd then need to draw a diagram that is an associahedron and check that it commutes. But it turns out that the two axioms above suffice. This is the statement of the Coherence Theorem.

Theorem 3.3 (Coherence Theorem)**.** *All diagrams built from only* α, λ, ρ *and* id *in a "free" monoidal category commute.*

In practice, this means for us that we can pretend that the monoidal product is associative and has a unit I. We will mostly ignore appearances of α, λ, and ρ.

Example 3.4.

(1) $\mathbf{C} = \mathbf{Set}$. The monoidal structure is the Cartesian product of sets, $X \times Y$. We pick a particular one-element set $\{*\}$ to be the monoidal unit.

(2) $\mathbf{C} = \mathbf{Vec}_k$, the category of vector spaces over a field k. Then the monoidal structure is the tensor product over k, $V \otimes_k W$. The unit object is k itself. Here, $\alpha\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is the canonical isomorphism.

(3) Let G be a group and k a field. Fix a normalized 3-cocycle $\phi\colon G^3 \to k^\times$. This means that ϕ satisfies the equations

$$\phi(g_2, g_3, g_4)\phi(g_1, g_2g_3, g_4)\phi(g_1, g_2, g_3) = \phi(g_1g_2, g_3, g_4)\phi(g_1, g_2, g_3g_4)$$

$$\phi(1, g, h) = \phi(g, 1, h) = \phi(g, h, 1) = 1$$

for all $g_1, g_2, g_3, g_4, g, h \in G$.

Let $\mathbf{C}$ be the category of G-graded vector spaces. The objects of $\mathbf{C}$ are collections $V = (V_g)_{g\in G}$ where each V_g is a k-vector space. The morphisms $f\colon V \to W$ are collections $f = (f_g)_{g\in G}$ where each $f_g\colon V_g \to W_g$ is a linear transformation.

The monoidal product of two objects V, W of $\mathbf{C}$ is given in components by

$$(V \bullet W)_g = \bigoplus_{g=xy} (V_x \otimes_k W_y)\,.$$

The monoidal unit I is

$$I = \begin{cases} k & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$$

The associativity constraint $\alpha_{U,V,W}\colon (U \bullet V) \bullet W \to U \bullet (V \bullet W)$ has components

$$\begin{tikzcd}
((U \bullet V) \bullet W)_g \arrow[r, "(\alpha_{U,V,W})_g"] \arrow[d, equal] & (U \bullet (V \bullet W))_g \arrow[d, equal] \\
\bigoplus_{g=xyz} (U_x \otimes V_y \otimes W_z) & \bigoplus_{g=xyz} (U_X \otimes V_y \otimes W_z) \\
U_x \otimes V_y \otimes W_z \arrow[u, hook] \arrow[r] & U_x \otimes V_y \otimes W_z \arrow[u, hook] \\
u \otimes v \otimes w \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & \phi(x, y, z) u \otimes v \otimes w \arrow[u, phantom, "\Cup"]
\end{tikzcd}$$

Definition 3.5. Let $\mathbf{C}$ be a monoidal category with monoidal unit I and monoidal structure $\bullet$. A triple (M, μ, ι) is a called a **monoid** in $\mathbf{C}$ if

(a) M is an object in $\mathbf{C}$,

(b) $\mu\colon M \bullet M \to M$ is a morphism in **C**

(c) $\iota\colon I \to M$ is a morphism in **C**.

such that the following diagrams commute.

$$\begin{tikzcd}
M \bullet M \bullet M \arrow[r, "\mu\bullet \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \bullet \mu"] & M \bullet M \arrow[d, "\mu"] \\
M \bullet M \arrow[r, "\mu"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
I \bullet M \arrow[r, "\iota\bullet \mathrm{id}_M"] \arrow[dr, equal, "\lambda"'] & M \bullet M \arrow[d, "\mu"] & M \bullet I \arrow[l, "\mathrm{id}_M \bullet \iota"'] \arrow[dl, equal, "\rho"] \\
& M &
\end{tikzcd}$$

Definition 3.6. A **morphism of monoids** $f\colon M \to N$ is a morphism in **C** such that

$$\begin{tikzcd}
M \bullet M \arrow[r, "f\bullet f"] \arrow[d, "\mu_M"] & N \bullet N \arrow[d, "\mu_N"] \\
M \arrow[r, "f"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[rr, "f"] & & N \\
& I \arrow[ul, "\iota_M"] \arrow[ur, "\iota_N"'] &
\end{tikzcd}$$

Example 3.7.

(1) In **Set**, with monoidal structure $\times$, the monoids are ordinary monoids.

(2) In $\mathbf{Vec}_k$, monoids are k-algebras.

(3) When **C** is the category of G-graded vector spaces, as above, and the cocycle ϕ is trivial (meaning $\phi(g,h,k) = 1$), then the monoids are G-graded k-algebras. If $G = \mathbb{Z}_2$, then these are called **superalgebras**.

Definition 3.8. A triple (C, Δ, ε) is a **comonoid** in **C** if

(a) C is an object of **C**

(b) $\Delta\colon C \to C \bullet C$ is a morphism of **C**

(c) $\varepsilon\colon C \to I$ is a morphism of **C**

these must satisfy the dual axioms to Definition 3.5.

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \bullet C \arrow[d, "\Delta\bullet \mathrm{id}_C"] \\
C \bullet C \arrow[r, "\mathrm{id}_C \bullet C"] & C \bullet C \bullet C
\end{tikzcd}
\qquad
\begin{tikzcd}
I \bullet C \arrow[dr, equal, "\lambda"'] & C \bullet C \arrow[l, "\varepsilon\bullet \mathrm{id}_C"'] \arrow[r, "\mathrm{id}_C \bullet \varepsilon"] & C \bullet I \arrow[dl, equal, "\rho"] \\
& C \arrow[u, "\Delta"] &
\end{tikzcd}$$

Likewise, a **morphism of comonoids** is defined dually to Definition 3.6.

Example 3.9.

(1) In $(\mathbf{Set}, \times)$, every object has a unique comonoid structure. What is this structure? Well, we have $\Delta\colon C \to C \times C$ and $\varepsilon\colon C \to \{*\}$. Write $\Delta(x) = (\Delta_1(x), \Delta_2(x))$ for functions $\Delta_1, \Delta_2\colon C \to C$. The counit axiom says that

$$x = (\varepsilon \times \mathrm{id})(\Delta(x)) = (*, \Delta_2(x)),$$

so it must be that $\Delta_2(x) = x$, and likewise $\Delta_1(x) = x$. So Δ must be the diagonal map.

(2) In $(\mathbf{Vec}_k, \otimes)$, comonoids are by definition k**-coalgebras**.

(3) In G-graded vector spaces, comonoids are G-graded coalgebras when ϕ is a trivial cocycle.

3.1 Convolution Monoids

Definition 3.10. Let $\mathbf{C}$ be a monoidal category and M a monoid in $\mathbf{C}$, C a comonoid in $\mathbf{C}$. Then given $f, g \in \mathrm{Hom}_{\mathbf{C}}(C, M)$, define the **convolution product** $f * g \in \mathrm{Hom}_{\mathbf{C}}(C, M)$ by

$$f * G\colon C \xrightarrow{\Delta} C \bullet C \xrightarrow{f \bullet g} M \bullet M \xrightarrow{\mu} M.$$

Further, define $u \in \mathrm{Hom}_{\mathbf{C}}(C, M)$ by

$$u\colon C \xrightarrow{\varepsilon} I \xrightarrow{\iota} M.$$

Proposition 3.11. $\mathrm{Hom}_{\mathbf{C}}(C, M)$ *is an ordinary monoid under convolution.*

Proof. First, we want to check that the convolution product $*$ is associative. Take $f, g, h \in \mathrm{Hom}_{\mathbf{C}}(C, M)$; we want to know if $f * (g * h)$ is equal to $(f * g) * h$. This follows from the commutativity of the following diagram.

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[rrrddd, bend left, "(f*g)*h"] \arrow[rrrddd, bend right, "f*(g*h)"'] & C \bullet C \arrow[d, "\Delta\bullet\mathrm{id}"] \arrow[rrdd, "(f*g)\bullet h"] & & \\
C \bullet C \arrow[r, "\mathrm{id}\bullet\Delta"] \arrow[rrdd, "f\bullet(g*h)"] & C \bullet C \bullet C \arrow[rd, "f\bullet g\bullet h"] & & \\
& & M \bullet M \bullet M \arrow[r, "\mu\bullet\mathrm{id}"] \arrow[d, "\mathrm{id}\bullet\mu"] & M \bullet M \arrow[d, "\mu"] \\
& & M \bullet M \arrow[r, "\mu"] & M
\end{tikzcd}$$

Similarly, we can check that u is a unit for this monoid. $\square$

Definition 3.12. Given monoids M, N and comonoids C, D, and a morphism of monoids $\phi\colon M \to N$, let

$$\begin{array}{rccc} \phi_{\#}\colon & \mathrm{Hom}_{\mathbf{C}}(C, M) & \to & \mathrm{Hom}_{\mathbf{C}}(C, N) \\ & f & \mapsto & \phi \circ f \end{array}$$

Similarly, given a morphism of comonoids $\psi\colon C \to D$, let

$$\begin{array}{rccc} \psi^{\#}\colon & \mathrm{Hom}_{\mathbf{C}}(D, M) & \to & \mathrm{Hom}_{\mathbf{C}}(C, M) \\ & f & \mapsto & f \circ \psi \end{array}$$

Proposition 3.13. *$\phi_{\#}$ and $\psi^{\#}$ are morphisms of monoids.*

Proof. Observe that the following diagram commutes. This shows that $\phi_{\#}(f * g) = \phi_{\#}(f) * \phi_{\#}(g)$

$$\begin{tikzcd}
 & & M \bullet M \arrow[r, "\mu"'] \arrow[d, "\phi\bullet\phi"] & M \arrow[d, "\phi"] \\
C \arrow[r, "\Delta"] \arrow[rrru, bend left, "f*g"] \arrow[rrrd, bend right, "\phi_{\#}(f)*\phi_{\#}(g)"'] & C \bullet C \arrow[ru, "f\bullet g"] \arrow[rd, "(\phi\circ f)\bullet(\phi\circ g)"'] & & \\
 & & N \bullet N \arrow[r, "\mu"] & N
\end{tikzcd}$$

□

4 Braided Monoidal Categories

To talk about bimonoids in general, we need to work in the slightly more specific setting of Braided monoidal categories. This is also the setting in which we can talk about commutative monoids and cocommutative comonoids. The braiding is an extra structure on monoidal categories that lets us switch the two tensor factors. Essentially, a braided monoidal category is a monoidal category that is commutative up to the coherence axiom.

Definition 4.1. A **braided monoidal category** consists of

(a) A monoidal category $(\mathbf{C}, \bullet, I)$

(b) A natural isomorphism β, called the **braiding**

$$\beta_{A,B}\colon A \bullet B \to B \bullet A.$$

These data must satisfy the following axioms.

$$\begin{tikzcd}
A \bullet B \bullet C \arrow[rr, "\beta_{A,B\bullet C}"] \arrow[dr, "\beta_{A,B}\bullet \mathrm{id}_C"'] & & B \bullet C \bullet A \\
& B \bullet A \bullet C \arrow[ur, "\mathrm{id}_B \bullet \beta_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet B \bullet C \arrow[rr, "\beta_{A\bullet B,C}"] \arrow[dr, "\mathrm{id}_A \bullet \beta_{B,C}"'] & & C \bullet A \bullet B \\
& A \bullet C \bullet B \arrow[ur, "\beta_{A,C}\bullet \mathrm{id}_B"'] &
\end{tikzcd}$$

Definition 4.2. A **symmetric monoidal category** is a braided monoidal category such that $\beta_{B,A} \circ \beta A, B = \mathrm{id}_{A \bullet B}$.

Proposition 4.3. *In a braided monoidal category, the following diagrams commute.*

$$\begin{tikzcd}
A \bullet I \arrow[rr, "\beta_{A,I}"] & & I \bullet A \\
& A \arrow[ul, "\rho_A"] \arrow[ur, "\lambda_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \bullet A \arrow[rr, "\beta_{I,A}"] & & A \bullet I \\
& A \arrow[ul, "\lambda_A"] \arrow[ur, "\rho_A"'] &
\end{tikzcd}$$

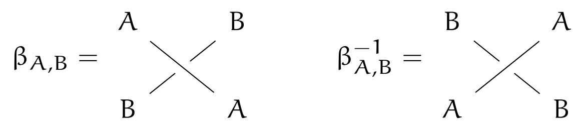
$$\begin{tikzcd}
& A \bullet B \bullet C \arrow[dl, "\beta_{A,B}\bullet \mathrm{id}_C"'] \arrow[dr, "\mathrm{id}_A \bullet \beta_{B,C}"] & \\
B \bullet A \bullet C \arrow[d, "\mathrm{id}_B \bullet \beta_{A,C}"] & & A \bullet C \bullet B \arrow[d, "\beta_{A,C}\bullet \mathrm{id}_B"] \\
B \bullet C \bullet A \arrow[dr, "\beta_{B,C}\bullet \mathrm{id}_A"'] & & C \bullet A \bullet B \arrow[dl, "\mathrm{id}_C \bullet \beta_{A,B}"] \\
& C \bullet B \bullet A &
\end{tikzcd}$$

Proof. The first two aren't hard; we will only show the hexagon to illustrate how to apply naturality of β.

$$\begin{tikzcd}
& A \bullet B \bullet C \arrow[dl, "\beta_{A,B}\bullet \mathrm{id}_C"'] \arrow[dr, "\mathrm{id}_A \bullet \beta_{B,C}"] \arrow[ddl, dashed, "\beta_{A,B\bullet C}"] & \\
B \bullet A \bullet C \arrow[d, "\mathrm{id}_B \bullet \beta_{A,C}"'] & & A \bullet C \bullet B \arrow[d, "\beta_{A,C}\bullet \mathrm{id}_B"] \arrow[ddl, dashed, "\beta_{A,C\bullet B}"'] \\
B \bullet C \bullet A \arrow[dr, "\beta_{B,C}\bullet \mathrm{id}_A"'] & & C \bullet A \bullet B \arrow[dl, "\mathrm{id}_C \bullet \beta_{A,B}"] \\
& C \bullet B \bullet A &
\end{tikzcd}$$

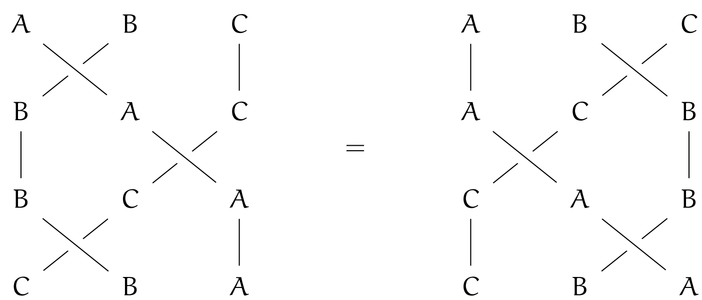
Adding in the two arrows $\beta_{A,B\bullet C}$ and $\beta_{A,C\bullet B}$, we see that the two triangles are the axioms that are satisfied by the braiding and the larger square is naturality of β. $\square$

We may interpret $\beta_{A,B}$ as a **decorated braid**, that looks like

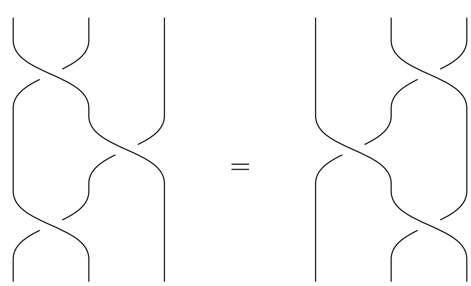


Theorem 4.4 (Coherence)**.** *A diagram constructed out of $\beta, \mathrm{id}, \alpha, \lambda, \rho$ and the monoidal product $\bullet$ commutes if and only if each side of the diagram defines the same element of the braid group.*

Example 4.5. Let's draw the hexagon from Proposition 4.3. We have that



because these have the same braids in the braid group B_3.



Moreover, these have the same underlying permutation in S_3, namely (13). Hence, this guarantees commutativity of the given diagram in a symmetric monoidal category as well.

Example 4.6.

(1) $(\mathbf{Set}, \times)$ is a symmetric monoidal category under $\beta_{X,Y} \colon X \times Y \to Y \times X$ given by $(x, y) \mapsto (y, x)$.

(2) $(\mathbf{Vec}_k, \otimes)$ is symmetric under $\beta_{V,W} \colon V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$.

(3) Let G be an abelian group. Fix $\gamma \colon G \times G \to k^\times$ that is **bimultiplicative**, that is

$$\gamma(xy, z) = \gamma(x, z)\gamma(y, z)$$
$$\gamma(x, yz) = \gamma(x, y)\gamma(x, z)$$

(Note that these laws imply $\gamma(x, 1) = 1 = \gamma(1, x)$). Let $\mathbf{C}$ be the category of G-graded vector spaces. View it as a monoidal category under $\bullet$ with trivial associativity constraint α.

Define $\beta_{V,W} \colon V \bullet W \to W \bullet V$ as follows. It's components are

$$\begin{tikzcd}
(V \bullet W)_g \arrow[r, dashed] \arrow[d, equal] & (W \bullet V)_g \arrow[d, equal] \\
\bigoplus_{xy=g} V_x \otimes W_y & \bigoplus_{x'y'=g} W_{x'} \otimes V_{y'} \\
V_x \otimes W_y \arrow[u, hook] \arrow[r, dashed] & W_y \otimes V_x \arrow[u, hook] \\
v \otimes w \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & \gamma(x, y) w \otimes v \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Given $x, y \in G$ such that $xy = g$, G abelian means that $yx = g$ as well. So we may choose $x' = y$, $y' = x$.

A specific instance of this is $G = \mathbb{Z}_n$, with $q \in k^\times$ such that $k^n = 1$. Then define $\gamma \colon \mathbb{Z}_n \times \mathbb{Z}_n \to k^\times$ by $(i, j) \mapsto q^{ij}$.

4.1 Monoids and Comonoids in Braided Categories

Let $(C, \bullet, I, \beta)$ be a braided monoidal category.

Proposition 4.7. *Let A and B be monoids in $\mathbf{C}$. Then $A \bullet B$ is again a monoid under*

$$\mu_{A \bullet B} \colon (A \bullet B) \bullet (A \bullet A) \xrightarrow{\mathrm{id} \bullet \beta_{B,A} \bullet \mathrm{id}} A \bullet A \bullet B \bullet B \xrightarrow{\mu_A \bullet \mu_B} A \bullet B$$

$$\iota_{A \bullet B} \colon I = I \bullet I \xrightarrow{\iota_A \bullet \iota_B} A \bullet B$$

Example 4.8. In $(\mathbf{Set}, \times)$, if A and B are monoids, then so is $A \times B$ via $(a, b) \cdot (a', b') = (aa', bb')$.

Proposition 4.9. *If* C, D *are comonoids, then so is* $C \bullet D$*, with*

$$\Delta_{C\bullet D} \colon C \bullet D \xrightarrow{\Delta_C \bullet \Delta_D} C \bullet C \bullet D \bullet D \xrightarrow{\mathrm{id}\bullet\beta\bullet\mathrm{id}} (C \bullet D) \bullet (C \bullet D)$$

$$\varepsilon_{C\bullet D} \colon C \bullet D \xrightarrow{\varepsilon_C \bullet \varepsilon_D} I \bullet I = I.$$

Definition 4.10. A monoid (A, μ, ι) is **commutative** if

$$\begin{tikzcd} A \bullet A \arrow[rr, "\beta_{A,A}"] \arrow[dr, "\mu_A"'] & & A \bullet A \arrow[dl, "\mu_A"] \\ & A & \end{tikzcd}$$

commutes. Dually, a comonoid (C, Δ, ε) is **cocommutative** if

$$\begin{tikzcd} C \bullet C \arrow[rr, "\beta_{C,C}"] & & C \bullet C \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

commutes.

Proposition 4.11. *Let* B *be both a monoid and a comonoid in* $\mathbf{C}$*, a braided monoidal category. Then the following are equivalent.*

(i) $\Delta \colon B \to B \bullet B$ *and* $\varepsilon \bullet B \to I$ *are morphisms of monoids.*

(ii) $\mu \colon B \bullet B \to B$ *and* $\iota \colon I \to B$ *are morphisms of comonoids.*

(iii) The following diagrams commute:

$$\begin{tikzcd} B \bullet B \arrow[r, "\mu"] \arrow[d, "\Delta\bullet\Delta"] & B \arrow[r, "\Delta"] & B \bullet B \\ B \bullet B \bullet B \bullet B \arrow[rr, "\mathrm{id}\bullet\beta_{B,B}\bullet\mathrm{id}"] & & B \bullet B \bullet B \bullet B \arrow[u, "\mu\bullet\mu"'] \end{tikzcd}$$

$$\begin{tikzcd} B \bullet B \arrow[r, "\varepsilon\bullet\varepsilon"] \arrow[d, "\mu"] & I \bullet I \arrow[d, equal] \\ B \arrow[r, "\varepsilon"] & I \end{tikzcd} \qquad \begin{tikzcd} I \bullet I \arrow[r, "\iota"] \arrow[d, equal] & B \bullet B \\ I \arrow[r, "\iota"] & B \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\iota"] \arrow[dr, "\mathrm{id}"'] & B \arrow[d, "\varepsilon"] \\ & I \end{tikzcd}$$

Definition 4.12. If any of the equivalent conditions in Proposition 4.11 is satisfied, then we call B a **bimonoid**.

Definition 4.13. We say that $f \colon B \to B'$ is a **morphism of bimonoids** if it is both a morphism of monoids and comonoids. We say that $f \colon H \to H'$ is a **morphism of Hopf monoids** if it is a morphism of bimonoids such that

$$\begin{tikzcd} H \arrow[r, "S"] \arrow[d, "f"] & H \arrow[d, "f"] \\ H' \arrow[r, "S"] & H' \end{tikzcd}$$

4.2 Hopf Monoids

Let $(B, \mu, \iota, \Delta, \varepsilon)$ be a bimonoid in the braided monoidal category $(\mathbf{C}, \bullet, I, \beta)$. Consider the set $\mathrm{Hom}_{\mathbf{C}}(B, B)$; as noted before, this is a monoid in **Set** under the convolution product, with unit $u \colon B \xrightarrow{\varepsilon} I \xrightarrow{\iota} B$. Note that the unit is *not* the identity map $\mathrm{id}_B \colon B \to B$, since $\iota \circ \varepsilon \neq \mathrm{id}_B$. But we can ask for it to be invertible.

Definition 4.14. We say that H is a **Hopf monoid** if id_H is convolution-invertible in $\mathrm{Hom}_{\mathbf{C}}(H, H)$. When it is, the convolution inverse is denoted by S and called the **antipode** of H.

$$\begin{tikzcd}
H \bullet H \arrow[rr, "S \bullet \mathrm{id}"] & & H \bullet H \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[d, "\Delta"] \arrow[r, "\varepsilon"] & I \arrow[r, "\iota"] & H \\
H \bullet H \arrow[rr, "\mathrm{id} \bullet S"] & & H \bullet H \arrow[u, "\mu"']
\end{tikzcd}$$

Remark 4.15. Let's organize everything we've defined so far. Given a braided monoidal category $(C, \bullet, I, \beta)$, we have

$$\begin{tikzcd}
 & \text{objects} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{monoids} \arrow[dr, dash] & & \text{comonoids} \arrow[dl, dash] \\
 & \text{bimonoids} \arrow[d, dash] & \\
 & \text{Hopf monoids} &
\end{tikzcd}$$

Going down in the diagram adds more structure.

Example 4.16. What does this diagram look like in $(\mathbf{Set}, \times)$?

$$\begin{tikzcd}
 & \text{objects = sets} \arrow[dl, dash] \arrow[dr, equal] & \\
\text{monoids = ordinary monoids} \arrow[dr, dash] & & \text{comonoids = sets} \arrow[dl, dash] \\
 & \text{bimonoids = ordinary monoids} \arrow[d, dash] & \\
 & \text{Hopf monoids = groups} &
\end{tikzcd}$$

So in the category of sets, comonoids are just sets because each set has a unique comonoid structure given by the diagonal. We also see that every monoid is a bimonoid, and it turns out that Hopf monoids are groups.

Why is a Hopf monoid in **Set** a group? Suppose that H is a Hopf monoid in **Set**, with antipode S. We have that $S * \mathrm{id} = u = \mathrm{id} * S$. What does this mean? Well, we have

$$S * \mathrm{id}: \qquad \begin{array}{ccccccc} B & \xrightarrow{\Delta} & B \times B & \xrightarrow{S \times \mathrm{id}} & B \times B & \xrightarrow{\mu} & B \\ \Cup & & \Cup & & \Cup & & \Cup \\ x & \longmapsto & (x, x) & \longmapsto & (S(x), x) & \longmapsto & S(x) \cdot x \end{array}$$

But we also have

$$\mu: \qquad \begin{array}{ccccc} B & \xrightarrow{\varepsilon} & I & \xrightarrow{\iota} & B \\ \Cup & & \Cup & & \Cup \\ x & \longmapsto & * & \longmapsto & 1 \end{array}$$

So $S * \mathrm{id} = u \iff S(x) \cdot x = 1$ for all $x \in B$, and similarly, we see that $x \cdot S(x) = 1$ for all $x \in B$. Hence, H is a group.

Recall that if G and G' are groups and $f\colon G \to G'$ preserves products and units, then $f(x^{-1}) = f(x)^{-1}$.

Proposition 4.17. *If* H *and* H' *are Hopf monoids and* $\phi\colon H \to H'$ *is a morphism of bimonoids, then* ϕ *preserves antipodes (that is,* ϕ *is a morphism of Hopf monoids).*

Proof. ϕ is a morphism of monoids, so $\phi^{\#}\colon \mathrm{Hom}(H', H') \to \mathrm{Hom}(H, H')$ is a morphism of monoids.

ϕ is a morphism of comonoids, so $\phi_{\#}\colon \mathrm{Hom}(H, H) \to \mathrm{Hom}(H, H')$ is a morphism of monoids.

We want to show that $\phi \circ S_H = S_{H'} \circ \phi$, or equivalently, $\phi_{\#}(S_H) = \phi^{\#}(S_{H'})$. Since $\phi_{\#}$ and $\phi^{\#}$ are morphisms of monoids then they preserve inverses. So $\phi_{\#}(\mathrm{id}_H)$ is the convolution inverse of $\phi_{\#}(S_H)$, and $\phi^{\#}(\mathrm{id}_{H'})$ is the convolution inverse of $\phi^{\#}(S_{H'})$. But these are both just ϕ.

$$\phi \circ S_H = \phi_{\#}(S_H) = \text{inverse of } \phi_{\#}(\mathrm{id}_H)$$
$$S_{H'} \circ \phi = \phi^{\#}(S_{H'}) = \text{inverse of } \phi^{\#}(\mathrm{id}_{H'})$$

But then $\phi^{\#}(\mathrm{id}_{H'}) = \phi = \phi_{\#}(\mathrm{id}_H)$. □

Proposition 4.18. *The antipode of a Hopf monoid reverses products and co-*

products. That is, the following diagrams commute.

$$
\begin{tikzcd}
H \bullet H \arrow[r, "S \bullet S"] \arrow[dd, "\mu"] & H \bullet H \arrow[d, "\beta"] \\
& H \bullet H \arrow[d, "\mu"] \\
H \arrow[r, "S"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "S"] \arrow[d, "\Delta"] & H \arrow[dd, "\Delta"] \\
H \bullet H \arrow[d, "\beta"] & \\
H \bullet H \arrow[r, "S \bullet S"] & H \bullet H
\end{tikzcd}
\tag{2}
$$

Example 4.19. In $(\mathsf{Set}, \times)$, H is a group, and this proposition easily verified.

$$
\begin{tikzcd}
(x, y) \arrow[r, mapsto] \arrow[d, mapsto] & (x^{-1}, y^{-1}) \arrow[d, mapsto] \\
x \cdot y \arrow[d, mapsto] & (y^{-1}, x^{-1}) \arrow[d, mapsto] \\
(x \cdot y)^{-1} \arrow[r, equal] & y^{-1} \cdot x^{-1}
\end{tikzcd}
$$

Proposition 4.20. *Antipode preserves units and counits. That is, the following diagrams commute.*

$$
\begin{tikzcd}
H \arrow[rr, "S"] & & H \\
& I \arrow[ul, "\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[rr, "S"] \arrow[dr, "\varepsilon"'] & & H \arrow[dl, "\varepsilon"] \\
& I &
\end{tikzcd}
$$

Proposition 4.21. *If* H *is commutative or cocommutative, then* $S^2 = \mathrm{id}$.

Proof. The statements are formally dual, so if one holds, then the other one does as well if we apply it in the opposite category. We will prove that if H is commutative. This tells us that S is a morphism of monoids (we know that $\mu \circ \beta = \mu$, see (2)), so there is a map

$$
\begin{array}{rccc}
S_{\#}\colon & \mathrm{Hom}_{\mathsf{C}}(H, H) & \longrightarrow & \mathrm{Hom}_{\mathsf{C}}(H, H) \\
& f & \longmapsto & S \circ f
\end{array}
$$

Hence, $S_{\#}(S)$ is the convolution-inverse of $S_{\#}(\mathrm{id}) = S$, but S is the convolution inverse of id and convolution inversion is involutive. Hence, $S \circ S = \mathrm{id}$. $\square$

5 Hopf Monoids in Species

5.1 Linearization

Definition 5.1. Given a set X and a field k, let kX denote the vector space consisting of formal linear combinations of elements of X with coefficients in k.

We call this the **linearization** of X. So X is a basis for kX.

Notice that every vector space V is the linearization of a set – pick a basis X of V, and then $V \cong kX$.

Proposition 5.2.

(a) $k(X \sqcup Y) \cong kX \oplus kY$

(b) $k(X \times Y) \cong kX \otimes kY$.

We can elaborate on Proposition 5.2(b) to say that linearization is a monoidal functor $\mathbf{Set} \to \mathbf{Vec}_k$.

Example 5.3. If a group G acts on a set X, then G acts linearly on kX. So X is a G-set implies that kX is a kG-module.

Note that not every kG-module is the linearization of a G-set, because there need not be any basis stable under G. In particular, if $G = S_n$, G acts linearly on k by $\sigma \cdot \lambda = \mathrm{sgn}(\sigma)\lambda$, and nothing is stable under the action of G.

Remark 5.4. The dual $(kX)^*$ has basis $\{x^* \mid x \in X\}$, where

$$x^*(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if not} \end{cases}$$

A map $f\colon A \to B$ of sets induces a linear map $f\colon kA \to kB$ and then another map

$$\begin{aligned} f^*\colon \quad (kB)^* &\longrightarrow (kA)^* \\ b^* &\longmapsto \sum_{\substack{a \in A \\ f(a)=b}} a^* \end{aligned}$$

because

$$f^*(b^*)(a) = b^*(f(a)) = \begin{cases} 1 & \text{if } b = f(a) \\ 0 & \text{if not} \end{cases}$$

5.2 Vector Species

We've so far been working with **set species**: functors $\mathbf{set}^\times \to \mathbf{Set}$. Let's linearize that.

Definition 5.5. A **vector species** is a functor $P\colon \mathbf{set}^\times \to \mathbf{Vec}$:

- one vector space $P[I]$ for each finite set I
- one linear map $\sigma^*\colon P[I] \to P[J]$ for each bijection $\sigma\colon I \xrightarrow{\sim} J$.

Definition 5.6. If A is a set species, then the **linearization** kA is a vector species, defined by

$$(kA)[I] = k(A[I])$$

Definition 5.7. We define the **dual species** by $H^*[I] = H[I]^*$ and on $\sigma\colon I \to J$ a bijection, $H[\sigma]^* = H[(\sigma^*)^{-1}]$.

Remark 5.8. Not every vector species is the linearization of a set species. Given a set species A, this is the same as a sequence of sets $A[n]$, each with an action of S_n.

A vector species P is now a sequence of vector species $P[n]$, each with the structure of a S_n-representation.

A (linearized) vector species $P = kA$ is a sequence, as with set species, of sets $P[n] = kA[n]$, where each is the linearization of an S_n-set.

Definition 5.9. Given vector species P and Q, we can define their **Cauchy product** $(P \bullet Q)$ by

$$(P \bullet Q)[I] = \bigoplus_{I=S\sqcup T} P[S] \otimes Q[T]$$

Definition 5.10. The **unit species** 1 is defined by

$$1[I] = \begin{cases} k & \text{if } I = \emptyset \\ 0 & \text{if not.} \end{cases}$$

Definition 5.11. Let $\mathbf{Sp}_k$ denote the category of vector species over k.

Proposition 5.12. $(\mathbf{Sp}_k, \bullet, 1)$ *is a monoidal category.*

Proof sketch. We will show that $(A \bullet B) \bullet C$ and $A \bullet (B \bullet C)$ are canonically isomorphic. Both have components $A[R] \otimes B[S] \otimes C[T]$, where $I = R \sqcup S \sqcup T$. Verifying the other axioms of a monoidal category are left to the reader, should she/he be sufficiently bored. $\square$

Now that we know that $\mathbf{Sp}_k$ is monoidal, we can speak of monoids and comonoids in $\mathbf{Sp}_k$. What do these look like?

A monoid A consists of k-linear maps

$$\mu_{S,T}\colon A[S] \otimes A[T] \to A[I].$$

A comonoid C consists of k-linear maps

$$\Delta_{S,T}\colon C[I] \to C[S] \otimes C[T].$$

Unlike in the category of set species, we no longer have the notions of restriction and corestriction; a general element of $C[S] \otimes C[T]$ is a linear combination of simple tensors, and looks like $\sum_i z_i \otimes z_i'$. So there is more to work with in vector species.

Remark 5.13. However, if A is a monoid in the category of set-species, then kA is a monoid in $\mathbf{Sp}_k$ with multiplication.

$$\begin{array}{cccc} \mu_{S,T}\colon & kA[S] \otimes kA[T] & \longrightarrow & kA[I] \\ & x \otimes y & \mapsto & x \cdot y \end{array}$$

for $x \in A[S]$ and $y \in A[T]$.

Similarly, if A is a comonoid in the category of set-species, then kA is a comonoid in $\mathbf{Sp}_k$ with comultiplication

$$\begin{array}{cccc} \Delta_{S,T}\colon & kA[I] & \longrightarrow & kA[T] \\ & z & \longmapsto & z|_S \otimes z/_S \end{array}$$

for $z \in A[I]$.

5.3 Braidings on $\mathbf{Sp}_k$

Fix $q \in k^\times$. Given vector species P, Q in $\mathbf{Sp}_k$, define a morphism $\beta_q \colon P \bullet Q \to Q \bullet P$. It's components are

$$\begin{tikzcd}
(P \bullet Q)[I] \arrow[r, "\beta_q"] \arrow[d, equal] & (Q \bullet P)[I] \\
\bigoplus_{I=S\sqcup T} P[S] \otimes Q[T] \arrow[r, dashed] & \bigoplus_{I=S'\sqcup T'} Q[S'] \otimes P[T'] \\
P[S] \otimes Q[T] \arrow[u, hook] \arrow[r] & Q[T] \otimes P[S] \arrow[u, hook] \\
x \otimes y \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & q^{|S||T|} y \otimes x \arrow[u, phantom, "\Cup"]
\end{tikzcd}$$

Remark 5.14. This braiding β_q is a symmetry precisely when $q = \pm 1$. In the case $q = 1$, we write $\beta = \beta_1$.

Let's check some of the axioms of a braiding. Let's verify that

$$\begin{tikzcd}
A \bullet B \bullet C \arrow[rr, "\beta_{A,B\bullet C}"] \arrow[dr, "\beta\bullet\mathrm{id}"'] & & B \bullet C \bullet A \\
& B \bullet A \bullet C \arrow[ur, "\mathrm{id}\bullet\beta"'] &
\end{tikzcd}$$

Look at the components:

$$\begin{tikzcd}
A[R] \otimes B[S] \otimes C[T] \arrow[rr] \arrow[dr] & & B[S] \otimes C[T] \otimes A[R] \\
& B[S] \otimes A[R] \otimes C[T] \arrow[ur] &
\end{tikzcd}$$

Let $x \in A[R]$, $y \in B[S]$, and $z \in C[T]$. Then we can chase the diagram:

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, mapsto] \arrow[d, mapsto] & q^{|R||S \sqcup T|} y \otimes z \otimes x \arrow[d, equal] \\
q^{|R||S|} y \otimes x \otimes z \arrow[r, mapsto] & q^{|R||S|} q^{|R||T|} y \otimes z \otimes x
\end{tikzcd}$$

Now that we've defined a braiding on $\mathbf{Sp}_k$, it makes sense to talk about bimonoids and Hopf monoids in $\mathbf{Sp}_k$.

Proposition 5.15. *If* H *is a finite-dimensionally valued bimonoid in* $\mathbf{Sp}_k$ *(each space* $H[I]$ *is finite dimensional), then it's dual* H^* *is a bimonoid in* $\mathbf{Sp}_k$*.*

Proof. Define $H^*[S] \otimes H^*[T] \to H^*[I]$ as the dual of $\Delta_{S,T}\colon H[I] \to H[S] \otimes H[T]$. Define $H^*[I] \to H^*[I] \otimes H^*[T]$ as the dual of $\mu_{S,T}\colon H[S] \otimes H[T] \to H[I]$. □

Example 5.16. Recall that L is the set-theoretic species of linear orders on a set I. We saw that L is a bimonoid in $\mathbf{Sp}_k$. This implies that $H = kL$ is a bimonoid in $\mathbf{Sp}_k$, and by the previous proposition, so is $H^* = (kL)^*$.

What does this look like?

$$\begin{aligned} H^*[S] \otimes H^*[T] &\longrightarrow H^*[I] \\ \ell_1^* \otimes \ell_2^* &\longmapsto \sum_{\substack{\ell \in L[I] \\ \ell|_S = \ell_1, \ell|_T = \ell_2}} \ell^* \end{aligned}$$

This is called the **shuffle product**.

More concretely, let $I = \{a, b, c\}$. Let $S = \{a, b\}$ and $T = \{c\}$. Then if $\ell_1 = ab$, $\ell_2 = c$, we have that

$$(ab)^* \cdot c^* = (abc)^* + (acb)^* + (acb)^* + (cab)^*$$

What is the coproduct?

$$\begin{aligned} H^*[I] &\longrightarrow H^*[S] \otimes H^*[T] \\ \ell^* &\longmapsto \sum_{\substack{\ell_1 \in L[S], \ell_2 \in L[T] \\ \ell_1 \cdot \ell_2 = \ell}} \ell_1^* \otimes \ell_2^* \end{aligned}$$

where $\ell \in L[I]$. This is dual to $\mu_{S,T}\colon H[S] \otimes H[T] \to H[I]$, where $\mu_{S,T}(\ell_1 \otimes \ell_2) = \ell_1 \ell_2$ is the concatenation of orders. Notice that we can simplify our description of this coproduct, because

$$\sum_{\substack{\ell_1 \in L[S], \ell_2 \in L[T] \\ \ell_1 \cdot \ell_2 = \ell}} \ell_1^* \otimes \ell_2^* = \begin{cases} \ell|_S \otimes \ell|_T & \text{if } S \text{ is initial segment under } \ell \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.17. $H = kL$ *is a Hopf monoid. It's antipode* $\mathbb{S}\colon H \to H$ *has components*

$$\begin{array}{cccc} \mathbb{S}_I\colon & H[I] & \longrightarrow & H[I] \\ & \ell & \longmapsto & (-1)^{|I|}\overline{\ell}, \end{array}$$

where $\overline{\ell}$ *is the reversal of the order* $\ell \in L[I]$.

Proof. If $I = \varnothing$, then there's nothing really to check.

Now we need to show that the proposed antipode is a convolution inverse for id_H.

$$\begin{tikzcd} H \bullet H \arrow[rr, "\mathbb{S}\bullet\mathrm{id}"] & & H \bullet H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & 1 \arrow[r, "\iota"] & H \end{tikzcd}$$

To that end, pass to components. Around the top of the diagram, we get

$$H[I] \xrightarrow{\sum_{I=S\sqcup T} \Delta_{S,T}} \bigoplus_{I=S\sqcup T} H[S]\otimes H[T] \xrightarrow{\bigoplus \mathbb{S}_S \otimes \mathrm{id}_T} \bigoplus_{I=S\sqcup T} H[S]\otimes H[T] \xrightarrow{\sum_{I=S\sqcup T} \mu_{S,T}} H[I] \tag{3}$$

Along the bottom of the diagram, we have just the zero map for all components $I \neq \varnothing$.

Hence, we want to show that (3) is zero for all $I \neq \varnothing$. We are asking if, for all $\ell \in L[I]$, do we have

$$\sum_{I=S\sqcup T} \mu_{S,T} \circ (\mathbb{S}_S \otimes \mathrm{id}_T) \circ \Delta_{S,T}(\ell) = 0?$$

Or equivalently, does

$$\sum_{I=S\sqcup T} (-1)^{|S|}\overline{\ell|_S} \cdot \ell|_T = 0?$$

We can pair up (S,T) with (S',T') such that if $\#S$ is even, then $\#S'$ is odd, and $\overline{\ell_| S} \cdot \ell|_T = \overline{\ell|_{S'}} \cdot \ell|_{T'}$ (a **sign-reversing** involution). ☐

Example 5.18. A small example to illustrate the last line of the previous proof. If $I = \{a,b,c\}$, and $\ell = abc$, then

S	T	$(-1)^{\#S}\overline{\ell|_S} \cdot \ell|_T$
$\varnothing$	a,b,c	$+abc$
a	b,c	$-abc$
b	a,c	$-bac$
c	a,b	$-cab$
a,b	c	$+bac$
a,c	b	$+bac$
b,c	a	$+cba$
a,b,c	$\varnothing$	$-cba$

The sum down the right-most column is zero. We pair up sets with their compliments and the matching terms cancel.

Example 5.19. A deformation of $H = kL$. The species is the same as before in Example 5.16. The product is the same as in Example 5.16 as well, but now there is a new coproduct:

$$\begin{array}{rcl} H[I] & \longrightarrow & H[S] \otimes H[T] \\ \ell & \longmapsto & q^{a_{S,T}(\ell)} \ell|_S \otimes \ell|_T \end{array}$$

where

$$a_{S,T}(\ell) = \#\{(i,j) \in S \times T \mid i > j \text{ in } \ell\}.$$

We can check coassociativity for this.

$$\begin{tikzcd} H[I] \arrow[r, "\Delta_{R,S\sqcup T}"] \arrow[d, "\Delta_{R\sqcup S,T}"] & H[R] \otimes H[S \sqcup T] \arrow[d, "\mathrm{id}\otimes\Delta_{S,T}"] \\ H[R \sqcup S] \otimes H[T] \arrow[r, "\Delta_{R,S}\otimes \mathrm{id}"] & H[R] \otimes H[S] \otimes H[T] \end{tikzcd}$$

$$\begin{tikzcd} \ell \arrow[r, mapsto] \arrow[d, mapsto] & q^{a_{R,S\sqcup T}(\ell)} \ell|_R \otimes \ell|_{S\sqcup T} \arrow[d, mapsto] \\ q^{a_{R\sqcup S,T}(\ell)} q^{a_{R,S}(\ell|_{R\sqcup S})} \ell|_R \otimes \ell|_S \otimes \ell|_T \arrow[r, equal, "?"] & q^{a_{R,S\sqcup T}(\ell)} q^{a_{S,T}(\ell|_{S\sqcup T}} \ell|_R \otimes \ell|_S \otimes \ell|_t \end{tikzcd}$$

To resolve whether or not the two resulting values are equal, we need to verify that

$$a_{R,S\sqcup T}(\ell) + a_{S,T}(\ell|_{S\sqcup T}) = a_{R\sqcup S,T}(\ell) + a_{R,S}(\ell|_{R\sqcup S}) \tag{4}$$

The left hand side of (4) counts the pairs (i,j) such that $i > j$ in ℓ and $i \in R, j \in S$ or $i \in R, j \in T$ or $i \in S, j \in T$.

The right hand side of (4) counts pairs (i,j) such that $i > j$ in ℓ and $i \in R$, $j \in T$ or $i \in S$, $j \in T$, or $i \in R$, $j \in S$.

These are the same! So coassociativity holds.

Remark 5.20. This number $a_{S,T}(\ell)$ is also known as the **Schubert statistic.** It's the dimension of the Schubert cell indexed by S in $\mathrm{Gr}_k(\mathbb{R}^n)$ where $n = |T|$ and $k = |S|$.

5.4 Exercises

Exercise 5.21.

(a) Check the compatibility condition for the bimonoid H defined in Example 5.19. That is, verify that the coproduct is a morphism of monoids and the product is a morphism of bimonoids.

(b) H is in fact a Hopf monoid in $(\mathbf{Sp}_k, \bullet, \beta_q)$. Show that

$$S_I(\ell) = q^{\binom{|I|}{2}}(-1)^{|I|}\overline{\ell}$$

Exercise 5.22. Let P be the set species of partial orders. Claim that P is a bimonoid in the category of set species. The product is

$$\begin{aligned} P[S] \times P[T] &\longrightarrow P[I] \\ (p_1, p_2) &\longmapsto p_1 \cdot p_2 \end{aligned}$$

where $p_1 \cdot p_2$ is the **ordinal sum**. That is, we consider everything in p_2 to be larger than anything in p_1. The coproduct is

$$\begin{aligned} P[I] &\longrightarrow P[S] \times P[T] \\ p &\longmapsto (p|_S, p|_T). \end{aligned}$$

(a) Check the axioms to verify that P is a bimonoid.

(b) Let $H = kP$ be the linearization of P. What is the antipode S? What is an explicit description?

Exercise 5.23. Note that $kP[I]$ has for a basis the set $P[I]$. Let's write x_p for the basis element corresponding to $p \in P[I]$. Introduce a second basis $\{y_p\}_{p \in P[I]}$ by

$$x_p = \sum_{q \subseteq p} y_q$$

for all $p \in P[I]$, where we say that $q \subseteq p$ if we view a partial order on I as a subset of $I \times I$. The q's are uniquely determined by the p's via a recursive system of equations.

Now consider the dual H^*, where $H^*[I] = (kP[I])^*$.

(a) Show that for $p_1 \in P[S]$, $p_2 \in P[T]$,

$$y^*_{p_1} \cdot y^*_{p_2} = y^*_{p_1 \sqcup p_2},$$

where $p_1 \sqcup p_2$ means the partial order on $S \sqcup T$ determined by p_1 and p_2 with no relations between the two.

(b) For $p \in P[I]$,

$$\Delta(y^*_p) = \begin{cases} y^*_{p|_S} \otimes y^*_{p|_T} & \text{if } S \text{ is a } \textbf{lower set} \text{ of } p, \\ 0 & \text{otherwise.} \end{cases}$$

We say that S is a lower set of p if $i \in S$ and $j \leq i$ under the order p, then $j \in S$. We might also say that S is **downward closed**.

Exercise 5.24. Show that the inclusion $L \hookrightarrow P$ is a morphism of bimonoids. Then show that this gives

$$\begin{array}{rcl} (kP)^* & \longrightarrow & (kL)^* \\ y_p^* & \longmapsto & \sum_{\ell \in \mathcal{L}(p)} \ell^*, \end{array}$$

where $\mathcal{L}(p)$ is the **set of linear extensions of** p,

$$\mathcal{L}(p) := \{\ell \in L[I] \mid p \leq \ell\}$$

Example 5.25. Let P be the poset on the set $\{a, b, c\}$ with relations $a \leq b$ and $a \leq c$. Then

$$\mathcal{L}\left(P\right) = \{abc, acb\}$$

5.5 Connected Species

Definition 5.26. A set species P is **connected** if $P[\emptyset] = \{*\}$. A vector species P is **connected** if $P[\emptyset] = k$.

Let M be a monoid in $(\mathbf{Sp}_k, \bullet)$. Then associativity guarantees that there is a well-defined map

$$M[R] \otimes M[S] \otimes M[T] \xrightarrow{\mu_{R,S,T}} M[I]$$

whenever $I = R \sqcup S \sqcup T$. More generally, if $I = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_k$, then there is a well-defined map.

$$M[S_1] \otimes \ldots \otimes M[S_k] \xrightarrow{\mu_{S_1,\ldots,S_k}} M[I]$$

Similarly, for a comonoid C, we have a well-defined map

$$C[I] \xrightarrow{\Delta_{S_1,\ldots,S_k}} C[S_1] \otimes \ldots \otimes C[S_k].$$

Now assume that M and C are connected. The unit axiom says that

$$\begin{tikzcd} M[I] \otimes M[\emptyset] \arrow[r, "\mu_{I,\emptyset}"] \arrow[d, equal] & M[I] \\ M[I] \arrow[ur, equal, "\mathrm{id}"'] & \end{tikzcd}$$

So we have that $\mu_{I,\emptyset} = \mathrm{id} = \mu_{\emptyset,I}$. More generally,

$$\mu_{S_1,\ldots,S_k} = \mu_{T_1,\ldots,T_h}$$

where $(T_1, \dots, T_h)$ is the sequence obtained from $(S_1, \dots, S_k)$ by removing all empty sets. Similarly,

$$\Delta_{S_1,\dots,S_k} = \Delta_{T_1,\dots,T_k}$$

Recall that if $f\colon C \to M$ is a morphism of species, then f consists of a collection of maps $f_I\colon C[I] \to M[I]$, one for each I.

Definition 5.27. Let $f \in \mathrm{Hom}_{\mathbf{Sp}_k}(C, M)$. Then f is **locally nilpotent** if $(f^{*k})_I = 0$ for all $k > |I|$, where f^{*k} denotes the k-fold iterated convolution product with itself.

Lemma 5.28. *Let M be a monoid, C a comonoid. Assume that both are connected. Let $f \in \mathrm{Hom}_{\mathbf{Sp}_k}(C, M)$ such that $f_\emptyset = 0$. Then*

(i) f is locally nilpotent, and

(ii) $u - f$ is invertible in $\mathrm{Hom}_{\mathbf{Sp}_k}(C, M)$, where $u = \iota\varepsilon$.

Proof. (i) The k-fold convolution product of f is

$$f^{*k}\colon C \xrightarrow{\Delta^{k-1}} \underbrace{C \bullet C \bullet \cdots \bullet C}_{k} \xrightarrow{f \bullet \cdots \bullet f} \underbrace{M \bullet \cdots \bullet M}_{k} \xrightarrow{\mu^{k-1}} M$$

with components

$$(f^{*k})_I = \sum_{I = S_1 \sqcup \ldots \sqcup S_k} \mu_{S_1,\dots,S_k}(f_{S_1} \otimes \cdots \otimes f_{S_k})\Delta_{S_1,\dots,S_k}.$$

If $k > |I|$, then at least one S_j is empty, which implies that $f_{S_j} = 0$, so $(f^{*k})_I = 0$.

(ii) Consider

$$\sum_{k \geq 0} f^{*k} \in \mathrm{Hom}_{\mathbf{Sp}_k}(C, M).$$

This is well-defined because

$$\left(\sum_{k \geq 0} f^{*k}\right)_I = \sum_{k=0}^{|I|} (f^{*k})_I.$$

We can check that this is the inverse of $u - f$.

□

Proposition 5.29. *Let H be a bimonoid in $(\mathbf{Sp}, \bullet, \beta_q)$. If H is connected, then H is a Hopf monoid.*

Proof. We have that $\mathrm{id} = u - (u - \mathrm{id})$. Let $f = u - \mathrm{id}$. Then $f_\emptyset = u_\emptyset - \mathrm{id}_\emptyset = 0$ because H is connected. Therefore, the following diagram commutes.

$$\begin{tikzcd} H[\emptyset] \arrow[r, "\varepsilon_\phi"] \arrow[rr, bend left, "u_\emptyset"] \arrow[rr, bend right, "\mathrm{id}_\emptyset"'] & k = 1[\emptyset] \arrow[r, "\iota_\emptyset"] & H[\emptyset] \end{tikzcd}$$

□

5.6 Antipode Formulas

Let H be a connected Hopf monoid in $\mathbf{Sp}_k$ with antipode $S\colon H \to H$. Then we have the following facts:

(1) $S_\emptyset = \mathrm{id}_k$, and $H[\emptyset] = k$, because H is connected. In general, we see that $H[\emptyset]$ is a Hopf algebra and $S_\emptyset$ is it's antipode.

(2) $\displaystyle\sum_{I=S\sqcup T} \mu_{S,T}(\mathrm{id}_S \otimes S_T)\Delta_{S,T} = 0$

(3) $\displaystyle\sum_{I=S\sqcup T} \mu_{S,T}(S_S \otimes \mathrm{id}_T)\Delta_{S,T} = 0$

whenever $I \neq \emptyset$.

Remark 5.30 (Preliminaries)**.** Let V be a k-vector space. Then $V \cong k \otimes V$ naturally in V, via the map $v \mapsto 1 \otimes v$ for $v \in V$.

Naturality means that given a linear map $f\colon V \to W$, the following commutes.

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[d, "f"] & k \otimes V \arrow[d, "\mathrm{id}\otimes f"] \\ W \arrow[r, "\sim"] & k \otimes V \end{tikzcd}$$

In the case of our connected Hopf monoid H, this natural isomorphism takes the form

$$\begin{tikzcd} H[I] \arrow[r, shift left, "\Delta_{\emptyset,I}"] & H[\emptyset] \otimes H[I] \arrow[l, shift left, "\mu_{\emptyset,I}"] \end{tikzcd}$$

and naturality looks like

$$\begin{tikzcd} H[I] \arrow[r, "\Delta_{\emptyset,I}"] \arrow[d, "f"] & H[\emptyset] \otimes H[I] \arrow[d, "\mathrm{id}\otimes f"] \\ H[I] & H[\emptyset] \otimes H[I] \arrow[l, "\mu_{\emptyset,I}"] \end{tikzcd}$$

for $f\colon H[I] \to H[I]$ is any linear map. In particular, if $f = \mathcal{S}_I\colon H[I] \to H[I]$, then this diagram above commutes. We can apply this to the antipode axioms:

$$\sum_{I=S\sqcup T} \mu_{S,T}(\mathrm{id}_S \otimes \mathcal{S}_T)\Delta_{S,T} = 0$$

Rewrite this as the partition of I into $S = I, T = \emptyset$ and a bunch of other terms:

$$\underbrace{\mu_{\emptyset,I}(\mathrm{id}_\emptyset \otimes \mathcal{S}_I)\Delta_{\emptyset,I}}_{\mathcal{S}_I} + \sum_{\substack{I=S\sqcup T\\ T\neq I}} \mu_{S,T}(\mathrm{id}_S \otimes \mathcal{S}_T)\Delta_{S,T} = 0$$

Now rearranging terms, we get

$$\mathcal{S}_I = -\sum_{\substack{I=S\sqcup T\\ T\neq I}} \mu_{S,T}(\mathrm{id}_S \otimes \mathcal{S}_T)\Delta_{S,T}$$

We have proved the following proposition

Proposition 5.31 (Milnor-Moore Formula)**.** *Let* H *be a connected Hopf monoid in* $\mathbf{Sp}_k$ *with antipode* $\mathcal{S}$*. Then the following recursive formulas hold*

$$\mathcal{S}_I = -\sum_{\substack{I=S\sqcup T\\ T\neq I}} \mu_{S,T}(\mathrm{id}_S \otimes \mathcal{S}_T)\Delta_{S,T}$$

$$\mathcal{S}_I = -\sum_{\substack{I=S\sqcup T\\ S\neq I}} \mu_{S,T}(\mathcal{S}_S \otimes \mathrm{id}_T)\Delta_{S,T}$$

Remark 5.32. Milnor-Moore are famous for many things in the study of Hopf algebras, and their names are more commonly attached to a theorem on the structure of Hopf algebras than the above formula.

The Milnor-Moore formula is often useful, but we would like to have a more explicit one that isn't recursive. Recall that in Lemma 5.28 we determined that for our connected Hopf monoid H in $\mathbf{Sp}_k$,

$$\mathcal{S} = \mathrm{id}^{-1} = (u - (u - \mathrm{id}))^{-1} = \sum_{k\geq 0} (u - \mathrm{id})^{*k}. \tag{5}$$

Note that this is well-defined because by Lemma 5.28, $u - \mathrm{id}$ is locally nilpotent with respect to convolution powers.

Now let $f = u - \mathrm{id}$. We know two things about components of f.

(1) $f_\emptyset = u_\emptyset - \mathrm{id}_\emptyset = \iota_\emptyset\varepsilon_\emptyset - \mathrm{id}_\emptyset$. Since H is connected, we see that $\iota_\emptyset\varepsilon_\emptyset = \mathrm{id}_\emptyset$. Hence, $f_\emptyset = 0$.

(2) $f_I = u_I - \mathrm{id}_I = \iota_I \varepsilon_I - \mathrm{id}_I$ for all $I \neq \emptyset$. But since the composition $H[I] \xrightarrow{\varepsilon_I} 1[I] \xrightarrow{\iota_I} H[I]$ is zero unless $I = \emptyset$, we get that $f_I = -\mathrm{id}_I$.

We will use these facts to compute the I-component of the convolution powers of f.

$$(f^{*k})_I = \sum_{I = S_1 \sqcup \ldots \sqcup S_k} \mu_{S_1,\ldots,S_k}(f_{S_1} \otimes \ldots \otimes f_{S_k})\Delta_{S_1,\ldots,S_k}. \tag{6}$$

and the species 1 is concentrated at $\emptyset$; that is, $1[I] = 0$ for $I \neq \emptyset$. Therefore, we can simplify (6) by

$$(f^{*k})_I = \sum_{\substack{I = S_1 \sqcup \ldots \sqcup S_k \\ S_i \neq \emptyset \forall\ i}} (-1)^k \mu_{S_1,\ldots,S_k}\Delta_{S_1,\ldots,S_k}.$$

This in turn can be used to simplify (5), as

$$\begin{aligned} \mathrm{S}_I &= \sum_{k \geq 0} (-1)^k \sum_{\substack{I = S_1 \sqcup \ldots \sqcup S_k \\ S_i \neq \emptyset \forall i}} \mu_{S_1,\ldots,S_k}\Delta_{S_1,\ldots,S_k} \\ &= \sum_{F \in \Sigma[I]} (-1)^{\ell(F)} \mu_F \Delta_F \end{aligned}$$

Here, $F \in \Sigma[I]$ is a composition of I. We have proved

Proposition 5.33 (Takeuchi's Formula). *If* H *is a connected Hopf monoid in* $\mathbf{Sp}_k$*, then*

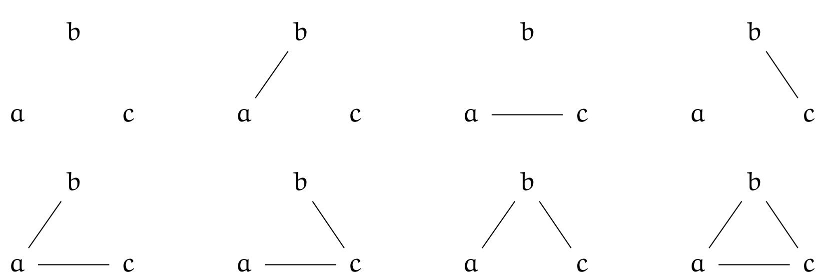
$$\mathrm{S}_I = \sum_{F \in \Sigma[I]} (-1)^{\ell(F)} \mu_F \Delta_F.$$

Let's apply Proposition 5.33 and Proposition 5.31 to some examples.

Example 5.34. Let G be the set species of simple graphs. This means that

$$G[I] = \{g \mid g \text{ is a simple graph with vertex set } I\}$$

For example, the simple graphs on the vertex set $I = \{a, b, c\}$ are



Notice that

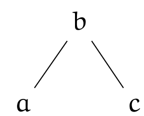
$$\#G[I] = 2^{\binom{\#I}{2}}.$$

This is a set sepecies, and a bimonoid in the category of set species. The product is the union of graphs, and the coproduct is

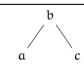
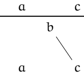
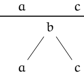
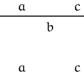
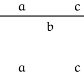
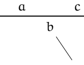
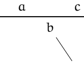
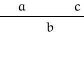
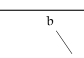
$$\begin{array}{rcl} G[I] & \longrightarrow & G[S] \times G[T] \\ g & \longmapsto & (g|_S, g|_T) \end{array}$$

where $g|_S$ is the induces subgraph on vertex set $S \subseteq I$. Hence, G is a bimonoid in the category of set species, which implies kG is a bimonoid in $\mathbf{Sp}_k$.

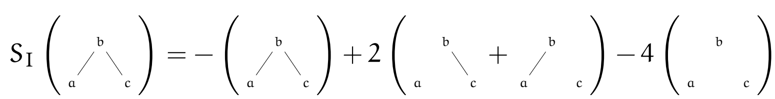
It is connected, so it is a Hopf monoid. So what is S? Let's compute. Consider

$g =$ 

Then, using Takeuchi's Formula, we see that

F	$(-1)^{\ell(F)}$	$\mu_F \Delta_F(g)$
abc	-1	
$a\|bc$	$+1$	
$bc\|a$	$+1$	
$b\|ac$	$+1$	
$ac\|b$	$+1$	
$c\|ab$	$+1$	
$ab\|c$	$+1$	
$a\|b\|c$	-1	
$\vdots$	$\vdots$	$\vdots$
$c\|b\|a$	-1	

Therefore, we get



We can use this example to figure out what the antipode looks like for $G[I]$ in general, although it would take a lot of work to prove this right now.

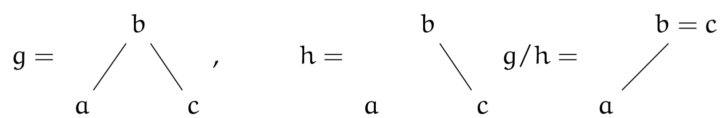
Proposition 5.35. *For any $\mathfrak{g} \in \mathrm{G}[\mathrm{I}]$,*

$$\mathrm{S}_{\mathrm{I}}(\mathfrak{g}) = \sum_{\mathfrak{h} \text{ flat of } \mathfrak{g}} (-1)^{c(\mathfrak{h})} a(\mathfrak{g}/\mathfrak{h})\, \mathfrak{h},$$

where

- $c(\mathfrak{g})$ *is the number of connected components of* $\mathfrak{g}$*;*
- $a(\mathfrak{g})$ *is the number of acyclic orientations of* $\mathfrak{g}$*;*
- $\mathfrak{g}/\mathfrak{h}$ *is the graph obtained by contracting each edge in* $\mathfrak{h} \subseteq \mathfrak{g}$ *(and removing multiple edges);*
- $\mathfrak{h}$ *is a **flat** of* $\mathfrak{g}$ *according to Definition 5.36.*

For example,



Definition 5.36. A graph $\mathfrak{h}$ is a **flat** of a graph $\mathfrak{g}$ if both $\mathfrak{g}, \mathfrak{h} \in \mathrm{G}[\mathrm{I}]$ and for each connected component C of $\mathfrak{h}$, $\mathfrak{h}|_C = \mathfrak{g}|_C$.

Equivalently, if two vertices are connected by a path in $\mathfrak{h}$ and an edge in $\mathfrak{g}$, then that edge belongs to $\mathfrak{h}$.

Example 5.37. Let's calculate the antipode of $\mathrm{k}\Sigma$, where Σ is the species of compositions. THis is easier than the computation for G, because $\Sigma = \mathrm{L} \circ \mathrm{E}_+$ whereas $\mathrm{G} = \mathrm{E} \circ \mathrm{G}_{\text{conn}}$, where G_{conn} is the species of simple connected graphs.

Suppose that $F = (S_1, \dots, S_k) \in \Sigma[\mathrm{I}]$. This implies that $F = \perp_{S_1} \cdot \dots \cdot \perp_{S_k}$, where $\perp_{\mathrm{I}} = (\mathrm{I})$ is the unique composition of I as a single block.

So we see that

$$\mathrm{S}_{\mathrm{I}}(F) = \mathrm{S}_{S_k}(\perp_{S_k}) \cdots \mathrm{S}_{S_1}(\perp_{S_1}). \tag{7}$$

Hence, we only need to compute $S_{\mathrm{I}}(\perp_{\mathrm{I}})$. We can do this using Takeuchi. First, recall that

$$\Delta_{S,T}(F) = F|_S \otimes F|_T,$$

where $F|_S = (S_1 \cap S, \dots, S_k \cap S)$, removing any empty intersections.

Now we apply Takeuchi for $\perp_{\mathrm{I}}$.

$$\begin{aligned} \mathrm{S}_{\mathrm{I}}(\perp_{\mathrm{I}}) &= \sum_{F \in \Sigma[\mathrm{I}]} (-1)^{\ell(F)} \mu_F \Delta_F(\perp_{\mathrm{I}}) \\ &= \sum_{F \in \Sigma[\mathrm{I}]} (-1)^{\ell(F)} F \end{aligned}$$

Then feeding this back into (7), we get

$$\mathbb{S}_I(F) = \sum_{\substack{F_i \in \Sigma[S_i] \\ i=1,\ldots,k}} (-1)^{\ell(F_1)} \cdots (-1)^{\ell(F_k)} F_k \cdot \cdots \cdot F_1.$$

We can simplify this if we set $G = F_k \cdot \cdots \cdot F_1$, then

$$\mathbb{S}_I(F) = \sum_{G\colon \overline{F} \leq G} (-1)^{\ell(G)} G$$

where $\overline{F}$ is the reversal of F (that is, the sum is over all refinements G of the reversal $\overline{F}$ of F).

Example 5.38. Recall that $\Pi = S(E^+)$, that is, Π is the free monoid on E_+. The generators are the one-block partitions $\perp_I = \{I\} \in \Pi[I]$.

Let's compute the antipode of $k\Pi$, where Π is the species of partitions. Let $X \in \Pi[I]$. Then

$$\mathbb{S}_I(X) = \sum_{Y\colon X \leq Y} (-1)^{\ell(Y)} \left({}^{Y}\!/_{X}\right)! \, Y$$

where $\left({}^{Y}\!/_{X}\right)!$ is the integer

$$\left({}^{Y}\!/_{X}\right)! = \prod_{B \in X} m_B!$$

where m_B is the number of blocks of Y refining the block B of X.

Alternatively, we can find the antipode using the surjective morphism $\Sigma \xrightarrow{S} \Pi$ given by taking a composition F to the underlying partition of F.

Exercise 5.39. Let $f\colon \Pi \to G$ be the unique morphism of monoids such that

$$f(\perp_I) = k_I,$$

where k_I is the complete graph on I.

(a) Show that f is a morphism of Hopf monoids.

(b) Deduce the antipode formula for Π from that for G.

5.7 Convolution Monoids

In $(\mathbf{Set}, \times)$, let X be a monoid. Consider $\mu\colon X \times X \to X$, given by $(x, y) \mapsto xy$. Is μ a morphism of monoids?

$$\mu(x,y)\mu(x',y') = xyx'y'$$

$$\mu\left((x,y)(x',y')\right) = \mu(xx',yy') = xx'yy'$$

This is a morphism of monoids if and only if X is commutative.

Lemma 5.40. *Let* (M, ι, μ) *be a monoid in the braided monoidal category* $(\mathbf{C}, \bullet, \beta)$. *Then* μ *is a morphism of monoids if and only if* M *is commutative.*

Proof. $(\Longrightarrow)$. Exercise.

$(\Longleftarrow)$. Let's write down what it means for μ to be a morphism of monoids. It must preserve respect the multiplication structure on $M \bullet M$,

$$\begin{tikzcd}
(M \bullet M) \bullet (M \bullet M) \arrow[r, "\mu\bullet\mu"] \arrow[d, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"] & M \bullet M \arrow[dd, "\mu"] \\
M \bullet M \bullet M \bullet M \arrow[d, "\mu\bullet\mu"] & \\
M \bullet M \arrow[r, "\mu"] & M
\end{tikzcd} \tag{8}$$

and it must also respect the identity.

$$\begin{tikzcd}
M \bullet M \arrow[r, "\mu"] & M \\
I \bullet I \arrow[u, "\iota\bullet\iota"] \arrow[r, equal] & I \arrow[u, "\iota"']
\end{tikzcd} \tag{9}$$

To show that the first diagram (8) commutes, we can fill it in

$$\begin{tikzcd}
(M \bullet M) \bullet (M \bullet M) \arrow[rr, "\mu\bullet\mu"] \arrow[d, "\mathrm{id}\bullet\beta\bullet\mathrm{id}"] \arrow[dr, "\mathrm{id}\bullet\mu\bullet\mathrm{id}"] & & M \bullet M \arrow[dd, "\mu"] \\
M \bullet M \bullet M \bullet M \arrow[r, "\mathrm{id}\bullet\mu\bullet\mathrm{id}"] \arrow[d, "\mu\bullet\mu"] & M \bullet M \bullet M \arrow[dr, "\mu^{(2)}"] & \\
M \bullet M \arrow[rr, "\mu"] & & M
\end{tikzcd}$$

where $\mu^{(2)} = \mu(\mu \bullet \mathrm{id}) = \mu(\mathrm{id} \bullet \mu)$. The triangle commutes by commutativity of M, and the two squares commute by associativity. Hence Eq. (8) commutes. To show that Eq. (9) commutes, we fill it in.

$$\begin{tikzcd}
M \bullet M \arrow[rr, "\mu"] & & M \\
& I \bullet M \arrow[ul, "\iota\bullet\mathrm{id}"] \arrow[ur, equal] & \\
I \bullet I \arrow[uu, "\iota\bullet\iota"] \arrow[ur, "\mathrm{id}\bullet\iota"] \arrow[rr, equal] & & I \arrow[uu, "\iota"']
\end{tikzcd}$$

The left triangle commutes by functoriality of $\bullet$, and the top triangle commutes by a unit law, and the square commutes by naturality. Hence Eq. (9) commutes. $\square$

Remark 5.41 (Recall)**.** If C is a comonoid in $\mathbf{C}$ and M is a monoid in $\mathbf{C}$, then $\mathrm{Hom}_{\mathbf{C}}(C, M)$ is a monoid in $(\mathbf{Set}, \times)$ under convolution.

Now let B be a bimonoid. Look at

$$\mathrm{Hom}_{\mathbf{Mon}(\mathbf{C})}(B, M) \subseteq \mathrm{Hom}_{\mathbf{C}}(B, M),$$

where $\mathbf{Mon}(\mathbf{C})$ is the category of monoids in $\mathbf{C}$ and monoid morphisms. We can ask when this is a submonoid of the convolution monoid $\mathrm{Hom}_{\mathbf{C}}(B, M)$.

The answer is not always.

Exercise 5.42. If $f, g\colon B \to M$ are morphisms of monoids, then $f \bullet g$ is a morphism of monoids as well.

Proposition 5.43. *If M is commutative, then $\mathrm{Hom}_{\mathbf{Mon}(\mathbf{C})}(B, M)$ is a submonoid of $\mathrm{Hom}_{\mathbf{C}}(B, M)$*

Proof. Take $f, g\colon B \to M$ a morphism of monoids. Then $f * g = \mu \circ (f \bullet g) \circ \Delta$. We know that

- $\to$ $\Delta\colon B \to B \bullet B$ is a morphism of monoids by a bimonoid axiom
- $\to$ f, g are morphisms of monoids, so $f \bullet g$ is a morphism of monoids as well by Exercise 5.42
- $\to$ M commutative, so $\mu\colon M \bullet M \to M$ is a morphism of monoids by Lemma 5.40.

Therefore, $f * g$ is a morphism of monoids.

We also need to know that the unit $u = \iota \circ \varepsilon$ of $\mathrm{Hom}_{\mathbf{C}}(B, M)$ is a morphism of monoids. But we know that

- $\to$ $\varepsilon\colon B \to I$ is a morphism of bimonoids by a bimonoid axiom
- $\to$ ι is always a morphism of monoids.

Therefore, μ is a morphism of monoids. □

Definition 5.44. If M is a monoid in $\mathbf{Set}$, let $M^\times$ denote the group of invertible elements in M.

Proposition 5.45. *If H is a Hopf monoid with antipode S, M a monoid, then*

$$\mathrm{Hom}_{\mathbf{Mon}(\mathbf{C})}(H, M) \subseteq \mathrm{Hom}_{\mathbf{C}}(H, M)^\times$$

In fact, the inverse of a morphism of monoids $\phi\colon H \to M$ is $\phi \circ S$.

Proof.

$$\begin{aligned} \phi_{\#} \colon \mathrm{Hom}_{\mathbf{C}}(H, H) &\longrightarrow \mathrm{Hom}_{\mathbf{C}}(H, M) \\ f &\longmapsto \phi \circ f \end{aligned}$$

This is a morphism of ordinary monoids, so we see that $\phi_{\#}(S)$ is the convolution inverse of $\phi_{\#}(\mathrm{id})$. Hence, $\phi \circ S$ is the convolution inverse of ϕ in $\mathrm{Hom}_{\mathbf{MonC}}(H, M)$. □

Corollary 5.46. *If* H *is a Hopf monoid and* M *is a commutative monoid, then* $\mathrm{Hom}_{\mathbf{Mon(C)}}(H, M)$ *is a subgroup of* $\mathrm{Hom}_{\mathbf{C}}(H, M)^{\times}$*.*

Proof. We need to show that $\phi \circ S$ is again a morphism of monoids when ϕ is. We know that ϕ is a morphism of monoids, and so preserves products. But S is an antimorphism of monoids, and therefore reverses products. So $\phi \circ S$ is an antimorphism as well. But M is commutative, so $\phi \circ S$ preserves products and is therefore a morphism of monoids. □

6 Characters

Definition 6.1. Let H be a Hopf monoid in $\mathbf{Sp}_k$. A **character on** H is a morphism of monoids $\phi \colon H \to kE$.

This amounts to a collection of k-linear maps $\phi_I \colon H[I] \to kE[I] \cong k$, such that

$$\phi_{\varnothing}(1) = 1,$$

$$\phi_I(x \cdot y) = \phi_S(x)\phi_T(y)$$

for all $x \in H[S]$ and $y \in H[T]$ such that $I = S \sqcup T$. The multiplication above is multiplication in k. Also

$$\phi_J(\sigma^*(x)) = \phi_I(x)$$

for all $\sigma \colon I \xrightarrow{\sim} J$ and all $x \in H[I]$.

Definition 6.2. Let $\mathbb{X}(H)$ be the set of characters on H.

By Corollary 5.46, $\mathbb{X}(H)$ is a group under convolution. Explicitly, the convolution of two characters is given by

$$(\phi * \psi)_I = \sum_{I = S \sqcup T} (\phi_S \otimes \psi_T) \circ \Delta_{S,T}$$

We omit the $\mu \colon kE[S] \otimes kE[T] \to kE[I]$ because $kE[J] \cong k$ for all J and $k \otimes k$ is canonically isomorphic to k. So the codomain is $k \otimes k \cong k$.

The unit u is then

$$u_I = \begin{cases} \varepsilon_\varnothing \colon H[\varnothing] \to k & \text{if } I = \varnothing \\ 0 & \text{if } I \neq \varnothing \end{cases}$$

The convolution inverse of ϕ is $\phi^{-1} = \phi \circ S$, as in Proposition 5.45. So

$$(\phi^{-1})_I = \phi_I \circ S_I$$

Example 6.3. What is $\mathbb{X}(kE)$? Let's take $\phi \in \mathbb{X}(kE)$. Because ϕ is multiplicative, it suffices to consider only singleton sets. Let $X = \{*_X\}$ We have that $kE[\{*_X\}] = k\{*_X\}$, so write $\phi_X(*_X) = \lambda \in k$, independent of X.

Then for any set I,

$$\phi_I(*_I) = \phi_I\left(\prod_{i\in I} *_{\{i\}}\right) = \lambda^{\#I}.$$

Define $\mathbb{X}(kE) \longrightarrow (k, +)$ by $\phi \mapsto \lambda$. If ϕ corresponds to $\lambda \in k$ and ψ corresponds to $\mu \in k$, then

$$(\phi * \mu)_I(*_X) = \phi_X(*_X)\psi_\varnothing(*_\varnothing) + \phi_\varnothing(*_\varnothing)\psi_X(*_X) = \lambda \cdot 1 + 1 \cdot \mu.$$

Proposition 6.4. $\mathbb{X}(k\Pi) \cong \left\{ \sum_{m \geq 0} a_n \frac{x^m}{m!} \in k[[x]] \;\middle|\; a_0 = 1 \right\}.$

The right-hand side of the above is a group under multiplication.

Proof. Define the map

$$\begin{array}{ccc} \mathbb{X}(k\Pi) & \xrightarrow{\cong} & \left\{ \sum_{m\geq 0} a_n \frac{x^m}{m!} \in k[[x]] \;\middle|\; a_0 = 1 \right\} \\ \phi & \longmapsto & \sum_{m\geq 0} a_n \frac{x^m}{m!} \end{array}$$

where $a_m = \phi_{[m]}(\perp_{[m]})$.

As we discussed long ago in our discussion of free commutative monoids, Π is free on the generators $\perp_I$. This tells us that the map above is a bijection. Now we have to check that the convolution product on $\mathbb{X}(k\Pi)$ corresponds to multiplication of power series.

$$\begin{aligned} (\phi * \psi)_{[m]}(\perp_{[m]}) &= \sum_{[m] = S \sqcup T} (\phi_S \otimes \phi_T)\Delta_{S,T}(\perp_{[m]}) \\ &= \sum_{[m] = S \sqcup T} \phi_S(\perp_S)\psi_T(\perp_T) \\ &= \sum_{m = i+j} \binom{m}{i} \phi_{[i]}(\perp_{[i]})\psi_j(\perp_{[} j]) \\ &= \sum_{m=i+j} \binom{m}{i} a_i b_j \end{aligned}$$

Therefore, if

$$\phi \mapsto \sum_{m \geq 0} a_m \frac{x^m}{m!} \qquad \text{and} \qquad \psi \mapsto \sum_{m \geq 0} b_m \frac{x^m}{m!}$$

Then

$$\begin{aligned}(\phi * \psi) \mapsto \sum_{m \geq 0} \left(\sum_{m=i+j} \binom{m}{i} a_i b_j \right) \frac{x^m}{m!} \\ &= \sum_{i,j \geq 0} a_i \frac{x^i}{i!} b_j \frac{x^j}{j!} \\ &= \left(\sum_{m \geq 0} a_m \frac{x^m}{m!} \right) \left(\sum_{n \geq 0} b_n \frac{x^n}{n!} \right) \end{aligned}$$

□

Example 6.5 (An Application)**.** Let

$$a(x) = \sum_{m \geq 0} a_m \frac{x^m}{m!}$$

with $a_0 = 1$. Let

$$b(x) = \frac{1}{a(x)} = \sum_{m \geq 0} b_m \frac{x^m}{m!}.$$

We want an explicit expression for b_m in terms of the a_m. We could do it by hand, but that's gross. If we did it anyway, we'd see that

$$\begin{aligned} b_0 &= 1 \\ b_1 &= -a_1 \\ \tfrac{b_2}{2!} &= a_1^2 - \tfrac{a_2}{2} \\ \tfrac{b_3}{3!} &= -a_1^3 - a_1 a_2 - a_3 \\ &\vdots \end{aligned}$$

Fortunately, there's a better way.

Let $\phi \in \mathbb{X}(\mathrm{k}\Pi)$ correspond to $a(x)$. Then $\phi^{-1} = \phi \circ \mathsf{S}$ corresponds to $b(x)$. Hence,

$$b_m = (\phi^{-1})_{[m]}(\perp_{[m]}) = \phi_{[m]}(\mathsf{S}_{[m]}(\perp_{[m]}))$$

Recall that for $X \in \Pi[I]$,

$$\mathsf{S}_I(X) = \sum_{Y\colon X \leq Y} (-1)^{\ell(Y)} \left({}^{Y}\!/_{X} \right)! \, Y$$

$$\mathbb{S}_{[n]}(\perp_{[n]}) = \sum_{Y \in \Pi[n]} (-1)^{\ell(Y)} \ell(Y)!\, Y \tag{10}$$

$$b_0 = a_0 \phi_{[n]}(Y) = \prod_{B \in Y} \phi_B(\perp_B) = \prod_{B \in Y} a_{\#B} \tag{11}$$

Therefore, combining (10) and (11), we see that

$$b_n = \sum_{Y \in \Pi[n]} (-1)^{\ell(Y)} \ell(Y)! \prod_{B \in Y} a_{\#B}$$

For example,

$$n = 1: \qquad Y = \perp_{[1]} \implies b_1 = -a_1$$

$$n = 2: \qquad \left.\begin{aligned} Y &= \perp_{[2]} = 12 \\ Y &= 1|2 \end{aligned}\right\} \implies b_2 = -a_2 + 2a_1^2$$

$$n = 3: \qquad \left.\begin{aligned} Y &= 123 \\ Y &= 1|23 \\ Y &= 2|13 \\ Y &= 3|13 \\ Y &= 1|2|3 \end{aligned}\right\} \implies b_3 = -a_3 + 3 \cdot 2! a_1 a_2 - 3! a_1^3$$

Remark 6.6. $\mathbb{X}(kL) \cong \mathbb{X}(kE)$
$\mathbb{X}(k\Sigma) \cong \mathbb{X}(k\Pi)$
The reason is that characters factor, so the following diagrams commute

$$\begin{tikzcd} kL \arrow[r, "\phi"] \arrow[d, "s"] & kE \\ kE \arrow[ur, dashed, "\widehat{\phi}"] & \end{tikzcd} \qquad \begin{tikzcd} k\Sigma \arrow[r, "\phi"] \arrow[d, "s"] & kE \\ k\Pi \arrow[ur, dashed, "\widehat{\phi}"] & \end{tikzcd}$$

6.1 Lagrange Inversion

Recall the inverse function theorem from analysis.

Theorem 6.7 (Inverse Function Theorem). *Suppose that* $f\colon \mathbb{R} \to \mathbb{R}$ *is differentiable around* a*,* $f(a) = b$*,* $f'(a) \neq 0$*. Then* f^{-1} *exists in a neighborhood of* b *and*

$$f^{-1}(b) = a, \qquad (f^{-1})'(b) = \frac{1}{f'(a)}.$$

There is an analogous statement for formal power series, which we will recover using Hopf monoids in the category of species.

6.2 Formal Diffeomorphisms

Definition 6.8. The set of **formal diffeomorphisms of the line** is

$$\mathrm{Diff}^1 = \left\{ \sum_{n\geq 0} a_n x^n \in k[[x]] \;\middle|\; a_0 = 0, a_1 = 1 \right\}.$$

Definition 6.9 (Notation)**.** If we are interested in the coefficient of x^n in the power series $f(x) = \sum_{n\geq 0} a_n x^n$, then we use the notation

$$[x^n]f(x) = a_n.$$

Definition 6.10. Let $f(x) = x + a_2x^2 + a_3x^3 + \ldots \in \mathrm{Diff}^1$. Then denote the coefficients of the i-th power of $f(x)$ by

$$f(x)^i = x^i + a_{i,i+1}x^{i+1} + a_{i,i+2}x^{i+2} + \ldots$$

Definition 6.11. Given $f(x) = \sum_{n\geq 1} a_n x^n$, $g = \sum_{m\geq 1} b_m x^m$, define their **composition** $(f \circ g)(x) \in \mathrm{Diff}^1$ by

$$(f \circ g)(x) = \sum_{i\geq 1} a_i g(x)^i$$

The right hand side of $(f \circ g)(x)$ in the definition above is an infinite sum of power series, we need to check that it's well-defined. But since $g(x)^i$ starts with x^i, the coefficient of x^n in $(f \circ g)(x)$ is

$$[x^n](f \circ g)(x) = [x^n]\sum_{i=1}^{n} a_i g(x)^i = \sum_{i=1}^{n} a_i \sum_{j_1+\ldots+j_i=m} b_{j_1}\cdots b_{j_i}.$$

Hence the coefficient of x^n is a finite sum, for all n. Hence, the composite is well-defined.

Proposition 6.12 (Formal inverse function theorem)**.** Diff^1 *is a group with unit* x *under composition.*

Proof. The proof that x is a unit is easy. It's slightly harder to see associativity, and a bit messy, but not hard.

The meat of this theorem is producing the inverse for any given power series $f(x) = \sum_{n\geq 0} a_n x^n$. It's enough to show that every $g(x) \in \mathrm{Diff}^1$ has a left-inverse $f(x)$. (This is because, once every element has a left inverse, then the left inverse f of an element g has a left inverse h, $hf = 1$, then we can show that $h = g$.)

To see that $f(x)$ has a left-inverse, we can solve the system of equations

$$\sum_{i=1}^{m} a_i \sum_{j_1+\ldots+j_i=m} b_{j_1}\cdots b_{j_i} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

for a_n in terms of the sequence b_n. We know that we can always solve this system because there is only one term with a_n in it in the n-th equation, and knowing b_i for all i and a_i for $i < n$ lets us solve for a_n.

For example, when $n = 1$,

$$a_1 b_1 = 1 \implies a_1 = \frac{1}{b_1} = 1$$

When $n = 2$,

$$a_1 b_2 + a_2 b_1^2 = 0 \implies a_2 = -b_2$$

When $n = 3$,

$$a_1 b_2 + a_2(b_1 b_2 + b_2 b_1) + a_3 b_1^3 \implies a_3 = -b_3^3 + 2b_2^2.$$

□

Definition 6.13 (Notation)**.** If $g(x)$ has composition-inverse $f(x)$, then we write $g(x)^{\langle -1 \rangle} = f(x)$.

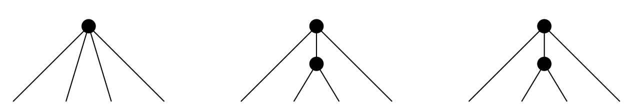
Theorem 6.14 (Combinatorial Lagrange Inversion)**.** *Let* $g(x) \in \mathrm{Diff}^1$ *with composition-inverse* $g(x)^{\langle -1 \rangle}$*. Then the coefficient of* x^n *in* $g(x)^{\langle -1 \rangle}$ *is given by the formula*

$$[x^n] g(x)^{\langle -1 \rangle} = \sum_{t \in \mathrm{PRT}(n)} (-1)^{i(t)} \prod_{v \in I(t)} b_{c(v)},$$

where

- $\mathrm{PRT}(n)$ *is the set of planar rooted trees with* n *leaves*
- $I(t)$ *is the set of internal nodes (non-leaves)*
- $i(t) = |I(t)|$
- $c(v)$ *is the number of children of a node* v *(internal or root).*

Example 6.15. PRT(4) consists of all planar rooted trees with four leaves. The trees below are some examples of planar rooted trees with four leaves. (Note to reader: I refuse to draw all of them).



We can use this to compute

$$[x^4] g(x)^{\langle -1 \rangle} = -b_4 + 5b_3 b_2 - 5b_2^3$$

6.3 Sewing and Ripping

Let W be the species of simple graphs (we called it G previously), but with a new Hopf monoid structure.

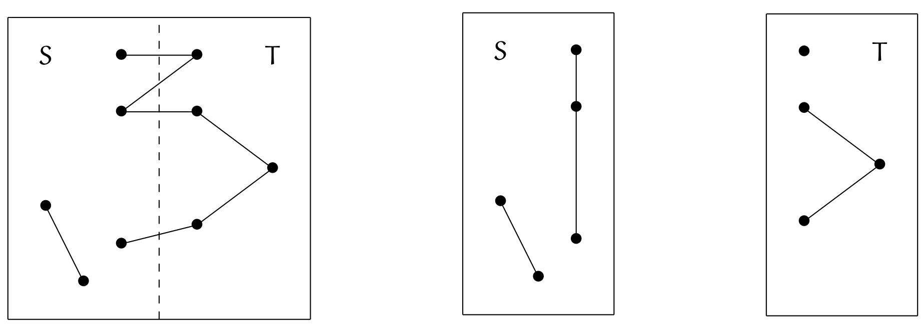
$$\begin{aligned} \mu_{S,T}\colon W[S] \times W[T] &\longrightarrow W[I] \\ (g_1, g_2) &\longmapsto g_1 \cup g_2 \end{aligned}$$

$$\begin{aligned} \Delta_{S,T}\colon W[I] &\longrightarrow W[S] \times W[T] \\ g &\longmapsto (g|_S, g/_S) \end{aligned}$$

where

- $g/_S$ is g but only with the edges incident to T (**rip off** S).
- $g|_S$ is g where we keep edges incident to S and **sew in** edges between u and v when there is a T-**thread** between u and v.

A T-**thread** is a path connecting 2-vertices u, v in S through vertices in T.

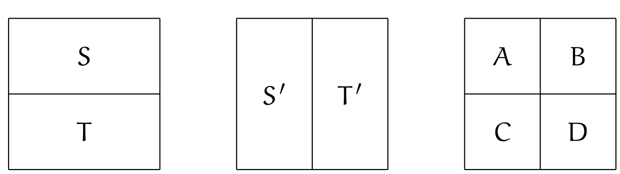


g $g|_S$ $g/_S$

Proposition 6.16. kW *is a connected Hopf monoid.*

Proof. It is enough to check that, if $I = S \sqcup T = S' \sqcup T'$, $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, $D = T \cap T'$. then

$$(g_1 \cdot g_2)|_{S'} = g_1|_A \cdot g_2|_B.$$

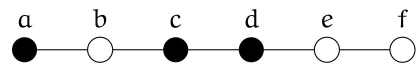


But this is relatively easy to check. $\square$

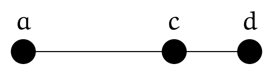
Definition 6.17. Now let W_p be the submonoid of W generated by paths.

This next example will be useful in the next definition.

Example 6.18. Let

$p =$ 

and $S = \{a, c, d\}$, $T = \{b, e, f\}$.

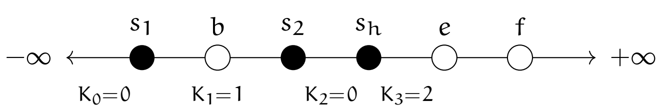
$p|_S =$ 

$p/_S =$ 

Example 6.19. More generally, write $S = \{s_1, \dots, s_h\}$ with $s_1 < \dots < s_h$ along the path. Count elements in T between each pair of consecutive elements of S, including to the left of s_1 and to the right of s_h. Then let

$$\begin{aligned} K_0 &= |T \cap (-\infty, s_1)| \\ K_1 &= |T \cap (s_1, s_2)| \\ &\vdots \\ K_h &= |T \cap (s_h, +\infty)| \end{aligned}$$

Then $p|_S$ is a path on h vertices, while $p/_S$ is a product of paths in K_0, K_1, $\dots$, K_h elements.



Note that

$$K_0 + K_1 + \dots + K_h = |T| = n - 1 - h$$

and moreover, the values $K_0, \dots, K_h$ determine S and T uniquely.

Proposition 6.20.

(i) *W_p is free commutative on paths.*

(ii) *W_p is a Hopf submonoid.*

(iii) $\mathbb{X}(W_p) \cong \mathrm{Diff}^1$.

Proof.

(i) Clear.

(ii) Let p be a path on $I = S \sqcup T$. Then $p|_S$ is again a path, while $p/_S$ is a product of paths. In any case, both terms are in W_p. Since Δ is a morphism of monids, then this implies $\Delta(W_p) \subseteq W_p \bullet W_p$.

(iii) Define $\mathbb{X}(W_p) \to \mathrm{Diff}^1$ by

$$\phi \longmapsto \sum_{n\geq 1} a_n x^n$$

by saying that $a_n = \phi(p)$ where p is *any* path on $n-1$ vertices. This is well-defined by naturality because any 2 paths on the same number of vertices are isomorphic.

In particular, $a_1 = \phi(1) = 1$.

Since W_p is free commutative on paths, then this is a bijection. We need to verify that this is a group homomorphism.

Suppose that

$$\phi \longmapsto f(x) = \sum_{n\geq 1} a_n x^n$$
$$\psi \longmapsto g(x) = \sum_{n\geq 1} b_n x^n$$

Suppose also that

$$\phi * \psi \longmapsto \sum_{n\geq 1} c_n x^n$$

Then

$$c_n = (\phi * \psi)(p) = \sum_{I = S\sqcup T} \phi(p|_S)\psi(p/_S)$$

Make the change of variables $h = |S|$ and $m - 1 - h = |T|$, (see Example 6.19) to get

$$c_n = \sum_{h=0}^{n-1} \sum_{K_0+K_1+\ldots+K_h = n-1-h} a_{h+1} b_{K_0+1} b_{K_1+1} \cdots b_{K_h+1}$$

And then again making a change of variables with $i = h+1$ and $j_r = K_{r-1}+1$, we have

$$c_n = \sum_{i=1}^{m} \sum_{j_1+\ldots+j_i = m} a_i b_{j_1} \cdots b_{j_i} = [x^n](f \circ g)(x),$$

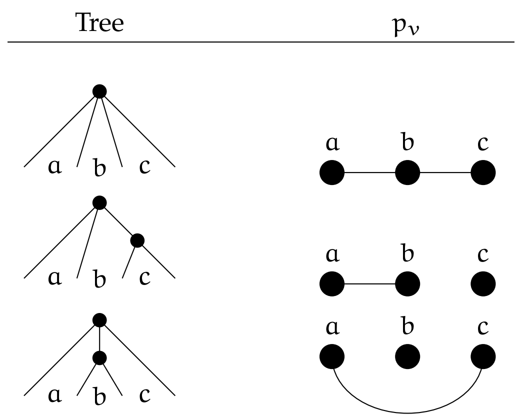
as we wanted. $\square$

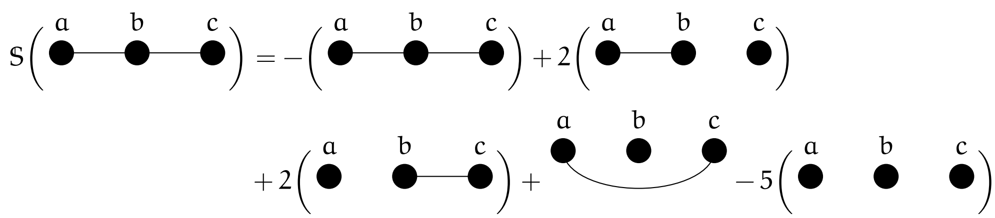
Theorem 6.21. *Let* p *be a path on* $n-1$ *vertices. Then the antipode* S *of* W_p *is*

$$S(p) = \sum_{t \in PRT(n)} (-1)^{i(t)} \prod_{v \in I(t)} p_v$$

where p_v *is the path on the labels visible from* v*, when we label the* $n-1$ *regions between leaves of* t *with the elements of* I*, according to* p*.*

Example 6.22.





Corollary 6.23 (Lagrange Inversion)**.** *Let* $g(x) = \sum_{n \geq 1} b_n x^n$*, and suppose that* $\psi \in \mathbb{X}(W_p)$ *is such that* $\psi \mapsto g(x)$ *under the isomorphism of groups* $W_p \cong \mathbb{X}(W_p)$*. Then*

$$[x^n] g(x)^{\langle -1 \rangle} = (\psi \circ S)(p)$$

and $\psi(p_v) = b_{c(v)}$*.*

6.4 Invariants from Characters

Remark 6.24 (Convention)**.** Throughout this section, assume that k is a field of characteristic zero.

Definition 6.25. Let H be a connected Hopf monoid in $\mathbf{Sp}_k$, and $\phi \in \mathbb{X}(\mathrm{H})$ a character. Define $\chi\colon \mathrm{H} \times \mathbb{N} \to k$ by

$$\begin{array}{rcl} \chi_I\colon \mathrm{H}[I] \times \mathbb{N} & \longrightarrow & k \\ (x, n) & \longmapsto & \phi_I^{*n}(x) \end{array}$$

We call χ the **invariant associated to the character** ϕ.

For each $n \in \mathbb{N}$, consider the function

$$\chi(\,\cdot\,, n) = \phi^{*n}\colon \mathrm{H} \to \mathrm{E},$$

that is, χ takes the n-th convolution power of ϕ. The symbol $\cdot$ represents an empty spot where we will later plug in inputs. Note that $\chi(\,\cdot\,, 1) = \phi$ and $\chi(\,\cdot\,, 0) = u$. Moreover, for each finite set I, we have

$$\chi_I(\,\cdot\, n)\colon \mathrm{H}[I] \to k$$

a linear functional. Fix $x \in \mathrm{H}[I]$, and let n vary; then we obtain a function

$$\chi_I(x, \,\cdot\,)\colon \mathbb{N} \to k.$$

Explicitly, we have a function $\chi_I\colon \mathrm{H}[I] \times \mathbb{N} \to k$ for each finite set I.

$$\chi_I(x, n) = \sum_{I = S_1 \sqcup \ldots \sqcup S_n} \left(\phi_{S_1} \otimes \cdots \otimes \phi_{S_m}\right) \Delta_{S_1, \ldots, S_n}(x).$$

There are $n^{\#I}$ summands, in bijection with functions $f\colon I \to [n]$ such that $S_i = f^{-1}(i)$.

Definition 6.26. We say that $x \in \mathrm{H}[I]$ and $y \in \mathrm{H}[J]$ are **isomorphic** if there exists $\sigma\colon I \xrightarrow{\sim} J$ such that $\sigma^*(x) = y$.

This next proposition explains the term **invariant**.

Proposition 6.27. *If $x, y \in \mathrm{H}[I]$ are isomorphic, then $\chi_I(x, n) = \chi_I(y, n)$.*

Proof. Indeed,

$$\chi_J(y, n) = (\phi^{*n})_J(y) = (\phi^{*n})_J(\sigma^*(x)) = (\phi^{*n})_I(x) = \chi_I(y, n) \qquad \square$$

Proposition 6.28. *$\chi_I(x, \,\cdot\,)$ is a polynomial function of n: there is $p(t) \in k[t]$ such that $p(n) = \chi_I(x, n)$ and $\deg p(t) \leq |I|$.*

Proof. We can use the binomial theorem in $\mathrm{Hom}_{\mathbf{Sp}_k}(\mathrm{H}, \mathrm{E})$, which is an algebra under convolution.

$$\chi(\,\cdot\,, n) = \phi^{*n} = (\phi - u + u)^{*n} = \sum_{\ell=0}^{n} \binom{n}{\ell} (\phi - u)^{*\ell} * u^{*(n-\ell)}$$

But recall that u is the convolution identity, so we can omit the term $u^{*(n-\ell)}$. Recall also that $\phi - u$ is locally nilpotent by Lemma 5.28, so

$$\chi_I(x, n) = \sum_{\ell=0}^{n} \binom{n}{\ell} (\phi - u)^{*\ell}(x) = \sum_{\ell=0}^{\#I} (\phi - u)^{*\ell}$$

But we have here that $(\phi - u)_I^{*\ell}(x) \in k$, and $\binom{n}{k}$ is a polynomial function of n. So we take

$$p(t) = \sum_{\ell=0}^{|I|} (\phi - u)_I^{*\ell}(x) \binom{t}{k},$$

where

$$\binom{t}{\ell} = \frac{1}{\ell!} t(t-1) \cdot (t - \ell + 1) \in k[t]. \qquad \square$$

Example 6.29. Let $H = kG$. Let $\phi \colon H \to E$ be the morphism of species defined by

$$\phi_I(g) = \begin{cases} 1 & \text{if } g \text{ is discrete} \\ 0 & \text{otherwise} \end{cases}$$

Then $\phi \in \mathbb{X}(H)$ because, for any two graphs, $g_1 \cup g_2$ is discrete if and only if g_1 and g_2 are discrete. We have that

$$\begin{aligned}
\chi_I(g, n) &= \sum_{I = S_1 \sqcup \ldots \sqcup S_n} (\phi_{S_1} \otimes \ldots \otimes \phi_{S_n}) \Delta_{S_1, \ldots, S_n}(g) \\
&= \sum_{I = S_1 \sqcup \ldots \sqcup S_n} \phi_{S_1}(g|_{S_1}) \cdots \phi_{S_n}(g|_{S_n}) \\
&= \#\left\{ (S_1, \ldots, S_n) \;\middle|\; I = S_1 \sqcup \ldots \sqcup S_n, g|_{S_i} \text{ is discrete for all } i \right\} \\
&= \#\left\{ f \colon I \to [n] \;\middle|\; g|_{f^{-1}(i)} \text{ discrete for all } i \right\} \\
&= \#\left\{ \text{proper } n\text{-colorings of } g \right\}
\end{aligned}$$

Then $\chi_I(g, \cdot\,)$ is the **chromatic polynomial** of g.

Exercise 6.30. Fix $q \in k$. If instead we let $\phi \colon kG \to E$ be

$$\phi_I(g) = q^{\#\text{ edges of } g}.$$

Calculate $\chi_I(g, \cdot\,)$ and check that it is (a reparameterization of) the Tuttle polynomial of g.

Example 6.31. Let $H = (kP)^*$, where P is the species of partial orders as in Exercise 5.22. Recall the operations on the basis y_p^*, which we will instead denote by p for this example. We write $p_1 \cdot p_2$ for the union of partial orders. Recall

$$\Delta_{S,T}(p) = \begin{cases} p|_S \otimes p|_T & \text{if } S \text{ is a lower set of } p \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\Delta R, S, T(p) = \begin{cases} p|_R \otimes p|_S \otimes p|_T & \text{if } R \sqcup S, R \text{ are lower sets of } p \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi\colon (kP)^* \to E$ be

$$\phi_I(p) = \begin{cases} 1 & \text{if } p \text{ is discrete} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \chi_I(p, n) &= \sum_{\substack{I = S_1 \sqcup \ldots \sqcup S_n \\ S_1 \sqcup \ldots \sqcup S_i \text{ is a lower set of } p}} \phi_{S_1}(p|_{S_1}) \cdots \phi_{S_n}(p|_{S_n}) \\ &= \#\left\{ f\colon I \to [n] \;\middle|\; f^{-1}([i]) \text{ lower set of } p \text{ for all } i, f^{-1}(i) \text{ discrete} \right\} \\ &= \#\left\{ \#f\colon I \to [n] \;\middle|\; f \text{ strictly order preserving} \right\} \end{aligned}$$

This means that $x < y$ in p implies $f(x) < f(y)$ in $[n]$.

(We can see the last two lines are equivalent as follows. Take $y \in f^{-1}([i])$ and $x \leq y$ in p. Then $f(x) \leq f(y) \in [i] \implies f(x) \in [i] \implies x \in f^{-1}([i])$.)

So $\chi_I(p\,\cdot\,)$ is the **strict order polynomial** of p.

Exercise 6.32. Given $g \in G[I]$, and an *acyclic* orientation α of g, let $g_\alpha \in P[I]$ be the *partial order* on I given by $x \leq y$ if there is a path $x \to \ldots \to y$ in g, with edges directed according to α. Define $\rho\colon kG \to (kP)^*$ by

$$\phi_I(g) = \sum_{\alpha \in AO(G)} g_\alpha,$$

where $AO(G)$ is the set of acyclic orders on G.

Show that ρ is a morphism of Hopf monoids and, for ζ a character of kG and ϕ a character of $(kP)^*$ as before, the following commutes.

$$\begin{tikzcd} kG \arrow[rr, "\rho"] \arrow[dr, "\zeta"'] & & (kP)^* \arrow[dl, "\phi"] \\ & E & \end{tikzcd}$$

Proposition 6.33 (Functoriality of χ). *Let $\rho\colon \mathrm{H} \to \mathrm{H}'$ be a morphism of Hopf monoids and ϕ, ϕ' characters such that*

$$\begin{tikzcd} \mathrm{H} \arrow[rr, "\rho"] \arrow[dr, "\phi"'] & & \mathrm{H}' \arrow[dl, "\phi'"] \\ & \mathrm{E} & \end{tikzcd}$$

Then $\chi_{\mathrm{I}}(x,t) = \chi'_{\mathrm{I}}(\rho_{\mathrm{I}}(x),t)$ for all $x \in \mathrm{H}[\mathrm{I}]$ for all I.

Proof.

$$\begin{aligned}
\chi'_{\mathrm{I}}(\rho_{\mathrm{I}}(x),n) &= (\phi'^{*n})_{\mathrm{I}}(\rho_{\mathrm{I}}(x)) \\
&= (\phi'^{*n} \circ \rho)_{\mathrm{I}}(x) \\
&= \rho^{\#}(\phi'^{*n})_{\mathrm{I}}(x) \\
&= (\phi^{\#}(\phi')^{*n})_{\mathrm{I}}(x) \\
&= (\phi' \circ \rho)^{*n}_{\mathrm{I}}(x) \\
&= (\phi^{*n})_{\mathrm{I}}(x) = \chi_{\mathrm{I}}(x,n)
\end{aligned}$$

Note that $\rho^{\#}$ preserves convolution because ρ is a morphism of comonoids. $\square$

Example 6.34. A consequence of this is that the chromatic polynomial of G is the sum over all acyclic orderings α of G of the strict order polynomial of the orderings α

$$\chi^{\mathrm{kG}}_{\mathrm{I}}(g,t) = \sum_{\alpha \in \mathrm{AO}(\mathrm{G})} \chi^{(\mathrm{kP})^*}_{\mathrm{I}}(g_\alpha, t)$$

Proposition 6.35. *(i) Let $x \in \mathrm{H}[\mathrm{S}], y \in \mathrm{H}[\mathrm{T}]$. Then $\chi_{\mathrm{I}}(x \cdot y, t) = \chi_{\mathrm{S}}(x,t)\chi_{\mathrm{T}}(y,t)$ as polynomials in $\mathrm{k}[t]$.*

(ii) Let $z \in \mathrm{H}[\mathrm{I}]$. Then, as polynomials in $\mathrm{k}[s,t]$,

$$\chi_{\mathrm{I}}(z, t+s) = \sum_{\mathrm{I}=\mathrm{S}\sqcup\mathrm{T}} \chi_{\mathrm{S}}(z|_{\mathrm{S}}, s)\chi_{\mathrm{T}}(z/_{\mathrm{S}}, t).$$

Proof.

(i) $\chi_{\mathrm{I}}(x \cdot y, n) = (\phi^{*n})_{\mathrm{I}}(x \cdot y) = (\phi^{*n})_{\mathrm{S}}(x)(\phi^{*n})_{\mathrm{T}}(y) = \chi_{\mathrm{S}}(x,n)\chi_{\mathrm{T}}(y,n)$

(ii)

$$\begin{aligned}
\chi_{\mathrm{I}}(z, m+n) &= (\phi^{*(m+n)})_{\mathrm{I}}(z) \\
&= (\phi^{*m} * \phi^{*n})_{\mathrm{I}}(z) \\
&= \sum_{\mathrm{I}=\mathrm{S}\sqcup\mathrm{T}} (\phi^{*m})_{\mathrm{S}}(z|_{\mathrm{S}})(\phi^{*n})_{\mathrm{T}}(z/_{\mathrm{T}}) \\
&= \sum_{\mathrm{I}=\mathrm{S}\sqcup\mathrm{T}} \chi_{\mathrm{S}}(z|_{\mathrm{S}}, m)\chi_{\mathrm{T}}(z/_{\mathrm{S}}, n).
\end{aligned}$$ $\square$

Fix $x \in H[I]$, and consider the polynomial $\chi_I(x,t) \in k[t]$. By construction, the values of $n \in \mathbb{N}$ have a combinatorial interpretation when plugged into $\chi_I(x,\cdot)$, because

$$\chi_I(x,n) = \sum_{I=S_1 \sqcup \ldots \sqcup S_n} (\phi_{S_1} \otimes \cdots \otimes \phi_{S_n})\Delta_{S_1,\ldots,S_n}(x).$$

But since $\chi_I(x,t)$ a polynomial, we can evaluate it at any scalar, in particular for any $t \in \mathbb{Z}$ including $t < 0$.

What is the combinatorial interpretation for $\chi_I(x,-n)$?

Lemma 6.36. *In a ring, if a is nilpotent, then $a+1$ is invertible and*

$$(a+1)^{-1} = \sum_{k \geq 0} (-1)^k a^k.$$

More generally,

$$(a+1)^{-n} = \sum_{k \geq 0} \binom{-n}{k} a^k.$$

Proposition 6.37 (Reciprocity)**.**

(i) $\chi_I(x,-1) = \phi_I(S_I(x))$

(ii) $\chi_I(x,-n) = \sum_{I=S_1 \sqcup \ldots \sqcup S_n} (\phi_{S_1} \otimes \cdots \otimes \phi_{S_n})(S_{S_1} \otimes \ldots \otimes S_{S_n})\Delta_{S_1,\ldots,S_n}(x)$

Proof. Note that (i) is a special case of (ii), so we will only prove (ii).

Recall that $\chi_I(x,t) = p(t)$ where

$$p(t) = \sum_{k \geq 0} (\phi - u)_I^{*k}(x) \binom{t}{k}$$

This implies that

$$\chi_I(x,-n) = \sum_{k \geq 0} (\phi - u)_I^{*k}(x) \binom{-n}{k}.$$

Then apply Lemma 6.36. So

$$\begin{aligned}\chi_I(x,-n) &= \sum_{k \geq 0} (\phi - u)_I^{*k}(x) \binom{-n}{k} \\ &= (\phi - u + u)_I^{*(-n)}(x) \\ &= (\phi^{*(-n)})_I \\ &= (\phi^{-1})_I^{*n} = ((\phi \circ S)^{*n})_I\end{aligned}$$

□

Remark 6.38 (Recall).

$$\begin{tikzcd} kG \arrow[rr, "\rho"] \arrow[dr, "\zeta"'] & & (kP)^* \arrow[dl, "\phi"] \\ & E & \end{tikzcd}$$

$$\phi_I(p) = \begin{cases} 1 & \text{if } p \text{ discrete} \\ 0 & \text{otherwise.} \end{cases}$$

$$\zeta_I(p) = \begin{cases} 1 & \text{if } g \text{ is discrete} \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho_I(g) = \sum_{\alpha \in AO(g)} g_\alpha$$

Proposition 6.39.

(i) $\phi_I^{-1}(p) = (-1)^{|I|}$ *for all* $p \in P[I]$.

(ii) $\zeta_I^{-1}(g) = (-1)^{|I|} a(g)$, *where* $a(g) = \#AO(g)$.

Proof.

(i) Define $\psi \colon (kP)^* \to E$ by $\psi_I(p) = (-1)^{|I|}$.

$$\begin{aligned} (\phi * \psi)_I(p) &= \sum_{\substack{I = S \sqcup T \\ S \text{ lower set of } p}} \phi(p|_S)\psi(p|_T) \\ &= \sum_{\substack{I = S \sqcup T \\ S \text{ lower set of } p \\ p|_S \text{ discrete}}} (-1)^{|T|} \\ &= (-1)^{|I|} \sum_{S \subseteq \min(p)} (-1)^{|S|} = \begin{cases} (-1)^{|I|} & \text{if } \min(p) = \emptyset \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here $\min(p)$ is the set of minimal elements of the poset p. So we have shown that ψ is the convolution inverse of ψ.

(ii) $\rho^{\#} \colon \mathbb{X}((kP)^*) \to \mathbb{X}(kG)$ is a morphism of groups. We have that

$$\rho^{\#}(\phi) = \phi \circ \rho = \zeta$$

Which then implies that $\phi^{\#}(\rho^{-1}) = \zeta^{-1}$, so $\zeta^{-1} = \phi^{-1} \circ \rho$. Thus,

$$\zeta_I^{-1}(g) = \phi_I^{-1}\left(\sum_{\alpha \in AO(g)} g_\alpha \right) = (-1)^{|I|}(\#AO(g)) = (-1)^{|I|} a(g). \quad \square$$

Exercise 6.40.

(a) Derive an antipode formula for $(\mathrm{k}P)^*$ and deduce from it Proposition 6.39(i).

(b) Deduce Proposition 6.39(ii) from the antipode formula for $\mathrm{k}G$.

Recall from Example 6.31 that the invariant associated to the character $\phi\colon (\mathrm{k}P)^* \to E$ is

$$\chi_I(p)(n) = \#\{f\colon I \to [n] \mid f \text{ is strictly order preserving}\}.$$

We say that $f\colon I \to [n]$ is **strictly order preserving** if $i < j$ in p implies $f(i) < f(j)$ in $[n]$.

Proposition 6.41.

$$\chi_I(p)(-n) = (-1)^{|I|}\#\{f\colon I \to [n] \mid f \textit{ is order preserving} \}.$$

Notice that here, f need not be strictly order preserving.

Proof.

$$\begin{aligned}
\chi_I(p)(-n) &= \phi^{*(-n)}(p) \\
&= (\phi^{-1})_I^{*n}(p) \\
&= \sum_{I = S_1 \sqcup \ldots \sqcup S_n} (\phi_{S_1}^{-1} \otimes \cdots \otimes \phi_{S_n}^{-1})\Delta_{S_1,\ldots,S_n}(p) \\
&= \sum_{\substack{I = S_1 \sqcup \ldots \sqcup S_n \\ S_1 \sqcup \ldots \sqcup S_i \text{ is a lower set of } p \text{ for all } i}} \phi_{S_1}^{-1}(p|_{S_1}) \cdots \phi_{S_n}^{-1}(p|_{S_n}) \\
&= (-1)^{|I|}\#\left\{ (S_1, \ldots, S_n) \;\middle|\; \begin{array}{l} I = S_1 \sqcup \ldots \sqcup S_n, \\ S_1 \sqcup \ldots \sqcup S_i \text{ is a lower set of } p \text{ for all } i \end{array} \right\}
\end{aligned}$$

To make the translation between this and what we want, set $f^{-1}([i]) = S_1 \sqcup \ldots \sqcup S_i$, and then we have exactly that

$$\chi_I(p)(-n) = (-1)^{|I|}\#\{f\colon I \to [n] \mid f \text{ is order preserving} \}.$$

$\square$

Remark 6.42. In Stanley's book, the notation that he uses is $\chi_I(p, n) = \overline{\Omega}(p, n)$ for the strict order polynomial, and $\chi_I(p, -n) = \Omega(p, -n)$ for the order polynomial.

Proposition 6.43 (Stanley's Negative One Color Theorem)**.** *Consider the invariant associated to ζ:*

$$\chi_I(g)(n) = \#\{f\colon I \to [n] \mid f \textit{ proper coloring} \}.$$

Then $\chi_I(g)(-1) = (-1)^{|I|}a(g)$.

Proof. By combining Proposition 6.39 and Proposition 6.41, we see that

$$\chi_I(g)(-1) = (\zeta^{-1})_I(g) = (-1)^{|I|} a(g). \qquad \square$$

7 Combinatorial Topology

7.1 The antipode of $\mathbb{E}$

Recall the formula for the antipode of $k\mathbb{E}$:

$$\mathcal{S}_I(*_I) = (-1)^{|I|} *_I .$$

When $I = \{a\}$ is just a singleton, then the formula

$$0 = \sum_{I = S \sqcup T} \mu_{S,T}(\mathcal{S}_S \otimes \mathrm{id}_T)\Delta_{S,T}$$

gives that

$$\mathcal{S}_{\{a\}}(*_{\{a\}}) *_\emptyset + \mathcal{S}_\emptyset(*_\emptyset) *_{\{a\}} = 0.$$

Then we can get

$$*_I = \sum_{a \in I} *_{\{a\}}.$$

Now according to Milnor-Moore recursion,

$$\begin{aligned}
\mathcal{S}_I(*_I) &= - \sum_{\substack{I = S \sqcup T \\ T \neq I}} *_S \mathcal{S}_T(*_T) \\
&= - \sum_{T \subsetneq I} (-1)^{|T|} *_I
\end{aligned}$$

Or equivalently,

$$(-1)^{|I|} = - \sum_{T \subsetneq I} (-1)^{|T|}.$$

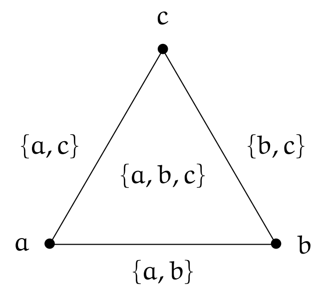
But there's another way to think about this. If we let the subsets of I index the faces of the simplex $\Delta^{(I)}$, then the dimension of the face indexed by T is $|T| - 1$. So we see that

$$(-1)^{|I|} = \sum_{F \text{ face of } \partial\Delta^{(I)}} (-1)^{\dim F} = \overline{\chi}(\partial \Delta^{(I)})$$

is the **reduced Euler characteristic** of the simplex. Now, $\partial\Delta^{(I)} \cong S^{|I|-2}$. Hence,

$$\chi(S^n) = (-1)^n + 1, \qquad \overline{\chi}(S^n) = (-1)^n = (-1)^{|I|}$$

Example 7.1. If $I = \{a, b, c\}$, then we get the picture



$\emptyset$ indexes the empty face.

Let's see what Takeuchi's formula gives us.

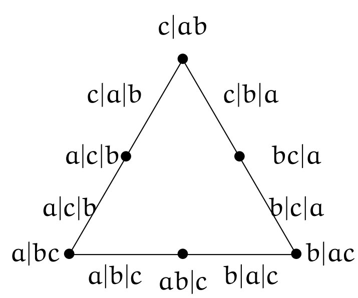
$$\begin{aligned} \mathbb{S}_I(*_I) &= \sum_{F \in \Sigma[I]} (-1)^{\ell(F)} \overbrace{\mu_F \Delta_F(*_F)}^{*_I} \\ (-1)^{|I|} *_I &= \sum_{F \in \Sigma[I]} (-1)^{\ell(F)} *_I \\ (-1)^{|I|} &= \sum_{F \in \Sigma[I]} (-1)^{\ell(F)} \end{aligned}$$

Using a little combinatorial topology,

$$\mathbb{S}_I(*_I) = \sum_{F \in \Sigma[I]} (-1)^{\dim F} = \overline{\chi}(\Sigma[I])$$

where $\Sigma[I]$, as a space, is the barycentric subdivision of $\partial\Delta^{|I|}$.

Example 7.2. The elements of $\Sigma[I]$ index the faces of the barycentric subdivision of $\partial\Delta^{(I)}$.



The empty face is labelled by abc.

Note that compositions of I are in bijection with strict chains of subsets of I, via

$$(S_1, \ldots, S_n) \in \Sigma[I] \longmapsto (S_1 \leq S_1 \sqcup S_1 \leq \ldots \bigsqcup_{i=1}^{n} S_i = I).$$

The dimension of the face indexed by the composition F is $\ell(F) - 2$.

7.2 The antipode of L

$\mathbb{S}_I(\ell) = (-1)^{|I|}\bar{\ell}$. So using Takeuchi,

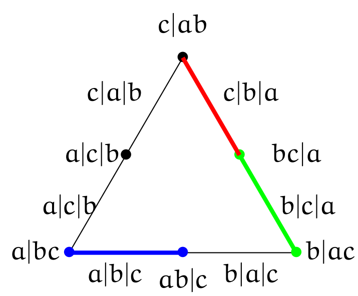
$$\begin{aligned} \mathbb{S}_I(\ell) &= \sum_{F \in \Sigma[I]} (-1)^{\ell(F)} \mu_F \Delta_F(\ell) \\ &= \sum_{\ell' \in L[I]} \left(\sum_{\substack{F \in \Sigma[I] \\ \mu_F \Delta_F(\ell) = \ell'}} (-1)^{\ell(F)} \right) \ell' \end{aligned}$$

Definition 7.3. In general, $A(\ell, \ell')$ Let $A(\ell, \ell') = \left\{ F \in \Sigma[I] \mid \mu_F \Delta_F(\ell) = \ell' \right\}$.

Example 7.4. Fix $\ell = a|b|c$.

ℓ'	$A(\ell, \ell')$
$b\|c\|a$	$\{bc\|a, b\|c\|a\}$
$c\|b\|a$	$\{c\|b\|a\}$
$a\|b\|c$	$\{abc, a\|bc, ab\|c, a\|b\|c\}$

These are faces of the previous picture:



Remark 7.5. In general, $A(\ell, \ell')$ is a Boolean poset. The minimal element is $(S_1, \ldots, S_k)$ where

- S_1 is the longest initial segment of ℓ' $\ell'|_{S_1}$ such that $\ell'|_{S_1} = \ell|_{S_1}$.
- S_2 is the next-longest segment of ℓ' such that $\ell'_{S_2} = \ell|_{S_2}$.

$\vdots$

- S_k is the last segment of ℓ' such that $\ell'|_{S_k} = \ell|_{S_k}$.

The maximal element is ℓ'.

Remark 7.6. Note that the maximum and the minimum elements are the same precisely when $\ell' = \bar{\ell}$. The coefficient of ℓ' in $\mathsf{S}_I(\ell)$ is

$$\sum_{F \in A(\ell,\ell')} (-1)^{\ell(F)} = \begin{cases} (-1)^{|I|} & \text{if } \ell' = \bar{\ell} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 7.7 (Goal)**.** We want to prove the following. Let B be a set-theoretic bimonoid. Assume that it is either commutative or cocommutative. Consider the antipode $\mathsf{S}\colon kB \to kB$. Then pick $x, y \in B[I]$. The coefficient of y in $\mathsf{S}_I(x)$ is $\chi(X) - \chi(A)$, where $A \subseteq X$ are simplicial subcomplexes of $\Sigma[I]$, and X is always a ball or sphere.

7.3 The Coxeter Complex

Consider $\Sigma[I]$, which is the set of compositions F of I. A composition F of I is $F = (S_1, \dots, S_k)$ where $I = S_1 \sqcup \dots \sqcup S_k$ and $S_i \neq \emptyset$ for all i.

Remark 7.8. $\Sigma[I]$ is a poset under **refinement**. If $F, G \in \Sigma[I]$, then $F \leq G$ if each S_i in F is obtained by merging a number of contiguous blocks of G. Equivalently, $G = G_1 \cdot G_2 \cdots G_k$ (concatenation) where $G_i \in \Sigma[S_i]$.

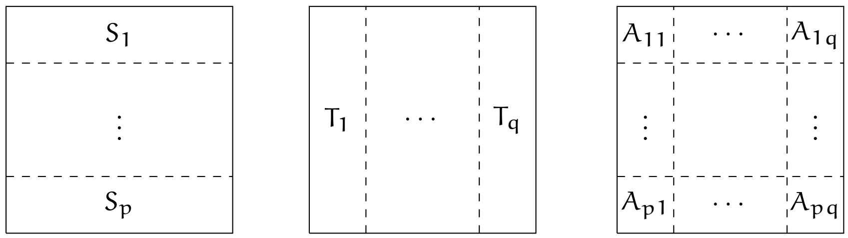
The composition $\perp = (I)$ is the unique minimal element of $\Sigma[I]$. The maximal elements are the linear orders.

Definition 7.9. $\Sigma[I]$ is a simplicial complex, called the **Coxeter complex of type** A. In particular, each interval $[F, G]$ is (isomorphic to) a Boolean poset.

Remark 7.10. $\Sigma[I]$ is the barycentric subdivision of $\partial\Delta^{(I)}$, where $\Delta^{(I)}$ is the simplex of dimension $|I| - 2$.

7.4 The Tits Product

Definition 7.11. Let $F = (S_1, \dots, S_p) \in \Sigma[I]$ and $G = (T_1, \dots, T_q) \in \Sigma[I]$. Consider the pairwise intersections $A_i = S_i \cap T_j$.The **Tits product** is $FG = \widehat{(A_{11}, \dots, A_{1q}, \dots, A_{p1}, \dots, A_{pq})}$, where the hat denotes that we remove any empty A_{ij}.



Fact 7.12.

(a) The Tits product is associative and unital, with unit $\perp$. So $\Sigma[I]$ is a monoid.

*(b) It is **not** commutative. FG and GF consist of the same blocks, but in a different order.*

(c) $F \leq FG$

(d) $F \leq G \iff FG = G$

(e) $F \leq G \implies GF = G$.

(f) $F^2 = F$

(g) $FGF = FG$

Definition 7.13. A monoid M satisfying property Fact 7.12(f) is called a **band**. If is also satisfies Fact 7.12(g), then it is a **left regular band** (LRB).

7.5 The Partition Lattice

Remark 7.14. Recall that $\Pi[I]$ is the set of partitions of I. It is a poset under refinement. The partition $\perp = \{I\}$ is the minimum element, and the partition $\top$ into singletons is the maximum.

$\Pi[I]$ is a lattice. The join $X \vee Y$ of X and Y consists of the nonempty pairwise intersections between blocks of X and blocks of Y. We can determine the meet of two elements by knowing the join and the fact that this lattice is finite.

Fact 7.15. *$\Pi[I]$ is then a monoid under the join with unit $\perp$. It is a commutative left regular band (see Definition 7.13).*

Definition 7.16. The **support map** is

$$\begin{array}{rccc} \mathrm{supp}\colon & \Sigma[I] & \longrightarrow & \Pi[I] \\ & (S_1, \dots, S_k) & \longmapsto & \{S_1, \dots, S_k\} \end{array}$$

Fact 7.17. $\mathrm{supp}\colon \Sigma[I] \to \Pi[I]$ *is a morphism of monoids, and* $\mathrm{supp}\, F = \mathrm{supp}\, F' \iff F$ *and* F' *consists of the same blocks but possibly in different orders.*

7.6 Higher Hopf Monoid Axioms

Let H be a connected Hopf monoid. Let $F = (S_1, \ldots, S_k) \in \Sigma[I]$. Write

$$H(F) = H[S_1] \otimes \cdots \otimes H[S_k].$$

Recall the maps $H(F) \underset{\Delta_F}{\overset{\mu_F}{\rightleftarrows}} H[I]$. Now let $G \geq F$, and write $G = G_1 \cdot G_2 \cdots G_k$ with each $G_i \in \Sigma[S_i]$.

Definition 7.18. Define $\mu_F^G \colon H(G) \to H(F)$ by

$$\begin{tikzcd} H(G) \arrow[r, dashed, "\mu_F^G"] \arrow[d, equal] & H(F) \arrow[d, equal] \\ H(G_1) \otimes \cdots \otimes H(G_k) \arrow[r, "\mu_{G_1} \otimes \cdots \otimes \mu_{G_k}"] & H[S_1] \otimes \cdots \otimes H[S_k]. \end{tikzcd}$$

Definition 7.19. Similarly, define $\Delta_F^G \colon H(F) \to H(G)$ by

$$\begin{tikzcd} H(F) \arrow[r, dashed, "\Delta_F^G"] \arrow[d, equal] & H(G) \arrow[d, equal] \\ H[S_1] \otimes \cdots \otimes H[S_k] \arrow[r, "\Delta_{S_1} \otimes \cdots \otimes \Delta_{S_k}"] & H(G_1) \otimes \cdots \otimes H(G_k). \end{tikzcd}$$

With these generalizations, we can state generalized versions of the Hopf monoid axioms. These are not hard to prove, and mostly follow by induction from the standard Hopf monoid axioms.

Proposition 7.20 (Higher (co)commutativity)**.** *For any $F \leq G$ in $\Sigma[I]$, the following diagrams commute.*

$$\begin{tikzcd} H(G) \arrow[r, "\mu_G"] \arrow[d, "\mu_F^G"'] & H[I] \\ H(F) \arrow[ur, "\mu_F"'] & \end{tikzcd} \qquad \begin{tikzcd} H[I] \arrow[r, "\Delta_G"] \arrow[dr, "\Delta_F"] & H(G) \\ & H(F) \arrow[u, "\Delta_F^G"'] \end{tikzcd}$$

Proposition 7.21 (Higher (co)commutativity)**.** *Let F and $F' \in \Sigma[I]$ be such that* $\operatorname{supp} F = \operatorname{supp} F'$*. If H is commutative, then the following commutes.*

$$\begin{tikzcd} H(F) \arrow[rr, "\beta_{F,F'}"] \arrow[dr, "\mu_F"'] & & H(F') \arrow[dl, "\mu_{F'}"] \\ & H[I] & \end{tikzcd}$$

If H *is cocommutative, then the following commutes.*

$$\begin{tikzcd} & H[I] \arrow[dl, "\Delta_F"'] \arrow[dr, "\Delta_{F'}"] & \\ H(F) \arrow[rr, "\beta_{F,F'}"] & & H(F') \end{tikzcd}$$

Proposition 7.22 (Higher compatability)**.** *For any* $F, G \in \Sigma[I]$*, the following commutes.*

$$\begin{tikzcd} H(F) \arrow[r, "\mu_F"] \arrow[d, "\Delta_F^{FG}"] & H[I] \arrow[r, "\Delta_G"] & H(G) \\ H(FG) \arrow[rr, "\beta_{FG,GF}"] & & H(GF) \arrow[u, "\mu_G^{GF}"'] \end{tikzcd}$$

Recall that if $F \leq FG$, $G \leq GF$, then $\mathrm{supp}(FG) = \mathrm{supp}(GF)$. So the bottom arrow in the diagram above makes sense.

Notice that the Tits product allows us to nicely state the higher versions of the Hopf monoid axioms. So there is a nice interplay between the geometry and combinatorics.

7.7 The action of Σ

Let H be a connected Hopf monoid as before.

Definition 7.23. Given $F \in \Sigma[I]$ and $x \in H[I]$, let $F \cdot x := \mu_F \Delta_F(x) \in H[I]$. This defines an action of $\Sigma[I]$ on $H[I]$ by

$$\begin{aligned} \Sigma[I] \times H[I] &\longrightarrow H[I] \\ (F, x) &\longmapsto F \cdot x \end{aligned}$$

Proposition 7.24. *If* H *is cocommutative, this is a left action. If* H *is commutative, then this is a right action.*

Proof. Assume that H is cocommutative.First, observe that

$$\perp \cdot x = \mu_\perp \Delta_\perp(x) = \mathrm{id}(\mathrm{id}(x)) = x.$$

Then for any $F, G \in \Sigma[I]$, $x \in H[I]$, we have

$$\begin{aligned} G \cdot (F \cdot x) &= \mu_G \Delta_G \mu_F \Delta_F(x) \\ &= \mu_G \mu_G^{GF} \beta_{FG,GF} \Delta_F^{FG} \Delta_F(x) && \text{(by Proposition 7.22)} \\ &= \mu_{GF} \beta_{FG,GF} \Delta_{FG}(x) && \text{(by Proposition 7.20)} \\ &= \mu_{GF} \Delta_{GF}(x) && \text{(by Proposition 7.21)} \\ &= GF \cdot x \end{aligned}$$

The case when H is commutative is similar. $\square$

Remark 7.25. Earlier we said that Hopf monoids are somewhat like groups. But what about the other structure in the category of species? What does it correspond to? It turns out the analogy extends nicely.

Category of Species	**Category of Sets**
Hopf Monoid H	Group G
Cocommutative Hopf Monoid H	Abelian Group G
Σ	$\mathbb{Z}$
$F \in \Sigma$	$n \in \mathbb{Z}$
$H[I]$ is a $\Sigma[I]$-module	G is a $\mathbb{Z}$-module
$\mu_F \Delta_F(x)$	x^n
Σ is itself a Hopf monoid	$\mathbb{Z}$ itself is an Abelian group

Since $\mathbb{Z}$ itself is the initial element in the category of $\mathbb{Z}$-modules, this suggests that Σ should have some sort of universal property of an initial object. Indeed, Σ is the initial **near-ring** in the category of species.

Definition 7.26. Let B be a set-theoretic cocommutative connected bimonoid. Then, as before, the monoid $\Sigma[I]$ acts on the set $B[I]$. Fix $x, y \in B[I]$. Then define

$$\Sigma_{x,y} = \left\{F \in \Sigma[I] \mid F \cdot x = y\right\}$$

Notice that $\Sigma_{x,y}$ is a subset of the simplicial complex $\Sigma[I]$.

Lemma 7.27. *$\Sigma_{x,y}$ is a convex subposet of $\Sigma[I]$. This means if $H \le F \le G$ and $H, G \in \Sigma_{x,y}$, then $F \in \Sigma_{x,y}$.*

Proof. Notice that $H \le F \implies FH = F$ by Fact 7.12, and similarly $F \le G \implies FG = G$, again by Fact 7.12. Then

$$\begin{aligned} F \cdot x &= FH \cdot x \\ &= F \cdot (H \cdot x) && \text{(since } H \in \Sigma_{x,y}) \\ &= F \cdot y \\ &= F \cdot (G \cdot x) && \text{(since } G \in \Sigma_{x,y}) \\ &= FG \cdot x \\ &= G \cdot x = y \end{aligned}$$ $\square$

Example 7.28. If $B = L$, $I = \{a, b, c\}$, and $x = a|b|c$, $y = b|c|a$, then

$$\Sigma_{x,y} = \{b|c|a, bc|a\}.$$

Definition 7.29. The closure of $\Sigma_{x,y}$ is

$$\overline{\Sigma}_{x,y} = \left\{F \in \Sigma[I] \mid \exists\, G \ge F \text{ s.t. } G \cdot x = y\right\}.$$

Theorem 7.30. *Let* $X = \overline{\Sigma}_{x,y}$*, and let*

$$A = \overline{\Sigma}_{x,y} / \Sigma_{x,y} = \{F \in \Sigma[I] \mid \exists\, G \geq F \text{ s.t. } G \cdot x = y \text{ and } F \cdot x \neq y\}.$$

Then X *and* A *are subcomplexes of* $\Sigma[I]$ *(lower sets of the poset* $\Sigma[I]$*).*

Proof. Note that X is always a subcomplex. So we just need to show this for A. Take $F \in A$ and $H \leq F$. We need $H \in A$. Then $F \in A$ implies that there is some $G \geq F$ such that $G \in \Sigma_{x,y}$ and $F \notin \Sigma_{x,y}$. So we have $H \leq F \leq G$ and $G \in \Sigma_{x,y}$, so $H \in X$.

Now suppose for contradiction that $H \notin A$. Then $H \in X \setminus A = \Sigma_{x,y}$. But $H \leq F \leq G$ implies that $F \in \Sigma_{x,y}$, so this is a contradiction, since $F \in A = X \setminus \Sigma_{x,y}$. □

Recall that the coefficient of y in $\mathsf{S}_I(x)$ is

$$\sum_{F \in \Sigma_{x,y}} (-1)^{\dim F}.$$

Corollary 7.31. *The coefficient of* y *in* $\mathsf{S}_I(x)$ *is* $\chi(X) - \chi(A)$.

Proof.

$$\begin{aligned} \chi(X) - \chi(A) &= \sum_{\substack{F \in X \\ F \neq \perp}} (-1)^{\dim F} - \sum_{\substack{F \in A \\ F \neq \perp}} (-1)^{\dim F} \\ &= \sum_{F \in \Sigma_{x,y}} (-1)^{\dim F} \end{aligned}$$ □

Remark 7.32. There are many questions that we could ask about the relations between Hopf monoids and topology. For instance, when is the antipode a discrete Morse function? How do we translate between topological invariants and properties of the Hopf monoids? Lots of these questions haven't been explored.

8 Generalized Permutahedra

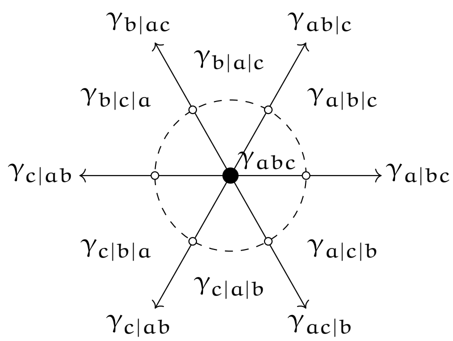
8.1 The Coxeter complex as a fan of cones

Consider the vector space $\mathbb{R}^I$. Its elements are functions $x \colon I \to \mathbb{R}$. For each $F \in \Sigma[I]$, let γ_F be the set of $x \in \mathbb{R}^I$ with the relations that $x_i = x_j$ if i and j belong to the same block of F, or $x_i < x_j$ if the block of i precedes the block of j in F.

Definition 8.1. γ_F is a **(polyhedral) cone**. The collection $\{\gamma_F : F \in \Sigma[I]\}$ is a (complete) fan of cones, meaning that

$$\mathbb{R}^I = \bigsqcup_{F \in \Sigma[I]} \gamma_F.$$

Example 8.2. Let $I = \{a, b, c\}$. Then we're working in $\mathbb{R}^3$ Then the picture looks like the following when projected onto the first two coordinate axes. The origin is $\gamma_\perp$, and the rays are γ_F where F has two blocks. The chambers are γ_F where F is a linear vector.



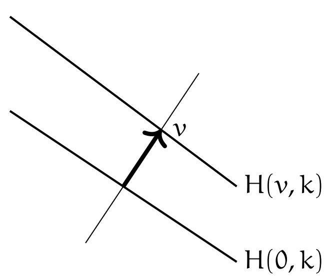
8.2 Polytopes

Let V be a real vector space with an inner product $\langle\ ,\ \rangle$.

Definition 8.3. A **hyperplane** in V is a subset of the form

$$H(v, k) = \{x \in V \mid \langle x, v \rangle = k\}$$

for some $v \in V$, $k \in \mathbb{R}$.



Definition 8.4. The **half spaces bound by** $H(v, k)$ are

$$H(v, k)^- = \{x \in V \mid \langle x, v \rangle \leq k\},$$

$$H(v, k)^+ = \{x \in V \mid \langle x, v \rangle \geq k\}.$$

Proposition 8.5 (Fundamental Theorem of Polytopes)**.** *Let* P *be a subset of* V*. Then the following are equivalent:*

(i) There is a finite subset X *of* V *such that* P *is the convex hull of* X*.*

(ii) P *is bounded and there is a finite set* $\mathcal{H}$ *of half-spaces such that* $P = \bigcap_{h \in \mathcal{H}} h$*.*

Definition 8.6. If either of the previous two cases in Proposition 8.5 hold, then P is a **polytope** in V.

Definition 8.7. A **face** Q of P is either P itself or the intersection of P with a **supporting** hyperplane (a hyperplane H that intersects P and such that $P \subseteq H^+$ or $P \subseteq H^-$). We write $Q \leq P$ for a face of P.

Definition 8.8. The **dimension** of P is the dimension of the affine subspace spanned by P.

Proposition 8.9. *If* $Q \leq P$ *then* Q *is itself a polytope.*

Definition 8.10. The **vertices** of P are the faces of dimension zero.

Proposition 8.11. *If* $Q \leq P$ *then the vertices of* Q *are among those of* P*.*

Given $x \in V$, let P_x be subset of P where the functional $\langle x, - \rangle \colon P \to \mathbb{R}$ achieves its maximum value.

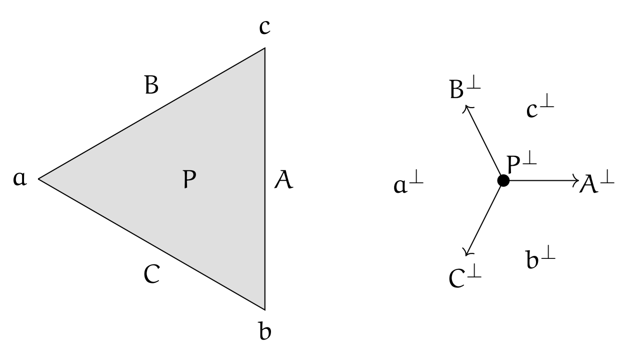
Proposition 8.12 (Fundamental Theorem of Linear Programming)**.** $P_x \leq P$ *for any* x*.*

Definition 8.13. Given a face Q of P, let $Q^\perp = \{x \in V \mid P_x = Q\}$. This is the **normal cone** of Q with respect to P.

Definition 8.14. The collection $N(P) = \{Q^\perp \mid Q \leq P\}$ is the **normal fan** of P.

Note that each $Q^\perp$ is relatively open, and $\dim Q^\perp = \dim V - \dim Q$.

Example 8.15. The polytope P on the left has normal fan $N(P)$ on the right.



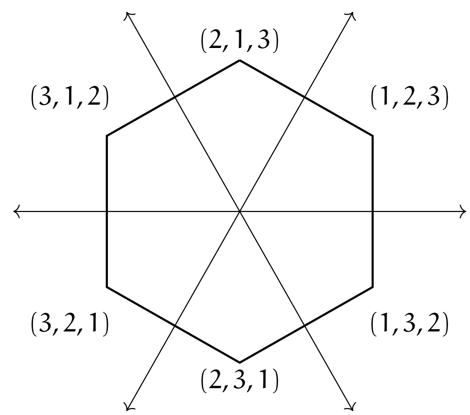
8.3 Permutahedra and Generalized Permutahedra

Now let $V = \mathbb{R}^I$ and $\langle x, y \rangle = \sum_i x_i y_i$ be the standard inner product.

Definition 8.16. The standard permutahedron P_I is the convex hull of

$$\{x\colon I \to [n] \subseteq \mathbb{R} \mid x \text{ is bijective}\}.$$

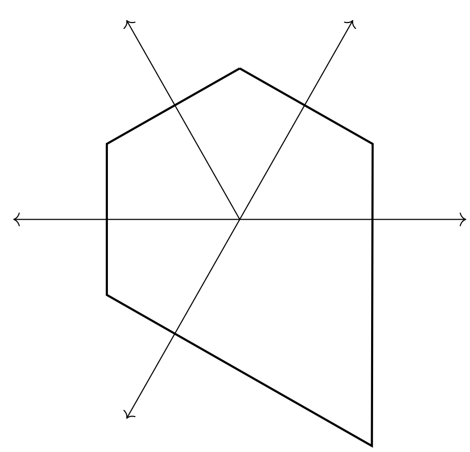
Example 8.17. If $I = \{a, b, c\}$, then P_I is drawn below. The region labelled $(1, 2, 3)$ corresponds to $x\colon I \to [3]$ with $x_a = 1$, $x_b = 2$, $x_c = 3$.



Definition 8.18. A polytope P in $\mathbb{R}^I$ is a **generalized permutahedra** if $N(P)$ is **coarser** than $N(P_I) = \Sigma[I]$. If $N(P)$ is coarser than $N(Q)$, then we write $N(P) \leq N(Q)$.

If $N(P) \leq N(Q)$, then each cone $\gamma \in N(P)$ is a union of some cones in $N(P_I)$, or equivalently, each cone γ_F is contained in a unique cone $Q^\perp$ for asome $Q \leq P$.

Example 8.19. If $I = \{a, b, c\}$, then the polytope drawn below is a generalized permuathedra because it's normal cone is coarser than $N(P_I)$.



Example 8.20. If $S \subseteq I$, let Δ_S be the convex hull of $\{e_i \mid i \in S\}$ (a **simplex**, for example $\{e_i \mid i \in I\}$ the standard basis of $\mathbb{R}^I$). Then Δ_S corresponds to a generalized permutahedra. It's enough to look at Δ_I because Δ_S is a face of Δ_I.

Proposition 8.21. *Let P be a generalized permutation in $\mathbb{R}^S$, and Q a generalized permutahedra in $\mathbb{R}^T$. Note that $\mathbb{R}^I = \mathbb{R}^S \times \mathbb{R}^T$. Then $P \times Q$ is a generalized permutahedra in $\mathbb{R}^I$.*

Proof sketch. $N(P \times Q) \cong N(P) \times N(Q)$. We can use also that $\Sigma[S] \times \Sigma[T]$ is coarser than $\Sigma[I]$. ☐

Proposition 8.22. *Let P and Q be generalized permutahedra in $\mathbb{R}^I$. Then $P + Q = \{v + w \mid v \in P, w \in Q\}$ is a generalized permutahedra in $\mathbb{R}^I$.*

Proof sketch. $N(P + Q)$ is the set of cones obtained by intersecting cones in $N(P)$ with cones in $N(Q)$. ☐

8.4 The species of generalized permutahedra

Let GP[I] be the (infinite) set of all generalized permutahedra in $\mathbb{R}^I$. Let $I = S \sqcup T$. Note that $\mathbb{R}^I = \mathbb{R}^S \times \mathbb{R}^T$. We will make GP into a species.

Proposition 8.23. *If $P_1 \in GP[S]$, $P_2 \in GP[T]$, then $P_1 \times P_2 \in GP[I]$.*

This allows us to define the product μ on the species GP. Define

$$\begin{aligned} \mu_{S,T} \colon \mathrm{GP}[S] \times \mathrm{GP}[T] &\longrightarrow \mathrm{GP}[I] \\ (P_1, P_2) &\longmapsto P_1 \times P_2 \end{aligned} \tag{12}$$

Proposition 8.24. *Assume $S, T \neq \emptyset$. Let $F = (S, T) \in \Sigma[I]$. Let $P \in GP[I]$. Let Q be the face of P such that $Q^\perp \supseteq \gamma_F$, which exists because $N(P) \leq N(P_I)$.*

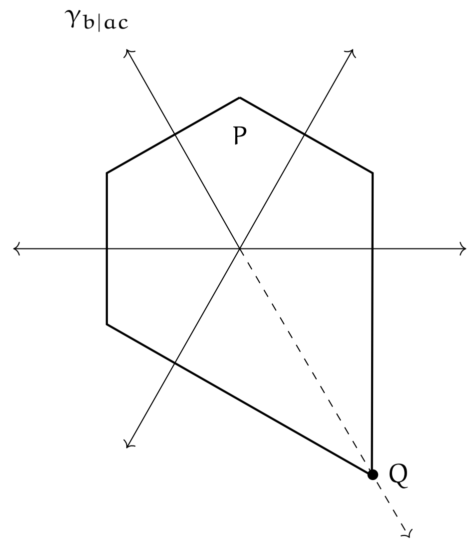
Then there exist $P_1 \in GP[S]$ and $P_2 \in GP[T]$ such that $Q = P_1 \times P_2$.

This proposition gives us the ingredients necessary to define a comultiplication on GP. Define

$$\begin{aligned} \Delta_{S,T} \colon \mathrm{GP}[I] &\longrightarrow \mathrm{GP}[S] \times \mathrm{GP}[T] \\ P &\longmapsto (P_1, P_2) \end{aligned} \tag{13}$$

Proposition 8.25. *With the structure as in (12) and (13), GP is a connected bimonoid in the category of set-species. It is commutative but not cocommutative.*

Example 8.26. Let $I = \{a, b, c\}$, $S = \{a, c\}$, $T = \{b\}$. Then $\Delta_{S,T}(P) = (\mathrm{pt}, \mathrm{pt})$, and $\Delta_{T,S}(P) = (\mathrm{segment}, \mathrm{pt})$.



Remark 8.27. $\mu_{S,T}\Delta_{S,T}(P) = Q$, where Q is the face of P such that $Q^{\perp} \supseteq \gamma_{S,T}$. More generally, for any $F \in \Sigma[I]$, $\mu_F\Delta_F(P)$ is the face Q of P such that $Q^{\perp} \supseteq \gamma_F$.

Theorem 8.28. *Consider the Hopf monoid* $\mathrm{k}(GP)$. *Its antipode satisfies*

$$\mathrm{S}_I(P) = \sum_{Q \leq P} (-1)^{|I| - \dim Q} Q. \tag{14}$$

Proof. Recall the coefficient of Q in $\mathrm{S}_I(P)$ is $\chi(X) - \chi(A)$, where

$$\Sigma_{P,Q} = \{F \in \Sigma[I] \mid F \cdot P = Q\}.$$

Note that $Q = F \cdot P = \mu_F\Delta_F(P)$ by Definition 7.23. Then we have

$$X = \overline{\Sigma}_{P,Q}, \qquad A = \overline{\Sigma}_{P,Q} \setminus \Sigma_{P,Q}.$$

If Q is not a face of P, $\Sigma_{P,Q} = \emptyset$ and the coefficient is zero.

Assume $Q \leq P$. Then

$$\Sigma_{P,Q} = \{F \in \Sigma[I] \mid Q^{\perp} \supseteq \gamma_F\}.$$

This in turn implies that

$$X = \{F \in \Sigma[I] \mid \overline{Q^{\perp}} \supseteq \gamma_F\}, \qquad A = \{F \in \Sigma[I] \mid \partial Q^{\perp} \supseteq \gamma_F\}.$$

Hence, X is a simplicial subdivision of $\overline{Q^{\perp}} \cap S^{n-1}$, and A is a simplicial subdivision of $\partial Q^{\perp} \cap S^{n-1}$.

If $Q < P$, then X is a ball of dimension $|I| - \dim Q - 2$, so $\chi(X) = 1$. In this situation, A is a sphere of dimension $|I| - \dim Q - 3$, so $\chi(A) = 1 + (-1)^{|I| - \dim Q - 1} = 0$. Then

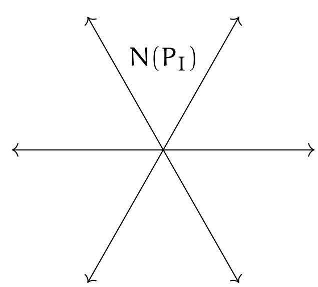
$$\chi(X) - \chi(A) = (-1)^{|I| - \dim Q}.$$

If $Q = P$ then $\perp \in \Sigma_{P,Q}$, so by convexity, we get that $\overline{\Sigma}_{P,Q} = \Sigma_{P,Q}$. Hence $A = \emptyset$. Therefore, $\chi(A) = 1$ (this is a convention that is necessary since we aren't using reduced Euler characteristic). In this case, X is a sphere of dimension $|I| - \dim P - 2$. So

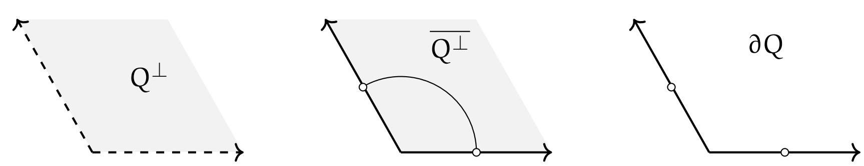
$$\chi(X) - \chi(A) = 1 + (-1)^{|I| + \dim P} - 1 = (-1)^{|I| - \dim P}.$$

So we have shown (14). $\square$

Example 8.29. To illustrate the ideas in the last proof, let $I = \{a, b, c\}$. Consider



An example of a cone $Q^\perp$ is as follows. We have $\overline{Q^\perp} \cap S^1 = B^1$.



Definition 8.30. Given $S \subseteq I$, the **standard simplex** Δ_S is the convex hull of $\{e_i \mid I \in S\} \subseteq \mathbb{R}^I$.

Fact 8.31.

(a) $\Delta_S \in \mathrm{GP}[I]$.

(b) $\mathrm{GP}[I]$ *is closed under Minkowski sums.*

Proof of Fact 8.31(b). $N(P_1 + P_2) = N(P_1) \vee N(P_2)$, where $\vee$ is the least common refinement of the two. If $P_1, P_2 \in \mathrm{GP}[I]$, then $N(P_i) \leq N(P_I)$. This in turn implies that $N(P_1) \vee N(P_2) \leq N(P_I)$. $\square$

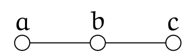
Definition 8.32. Two polytopes P_1 and P_2 in V are **normally equivalent** if $N(P_1) = N(P_2)$. Write $P_1 \sim P_2$, and let $\overline{\mathrm{GP}}[I] = {}^{\mathrm{GP}[I]}/_{\sim}$.

Definition 8.33. Let g be a simple graph on I. Its **graphic zonotope** is

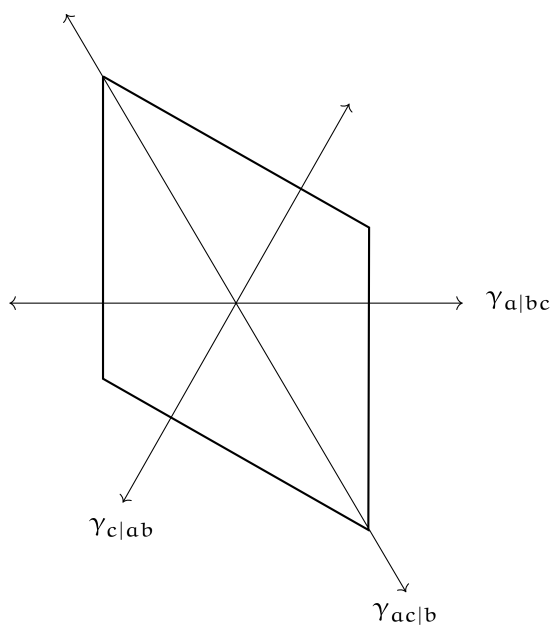
$$Z(g) = \sum_{\{a,b\} \text{ edge of } g} \Delta_{\{a,b\}}.$$

Remark 8.34. Faces of $Z(g)$ are indexed by pairs (X, α) where X is a **bond** of g (a partition of I into blocks B such that $g|_B$ is connected), and α is an acyclic orientation of the contraction of X in g. In particular, the vertices are indexed by acyclic orientations of g.

Example 8.35. Let

$g =$ 

Then $Z(g)$ is



Definition 8.36. Given a simple graph g with vertex set I, define a polytope

$$A(g) = \sum_{\substack{S \subseteq I \\ g|_S \text{ connected}}} \Delta_S$$

this is the **graphic associahedron** of g.

The **standard associahedron** is the graphic associahedron associated to a path on I.

Proposition 8.37. *The following maps are morphisms of Hopf monoids.*

$$\begin{array}{rcl} G & \longrightarrow & \overline{GP} \\ g & \longmapsto & \overline{Z(g)} \end{array} \qquad \begin{array}{rcl} W & \longrightarrow & \overline{GP} \\ g & \longmapsto & \overline{A(g)} \end{array}$$

Remark 8.38. The following diagram commutes (P is the species of posets).

$$\begin{tikzcd} & G \arrow[dr, "\rho"] & & & \\ \Pi \arrow[ur, "\psi"] \arrow[dr, "\phi"'] & & \overline{GP} \arrow[r, "\lambda"] & P^* \arrow[r] & L^* \\ & W \arrow[ur, "\sigma"'] & & & \end{tikzcd}$$

where the maps are given by $\phi(X) = k_X$, $\psi(X) = k_X$, (with $k_X = \sqcup_{B \in X} k_B$) and $\rho(k_X) = \sigma(k_X) = \prod_{B \in X} P_B$. λ is given by

$$\lambda(P) = \sum_{\nu \text{ vertex of } P} \nu^{\perp}.$$

The character $\chi\colon GP \to E$ is

$$\chi(P) = \begin{cases} 1 & \text{if } P \text{ is a poset} \\ 0 & \text{otherwise.} \end{cases}$$

$$\chi^{-1}(P) = (-1)^{|I|} \#(\text{vertices of } P).$$

9 Hyperplane Arrangements

Definition 9.1. Let V be a finite-dimensional real vector space. A **hyperplane arrangement** in V is a finite set $\mathcal{A}$ of hyperplanes in V.

Definition 9.2. A hyperplane arrangement is **linear** if all hyperplanes go through the origin, and **affine** if they need not contain the origin.

Remark 9.3. We will consider almost exclusively linear hyperplane arrangements, and if we say hyperplane arrangement without further qualification, we mean a linear arrangement.

Definition 9.4. The **center** O of a hyperplane arrangement $\mathcal{A}$ is the intersection of all hyperplanes in $\mathcal{A}$. The **rank** of $\mathcal{A}$ is $\dim V - \dim O$.

Note that the center of any (linear) arrangement contains the origin (recall that we assume linear arrangements unless otherwise stated).

Definition 9.5. The arrangement is **essential** if O is just the origin. The **essentialization** of $\mathcal{A}$ is the arrangement

$$\text{ess}(A) = \{{}^{H}/_{O} \mid H \in \mathcal{A}\}$$

in V/O.

9.1 Faces

Each $H \in \mathcal{A}$ decomposes $V = H^+ \sqcup H \sqcup H^-$, where H^+ and H^- are the open half spaces bound by H. Superimposing these decompositions, we obtain a finer decomposition of V into nonempty subsets called **faces**.

Definition 9.6. Equivalently, a **face** of A is a nonempty subset obtained by

(a) for each $H \in \mathcal{A}$, choose either H^+ or H^- or H itself.

(b) intersecting all of these choices.

Fact 9.7.

(a) $\Sigma[\mathcal{A}]$ *is finite, with at most* $3^{|A|}$ *elements.*

(b) We have $V = \bigsqcup_{F \in \Sigma[A]} F$.

(c) The center O *is always a face (when we choose the intersection of all of the hyperplanes themselves).*

Definition 9.8. The **chambers** are the faces that are intersections of only half-spaces. Let $L[A]$ be the set of chambers.

Definition 9.9. Given $F, G \in \Sigma[\mathcal{A}]$, we say $F \leq G$ when $\overline{F} \subseteq \overline{G}$. Then $\Sigma[A]$ is a poset.

Fact 9.10. O *is the minimum element of the poset* $\Sigma[\mathcal{A}]$ *and the chambers are the maximal elements.* $\Sigma[\mathcal{A}]$ *is graded with* $\text{rank}(F) = \dim(F) - \dim(O)$.

9.2 Flats

This is the companion notion to faces.

Definition 9.11. A **flat** if $\mathcal{A}$ is a subspace of V obtained by

(a) for each $H \in \mathcal{A}$, choosing either H or V.

(b) intersecting them all.

Notice that flats are always vector subspaces of V.

Definition 9.12. Let $\Pi[\mathcal{A}]$ be the set of flats.

Fact 9.13.

(a) $\Pi[\mathcal{A}]$ *is finite: it has at most* $2^{|A|}$ *elements.*

(b) $\Pi[\mathcal{A}]$ *is also a poset: given* $X, Y \in \Pi[\mathcal{A}]$, $X \leq Y$ *if* $X \subseteq Y$.

(c) The center O *is always a flat, denoted by* $\perp$, *which is the unique minimum element.*

(d) The space V *is also a flat, we denote it by* $\top$, *which is the unique maximum element.*

(e) The intersection of two flats is a flat, so $\Pi[\mathcal{A}]$ *is a semilattice with* ***meets:*** *greatest lower bounds given by intersections.*

(f) Hence, $\Pi[\mathcal{A}]$ *is a lattice because it is a finite semilattice with meets.*

(g) The lattice $\Pi[\mathcal{A}]$ *is graded with* $\operatorname{rank}(X) = \dim(X) - \dim(\perp)$.

Definition 9.14. The **support** of a face $F \in \Sigma[\mathcal{A}]$ is the flat

$$\operatorname{supp}(F) = \bigcap_{\substack{H \in \mathcal{A} \\ H \supseteq F}} H.$$

Fact 9.15. $\operatorname{supp}(F)$ *is also the subspace of* V *spanned by* F.

Fact 9.16. $\operatorname{supp}\colon \Sigma[\mathcal{A}] \to \Pi[\mathcal{A}]$ *is both order and rank preserving.*

Fact 9.17. *There are canonical poset isomorphisms as follows.*

(a) $\Sigma[\mathcal{A}] \xrightarrow{\sim} \Sigma[\operatorname{ess}(\mathcal{A})]$

(b) $\Pi[\mathcal{A}] \xrightarrow{\sim} \Pi[\operatorname{ess}(\mathcal{A})]$

Moreover, the following diagram commutes.

$$\begin{array}{ccc} \Sigma[\mathcal{A}] & \xrightarrow{\sim} & \Sigma[\operatorname{ess}(\mathcal{A})] \\ \big\downarrow{\scriptstyle \operatorname{supp}} & & \big\downarrow{\scriptstyle \operatorname{supp}} \\ \Pi[\mathcal{A}] & \xrightarrow{\sim} & \Pi[\operatorname{ess}(\mathcal{A})]. \end{array}$$

9.3 Examples

Example 9.18. One hyperplane, $\mathcal{A} = \{H\}$ in V, where $\operatorname{rank}(A) = 1$. The center of A is $O = H$. Then the essentialization of A is just a dimension zero subspace of a one-dimensional vector space $V/O = V/H \cong \mathbb{R}$, and there are three faces in $\Sigma[\mathcal{A}]$, corresponding to H, H^+, and H^-. There are two flats: H and V. So,

$$\Sigma[\mathcal{A}] = \{-, 0, +\}, \qquad \Pi[\mathcal{A}] = \{\bot, \top\}.$$

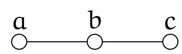
We have that $\operatorname{supp}(+) = \operatorname{supp}(-) = \top$ and $\operatorname{supp}(0) = \bot$.

Example 9.19 (Graphic Arrangements)**.** Let g be a simple graph with vertex set I. Then

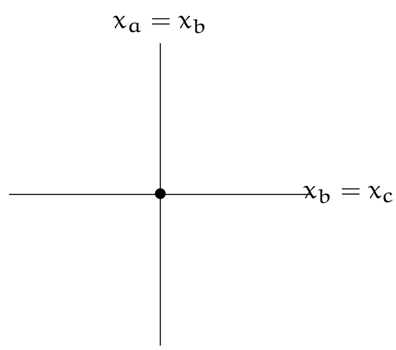
$$\mathcal{A}_g = \{H_{ij} \mid \{i, j\} \text{ is an edge of } g\}$$

in $\mathbb{R}^I$, where $H_{ij} = \{x \in \mathbb{R}^I \mid x_i = x_j\}$.

For example, if

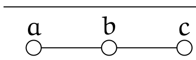
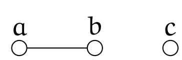
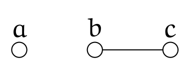
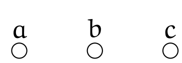
$g =$ 

Then $\operatorname{ess}(\mathcal{A}_g)$ is the following arrangement.



The center O is the line $x_a = x_b = x_c$ here, and in general $\dim(O) = c(g)$, the number of connected components of g. The rank of $\mathcal{A}_g$ is $|I| - c(g)$.

$\Pi[\mathcal{A}_g]$ is the **bond** lattice of g, where a **bond** of g is a partition of I such that the induced subgraph on each block is connected.

bond	flat
	$x_a = x_b = x_c$ (bottom flat)
	$x_a = x_b$
	$x_b = x_c$
	no equations (top flat).

The faces are a bit harder to describe.

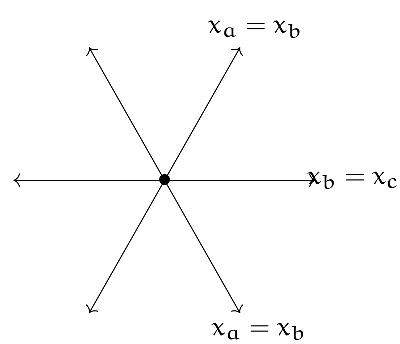
$\Sigma[\mathcal{A}_g] = \{(X, \alpha) \mid X \text{ is a bond and } \alpha \text{ is an acyclic orientation of the contraction } g/X\}$.

g/bond with orientation	face
abc	$x_a = x_b = x_c$ (center)
$\overset{ab}{\circ} \longrightarrow \overset{c}{\circ}$	$x_a = x_b < x_c$ (ray)
$\overset{ab}{\circ} \longleftarrow \overset{c}{\circ}$	$x_c < x_a = x_b$ (ray)
$\overset{a}{\circ} \longrightarrow \overset{bc}{\circ}$	$x_a < x_b = x_c$ (ray)
$\overset{ab}{\circ} \longleftarrow \overset{c}{\circ}$	$x_b = x_c < x_a$ (ray)
$\overset{a}{\circ} \longrightarrow \overset{b}{\circ} \longrightarrow \overset{c}{\circ}$	$x_a < x_b < x_c$ (chamber)
$\overset{a}{\circ} \longrightarrow \overset{b}{\circ} \longleftarrow \overset{c}{\circ}$	$x_a < x_b, x_c < x_b$ (chamber)
$\overset{a}{\circ} \longleftarrow \overset{b}{\circ} \longrightarrow \overset{c}{\circ}$	$x_b < x_a, x_b < x_c$ (chamber)
$\overset{a}{\circ} \longleftarrow \overset{b}{\circ} \longleftarrow \overset{c}{\circ}$	$x_a < x_b < x_c$ (chamber)

Example 9.20 (Braid Arrangment)**.** This is the graphic arrangement of the complete graph on a set I. We have

$$\mathcal{A} = \{H_{ij} \mid i, j \in I, i \neq j\}$$

in $\mathbb{R}^I$. Then $\dim O = 1$, $\operatorname{rank}(\mathcal{A}) = |I| - 1$. The picture for $\operatorname{ess}(\mathcal{A})$ is



Any contraction of a complete graph is again complete, and an acyclic orientation on a complete graph is the same as a linear order on the set of vertices. Hence,

$$\Pi[\mathcal{A}] = \Pi[I], \qquad \Sigma[\mathcal{A}] = \Sigma[I], \qquad \text{and } L[\mathcal{A}] = L[I].$$

Remark 9.21. $\Sigma[\mathcal{A}] = N(P_I)$, and more generally, $\Sigma[\mathcal{A}_g] = N(Z(g))$. For any arrangement $\mathcal{A}$, $\Sigma[\mathcal{A}] = N(Z(A))$.

9.4 Signed Sequence of a Face

For each $H \in \mathcal{A}$, choose $f_H \in V^*$ such that $H = \ker(f_H)$. Let $H^+ = \{x \in V \mid f_H(x) > 0\}$, and $H^- = \{x \in V \mid f_H(x) < 0\}$, and $H^0 = H$.

Fix $x \in V$. We have a function $\varepsilon_x \colon \mathcal{A} \to \{-, 0, +\}$ given by

$$\varepsilon_x(H) = \begin{cases} + & x \in H^+ \\ - & x \in H^- \\ 0 & x \in H^0. \end{cases}$$

Lemma 9.22. *Let $F \in \Sigma[\mathcal{A}]$. For any $x \in F$,*

$$F = \bigcap_{H \in \mathcal{A}} H^{\varepsilon_x(H)}.$$

Corollary 9.23. *Let $x, y \in V$. Then x and y belong to the same face if and only if $\varepsilon_x = \varepsilon_y$.*

We may then define a map $\varepsilon \colon \Sigma[\mathcal{A}] \to \{-, 0, +\}^{\mathcal{A}}$ by $\varepsilon(F) = \varepsilon_x$ where x is any point in F. It follows that ε is well-defined and injective.

Definition 9.24. Given two faces $F, G \in \Sigma[\mathcal{A}]$, define the **Tits product** FG as the first face entered when walking from a point $x \in F$ to a point $y \in G$ along a straight line.

Proposition 9.25. *This is a well-defined operation on the set of faces $\Sigma[\mathcal{A}]$, and moreover it turns $\Sigma[\mathcal{A}]$ into a monoid with unit O.*

Proof. For each $H \in \mathcal{A}$, $0 \le t \le 1$,

$$f_H((1-t)x + ty) = (1-t)f_H(x) + tf_H(y)$$

To see what happens in relation to H, we only care about the sign of the above. For small t, the sign of the right-hand side is determined by the sign of $f_H(x)$, provided $f_H(x) \neq 0$. For small t,

$$\text{sign(RHS)} = \begin{cases} f_H(x) & \text{if } f_H(x) \neq 0 \\ f_H(y) & \text{if } f_H(x) = 0. \end{cases}$$

In other words,

$$\text{sign(RHS)} = \begin{cases} \varepsilon_H(F) & \text{if } \varepsilon_H(F) \neq 0 \\ \varepsilon_H(G) & \text{if } \varepsilon_H(F) = 0 \end{cases}$$

This is independent of x and y, and only depends on x and y. Thus FG is well-defined and in fact we find that

$$\varepsilon_H(FG) = \begin{cases} \varepsilon_H(F) & \text{if } \varepsilon_H(F) \neq 0 \\ \varepsilon_H(G) & \text{if } \varepsilon_H(F) = 0. \end{cases}$$

for all $H \in \mathcal{A}$. It follows that $\varepsilon_H(FO) = \varepsilon_H(F) = \varepsilon_H(OF)$. It also follows that both $\varepsilon_H(E(FG))$ and $\varepsilon_H((EF)G)$ are given by

$$\begin{cases} \varepsilon_H(E) & \text{if } \varepsilon_H(E) \neq 0 \\ \varepsilon_H(F) & \text{if } \varepsilon_H(E) = 0,\ \varepsilon_H(F) \neq 0 \\ \varepsilon_H(G) & \text{if } \varepsilon_H(E) = \varepsilon_H(F) = 0. \end{cases}$$

Since ε is injective, then $FO = F = OF$ and $(EF)G = E(FG)$, so we have a monoid. $\square$

Corollary 9.26 (Consequence of the proof of Proposition 9.25). $\varepsilon\colon \Sigma[A] \to \{-,0,+\}^{\mathcal{A}}$ *is a morphism of monoids, where the right hand side has the structure of the product of copies of the monoid* $\{-,0,+\}$ *with the multiplication table*

		right		
		0	$+$	$-$
	0	0	$+$	$-$
left	$+$	$+$	$+$	$+$
	$-$	$-$	$-$	$-$

Remark 9.27. ε is also a morphism of graded posets. Moreover, the following commutes.

$$\begin{tikzcd} \Sigma[\mathcal{A}] \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{supp}"] & \{-,0,+\}^{\mathcal{A}} \arrow[d, "\mathrm{supp}"] & -,+ \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\ \Pi[\mathcal{A}] \arrow[r, "\varepsilon"] & \{\bot,\top\}^{\mathcal{A}} & \top & \bot \end{tikzcd}$$

$\{\bot,\top\}^{\mathcal{A}}$ is a Boolean poset (and therefore a lattice).

Proposition 9.28. $\mathrm{supp}\colon \Sigma[A] \to \Pi[A]$ *is a surjective monoid morphism.*

Proof. By the commutativity of the above diagram, it is enough to check that $\mathrm{supp}\colon \{-,0,+\} \to \{\bot,\top\}$ is a monoid morphism, since ε is an injective monoid morphism. This is straightforward.

For surjectivity, if a face F intersects a flat X, then $F \subseteq X$. Faces decompose V, so faces intersecting X decompose X, and X is covered by faces implies that some $F \supseteq X$ has the same dimension. Then $\mathrm{supp}(F) = X$. $\square$

Proposition 9.29. *Let* $F, G \in \Sigma[\mathcal{A}]$. *Then*

(i) $\mathrm{supp}(F) \leq \mathrm{supp}(G)$ *if and only if* $GF = G$.

(ii) $\mathrm{supp}(F) = \mathrm{supp}(G)$ *if and only if* $FG = F + GF = G$.

(iii) $F \leq G \implies \mathrm{supp}(F) \leq \mathrm{supp}(G)$. *So* supp *is order preserving.*

Proof.

(i) Both assertions are equivalent to the statement that for all $H \in \mathcal{A}$, if $\varepsilon_G(H) = 0$, then $\varepsilon_F(H) = 0$. This is easy to see.

(ii) This follows from (i).

(iii) $F \leq G \implies FG = G \implies GF = FGF \implies GF = FG = G \implies \operatorname{supp}(F) \leq \operatorname{supp}(G)$. □

Corollary 9.30. $\Pi[\mathcal{A}]$ *is the abelianization of* $\Sigma[\mathcal{A}]$ *via* supp.

Proof. We know that $\Pi[\mathcal{A}]$ is commutative and supp is a surjective monoid morphism. So take $\phi\colon \Sigma[\mathcal{A}] \to M$ any monoid morphism such that M is abelian. We will produce a unique morphism ψ as in the following diagram.

$$\begin{tikzcd} \Sigma[\mathcal{A}] \arrow[r, "\phi"] \arrow[d, "\operatorname{supp}"'] & M \\ \Pi[\mathcal{A}] \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Given $X \in \Pi[\mathcal{A}]$, pick F such that $\operatorname{supp}(F) = X$, since supp is surjective. Define $\psi(X) = \phi(F)$. Then this is well-defined, because

$$\phi(F) = \phi(FG) = \phi(F)\phi(G) = \phi(G)\phi(F) = \phi(GF) = \phi(G).$$

It is a tedious exercise to check all of the other necessary conditions on ψ. □

Definition 9.31. The **Janus Monoid** is

$$J[\mathcal{A}] = \Sigma[\mathcal{A}] \times_{\Pi[\mathcal{A}]} \Sigma[\mathcal{A}]^{\mathrm{op}}.$$

Elements of this monoid are $(F, G) \in \Sigma[\mathcal{A}]^2$ such that $\operatorname{supp}(F) = \operatorname{supp}(G)$. The product on $J[\mathcal{A}]$ is

$$(F, G) \cdot (F', G') = (FF', G'G)$$

and the unit is (O, O).

Fact 9.32. $J[\mathcal{A}]$ *is a band, although it is neither left nor right regular.*

$$\begin{tikzcd} & J[\mathcal{A}] \arrow[dl] \arrow[dr] & \\ \Sigma[\mathcal{A}] \arrow[dr] & & \Sigma[\mathcal{A}] \arrow[dl] \\ & \Pi[\mathcal{A}] & \end{tikzcd}$$

10 Species relative to a hyperplane arrangement

Definition 10.1. Let $\mathcal{A}$ be a hyperplane arrangement. Then an $\mathcal{A}$-species consists of

(a) vector spaces $P[F]$ for all $F \in \Sigma[\mathcal{A}]$.

(b) linear maps $\beta_{F,G} \colon P[F] \to P[G]$ whenever $\mathrm{supp}(F) = \mathrm{supp}(G)$.

Definition 10.2. A **morphism of $\mathcal{A}$-species** $f \colon P \to Q$ is a collection of linear maps $f_F \colon P[F] \to Q[F]$ such that

$$\begin{tikzcd} P[F] \arrow[r, "f_F"] \arrow[d, "\beta^P_{G,F}"] & Q[F] \arrow[d, "\beta^Q_{G,F}"] \\ P[G] \arrow[r, "f_G"] & Q[G] \end{tikzcd}$$

Definition 10.3. An **$\mathcal{A}$-monoid** is an $\mathcal{A}$-species M with a linear map $\mu^G_F \colon M[G] \to M[F]$ whenever $F \leq G$, such that the following axioms hold

(a) **Naturality:** Whenever $F \leq G$ and $\mathrm{supp}(F) = \mathrm{supp}(F)$, the following diagram commutes.

$$\begin{tikzcd} M[G] \arrow[r, "\mu^G_F"] \arrow[d, "\beta_{F'G,G}"] & M[F] \arrow[d, "\beta_{F,F'}"] \\ M[F'G] \arrow[r, "\mu^{F'}_{F'G}"] & M[F'] \end{tikzcd}$$

Note that $F' \leq F'G$ and

$$\mathrm{supp}(F'G) = \mathrm{supp}(F') \cup \mathrm{supp}(G) = \mathrm{supp}(F) \cup \mathrm{supp}(G) = \mathrm{supp}(G)$$

(b) **Associativity:**

$$\begin{tikzcd} & M[F] \arrow[dr, "\mu^F_E"] & \\ M[G] \arrow[ur, "\mu^G_F"] \arrow[rr, "\mu^G_E"] & & M[E] \end{tikzcd}$$

(c) **Unit law:** $\mu^F_F = \mathrm{id}_{M[F]}$.

Definition 10.4. A **morphism of $\mathcal{A}$-monoids** $f \colon M \to N$ is a morphism of $\mathcal{A}$-species such that

$$\begin{tikzcd} M[G] \arrow[r, "f_G"] \arrow[d, "\mu^G_F"] & N[G] \arrow[d, "\mu_{FG}"] \\ M[F] \arrow[r, "f_F"] & N[F] \end{tikzcd}$$

commutes.

Remark 10.5. We can similarly define $\mathcal{A}$-comonoids and $\mathcal{A}$-comonoid morphisms.

Definition 10.6. An $\mathcal{A}$-species H is an **$\mathcal{A}$-bimonoid** if it is both an $\mathcal{A}$-monoid and and $\mathcal{A}$-comonoid and these two structures are compatible: for $\mathsf{F} \leq \mathsf{G}$, we have

$$\begin{tikzcd} \mathsf{H}[\mathsf{G}] \arrow[r, shift left, "\mu_{\mathsf{F}}^{\mathsf{G}}"] & \mathsf{H}[\mathsf{G}] \arrow[l, shift left, "\Delta_{\mathsf{F}}^{\mathsf{G}}"] \end{tikzcd}$$

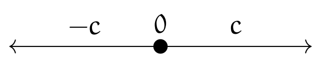
such that if $\mathsf{A} \leq \mathsf{F}$ and $\mathsf{A} \leq \mathsf{G}$, the following commutes.

$$\begin{tikzcd} \mathsf{H}[\mathsf{F}] \arrow[r, "\mu_{\mathsf{A}}^{\mathsf{F}}"] \arrow[d, "\Delta_{\mathsf{F}}^{\mathsf{FG}}"] & \mathsf{H}[\mathsf{A}] \arrow[r, "\Delta_{\mathsf{A}}^{\mathsf{G}}"] & \mathsf{H}[\mathsf{G}] \\ \mathsf{H}[\mathsf{FG}] \arrow[rr, "\beta_{\mathsf{GF},\mathsf{FG}}"] & & \mathsf{H}[\mathsf{GF}] \arrow[u, "\mu_{\mathsf{G}}^{\mathsf{GF}}"'] \end{tikzcd}$$

Note that $\operatorname{supp}(\mathsf{FG}) = \operatorname{supp}(\mathsf{GF})$.

Remark 10.7. Because everything we're working with here is connected, all $\mathcal{A}$-bimonoids are Hopf. We don't distinguish between bimonoids and Hopf monoids in this category.

Exercise 10.8. Let $\mathcal{A}$ be a rank 1 hyperplane arrangement



Show that an $\mathcal{A}$-bimonoid is the same as a vector space W with two idempotent operators $\mathsf{E}, \mathsf{F} \colon W \to W$ such that $\mathsf{EFE} = \mathsf{E}$ and $\mathsf{FEF} = \mathsf{F}$.

Lemma 10.9. *Let* H *be an* $\mathcal{A}$*-bimonoid. Then*

(i) For all $\mathsf{A} \leq \mathsf{F}$*, the following commutes.*

$$\begin{tikzcd} \mathsf{H}[\mathsf{F}] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mu_{\mathsf{A}}^{\mathsf{F}}"'] & & \mathsf{H}[\mathsf{F}] \\ & \mathsf{H}[\mathsf{A}] \arrow[ur, "\Delta_{\mathsf{A}}^{\mathsf{F}}"'] & \end{tikzcd}$$

In particular, $\mu_{\mathsf{A}}^{\mathsf{F}}$ *is injective and* $\Delta_{\mathsf{A}}^{\mathsf{F}}$ *is surjective.*

(ii) For all $\mathsf{A} \leq \mathsf{F}$ *and* $\mathsf{A} \leq \mathsf{G}$ *with* $\operatorname{supp}(\mathsf{F}) = \operatorname{supp}(\mathsf{G})$*, the following commutes.*

$$\begin{tikzcd} \mathsf{H}[\mathsf{F}] \arrow[rr, "\beta_{\mathsf{G},\mathsf{F}}"] \arrow[dr, "\mu_{\mathsf{A}}^{\mathsf{F}}"'] & & \mathsf{H}[\mathsf{G}] \\ & \mathsf{H}[\mathsf{A}] \arrow[ur, "\Delta_{\mathsf{A}}^{\mathsf{G}}"'] & \end{tikzcd}$$

Definition 10.10 (Notation)**.** When $\operatorname{supp}(F) = \operatorname{supp}(F')$, write $F \sim F'$.

Remark 10.11 (Recall)**.**

(a) $F \sim F' \iff F = FF'$ and $F' = F'F$.

(b) $F \leq G \iff FG = G$

Definition 10.12. If J is an $\mathcal{A}$-monoid, then a J**-module** is a module M over the algebra kJ.

Definition 10.13. An $\mathcal{A}$-monoid H is **commutative** if

$$\begin{tikzcd} H[F] \arrow[rr, "\beta"] \arrow[dr, "\mu"'] & & H[F'] \arrow[dl, "\mu"] \\ & H[A] & \end{tikzcd}$$

whenever $A \leq F, F'$ and $F \sim F'$. Similarly for cocommutative.

10.1 From modules to bimonoids

Proposition 10.14. *Let* M *be a left* $\Sigma[A]$*-module. For each* $F \in \Sigma[A]$ *set* $H_M[F] := F \cdot M$ *(this is a vector space!). Then*

(a) H_M *is a species.*

(b) H_M *is an* $\mathcal{A}$*-bimonoid, and it is cocommutative.*

(c) $M \mapsto H_M$ *is a functor from the category of left* $\mathcal{A}$*-modules to the category of cocommutative* $\mathcal{A}$*-bimodules.*

Proof.

(a) Define $\beta_{F,F'}$ by

$$\begin{tikzcd} H_M[F] \arrow[r, dashed, "\beta_{F,F'}"] \arrow[d, equal] & H_M[F'] \arrow[d, equal] \\ F \cdot M & F' \cdot M \\ x \arrow[u, phantom, "\cup" rotate=90] \arrow[r, mapsto] & F' \cdot x \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

We need to show that $\beta_{F,F} = \mathrm{id}$. But this follows because $F^2 = F$. Similarly, we get associativity; the crux of the argument is that if $F \sim F' \sim F''$, then $F'' = F''F'$.

(b) Suppose that $F \leq G$. Define

$$\begin{array}{ccc} H_M[F] & \overset{\mu}{\dashrightarrow} & H_M[F'] \\ \| & & \| \\ G\cdot M & & F\cdot M \\ \cup & & \cup \\ x & \longmapsto & F\cdot x \end{array} \qquad \begin{array}{ccc} H_M[F] & \overset{\Delta}{\dashrightarrow} & H_M[F'] \\ \| & & \| \\ F\cdot M & & G\cdot M \\ \cup & & \cup \\ y & \longmapsto & G\cdot y \end{array}$$

μ is well-defined since $FG = G \implies G\cdot M = F\cdot G\cdot M \subseteq FM$. Similarly for Δ.

To show that this is an $\mathcal{A}$-bimonoid, we need to show associativity and unitality for μ, coassociativity and counitality for Δ, cocommutativity and compatibility between Δ and μ.

We will check the last two.

Take $A \leq F, F'$ with $F \sim F'$. Then the following diagram commutes, because $y \mapsto F\cdot y \mapsto F'\cdot F\cdot y = F'\cdot y$

$$\begin{tikzcd} H_M[F] \arrow[rr, "\beta"] & & H_M[F'] \\ & H_M[A] \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

This shows cocommutativity.

For compatibility between Δ and μ, consider A such that $A \leq F$ and $A \leq G$. Then

$$\Delta(\mu(x)) = G\cdot x,$$

and

$$\mu(\beta(\Delta(x))) = \mu(GF\cdot FGx) = \mu(GFG\cdot x) = G\cdot x.$$

Hence, the two are the same, so we have established compatibility.

(c) This is a straightforward verification.

□

Example 10.15. Let $L[\mathcal{A}]$ be the chambers of the hyperplane arrangement $\mathcal{A}$, and $\Sigma[\mathcal{A}]$ the faces, $\Pi[\mathcal{A}]$ the flats, $E[\mathcal{A}]$ the exponential species $E[\mathcal{A}] = \{*\}$. Then the following is a diagram in the category of cocommutative $\mathcal{A}$-bimonoids; all are left-modules over $\Sigma[\mathcal{A}]$.

$$\begin{tikzcd} L[\mathcal{A}] \arrow[r, hook] \arrow[d] & \Sigma[\mathcal{A}] \arrow[d] \\ E[\mathcal{A}] \arrow[r, hook] & \Pi[\mathcal{A}] \end{tikzcd}$$

Take a face $F \in \Sigma[\mathcal{A}]$. The action of $\Sigma[\mathcal{A}]$ is

$$\begin{array}{ll} \text{on } \Sigma[\mathcal{A}]: & \mathsf{F} \cdot \mathsf{G} = \mathsf{FG} \\ \text{on } \mathsf{L}[\mathcal{A}]: & \mathsf{F} \cdot \mathsf{C} = \mathsf{FC} \\ \text{on } \Pi[\mathcal{A}]: & \mathsf{F} \cdot \mathsf{X} = \operatorname{supp}(\mathsf{F}) \vee \mathsf{X} \\ \text{on } \mathsf{E}[\mathcal{A}]: & \mathsf{F} \cdot * = * \end{array}$$

Remark 10.16. In each case of the above, taking $\mathcal{A}$ to be the braid arrangement recovers the usual species L, Σ, Π, or E from before!

Example 10.17. Both of the following are left $\Sigma[\mathcal{A}]$-modules, and hence give rise to cocommutative $\mathcal{A}$-bimonoids.

$$\mathsf{G}[\mathcal{A}] = \{\mathsf{B} \mid \mathsf{B} \leq \mathcal{A}\} = 2^{\mathcal{A}}$$

This is the $\mathcal{A}$-monoid of subarrangements of $\mathcal{A}$.

The action of $\Sigma[\mathcal{A}]$ on $\mathsf{G}[\mathcal{A}]$ is $\mathsf{F} \cdot \mathsf{B} = \{\mathsf{H} \in \mathsf{B} \mid \mathsf{F} \leq \mathsf{H}\}$. Note that this action depends only on the support of F, since $\mathsf{F} \subseteq \mathsf{H} \iff \operatorname{supp} \mathsf{F} \subseteq \mathsf{H}$. Therefore, the action of $\Sigma[\mathcal{A}]$ factors through the action of $\Pi[\mathcal{A}]$, and is therefore commutative.

Example 10.18. Consider

$$\mathsf{GP}[\mathcal{A}] = \{\text{ polytopes } \mathsf{P} \text{ in the ambient space of } \mathcal{A} \text{ such that } \mathsf{N}(\mathsf{P}) \leq \mathsf{N}(\mathsf{Z}(\mathsf{A}))\},$$

where $\mathsf{Z}(\mathcal{A})$ is the zonotope of $\mathcal{A}$, and $\mathsf{N}(\mathsf{P})$ denotes the normal fan of P. So this is all polytopes P in the ambient space of $\mathcal{A}$ that are coarser than the zonotope $\mathsf{Z}(\mathsf{A})$.

The action of $\Sigma[\mathcal{A}]$ on this is given by

$$\mathsf{F} \cdot \mathsf{P} = \mathsf{Q},$$

where Q is the face of P such that $\mathsf{Q}^\perp \supseteq \mathsf{F}$ with $\mathsf{Q}^\perp \in \mathsf{N}(\mathsf{P})$ and $\mathsf{F} \in \mathsf{N}(\mathsf{Z}(\mathcal{A}))$.

10.2 From bimonoids to modules

Proposition 10.19. *Let* H *be a cocommutative bimonoid. Define* $\mathsf{M}_\mathsf{H} = \mathsf{H}[\mathsf{O}]$*, where* O *is the central face of* $\mathcal{A}$*. Then*

(i) M_H *has the structure of a left* $\Sigma[\mathcal{A}]$*-module*

(ii) $\mathsf{H} \mapsto \mathsf{M}_\mathsf{H}$ *is a functor from the category of cocommutative* $\mathcal{A}$*-bimodules to left* $\Sigma[\mathcal{A}]$*-modules.*

Proof.

(i) Take $\mathrm{F} \in \Sigma[\mathcal{A}]$, $x \in \mathrm{M_H} = \mathrm{H}[\mathrm{O}]$. Then define $\mathrm{F} \cdot x = \mu\Delta(x)$. We have that $\mathrm{O} \cdot x = \mu_{\mathrm{O}}^{\mathrm{O}} \Delta_{\mathrm{O}}^{\mathrm{O}}(x) = x$ by the unit laws, and

$$\begin{aligned} \mathrm{G} \cdot \mathrm{F} &= \mu_{\mathrm{O}}^{\mathrm{C}} \Delta_{\mathrm{O}}^{\mathrm{G}} \mu_{\mathrm{O}}^{\mathrm{F}} \Delta_{\mathrm{O}}^{\mathrm{F}} \\ &= \mu_{\mathrm{O}}^{\mathrm{G}} \mu_{\mathrm{G}}^{\mathrm{GF}} \beta_{\mathrm{GF},\mathrm{FG}} \Delta_{\mathrm{F}}^{\mathrm{FG}} \Delta_{\mathrm{O}}^{\mathrm{F}}(x) \\ &= \mu_{\mathrm{O}}^{\mathrm{GF}} \beta_{\mathrm{GF},\mathrm{FG}} \mu_{\mathrm{O}}^{\mathrm{FG}}(x) \\ &= \mu_{\mathrm{O}}^{\mathrm{GF}} \Delta_{\mathrm{O}}^{\mathrm{GF}}(x) = \mathrm{GF} \cdot x \end{aligned}$$

(ii) Exercise.

$\square$

Remark 10.20. If instead H is commutative rather than cocommutative, we get a right action since, in the last step, we get $\mathrm{G} \cdot \mathrm{F} \cdot x = \mathrm{FG} \cdot x$.

Lemma 10.21. *Let $p \colon V \to W$ and $i \colon W \to V$ be linear maps between vector spaces such that $p \circ i = \mathrm{id}_W$ (a **splitting**). Then define $e = ip \colon V \to V$. e is an idempotent map such that $W \cong e(V)$, and moreover, the following commute*

$$\begin{tikzcd} & V \arrow[dl, "p"'] \arrow[dr, "e"] & \\ W \arrow[rr, "\cong"] & & e(V) \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "i"'] & \\ W \arrow[rr, "\cong"] & & e(V) \arrow[ul, hook] \end{tikzcd}$$

Theorem 10.22. *There is an equivalence of categories between the category $\Sigma[\mathcal{A}]$-**mod** of left $\Sigma[\mathcal{A}]$-modules and the category $(\mathcal{A}/\mathcal{A})$-$\mathbf{bim}^{cocomm.}$ of cocommutative $\mathcal{A}$-bimodules, given by*

$$\Sigma[\mathcal{A}]\text{-}\mathbf{mod} \rightleftarrows (\mathcal{A}/\mathcal{A})\text{-}\mathbf{bim}^{cocomm.}$$

$$\mathrm{M} \longmapsto \mathrm{H_M}$$

$$\mathrm{M_H} \longleftarrow\!\shortmid\ \mathrm{H}$$

Proof. On one hand, we have

$$\mathrm{M} \mapsto \mathrm{H_M} \mapsto \mathrm{M_{H_M}} = \mathrm{H_M}[\mathrm{O}] = \mathrm{O} \cdot \mathrm{M} = \mathrm{M}.$$

On the other hand, we have

$$\mathrm{H} \mapsto \mathrm{M_H} \mapsto \mathrm{H_{M_H}}.$$

We want to show that $\mathrm{H_{M_H}}$ is naturally isomorphic to H. Let $\mathrm{F} \in \Sigma[\mathcal{A}]$. Then $\mathrm{H_{M_H}}[\mathrm{F}] = \mathrm{F} \cdot \mathrm{M_H} = \mathrm{F} \cdot \mathrm{H}[\mathrm{O}]$. We need to show that $\mathrm{F} \cdot \mathrm{H}[\mathrm{O}] \cong \mathrm{H}[\mathrm{F}]$ as vector spaces. To do that, we apply Lemma 10.21 to the splitting

$$\mathrm{H}[\mathrm{O}] \underset{\mu}{\overset{\Delta}{\rightleftarrows}} \mathrm{H}[\mathrm{F}],$$

which is a splitting since $\Delta_{\mathrm{O}}^{\mathrm{F}} \mu_{\mathrm{O}}^{\mathrm{F}} = \mathrm{id}$ and $\mu_{\mathrm{O}}^{\mathrm{F}} \Delta_{\mathrm{O}}^{\mathrm{F}}$ *is* the action. $\square$

Remark 10.23. There are several other related equivalences of categories:

left or right $J[\mathcal{A}]$-modules	$\simeq$	$\mathcal{A}$-bimonoids
left $\Sigma[\mathcal{A}]$-modules	$\simeq$	commutative $\mathcal{A}$-bimonoids
right $\Sigma[\mathcal{A}]$-modules	$\simeq$	cocommutative $\mathcal{A}$-bimonoids
left or right $\Pi[\mathcal{A}]$-modules	$\simeq$	commutative and cocommutative $\mathcal{A}$-bimonoids

Remark 10.24. $J[\mathcal{A}]$ comes with a canonical involution that reversed products $(F, F') \mapsto (F', F)$.

Example 10.25. Consider $J[\mathcal{A}]$. Given an $\mathcal{A}$-bimonoid H, we define a different $M_H = H[O]$ with a different, twisted action. Then

$$(F, F') \cdot x = \mu_O^F \beta_{F,F'} \Delta_O^{F'}(x).$$

We can check that this is an action:

$$\begin{aligned}
(G, G')(F, F') \cdot x &= \mu_O^G \beta_{G,G'} \Delta_O^{G'} \mu_O^F \beta_{F,F'} \Delta_O^{F'}(x) \\
&= \mu_O^G \beta_{G,G'} \mu_O^{G'F} \beta_{G'F,FG'} \Delta_O^{FG'} \beta_{F,F'} \Delta_O^{F'}(x) \\
&= \mu_O^G \mu_O^{GF} \beta_{GF,G'F} \beta_{FG',F'G'} \Delta_{F'}^{F'G'} \Delta_O^{F'}(x) \\
&= \mu_O^{GF} \beta_{GF,F'G'} \Delta_O^{F'G'}(x) \\
&= (GF, F'G') \cdot x
\end{aligned}$$

10.3 Incidence algebras and Möbius functions

Definition 10.26. Let P be a finite poset and k a commutative ring. The **incidence algebra** of P is the set of all functions $f \colon \{(x, y) \in \P^2 \mid x \le y\} \to k$, with pointwise addition and product

$$(f \cdot g)(x, z) = \sum_{y \mid x \le y \le z} f(x, y) g(x, z).$$

The unit element δ is defined by

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$f^k(x, y) = \sum_{x = x_0 \le x_1 \le \ldots \le x_k = y} f(x_0, x_1) f(x_1, x_2) \cdots f(x_{k-1}, x_k)$$

Suppose that $f(x,x) = 0$ for all $x \in P$. Then

$$f^k(x,y) = \sum_{x=x_0<x_1<\ldots<x_k=y} f(x_0,x_1)f(x_1,x_2)\cdots f(x_{k-1},x_k)$$

is a sum over only strict chains. Since P is a finite poset, this means that f is nilpotent as there is a maximum length of chain.

Suppose that $f(x,x) = 1$ for all $x \in P$. Then

$$(\delta - f)(x,x) = 0$$

for all $x \in P$, so $\delta - f$ is nilpotent, which implies that f is invertible.

Definition 10.27. The **zeta function** of P is ζ, defined by $\zeta(x,y) = 1$ for all $x < y$ in P.

Definition 10.28. The **Möbius function** of P is $\mu = \zeta^{-1}$.

The fact that μ and ζ are inverse translates as follows. For all $x \in P$, $\mu(x,x) = 1$, and for all $x < z \in P$,

$$(\mu\cdot\zeta)(x,z) = \sum_{y|x\leq y\leq z} \mu(x,y) = 0 = \sum_{y|x\leq y\leq z} \mu(y,z) = (\zeta\cdot\mu)(x,z).$$

Either of these equations can be used to compute μ recursively.

Now let M be a k-module and M^P the set of all functions $m\colon P \to M$. Then M^P is a left $I(P)$-module under

$$(f\cdot m)(x) = \sum_{y\,:\,x\leq y} f(x,y)m(y).$$

Proposition 10.29 (Möbius Inversion)**.** *Let* $u, v\colon P \to M$ *be two functions. Then*

$$v(x) = \sum_{y\,:\,x\leq y} u(y) \iff u(x) = \sum_{y\,:\,x\leq y} \mu(x,y)v(y) \tag{M1}$$

for all $x \in P$.

Proof. The left-most equation holds if and only if $\zeta\cdot\mu = v$ if and only if $\mu\cdot v = \mu$ if and only if the right-hand side holds. □

Exercise 10.30. Define a right $I(P)$-module structure on M^P and deduce that for $u, v\colon P \to M$:

$$v(y) = \sum_{x\,:\,x\leq y} u(x) \iff u(y) = \sum_{x\,:\,x\leq y} v(x)\mu(x,y) \tag{M2}$$

for all $y \in P$.

Proposition 10.31 (Weigner's Formula)**.** *Let* P *be a finite lattice. Let* $\perp$ *and* $\top$ *be the bottom and top elements of this lattice.*

(a) Fix $y > \perp$ *and* $z \in P$*. Then* $\displaystyle\sum_{x\,:\,x \vee y = z} \mu(\perp, x) = 0.$

(b) Fix $y < \top$ *and* $z \in P$*. Then* $\displaystyle\sum_{x\,:\,x \wedge y = z} \mu(x, \top) = 0.$

Proof. Exercise. For (a), if $y \not\leq z$, then the left hand side is zero. If $x \leq z$, then

$$\{x \mid x \leq z\} = \{x \mid x \vee y = z\} \sqcup \{x \mid x \vee y < z\}.$$

Proceed by induction. □

Definition 10.32. A finite lattice P is **lower semimodular** if it is graded and $\operatorname{rank}(x) + \operatorname{rank}(y) \leq \operatorname{rank}(x \vee y) + \operatorname{rank}(x \wedge y)$ for all $x, y \in P$.

Uppser semimodular and **modular** are defined similarly.

Proposition 10.33. *Let* P *be lower semimodular. Then*

$$\mathit{sign}\mu(x,y) = (-1)^{\operatorname{rank}(y) - \operatorname{rank}(x)} \text{ } \mathit{OR} \text{ } \mu(x,y) = 0 \text{ } \mathit{for\ all} \text{ } x, y \in P.$$

Proof. Any interval $[x, y]$ in P is itself a lower-semimodular lattice, with bottom x and top y. So it's enough to show that

$$\operatorname{sign}(mu(\perp, \top)) = (-1)^{\operatorname{rank}(\top)} \text{ or } 0.$$

If $\perp = \top$, then $\mu(\perp, \top) = 1$ and $\operatorname{rank}(\top) = 0$, so we're done.

Otherwise, choose $y \in P$ covered by $\top$ ($y < \top$ and $\nexists y'$ such that $y < y' < \top$). Apply Proposition 10.31(b) with $z = 1$:

$$\sum_{x\,:\,x \wedge y = \perp} \mu(x, \top) = 0.$$

There are two cases: $x = \perp$ or $x \wedge y = \perp$, but $x \neq \perp$. If $x = \perp$, then

$$\mu(\perp, \top) = -\sum_{\substack{x\,:\,x \wedge y = \perp \\ x \neq 1}} \mu(x, \top).$$

Now choose x such that $x \wedge y = \perp$, $x \neq \perp$. We have

$$\begin{aligned} 1 &\leq \operatorname{rank}(x) \\ &\leq \operatorname{rank}(x \vee y) + \operatorname{rank}(x \wedge y) - \operatorname{rank}(x) \\ &= \operatorname{rank}(x \vee y) - \operatorname{rank}(y) \\ &\leq \operatorname{rank}(\top) - \operatorname{rank}(y) = 1 \end{aligned}$$

since $y < \top$. Hence, $\operatorname{rank}(x) = 1$, so $\operatorname{rank}[x, \top] = \operatorname{rank}(\top) - 1$.

Proceeding by induction on the rank of P, we may assume $\operatorname{sign}\mu(x, \top) = (-1)^{\operatorname{rank}(\top)-1}$. Hence,

$$\operatorname{sign}(\mu(\perp, \top)) = \begin{cases} -(-1)^{\operatorname{rank}(\top)-1} = (-1)^{\operatorname{rank}(\top)} \\ 0 \text{ if } \nexists x \text{ such that } x \wedge y = \perp, x \neq \top. \end{cases}$$

□

Remark 10.34. If in addition P is **relatively compliemented** (given $x \leq y \leq z$, $\exists y'$ such that $x \leq y' \leq z$ and $y \wedge y' = x$, $y \vee y' = z$), then $\mu(x, y) \neq 0$ for all $x \leq y$ in P. Such lattices are called **geometric**. In the above proof, we can find x such that $x \wedge y = \perp$, $x \vee y = \top$, hence $x \neq \perp$ since $y < \top$. So the set

$$\{x \mid x \wedge y = \perp, x \neq \perp\}$$

is nonempty, and induction yields $\operatorname{sign}(\mu(\perp, \top)) = (-1)^{\operatorname{rank}(\top)}$.

10.4 The algebra of a lattice

Let P be a finite lattice. We view it as a monoid under

$$x \cdot y = x \vee y. \tag{15}$$

The unit is $\perp$. Let kP be the associated algebra of k-linear combinations of elements of P. The elements of P form a basis of this algebra.

Lemma 10.35. *There is a second basis* $\{Q_x\}_{x \in P}$ *such that*

$$x = \sum_{y \,:\, x \leq y} Q_y \tag{16}$$

for all $x \in P$.

Proof. Define

$$Q_x = \sum_{y \,:\, x \leq y} \mu(x, y) \cdot y \in kP \tag{17}$$

for all $x \in P$. Then apply Möbius inversion with $M = kP$ (a k-module) and $u, v \colon P \to M$ given by

$$u(x) = Q_x, \qquad v(x) = x$$

for all $x \in P$. Then Eq. (17) implies that

$$u(x) = \sum_{y \,:\, x \leq y} \mu(x, y) v(y) \implies v(x) = \sum_{y \,:\, x \leq y} u(y)$$

by (M1). This shows (16).

Then (16) implies that $\{Q_x\}_{x \in P}$ spans kP, so $\{Q_x\}_{x \in P}$ is a basis for kP because it is of the right size. Uniqueness follows from properties of Möbius inversion. $\square$

Proposition 10.36. *The basis $\{Q_x\}_{x\in P}$ is also a complete system of orthogonal idempotents for kP. This means:*

$$\sum_{y \in P} Q_y = 1 \tag{18}$$

$$Q_x \cdot Q_y = \begin{cases} Q_x & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Proof. Notice that (16) with $x = \perp$ shows (18) since $\perp = 1$.

Now assume (18) holds. Then

$$\begin{aligned} x \cdot y &= \left(\sum_{s\colon x \le s} Q_s \right) \left(\sum_{t\colon y \le t} Q_t \right) && \text{by (16)} \\ &= \sum_{\substack{s\colon x \le s \\ y \le s}} Q_s && \text{by (15)} \\ &= \sum_{s\colon s \vee y \le s} Q_s \\ &= x \vee y && \text{by (16)} \end{aligned}$$

Hence, we have recovered (15): $x \cdot y = x \vee y$.

Now we claim that this shows (18) does hold. The reason is as follows. Let $A = kP$, and let $\mu\colon A \times A \to A$ be defined by (15), $\mu'\colon A \times A \to A$ defined by (18). What we saw is that

$$\mu'(x, y) = \mu(x, y)$$

for all $x, y \in P$, so $\mu = \mu'$, since P is a basis of A. Therefore,

$$\mu(Q_x, Q_y) = \mu'(Q_x, Q_y),$$

and this is (18).

So it remains to show (19). Claim that

$$y \cdot Q_x = \begin{cases} Q_x & \text{if } y \le x \\ 0 & \text{otherwise.} \end{cases}$$

But this holds because

$$y \cdot Q_x = \left(\sum_{z\colon y \le z} Q_z \right) Q_x = \begin{cases} Q_x & \text{if } y \le x \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $y \cdot Q_\perp = 0$ for all $y \geq \perp$. Let's expand this using (17).

$$0 = y \cdot Q_\perp = y \cdot \left(\sum_{z \in P} \mu(\perp, z) \cdot z \right) = \sum_{z \in P} \mu(\perp, z) y \vee z = \sum_{w \in P} \left(\sum_{\substack{z \in P \\ y \vee z = w}} \mu(\perp, z) \right) w$$

This implies that

$$\sum_{\substack{z \in P \\ y \vee z = w}} \mu(\perp, z) = 0.$$

This is just Proposition 10.31(a). □

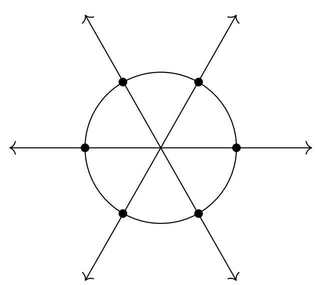
10.5 Zaslavsky's Formulas

Definition 10.37. For each $X \in \Pi[\mathcal{A}]$, let $c^X = \#\{F \in \Sigma[A] \mid \mathrm{supp}(F) = X\}$.

In particular, $c^\top = \#L[\mathcal{A}]$.

Remark 10.38 (Recall)**.** The faces of $\mathcal{A}$ form a decomposition of a sphere of dimension equal to the rank of $\mathcal{A}$ minus one.

Example 10.39. Consider the hyperplane arrangement of rank 2.



Each face corresponds to a cell of dimension $\mathrm{rank}(F) - 1$. In this case, we get a regular CW-complex structure on the sphere.

Fact 10.40. *In general, for each* $Y \in \Pi[\mathcal{A}]$*, the faces* $F \in \Sigma[\mathcal{A}]$ *with* $\mathrm{supp}(F) \leq Y$ *form a regular CW-decomposition of a sphere of dimension* $\mathrm{rank}(Y) - 1$.

Therefore,

$$\sum_{\substack{F \in \Sigma[\mathcal{A}] \\ \mathrm{supp}(F) \leq Y}} (-1)^{\mathrm{rank}(F)-1} = \overline{\chi}(S^{\mathrm{rank}(Y)-1}) = (-1)^{\mathrm{rank}(Y)-1}.$$

This implies that

$$\sum_{\substack{X\in\Pi[\mathcal{A}]\\ X\leq Y}} (-1)^{\operatorname{rank}(X)} c^X = (-1)^{\operatorname{rank}(Y)-1}$$

for each flat $Y \in \Pi[\mathcal{A}]$.

Now we can apply the Möbius inversion from last time. Let $u, v\colon \Pi[\mathcal{A}] \to k$ be given by

$$u(x) = (-1)^{\operatorname{rank}(X)} c^X, \qquad v(x) = (-1)^{\operatorname{rank}(X)}.$$

We have that

$$\sum_{\substack{X\in\Pi[\mathcal{A}]\\ X\leq Y}} u(X) = v(Y)$$

for all $y \in \Pi[\mathcal{A}]$. Then by (M1), we invert the formula

$$u(Y) = \sum_{\substack{X\in\Pi[\mathcal{A}]\\ X\leq Y}} v(x)\mu(X,Y)$$

to get

Proposition 10.41 (Zaslavsky's First Formula)**.** *For all* $Y \in \Pi[\mathcal{A}]$*,*

$$c^Y = \sum_{\substack{X\in\Pi[A]\\ X\leq Y}} (-1)^{\operatorname{rank}(Y)-\operatorname{rank}(X)} \mu(X,Y) \tag{Z1}$$

In particular, when we take $Y = T$, we get

Proposition 10.42 (Zaslavsky's Second Formula)**.**

$$\#L[\mathcal{A}] = \sum_{X\in\Pi[\mathcal{A}]} (-1)^{\operatorname{rank}(A)-\operatorname{rank}(X)} \mu(X,T) \tag{Z2}$$

Also, since

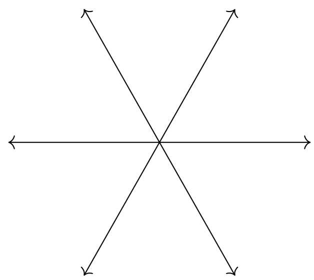
$$\Sigma[\mathcal{A}] = \bigsqcup_{Y\in\Pi[\mathcal{A}]} \{F \in \Sigma[\mathcal{A}] \mid \operatorname{supp}(F) = Y\},$$

we get

Proposition 10.43 (Zaslavsky's Third Formula)**.**

$$\#\Sigma[\mathcal{A}] = \sum_{\substack{X,Y\in\Pi[\mathcal{A}]\\ X\leq Y}} (-1)^{\operatorname{rank}(Y)-\operatorname{rank}(X)} \mu(X,Y) \tag{Z3}$$

Example 10.44. Again consider the following hyperplane arrangement $\mathcal{A}$ of rank 2.



Then

$$\Pi[\mathcal{A}] = \begin{tikzcd} & \top \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \bullet & \bullet & \bullet \\ & \bot \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

Therefore,

$$\mu(X, T) : \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ -1 & -1 & -1 \\ & 2 \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

Hence,

$$\#L[\mathcal{A}] = (-1)^2 2 + 3(-1)^1(-1) + (-1)^0 1 = 2 + 3 + 1 = 6,$$

and all the summands are positive.

Remark 10.45. We know that $\operatorname{sign}\mu(X, Y) = (-1)^{\operatorname{rank}(Y)-\operatorname{rank}(X)}$. Hence, we can rewrite the Zaslavsky formulas as follows:

$$\#L[\mathcal{A}] = \sum_{X \in \Pi[\mathcal{A}]} |\mu(X, T)| \tag{Z2}$$

$$\#\Sigma[\mathcal{A}] = \sum_{\substack{X, Y \in \Pi[\mathcal{A}] \\ X \leq Y}} |\mu(X, Y)| \tag{Z3}$$

10.6 Contraction and Restriction

Definition 10.46. Let $X \in \Pi[\mathcal{A}]$. The **contraction** of $\mathcal{A}$ to X is

$$\mathcal{A}^X = \{H \cap X \mid H \not\supseteq X,\ H \in \mathcal{A}\}.$$

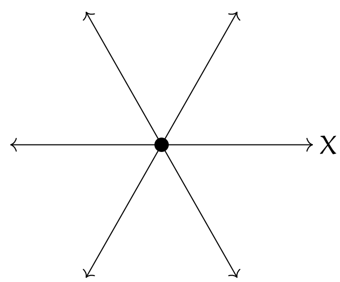
It is a hyperplane arrangement with ambient space X. It has the same center as $\mathcal{A}$.

Definition 10.47. Let $X \in \Pi[\mathcal{A}]$. The **restriction** of $\mathcal{A}$ to X is

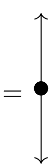
$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}.$$

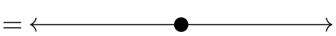
It is a hyperplane arrangement with ambient space X. It has center X.

Example 10.48. Again, we will use that hyperplane arrangement $\mathcal{A}$ we always choose:



Then

$\mathcal{A}_X =$  $\mathrm{ess}(\mathcal{A}_X) =$ 

$\mathcal{A}^X =$ 

Remark 10.49. We chose the notation $\mathcal{A}^X$ because $\mathcal{A}^X$ singles out the portion of $\mathcal{A}$ below X, and $\mathcal{A}_X$ singles out the portion of $\mathcal{A}$ above X.

More precisely,

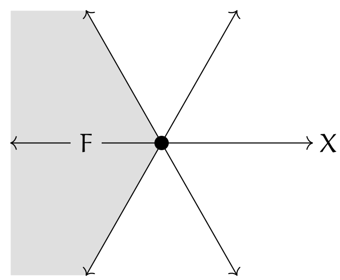
$$\begin{aligned}
\Pi[\mathcal{A}^X] &= \{Y \in \Pi[\mathcal{A}] \mid Y \leq X\} = [\bot, X] \\
\Sigma[\mathcal{A}^X] &= \{F \in \Sigma[\mathcal{A}] \mid \mathrm{supp}(F) \leq X\} \\
L[\mathcal{A}^X] &= \{F \in \Sigma[\mathcal{A}] \mid \mathrm{supp}(F) = X\}.
\end{aligned}$$

$$\Pi[\mathcal{A}_X] = \{Y \in \Pi[\mathcal{A}] \mid X \leq Y\} = [X, \top].$$

Fix a face F with $\mathrm{supp}(F) = X$. Then there are canonical bijections

$$\begin{aligned}
\Sigma[\mathcal{A}_X] &\cong \{G \in \Sigma[\mathcal{A}] \mid G \geq F\} \\
L[\mathcal{A}_X] &\cong \{C \in L[\mathcal{A}] \mid C \geq F\}
\end{aligned}$$

Example 10.50 (Continued from Example 10.48)**.** Pick the following face in $\mathcal{A}$ to see the canonical bijections in the previous remark.



11 Properties of Σ-modules

11.1 Characters of the Tits Monoid

Let M be a Σ-module over a field k. Let

$$\psi_M \colon \Sigma \to \mathrm{End}_k(M)$$

be the associated representation,

$$\psi_M(F)(x) = F \cdot x,$$

$F \in \Sigma$, $x \in M$.

Definition 11.1. The **character** of M is the function

$$\chi_M \colon \Sigma \to k, \qquad \chi_M(F) = \mathrm{tr}(\psi_M(F)).$$

Remark 11.2. This is *not* the same as the characters we were considering in Section 6.

Lemma 11.3. $\chi_M(F) = \dim(F \cdot M)$.

Proof. Notice that F is idempotent, so we get that $M = F \cdot M \oplus (1 - F) \cdot M$, and $\psi_M(F)|_{F \cdot M} = \mathrm{id}$, $\psi_M(F)|_{(1-F)\cdot M} = 0$. Therefore, the matrix of $\psi_M(F)$ is conjugate to

$$\psi_M(F) \sim \begin{bmatrix} \mathrm{id}_{F \cdot M} & 0 \\ 0 & 0 \end{bmatrix}.$$

Taking traces gives the result. ☐

Lemma 11.4. $F \sim F' \implies \chi_M(F) = \chi_M(F')$

First proof of Lemma 11.4. Recall that $F \sim F' \iff FF' = F$ and $F'F = F'$. So in $k\Sigma$, $(F - F')^2 = 0$. Hence, $\mathrm{tr}(\psi_M(F - F')) = 0$. □

Second proof of Lemma 11.4. We saw that there is an isomorphism $\beta\colon F \cdot M \to F' \cdot M$, so apply Lemma 11.3. □

Remark 11.5. Lemma 11.4 says that there is a function $\overline{\chi}_M\colon \Pi \to k$, such that

$$\begin{tikzcd} \Sigma \arrow[r, "\chi_M"] \arrow[d, two heads, "\mathrm{supp}"'] & k \\ \Pi \arrow[ur, dashed, "\overline{\chi}_M"'] & \end{tikzcd}$$

We may extend both χ_M to kM and $\overline{\chi}_M$ to $k\Pi$ by linearity. Fix $F \in \Sigma$, and let $X = \mathrm{supp}(F)$. Recall that

$$X = \sum_{Y \geq X} Q_Y \in k\Pi.$$

Then applying $\overline{\chi}_M$ to both sides of this expression, we get

$$\chi_M(F) = \overline{\chi}_M(X) = \sum_{Y \geq X} \overline{\chi}_M(Q_Y) \tag{20}$$

Remark 11.6 (Goal)**.** Our goal is to understand the characters $\overline{\chi}_M(Q_Y)$ for $Y \in \Pi[\mathcal{A}]$.

Here's the approach. Since

$$Q_X = \sum_{Y \geq X} \mu(X, Y) Y,$$

then we see that

$$\overline{\chi}_M(Q_X) = \sum_{Y \geq X} \mu(X, Y)\overline{\chi}_M(Y).$$

But $\mu(X, Y) \in \mathbb{Z}$, and $\overline{\chi}_M(Y) \in \mathbb{N}$, since $\overline{\chi}_M(Y) = \chi_M(G)$ for some G. We will actually show that $\overline{\chi}_M(Q_X) \in \mathbb{N}$ instead.

Example 11.7. Let $M = kL[\mathcal{A}]$. The action of $F \in \Sigma[\mathcal{A}]$ is $F \cdot C = FC$. Recall that $F \leq D \iff FD = D$. Hence

$$F \cdot M = k\{F \cdot C \mid C \in L[\mathcal{A}]\} = k\{D \in L[\mathcal{A}] \mid D \geq F\}.$$

Therefore,

$$\chi_M(F) = \dim(F \cdot M) = \#\{D \in L[\mathcal{A}] \mid D \geq F\} = \#L[\mathcal{A}_X],$$

where $X = \mathrm{supp}(F)$.

Now apply (Z2) to $\mathcal{A}_X$.

$$\#L[\mathcal{A}_X] = \sum_{Y \in \Pi[\mathcal{A}_X]} |\mu(Y, \top)| = \sum_{\substack{Y \in \Pi[\mathcal{A}] \\ X \leq Y}} |\mu(Y, \top)|$$

Now look at the right-hand-side of (20); which is of the same form as the right hand side above. By Möbius inversion,

$$\overline{\chi}_M(Q_Y) = |\mu(Y, \top)|.$$

Example 11.8. Let $M = k\Sigma[\mathcal{A}]$ with $F \cdot G = FG$. As before, $F \cdot M = k\{G \in \Sigma[\mathcal{A}] \mid F \leq G\}$. Therefore, $\chi_M(F) = \#\Sigma[\mathcal{A}_X]$ where $X = \operatorname{supp}(F)$.

$$\begin{aligned} \overline{\chi}_M(X) &= \#\Sigma[\mathcal{A}_X] \\ &= \sum_{\substack{Y,Z \in \Pi[\mathcal{A}_X] \\ Y \leq Z}} |\mu(Y, Z)| \\ &= \sum_{\substack{Y,Z \in \Pi[\mathcal{A}] \\ X \leq Y \leq Z}} |\mu(Y, Z)| \\ &= \sum_{Y \colon X \leq Y} \left(\sum_{Z \colon Y \leq Z} |\mu(Y, Z)| \right) \end{aligned}$$

Then again by Möbius inversion,

$$\overline{\chi}_M(Q_Y) = \sum_{Z \colon Y \leq Z} |\mu(Y, Z)|.$$

Remark 11.9. We saw that if $F \sim F'$, then $(F - F')^2 = 0$. This comes from the fact that

$$\ker(\operatorname{supp}) = k\{F - F' \mid F \sim F'\}.$$

This is an ideal of $k\Sigma$ linearly spanned by nilpotent elements, and in fact this ideal is nilpotent (although this is not always the case!)

Moreover, $\ker(\operatorname{supp})$ is the Jacobson radical of $k\Sigma$.

11.2 Primitive Elements

Let M be a left Σ-module.

Definition 11.10. An element $x \in M$ is **primitive** if $F \cdot x = 0$ for all $F \neq 0$. Let $P(M)$ be the space of primitive elements of M.

Definition 11.11. Let Lie $= P(kL)$ and Zie $= P(k\Sigma)$.

Remark 11.12. We call $P(k L)$ Lie by analogy with the Lie elements of a Hopf Algebra. We offer the name Zie with no justification. It is what it is.

Definition 11.13. We say that $z \in$ Zie is **special** if when we write $z = \sum z^F F$, with $F \in \Sigma$, we have $z^O = 1$.

Proposition 11.14. *If* $z \in$ Zie*, then the image of* $\psi_M(z)\colon M \to M$ *is contained in the space* $P(M)$ *of primitive elements of* M*. If, in addition,* z *is special, then* $\psi_M(z)$ *projects* M *onto* $P(M)$.

Proof. Take $x \in M$. We have to show that $\psi_M(z)(x) = z \cdot x \in P(M)$. For this, take $E \neq 0$ in Σ.

$$E \cdot (z \cdot x) = (E \cdot z) \cdot x = 0$$

because $z \in P(k\Sigma)$. Then $z \cdot x \in P(M)$.

Now suppose that z is special as well. Take $x \in P(M)$.

$$z \cdot x = \sum_F z^F (F \cdot x) = z^O x = x$$

because $F \cdot x = 0$ unless $F = O$. □

Corollary 11.15. *Special* Zie *elements are idempotent.*

Proof. $z^2 = \psi_{k\Sigma}(z)(z) = z$ since $z \in \text{Zie} = P(k\Sigma)$. □

Remark 11.16. Let F be the forgetful functor

$$F\colon \Sigma\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}_k$$

. Then $k\Sigma \cong \text{End}(F)$ can be recovered as the algebra of natural transformations $F \to F$, given by

$$\begin{array}{rcl} k\Sigma & \xrightarrow{\sim} & \text{End}(F) \\ z & \longmapsto & \psi_M(z)\colon F(M) \to F(M) \end{array}$$

Similarly, Zie $\cong \text{Hom}(F, P)$, given by

$$\begin{array}{rcl} \text{Zie} & \xrightarrow{\sim} & \text{Hom}(F, P) \\ z & \longmapsto & \psi_M(z)\colon M \to P(M) \end{array}$$

Now take $E \in \Sigma$, $z = \sum_F z^F F \in k\Sigma$. Then

$$E \cdot z = \sum_F z^F E \cdot F = \sum_G \left(\sum_{F\colon EF=G} z^F \right) G.$$

(If G intervenes here, then $E \leq G$) Therefore,

$$z \in \text{Zie} \iff 0 = \sum_{F\colon EF=G} z^F \text{ for all } E, G \in \Sigma \text{ such that } O < E \leq G.$$

Consider the special case $E = G$. Recall that $GF = G \iff \text{supp}(F) \leq \text{supp}(G)$. Then

$$z \in \text{Zie} \implies \sum_{F:\ \text{supp}(F) \leq X} z^F = 0 \text{ for all } X \in \Pi, X \neq \bot. \tag{21}$$

Lemma 11.17. *For $z \in k\Sigma$, the following are equivalent.*

(i) $z^O = 1$ *and* $\displaystyle\sum_{F:\ \text{supp}(F) \leq X} z^F = 0$ *for all* $X \neq \bot$.

(ii) $\displaystyle\sum_{F:\ \text{supp}(F) = X} z^F = \mu(\bot, X)$ *for all* $X \in \Pi$.

(iii) $\text{supp}(z) = O_\bot = \displaystyle\sum_X \mu(\bot, X) X$

Proof. To show that (ii) and (iii) are equivalent, observe.

$$\text{supp}(z) = \sum_F z^F \,\text{supp}(F) = \sum_X \left(\sum_{F:\ \text{supp}(F) = X} z^F \right) X$$

Comparing the coefficients shows (ii) $\iff$ (iii).

Now let

$$f(X) = \sum_{F:\ \text{supp}(F) = X} z^F$$

for $X \in \Pi$. Then (i) holds if and only if $f(\bot) = 1$ and

$$\sum_{Y:\ Y \leq X} f(Y) = O$$

for all $X \neq \bot$, if and only if

$$f(X) = \mu(\bot, X)$$

by the definition of the Möbius function. Therefore, we have shown (i) $\iff$ (ii). ☐

Remark 11.18. Notice that if z is a special Zie element, then Lemma 11.17(i) holds by (21). Therefore, Lemma 11.17(ii) and Lemma 11.17(iii) also hold for special Zie elements.

?

Remark 11.19 (Recall Remark 11.6)**.** We saw last time that

$$\begin{tikzcd} \Sigma \arrow[r, "\chi_M"] \arrow[d, two heads, "\text{supp}"'] & k \\ \Pi \arrow[ur, dashed, "\overline{\chi}_M"'] & \end{tikzcd}$$

where $\overline{\chi}_M(X) = \chi_M(F)$ for F such that $\operatorname{supp}(F) = X$.

$$\chi_M(F) = \operatorname{tr}(\psi_M(F)) = \dim(\operatorname{im}(\psi_M(F))).$$

We want to understand $\overline{\chi}_M(Q_X)$, where

$$X = \sum_{Y\colon X \leq Y} Q_Y.$$

Proposition 11.20. $\overline{\chi}_M(Q_\perp) = \dim P(M)$.

Proof. We use the fact that special Zie elements do exist, which we have not yet shown, but will show later.

Let z be a special Zie element. By Lemma 11.17(iii),

$$\begin{aligned}
\overline{\chi}_M(Q_\perp) &= \overline{\chi}_M(\operatorname{supp}(z)) && \text{by Lemma 11.17(iii)} \\
&= \chi_M(z) \\
&= \operatorname{tr}(\psi_M(z)) \\
&= \dim(\operatorname{im}(\psi_M(z))) && \text{by Corollary 11.15, } z \text{ is idempotent} \\
&= \dim P(M) && \text{by Proposition 11.14}
\end{aligned}$$

□

Corollary 11.21. *Let $\mathcal{A}$ be a hyperplane arrangement. Then*

$$\dim \operatorname{Lie}[\mathcal{A}] = |\mu(\perp, \top)|,$$
$$\dim \operatorname{Zie}[\mathcal{A}] = \sum_z |\mu(\perp, z)|.$$

Proof. By Proposition 11.20,

$$\dim \operatorname{Lie} = \dim P(kL) = \overline{\chi}_{kL}(Q_\perp) = |\mu(\perp, \top)|,$$
$$\dim \operatorname{Zie} = \dim P(k\Sigma) = \overline{\chi}_{k\Sigma}(Q_\perp) = \sum_z |\mu(\perp, z)|.$$

□

Remark 11.22. Similarly, one shows that

$$\overline{\chi}_M(Q_X) = \dim \left\{ x \in F \cdot M \;\middle|\; G \cdot x = 0 \text{ for all } G > F \right\} \tag{22}$$

where $F \in \Sigma$ is *any* face such that $\operatorname{supp}(F) = X$. For this, one applies the preceeding to the hyperplane arrangement $\mathcal{A}_X$.

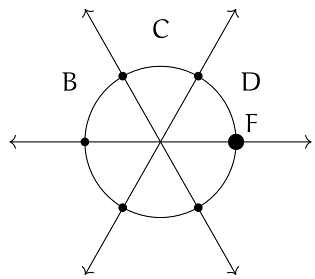
Noting that $[X, \top]_{\Pi[\mathcal{A}]} = \Pi[\mathcal{A}_X]$, and that $F \cdot M$ is a module over $F \cdot \Sigma[\mathcal{A}] \cong \Sigma[\mathcal{A}_X]$, with $P(F \cdot M)$ exactly the space on the right-hand side of (22). This accomplishes the goal in Remark 11.6.

11.3 An analysis of Lie $= P(kL)$

Definition 11.23. Given $F \in \Sigma$, $D \in L$, let

$$\ell(F, D) = \{C \in L \mid FC = D\}.$$

Example 11.24. For the hyperplane arrangement $\mathcal{A}$ below, where F and D are as labelled, $\ell(F, D)$ consists of D, C and B.

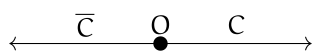


Given $z \in kL$, write $z = \sum_{C \in L} z^C C$. As for Zie elements, we have that

$$z \in \text{Lie} \iff F \cdot z = O \text{ for all } F \neq O \iff \sum_{C \in \ell(F,D)} z^c = 0 \text{ for all } O < F \leq D$$

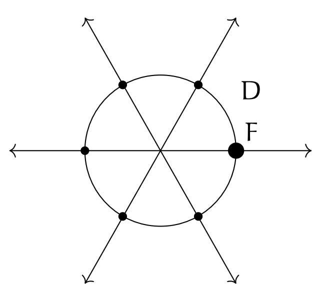
with $F \in \Sigma$ and $D \in L$.

Example 11.25. If $\mathcal{A}$ is the hyperplane arrangement below, $\Pi[\mathcal{A}] = \{\perp, \top\}$, $|\mu(\perp, \top)| = 1$, $\ell(C, C) = \{C, \overline{C}\} = \ell(\overline{C}, \overline{C})$.

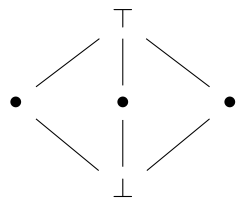


$$\text{Lie}[\mathcal{A}] = \{aC + b\overline{C} \in kL[\mathcal{A}] \mid a + b = 0\}.$$

Example 11.26 (Example 11.24, continued)**.** For the hyperplane arrangement $\mathcal{A}$ below, where F and D are as labelled, $\ell(F, D)$ consists of D, C and B.



$\Pi[\mathcal{A}]$ is the lattice below.

$\Pi[\mathcal{A}] =$ 

We have that $|\mu(\perp, \top)| = 2 = \dim \mathrm{Lie}[\mathcal{A}]$. The equations say that along each semicircle, the sum of the coefficients is zero. Then $\ell(F, D)$ is a semicircle and $\ell(D, D)$ is a full circle. We have that

$$\mathrm{Lie}[\mathcal{A}] = \{(a, b, c) \in k^3 \mid a + b + c = 0\}.$$

Below are some facts that we will not prove, but are true anyway.

Fact 11.27. *Consider* $\mathrm{supp}\colon k\Sigma \to k\Pi$. *Then*

(a) $\ker(\mathrm{supp})$ *is the Jacobson radical of* $k\Sigma$. *This ideal is nilpotent, and in fact*

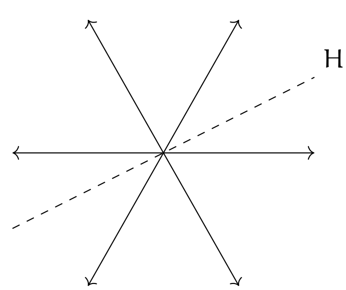
$$\mathrm{Jac}(k\Sigma)^{\mathrm{rank}(\mathcal{A})+1} = 0.$$

(b) $\mathrm{Jac}(k\Sigma)^{\mathrm{rank}(\mathcal{A})} = \mathrm{Lie}[\mathcal{A}]$.

11.4 Dynkin Idempotents

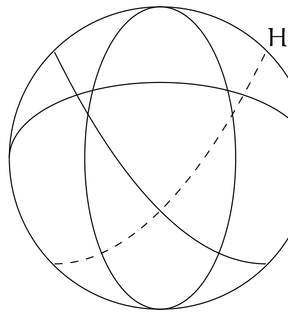
Definition 11.28. Let $\mathcal{A}$ be a hyperplane arrangement with center O and ambient space V. A hyperplane H in V is **generic** with respect to $\mathcal{A}$ if it contains O but does not contain any other face of $\mathcal{A}$. In particular, $H \notin \mathcal{A}$

Example 11.29. Consider the hyperplane

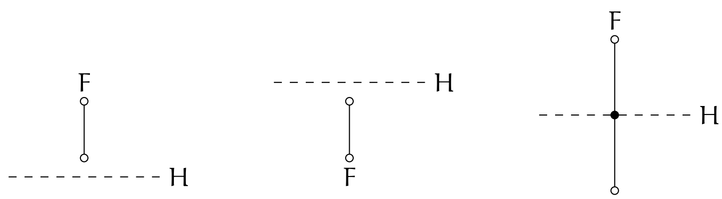


Example 11.30. If $\mathrm{rank}(\mathcal{A}) = 1$, then the only possible choice of generic hyperplane is $H = O$.

Example 11.31. For an example where $\operatorname{rank}(\mathcal{A}) = 3$, consider the hyperplane arrangement below; it is drawn projected onto the unit sphere in $\mathbb{R}^3$.



Let H be a generic hyperplane with respect to $\mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \cup \{H\}$. A face F of $\mathcal{A}$ is either completely contained in one of H^+ or H^-, or has points in both H^+ and H^-.



In the first two cases, F remains a face of $\mathcal{A}'$. In the last, F gives rise to 3 faces of $\mathcal{A}'$: namely $F \cap H^+$, $F \cap H^-$, and $F \cap H$.

Definition 11.32. Let h be a closed half space where the bounding hyperplane is generic with respect to $\mathcal{A}$. The associated **Dynkin element** is

$$\theta_h = \sum_{F:\, F \subseteq h} (-1)^{\operatorname{rank}(F)} F \in k\Sigma[\mathcal{A}].$$

Remark 11.33. This gives an example of a special Zie element; we proved Proposition 11.20 assuming the existence of special Zie elements, and this finally gives an example of such an element.

Note that

$$\theta_h = O + \sum_{F:\, F \subseteq \mathring{h}} (-1)^{\operatorname{rank}(F)} F$$

Proposition 11.34. θ_h *is a special Zie element. In particular, it is idempotent.*

Proof sketch[1]. Recall that

$$z \in \mathrm{Zie} \iff \sum_{F:\, EF = G} z^F = O$$

for $O < E \leq G$, if and only if

$$\sum_{F:\, EF \leq G} z^F = 0$$

for all $O < E \leq G$.

So we need to show that

$$\sum_{\substack{F:\, EF \leq G \\ F \subseteq h}} (-1)^{\operatorname{rank}(F)} = O$$

for all $O < E \leq G$.

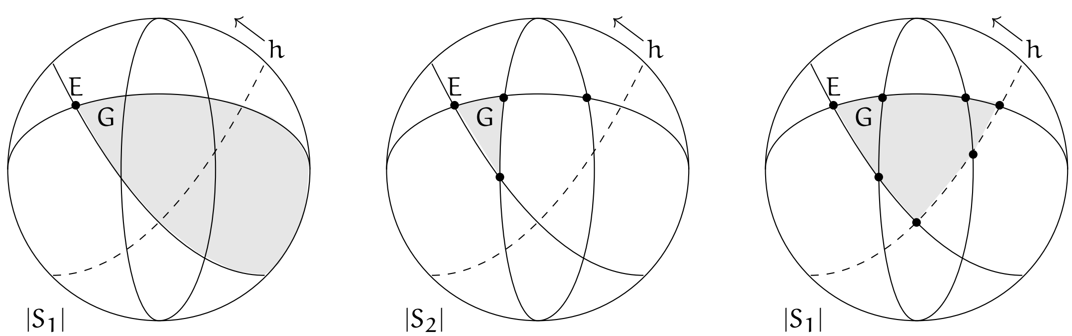
Recall that each face corresponds to a cell in a CW decomposition of a face. Consider the following sets.

$$S_1 = \{F \in \Sigma[\mathcal{A}] \colon EF \leq G\}$$

$$S_2 = \{F \in \Sigma[\mathcal{A}] \colon EF \leq G, F \subseteq h\}$$

$$S_3 = \{F \in \Sigma[\mathcal{A}] \colon EF \leq G, F \subseteq h\}$$

Here $\mathcal{A} \subseteq \mathcal{A} \cup \{H\}$ and $H = \partial h$.



The idea is that $|S_2|$ is complicated, but $|S_1|$ and $|S_3|$ are not.

$|S_3|$ is a **cone** of teh arrangement $\mathcal{A}'$. Hence, it is either a ball or a sphere. It is not a sphere in this case because H is generic. Therefore, $|S_3|$ is a ball and

$$\sum_{F \in S_3} (-1)^{\operatorname{rank}(F)} = 0.$$

Now the faces in S_3/S_2 are of the form $F \cap h$ or $F \cap H$, for some $F \in S_1$.

So they come in pairs with rank difference 1. Therefore,

$$\sum_{F \in S_3 \setminus S_2} (-1)^{\operatorname{rank}(F)} = 0.$$

It follows that

$$\sum_{F \in S_2} (-1)^{\mathrm{rank}(F)} = 0.$$

□

Corollary 11.35. *For any* h *as above,*

$$|\mu(\perp, \top)| = \#\{c \in L[\mathcal{A}] \colon C \subseteq h\}.$$

Proof. We saw that for a special Zie element z, and any $X \in \Pi[\mathcal{A}]$,

$$\sum_{F \colon \mathrm{supp}(F) = X} z^F = \mu(\perp, X).$$

Apply to $z = \theta_h$, $X = \top$. We get

$$\sum_{\substack{C \in L[\mathcal{A}] \\ C \subseteq h}} (-1)^{\mathrm{rank}(C)} = \mu(\perp, \top).$$

Then

$$\#\{C \in L[\mathcal{A}] \colon C \subseteq h\} = (-1)^{\mathrm{rank}(\top)} \mu(\perp, \top) = |\mu(\perp, \top)|.$$

□

Remark 11.36. Notice that, because each Dynkin element is a special Zie element, it is an idempotent.

Recall that θ_h is a special Zie element, which means that

$$\begin{array}{rcl} \psi_{kL}(\theta_h) \colon kL[\mathcal{A}] & \longrightarrow & \mathrm{Lie}[\mathcal{A}] \\ c & \longmapsto & \theta_h \cdot c \end{array}$$

Lemma 11.37. *Let* h *be as before. Let* C *be a chamber such that* $C \not\subseteq \overline{h}$*, where* $\overline{h}$ *is the opposite half-space. Then* $\theta_h \cdot C = 0$.

The proof of this lemma is hard, and so we will not prove it.

Proposition 11.38. *For any such* h*, and any* $C \in L[\mathcal{A}]$*, the set*

$$\{\theta_h \cdot C \mid C \subseteq \overline{h}\}$$

is a basis of the space $\mathrm{Lie}[\mathcal{A}]$.

Proof. The facts that $\psi(\theta_h)$ is onto and Lemma 11.37 imply that

$$\{\theta_h \cdot C \mid C \subseteq \overline{h}\}$$

spans $\mathrm{Lie}[\mathcal{A}]$. This set has the right-dimension, by Corollary 11.35 applied to $\overline{h}$. □

11.5 Application: another Zaslavsky's formula

We can use this to derive another form of Zaslavsky's formula, which we call **Zaslavsky's formula for bounded chambers.** Let $\mathcal{B}$ be an affine hyperplane arrangement. Then $\Pi[\mathcal{B}]$ is a join-semilattice, meaning it has meets and $\perp$ may not exist.

Proposition 11.39 (Zaslavsky)**.** *The number of bounded chambers in an affine hyperplane arrangment $\mathcal{B}$ is*

$$\left|\sum_{X \in \Pi[\mathcal{B}]} \mu(X, \top)\right|$$

Before proving this, we need a lemma. We will not prove the lemma.

Lemma 11.40.

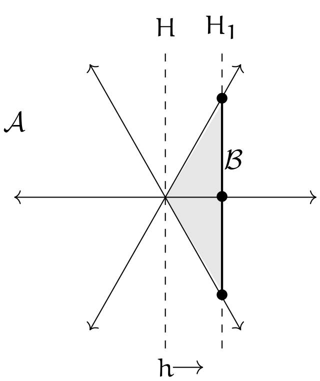
(1) $\Pi[\mathcal{A}] \cong \Pi[\mathcal{B}] \sqcup \{\perp\}$.

(2) Let h *be the half space bounded by* H *that contains* H_1*. Then*

$$\{C \in L[\mathcal{A}] : C \subseteq h\} \cong \{C \in L[\mathcal{B}] \mid C \text{ bounded}\}.$$

(3) Any affine hyperplane arrangement $\mathcal{B}$ arises in this manner, where $\mathcal{A}$ is the projectivization of $\mathcal{B}$.

Proof of Proposition 11.39. Let $\mathcal{A}$ be a linear hyperplane arrangemetn and H a generic hyperplane with respect to $\mathcal{A}$. Let H_1 be an affine hyperplane parallel to H, and different from H itself. Let $\mathcal{B} = \mathcal{A}^{H_1}$. This is an affine hyperplane arrangement with ambient space H_1.



Then, using Lemma 11.40, we can conclude that

$$\begin{aligned}
\#\{C \in L[\mathcal{B}] \mid C \text{ bounded}\} &= \#\{C \in L[\mathcal{A}] \mid C \subseteq h\} \\
&= |\mu_{\Pi[\mathcal{A}]}(\perp, \top)| && \text{by Corollary 11.35}
\end{aligned}$$

But we also have that

$$\mu_{\Pi[\mathcal{A}]}(\bot, \top) = -\sum_{X \in \Pi[\mathcal{A}], X \neq \bot} \mu_{\Pi[\mathcal{A}]}(X, \top) = -\sum_{X \in \Pi[\mathcal{B}]} \mu_{\Pi[\mathcal{B}]}(X, \top).$$

Hence we conclude the result. □

11.6 The radical of $k\Sigma$

Definition 11.41. A **nilpotent ideal** of an algebra A is an ideal I such that $I^m = 0$ for some m.

Note that this is stronger than the statement that each element of the ideal is nilpotent. In particular, any finite product of elements of the ideal of length longer than m is zero.

Definition 11.42. The **(Jacobson) radical** of an algebra A is the largest nilpotent ideal of A.

Let $K = \ker(\text{supp}\colon k\Sigma \to k\Pi)$. Our goal is to show that K is the radical of $k\Sigma$.

Definition 11.43.

An element $x \in k\Sigma$ is homogeneous if

$$x = \sum_{G\colon \text{ supp}(G) = X} x^G G$$

for some $x \in \Pi$. We write $\text{supp}(x) = X$.

Fact 11.44. K *is linearly spanned by homgeneous elements. In fact,* K *is linearly spanned by all elements of the form* $F - F'$ *where* $F \sim F'$*.*

Note that if $x \in K$ is homogeneous, then

$$\sum_{G\colon \text{ supp}(G) = X} x^G = 0,$$

where $X = \text{supp}(X)$.

Lemma 11.45. *Let* $x \in K$ *be homogeneous,* $\text{supp}(x) = X$*.*

(a) If $\text{supp}(F) \geq X$*, then* $F \cdot x = 0$*.*

(b) If $y \in k\Sigma$ *is another homogeneous element, with* $\text{supp}(y) \geq X$*, then* $yx = 0$*.*

Proof. (a) Recall that $\operatorname{supp}(F) \geq \operatorname{supp}(G) \iff FG = F$. Hence,

$$F \cdot x = \sum_{G:\ \operatorname{supp}(G)=X} x^G FG = \left(\sum_{G:\ \operatorname{supp}(G)=X} xG \right) F = 0$$

(b) This follows from (a), because if y is homogeneous then y is a linear combination of elements F with $\operatorname{supp}(F) \geq X$.

$\square$

Proposition 11.46. *Let* $r = \operatorname{rank}(\mathcal{A})$. *Then* $K^r \subseteq \text{Lie}$.

Proof. Take $x_1, \dots, x_r \in K$ and $F \in \Sigma, F > O$. We need to show

$$x_1 x_2 \cdots x_r \in kL \subseteq k\Sigma \tag{23}$$

$$Fx_1 x_2 \cdots x_r = 0 \tag{24}$$

If we show both of these, then $x_1, \dots, x_r \in P(kL) = \text{Lie}$.

We may assume that the x_i are homogeneous. Since

$$\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y),$$

we have that

$$\perp \leq \operatorname{supp}(x_1) \leq \operatorname{supp}(x_1 x_2) \leq \ldots \leq \operatorname{supp}(x_1 x_2 \cdots x_r) \leq \top.$$

This is a chain of length $r+1$ inside a lattice of rank r, so one of these must be an equality. Either the inequality comes in the middle or at the top.

Suppose that

$$\operatorname{supp}(x_1 x_2 \cdots x_{i-1}) = \operatorname{supp}(x_1 x_2 \cdots x_i)$$

for some i, $1 \leq i \leq r$. Then

$$\operatorname{supp}(x_i) \leq \operatorname{supp}(x_1 x_2 \cdots x_{i-1}).$$

Now let $x = x_i$, $y = x_1 \cdots x_{i-1}$. Then by Lemma 11.45, $yx = 0$. But $yx = x_1 x_2 \cdots x_i$, so we have

$$x_1 x_2 \cdots x_r = (x_1 x_2 \cdots x_i) x_{i+1} \cdots x_r = yx(x_{i+1} \cdots x_r) = 0$$

and we're done.

Otherwise, we have a strict chain, except at the top.

$$\perp < \operatorname{supp}(x_1) < \operatorname{supp}(x_1 x_2) < \ldots < \operatorname{supp}(x_1 x_2 \cdots x_r) \leq \top.$$

Because the rank is r, again we must have equality at the top otherwise the rank is larger. This implies

$$\mathrm{supp}(x_1 x_2 \cdots x_r) = \top \implies x_1 x_2 \cdots x_r \in kL.$$

This shows (25). To show (24), apply a similar argument for

$$\perp \leq \mathrm{supp}(F) \leq \mathrm{supp}(Fx_1) \leq \ldots \leq \mathrm{supp}(Fx_1 x_2 \cdots x_r) = \top.$$

This means that

$$\mathrm{supp}(Fx_1 x_2 \cdots x_{i-1}) = \mathrm{supp}(Fx_1 x_2 \cdots x_i)$$

for some $1 \leq i \leq r$. Therefore, $Fx_1 x_2 \cdots x_r = 0$. □

Remark 11.47. Later we'll see that $K^r = \mathrm{Lie}$.

Proposition 11.48. $K^{r+1} = 0$

Proof. Either use a similar argument to the proof of Proposition 11.46, or do the following.

We have $K^r \subseteq \mathrm{Lie} \subseteq \mathrm{Zie}$, so $F \cdot K^r = O$ for all $F > O$. Therefore, $(F - F') \cdot K^r = 0$ for all $F \sim F'$. Hence $K \cdot K^r = 0$. □

Corollary 11.49. $K = \mathrm{rad}(k\Sigma)$.

Proof. We know that K is nilpotent and $k\Sigma / K = k\Pi$ is **semisimple**, meaning it has no nontrivial nilpotent ideals.

Take $x \in k\Sigma$ nilpotent. Therefore $\bar{x} \in k\Sigma/K$ is nilpotent. Hence $\bar{x} = \bar{0}$. Therefore, $x \in K$.

Therefore, K consists precisely of the nilpotent elements of $k\Sigma$. So K is a nilpotent ideal (by Proposition 11.48) and contains each nilpotent element, so it must be the largest nilpotent ideal. Hence, $K = \mathrm{rad}(k\Sigma)$. □

Remark 11.50. The algebra $M_n(k)$ of $n \times n$ matrices over k is **simple**, meaning it has no proper nontrivial ideals. So $\mathrm{rad}(M_n(k)) = 0$. But there are many nilpotent elements of this algebra.

12 The Joyal-Klyachoko-Stanley isomorphism

12.1 Homology of Posets

Definition 12.1. Let P be a finite poset with minimum $\perp$ and maximum $\top$. Suppose $\perp < \top$. The **order complex** of P is the simiplicial complex $\Delta(P)$, whose i-simplicies are the strict chains of length i in $P \setminus \{\perp, \top\}$, or equivalently, the chains of length $i+2$ from $\perp$ to $\top$ in P,

$$\perp = x_0 < x_1 < \ldots < x_i < x_{i+1} < x_{i+2} = \top.$$

Definition 12.2. The **homology** of P is the homology of the chain complex

$$C_i(P) = k\Delta_i(P) = \text{ formal linear combinations of } i\text{-simplicies.}$$

$$\cdots \xrightarrow{\partial_{i+1}} C_i(P) \xrightarrow{\partial_i} C_{i-1}(P) \longrightarrow \cdots \longrightarrow C_0(P) \xrightarrow{\partial_0} C_{-1}(P),$$

$$\partial_i(\perp < x_1 < \ldots < x_{i+1} < \top) = \sum_{j=1}^{i+1} (-1)^j (\perp < x_1 < \ldots < \widehat{x_j} < \ldots < x_{i+1} < \top).$$

$$H_i(P) = {}^{\ker(\partial_i)}\!/\!_{\operatorname{im}(\partial_{i+1})}$$

Note that $C_0(P) = k\Delta_0(P) \cong kP$, $C_{-1}(P) = k\{\perp < \top\} \cong k$.

Remark 12.3. This is the **reduced homology** of the geometric realization of $\Delta(P)$. So

$$H_i(P) = \widetilde{H_i}(\Delta(P)).$$

Definition 12.4. A finite lattice P is **cogeometric** if P is lower semimodular and every element is the meet of elements of rank $r-1$, where $r = \operatorname{rank}(P)$.

Example 12.5. $\Pi[\mathcal{A}]$ is cogeometric.

Fact 12.6. *If P is a cogeometric lattice of rank r, then $\Delta(P)$ is homotopy equivalent to a wedge of spheres of dimension $r-2$.*

$$\Delta(P) \simeq S^{r-2} \vee S^{r-2} \cdots \vee S^{r-2}.$$

Therefore,

$$H_i(P) = \begin{cases} k^m & \text{if } i = r-2 \\ 0 & \text{if } i \neq r-2, \end{cases}$$

and we can compute the reduced Euler characteristic as

$$\widetilde{\chi}(P) = (-1)^{r-2} m.$$

This implies that

$$\mu(\perp, \top) = (-1)^r m.$$

So we can figure out m from the Möbius function:

$$m = |\mu(\perp, \top)|.$$

Example 12.7. Now let $P = \Pi[\mathcal{A}]$, where $\operatorname{rank}(\mathcal{A}) = r$. So

$$\dim H_{r-2}(\Pi[\mathcal{A}]) = |\mu(\perp, \top)|.$$

Recall also that $\dim \mathrm{Lie}[\mathcal{A}] = |\mu(\perp, \top)|$, so both $\mathrm{Lie}[\mathcal{A}]$ and $H_{r-2}(\Pi[\mathcal{A}])$ are vector spaces of the same dimension. Our goal is to find a deeper relation between these two.

Theorem 12.8 (Aguiar)**.** *There is a natural isomorphism (natural in $\mathcal{A}$)*

$$H^{r-2}(\Pi[\mathcal{A}]) \otimes E^0[\mathcal{A}] \cong \mathrm{Lie}[\mathcal{A}],$$

where $E^0[\mathcal{A}]$ is the ***space of orientations*** *of $\mathcal{A}$ (to be defined later).*

We will prove this theorem in this section.

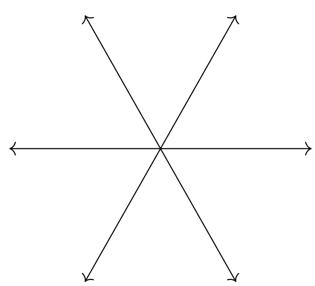
Example 12.9 (Joyal-Klyachko-Stanley, Barcelo-Wachs, Björner)**.** In particular, when $\mathcal{A}$ is the braid arrangement in $\mathbb{R}^n$ (having rank $r = n-1$), then there is an isomorphism

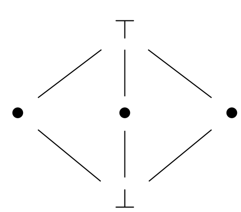
$$H^{n-3}(\Pi[n]) \otimes \mathcal{E}_n \cong \mathrm{Lie}[n]$$

not only as vector spaces, but also as S_n-modules. Here,

- $\Pi[n]$ is the lattice of partitions of $[n]$.
- $\mathcal{E}_n$ is the sign representation of S_n.
- $\mathrm{Lie}[n]$ is the space of classical Lie elements.
- S_n is the symmetric group.

Example 12.10. Let $\mathcal{A}$ be the braid arrangement in $\mathbb{R}^3$.



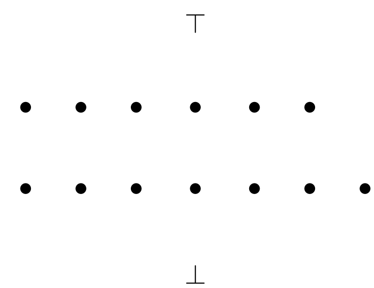
$\Pi[\mathcal{A}] =$ 

$$\Delta(\Pi[\mathcal{A}])\colon \bullet \qquad \bullet \qquad \bullet \simeq S^0 \vee S^0$$

$$m = 2 = \mu(\perp, \top)$$

Example 12.11. Let $\mathcal{A}$ be the braid arrangement in $\mathbb{R}^4$. Then

$\Pi[\mathcal{A}] =$



Then $\Delta(\Pi[\mathcal{A}])$ has 13 vertices and 18 edges. $\mu(\perp, \top) = -6$, and

$$\Delta(\Pi[\mathcal{A}]) \simeq S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1.$$

12.2 Orientations

Let V be a real vector space of dimension n. Given two ordered bases of V, $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{C} = (w_1, \ldots, w_n)$. Let $T\colon V \to V$ be the linear change of basis. Write $\mathcal{B} \sim \mathcal{C}$ if $\det(T) > 0$. This is an equivalence relation.

Definition 12.12. The equivalence class of ordered bases on V is an **orientation** of V. There are exactly two orientations if $\dim V \geq 1$.

Definition 12.13. Let $\mathcal{A}$ be a hyperplane arrangement in V. Let $E^0[\mathcal{A}]$ denote the k-vector space spanned by the two orientations σ_1, σ_2 of V/O modulo the relation $\sigma_1 + \sigma_2 = 0$.

Let $\Delta_{\max}(\Sigma[\mathcal{A}])$ be the set of maximal chains of faces

$$f = (0 < F_1 < \ldots < F_{r-1} < F_r).$$

Let $C_f = F_r$, and

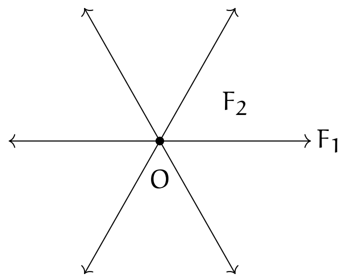
$$\operatorname{supp}(f) = (\perp < x_1 < \ldots < x_{r-1} < \top).$$

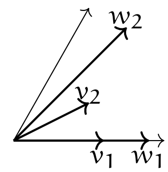
Note that $r = \operatorname{rank}(\mathcal{A})$, C_f is a chain, and

$$\operatorname{supp}(f) \in \Delta_{r-2}(\Pi[\mathcal{A}]).$$

Pick a vector $v_i \in F_i$ for each $i = 1, \ldots, r$. Then $(v_1, \ldots, v_r)$ is an ordered basis of V/O. Let $[f]$ be the equivalence class of this basis: it does not depend on the chosen vectors.

Example 12.14.





$w_1 = ar_1$, $w_2 = br_1 + cr_2$.

$$T \sim \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a, c > 0$.

Definition 12.15. Given an orientation T, define

$$(\sigma\colon f) = \begin{cases} 1 & \text{if } \sigma = [f] \\ -1 & \text{otherwise.} \end{cases}$$

Then, in $E^0[\mathcal{A}]$, $\sigma = (\sigma\colon f)[f]$.

Recall that $C_{r-2}(\Pi[\mathcal{A}]) = k\Delta_{r-2}(\Pi[\mathcal{A}])$.

Definition 12.16 (Wachs, Björner)**.** Given a chamber C and an orientation σ, define the **Wachs** elements

$$W_{C,\sigma} \in C_{r-2}(\Pi[\mathcal{A}])$$

by

$$W_{C,\sigma} = \sum_{\substack{f \in \Delta_{\max}(\Sigma[\mathcal{A}]) \\ C_f = C}} (\sigma\colon f)\,\mathrm{supp}(f).$$

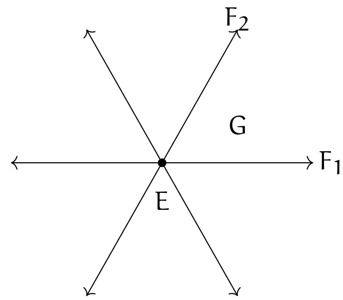
We will show that

$$W_{C,\sigma} \in H_{r-2}(\Pi[\mathcal{A}]) = \ker\left(C_{r-2}(\Pi[\mathcal{A}]) \xrightarrow{\partial_{r-2}} C_{r-1}(\Pi[\mathcal{A}]) \right).$$

Lemma 12.17. *Let E, G be faces of A such that $E \leq G$ and $\mathrm{rank}(G) = \mathrm{rank}(E) + 2$. Then there are exactly two faces F_1 and F_2 such that $E < F_i < G$ for $i = 1, 2$.*

$$\begin{tikzcd} & G & \\ F_1 \arrow[ur, dash] & & F_2 \arrow[ul, dash] \\ & E \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Proof. If $\mathrm{rank}(A) = 2$, this is clear.



In general, let $X = \mathrm{supp}(E)$, $Y = \mathrm{supp}(G)$. Then $\mathcal{A}_X^Y$ is of rank 2 and $[E, G]_{\Sigma[\mathcal{A}]}$ is in bijection with an interval from the center to a chamber of $\mathcal{A}_X^Y$. $\square$

Proposition 12.18. $W_{C,\sigma} \in H_{r-2}(\Pi[\mathcal{A}])$

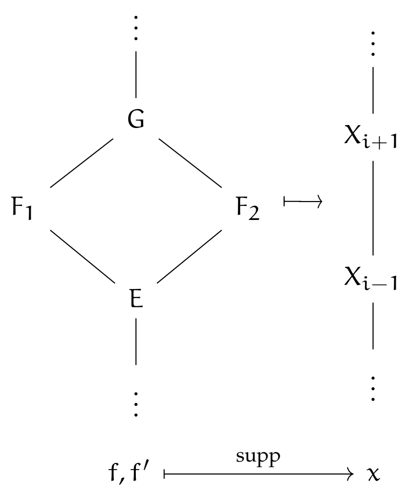
Proof.

$$\partial(W_{C,\sigma}) = \sum_{f\colon C_f = C} (\sigma\colon f)\partial(\mathrm{supp}(f)).$$

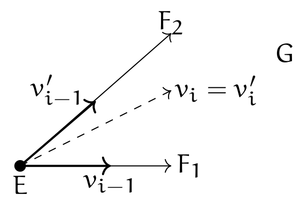
This is a linear combination of chains of flats of the form

$$x = (\bot < X_1 < \ldots < \widehat{X_i} < \ldots < X_{r-1} < \top).$$

Each chain of faces f such that $\mathrm{supp}(f) = x$, there is exactly one other chain f' such that $\mathrm{supp}(f')$ is x after removing the i-th element of the chain, and f' differs from f in exactly place i. This is by Lemma 12.17.



So it suffices to check that $(\sigma\colon f) + (\sigma\colon f') = 0$, that is, $[f] \neq [f']$.



$v'_{i-1} = a v_{i-1} + b v_i$ for $a < 0$. □

12.3 Cohomology of Posets

Definition 12.19. Now let $C^i(\Pi[\mathcal{A}]) = C_i(\Pi[\mathcal{A}])^*$ be the dual of the space of chains in P. This is called the space of **cochains**. We reverse all the arrows on the complex of spaces of chains to get a complex of cochains.

$$C_{r-2} \xrightarrow{\partial_{r-2}} C_{r-3} \xrightarrow{\partial_{r-3}} \cdots \longrightarrow C_0 \xrightarrow{\partial_0} C_{-1}$$

$$C^{r-2} \xleftarrow{\partial^{r-2}} C^{r-1} \xleftarrow{\partial^{r-1}} \cdots \longleftarrow C^0 \xleftarrow{\partial^0} C^{-1}$$

$$H_{r-2} = \ker(\partial_{r-2})$$
$$H^{r-2} = \operatorname{coker}(\partial^{r-2}) = C^{r-2} / \operatorname{im}(\partial^{r-2}).$$

Definition 12.20. Define a map

$$\begin{array}{rcl} J\colon C^{r-2}(\Pi[\mathcal{A}]) & \longrightarrow & L[\mathcal{A}] \otimes E^0[\mathcal{A}] \\ \phi & \longmapsto & \sum_{f \in \Delta_{\max}(\Sigma[\mathcal{A}])} \phi(\operatorname{supp}(f))\, C_f \otimes [f] \end{array}$$

natural in $\mathcal{A}$.

Lemma 12.21. J *factors through cohomology:*

$$\begin{tikzcd} C^{r-2} \arrow[r, "J"] \arrow[d, two heads] & L \otimes E^0 \\ H^{r-2} \arrow[ur, dashed, "J"'] & \end{tikzcd}$$

and

$$J(\phi) = \sum_{C \in L} \phi(W_{C,\sigma}) C \otimes \sigma. \tag{25}$$

Proof. Pick an orientation σ. By the definition of $(\sigma\colon f)$, we have

$$\begin{aligned} J(\phi) &= \sum_{f \in \Delta_{\max}} \phi(\operatorname{supp}(f))(\sigma\colon f) C_f \otimes \sigma \\ &= \sum_{C \in L} \left(\sum_{\substack{f \in \Delta_{\max} \\ C_f = C}} \phi(\operatorname{supp}(f))(\sigma\colon f) \right) C \otimes \sigma \\ &= \sum_{C \in L} \phi(W_{C,\sigma}) C \otimes \sigma. \end{aligned}$$

If $\phi = \partial^*(\psi)$ for some $\psi \in C^{r-3}$, then

$$\phi(W_{C,\sigma}) = \psi(\partial(W_{C,\sigma})) = \psi(0) = 0$$

for all C. So $J(\phi) = 0$. □

Theorem 12.22 (Björner-Wachs)**.** *Pick a generic hyperplane* H *for* $\mathcal{A}$*, and let* h *be one of its closed half-spaces,* $\overline{h}$ *the opposite half space. Then*

$$\{W_{C,\sigma}\colon C \subseteq \overline{h}\}$$

is a basis of $H_{r-2}(\Pi[\mathcal{A}])$*, and therefore*

$$\dim H_{r-2}(\Pi[\mathcal{A}]) = |\mu(\perp, \top)| = \#\{C\colon C \subseteq \overline{h}\}.$$

We won't prove this theorem, because it's a lot of work, but we will use it often.

Definition 12.23. Let $\{W_h^{C,\sigma}\colon C \subseteq \overline{h}\}$ be the dual basis of $H^{r-2}(\Pi[\mathcal{A}])$, that is, $W_h^{C,\sigma} = (W_{C,\sigma})^*$.

Lemma 12.24. *The vectors* $J(W_h^{C,\sigma})$ *are linearly independent.*

Proof. A computation.

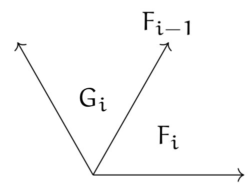
$$\begin{aligned} J(W_h^{C,\sigma}) &= \sum_{D \in L} W_h^{C,\sigma}(W_{D,\sigma}) D \otimes \sigma && \text{by Eq. (25)} \\ &= \sum_{\substack{D \in L \\ D \subseteq \overline{h}}} W_h^{C,\sigma}(W_{D,\sigma}) D \otimes \sigma + \sum_{\substack{D \in L \\ D \not\subseteq \overline{h}}} W_h^{C,\sigma}(W_{D,\sigma}) D \otimes \sigma \\ &= C \otimes \sigma + \sum_{\substack{D \in L \\ D \not\subseteq \overline{h}}} W_h^{C,\sigma}(W_{D,\sigma}) D \otimes \sigma \end{aligned}$$

Each $J(W_h^{C,\sigma})$ has a term $C \otimes \sigma$ that doesn't appear in any other $J(W_h^{C',\sigma})$ for any other chamber C'. Therefore, these vectors should be linearly independent. □

Lemma 12.25. *Let* $f = (O < F_1 < \ldots < F_r) \in \Delta_{\max}(\Sigma[\mathcal{A}])$. *For each* i, *let* G_i *be the* second *face, other than* F_i, *such that* $F_{i-1} < G_i$ *and* $\operatorname{supp}(G_i) = \operatorname{supp}(F_i)$.*Then*

$$\sum_{\substack{g \in \Delta_{\max} \\ \operatorname{supp}(g) = \operatorname{supp}(f)}} [g] C_g = (F_1 - G_1) \cdots (F_r - G_r)$$

where the product is taken in $k\Sigma[\mathcal{A}]$.



Assuming this lemma, each $F_i - G_i$ is an element of the radical of $k\Sigma = \ker(\operatorname{supp})$. Hence,

$$\prod_i (F_i - G_i) \in \operatorname{rad}(k\Sigma) \subseteq \operatorname{Lie}.$$

Theorem 12.26.

$$\begin{tikzcd} H^{r-2}(\Pi[\mathcal{A}]) \arrow[r, "J"] \arrow[dr, dashed] & L[\mathcal{A}] \otimes E^0[\mathcal{A}] \\ & \operatorname{Lie}[\mathcal{A}] \otimes E^0[\mathcal{A}] \arrow[u, hook] \end{tikzcd}$$

Proof. It suffices to show that

$$\begin{aligned} \widetilde{J} \colon H^{r-2}(\Pi[\mathcal{A}]) \otimes E^0[\mathcal{A}] &\longrightarrow L[\mathcal{A}] \\ \phi \otimes \sigma &\longmapsto \textstyle\sum_{f \in \Delta_{\max}} \phi(\operatorname{supp}(f))(\sigma \colon f) C_f \end{aligned}$$

has image in $\operatorname{Lie}[\mathcal{A}]$ (this is enough because $E^0[\mathcal{A}] \otimes E^0[\mathcal{A}] \cong k$). □

Corollary 12.27. $J \colon H^{r-2}(\Pi[\mathcal{A}]) \cong \mathrm{Lie}[\mathcal{A}] \otimes E^0[\mathcal{A}]$

13 Connections to classical algebra

Definition 13.1. Let V be a vector space. Then define the following:

(a) $\mathcal{T}(V)$ is the free associative algebra on V,

(b) $\mathcal{S}(V)$ is the free commutative algebra on V,

(c) $\mathcal{L}(V)$ is the free Lie algebra on V.

Fact 13.2.

(a) $\mathcal{T}(V) \cong \bigoplus_{n \geq 0} V^{\otimes n}$

(b) $\mathcal{S}(V)$ *is the abelianization of* $\mathcal{T}(V)$.

(c) $\mathcal{T}(V)$ *is the universal enveloping algebra of* $\mathcal{L}(V)$.

(d) $\mathcal{T}(V)$ *is a Hopf algebra.*

(e) $\mathcal{L}(V)$ *is the primitive elements of* $T(V)$.

$$\mathcal{L}(V) = \{x \in \mathcal{T}_+(V) \mid \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

Proof sketch.

(a) This object satisfies the universal property of the free associative algebra on V.

(b) Both objects satisfy the universal property of the free commutative algebra on V.

(c) Both objects satisfy the universal property of the free associative algebra on V.

(d) $\mathcal{T}(V)$ is the universal enveloping algebra of a Lie algebra, and therefore Hopf. Explicitly, the coproduct is given on a homogeneous element $v_1 v_2 \cdots v_m$ by

$$\Delta(v_1 v_2 \cdots v_m) = \sum_{S \sqcup T = [n]} v_S \otimes v_T.$$

(e) For any Lie algebra $\mathfrak{g}$, the set of the primitive elements of $U(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$ as Lie algebra. □

$\mathcal{L}(V)$ is the Lie subalgebra of $T(V)$ with the commutator bracket $[v, w] = vw - wv$. It is generated by V, and it's elements are finite brackets of elements of v.

Definition 13.3. The **Dynkin operator** is

$$\begin{array}{rcl} \Theta_V \colon \mathcal{T}(V) & \longrightarrow & \mathcal{L}(V) \\ v_1 \cdots v_n & \longmapsto & \frac{1}{n} [\cdots [[v_1, v_2], v_3], \cdots, v_n] \end{array}$$

Theorem 13.4 (Dynkin-Weber-Specht)**.** Θ_V *is an idempotent and* $\Theta_V|_{\mathcal{L}(V)} = \mathrm{id}$.

Example 13.5. When $n = 2$,

$$\Theta_V(vw) = \frac{1}{2}[v, w] = \frac{1}{2}(vw - wv)$$

$$\Theta_V^2(vw) = \frac{1}{2}\left(\frac{1}{2}(vw - wv) - \frac{1}{2}(wv - vw)\right) = \frac{1}{2}(vw - wv) = \Theta_V(vw).$$

13.1 The Schur functor of a species

Definition 13.6. Let P be a species. Its **Schur functor** $\mathcal{F}_P \colon \mathbf{Vec} \to \mathbf{Vec}$ is

$$\mathcal{F}_P(V) = \bigoplus_{n \geq 0} V^{\otimes n} \otimes_{kS_n} P[n].$$

Recall that $P[n]$ is a left S_n-module and $V^{\otimes n}$ is a right S_n-module via

$$v_1 \otimes \cdots \otimes v_n \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Example 13.7. We have that, as kS_n-modules,

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} \cong \bigoplus_{n \geq 0} V^{\otimes n} \otimes_{kS_n} kL[n],$$

since $kL[n]$ is the regular representation of S_n, hence $kS_n \cong kL[n]$. So $T = \mathcal{F}_{kL}$.

Definition 13.8. If G acts on W on the right, then the **space of coinvariants** of this action is

$$W_G = {}^{W}\big/_{k\{w - w \cdot g \mid w \in W, g \in G\}}.$$

As a kG-module, $W_G \cong W \otimes_{kG} k$, where k is the trivial G-module.

Example 13.9. As kS_n-modules,

$$S(V) = \bigoplus_{n \geq 0} (V^{\otimes n})_{S_n} \cong \bigoplus_{n \geq 0} V^{\otimes n} \otimes_{kS_n} k \cong \bigoplus_{n \geq 0} V^{\otimes n} \otimes_{kS_n} kE[n].$$

Hence, $S = \mathcal{F}_{kE}$.

Fact 13.10. *There is a species* Lie *such that* $\mathcal{L} = \mathcal{F}_{\mathrm{Lie}}$. *This means that*

$$\mathcal{L}(V) = \bigoplus_{n \geq 0} V^{\otimes n} \otimes_{kS_n} \mathrm{Lie}[n].$$

13.2 Schur-Weyl Duality

The classical Schur-Weyl Duality gives a relation between $GL(V)$-representations and S_n-representations, where $\dim V = n$.

There is an action of $GL(V)$ on V from the left, so we get an action of $GL(V)$ on $V^{\otimes n}$ diagonally;

$$A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (Av_1) \otimes (Av_2) \otimes \cdots \otimes (Av_n).$$

Definition 13.11. The **centralizer algebra** of this action is $\mathrm{End}_{GL(V)}(V^{\otimes n})$.

If $f \in \mathrm{End}_{GL(V)}(V^{\otimes n})$, then f is a linear map $f\colon V^{\otimes n} \to V^{\otimes n}$ such that

$$\begin{array}{ccc} V^{\otimes n} & \xrightarrow{f} & V^{\otimes n} \\ \Big\downarrow g & & \Big\downarrow g \\ V^{\otimes n} & \xrightarrow{f} & V^{\otimes n} \end{array}$$

for all $g \in GL(V)$. That is, f commutes with all $g \in GL(V)$.

The right-action of $\sigma \in S_n$ on $V^{\otimes n}$ by permuting the factors defines a map

$$S_n \to \mathrm{End}_{GL(V)}(V^{\otimes n}).$$

Moreover, this commutes with the action of $GL(V)$.

Theorem 13.12 (Schur-Weyl Duality). *If* $\dim V \geq n$*, then*

$$kS_n \xrightarrow{\cong} \mathrm{End}_{GL(V)}(V^{\otimes n}).$$

Theorem 13.13 (Categorical Schur-Weyl Duality). *The functor*

$$\begin{array}{rcl} \mathbf{Sp} & \longrightarrow & \mathrm{End}(\mathbf{Vect}) \\ P & \longmapsto & \mathcal{F}_P \end{array}$$

is full and faithful.

This says that given two species,

$$\mathrm{Hom}_{\mathbf{Sp}}(P, Q) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{F}_P, \mathcal{F}_Q)$$

Therefore, for any $f\colon P \to Q$ a morphism of species, we have a natural transformation $\mathcal{F}_P \to \mathcal{F}_Q$ given by

$$\mathcal{F}_P(V) \xrightarrow{\qquad\qquad} \mathcal{F}_Q(V)$$

$$V^{\otimes n} \otimes_{kS_n} P[n] \overset{\mathrm{id} \otimes_{kS_n} f}{\dashrightarrow} V^{\otimes n} \otimes_{kS_n} Q[n]$$

Why is this theorem called Categorical Schur-Weyl duality?

Example 13.14. In the case of $T = \mathcal{F}_{kL}$,

$$\mathrm{End}(T) \cong \mathrm{End}_{\mathbf{Sp}}(kL) \cong \prod_{n \geq 0} \mathrm{End}_{kS_n}(kL[n]) \cong \prod_{n \geq 0} kL[n].$$

On the right, Schur's lemma gives that $\mathrm{End}_{kS_n}(kL[n]) \cong kL[n]$, since $kL[n] \cong kS_n$.

In particular, we have by Categorical Schur-Weyl duality,

$$\mathrm{Hom}(\mathsf{T}, \mathcal{L}) \cong \mathrm{Hom}_{\mathbf{Sp}}(\Bbbk\mathrm{L}, \mathrm{Lie}) \cong \mathrm{Lie}$$

Extracting the componentwise isomorphisms for a vector space V, we see that for all n, the following holds.

$$\mathrm{Hom}(V^{\otimes n}, V^{\otimes n} \otimes_{\Bbbk S_n} \mathrm{Lie}[n]) \cong \mathrm{Hom}_{\Bbbk S_n}(\Bbbk\mathrm{L}[n], \mathrm{Lie}[n]) \cong \mathrm{Lie}[n].$$

Let's apply this to the Dynkin operator. We have

$$\Theta_V \colon \mathsf{T}(V) \to \mathcal{L}(V)$$

for each vector space V. Verifying naturality in V is the same as checking that the following diagram commutes for all $g \colon V \to W$ linear.

$$\begin{tikzcd}
\mathcal{T}(V) \arrow[r, "\Theta_V"] \arrow[d, "\mathsf{T}(g)"] & \mathcal{L}(V) \arrow[d, "\mathcal{L}(g)"] \\
\mathcal{T}(W) \arrow[r] & \mathcal{L}(W)
\end{tikzcd}$$

This is routine, and gives us a natural transformation $\Theta \in \mathrm{Hom}(\mathsf{T}, \mathcal{L})$.

This natural transformation Θ corresponds to a sequence of linear maps $(\Theta_n)_{n \geq 0}$ with $\Theta_n \in \mathrm{Lie}[n]$. We have

$$\Theta_V(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n \otimes_{S_n} \Theta_n \in V^{\otimes n} \otimes_{\Bbbk S_n} \mathrm{Lie}[n].$$

Example 13.15.

$$\begin{aligned}
\Theta_V(uvw) &= \frac{1}{3}\,[[u,v],w] \otimes_{S_3} \mathrm{id}_{S_3} \\
&= \frac{1}{3}\,(uvw - vuw - wuv + wvu) \otimes_{S_3} (1\ 2\ 3) \\
&= \frac{1}{3}\,(uvw - uvw \cdot (2\ 1\ 3) - uvw \cdot (3\ 1\ 2) + uvw \cdot (3\ 2\ 1)) \otimes_{S_n} (1\ 2\ 3) \\
&= \frac{1}{2} uvw \otimes_{S_3} ((1\ 2\ 3) - (2\ 1\ 3) - (3\ 1\ 2) + (3\ 2\ 1))
\end{aligned}$$

So in this case, we have

$$\Theta_3 = \frac{1}{3}\,((1\ 2\ 3) - (2\ 1\ 3) - (3\ 1\ 2) + (3\ 2\ 1)) \in \mathrm{Lie}[3]$$

We write this sometimes as

$$\Theta_3 = \frac{1}{3}\,[[\mathbf{1}, \mathbf{2}], \mathbf{3}]$$

where the boldface digits $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are formal symbols, *not* numbers.

Remark 13.16. Here Lie is a species, and so $\mathrm{Lie}[n]$ is an S_n-module; neither are Lie algebras. The Lie algebra in question is $\mathcal{L}(V)$. But just as $\Bbbk\mathrm{L}$ is a Hopf monoid in **Sp**, the species Lie is a **Lie monoid** in **Sp**.

13.3 Fock Functor

To define the Schur functor $\mathcal{F}_P$, we fixed P, but let V vary to get $\mathcal{F}_P\colon \mathbf{Vect} \to \mathbf{Vect}$. If we fix V but let P vary, then we get the Fock functor.

Definition 13.17. The **Fock functor** $K_V\colon \mathbf{Sp} \to \mathbf{gVec}$ from the category of species to the category of graded vector spaces is defined by

$$K_V(P) = \bigoplus_{n\geq 0} V^{\otimes n} \otimes_{kS_n} P[n].$$

Remark 13.18 (Recall)**.** **Sp** is a symmetric monoidal category under the Cauchy product, and so is the category **gVec** of graded vector spaces.

Fact 13.19. K_V *is a* ***strong*** *symmetric monoidal functor. Therefore, if* P *is a monoid in* **Sp***, then* $K_V(P)$ *is a graded algebra.*

If P *is Hopf, then* $k_V(P)$ *is a graded Hopf algebra, and if* P *is a Lie monoid, then* $K_V(P)$ *is a graded Lie algebra.*

Example 13.20. If $P = kL$, then $K_V(kL) = \mathcal{T}(V)$ is a cocommutative graded Hopf algebra.

If $P = kE$, then $K_V(kE) = \mathcal{S}(V)$ is a commutative and cocommutative graded Hopf algebra.

If $P = \mathrm{Lie}$, then $K_V(\mathrm{Lie}) = \mathcal{L}(V)$ is a graded Lie algebra.

Remark 13.21.

$$\mathcal{L}(V) = P(\mathcal{T}(V)) \iff \mathrm{Lie} = P(kL)$$

where P denotes taking the primitive elements.

$$\Delta_{S,T}\colon H[I] \to H[S] \otimes H[T]$$

$$\begin{aligned} P(H)[I] &= \bigcap_{\substack{I=S\sqcup T\\ S,T\neq\emptyset}} \ker(\Delta_{S,T}) \\ &= \bigcap_{F\in\Sigma[I]} \ker(\Delta_F) \\ &= \bigcap \ker(F\colon H[F] \to H[F]) = \bigcap \ker(\mu_F\Delta_F) \end{aligned}$$

14 Additional (Category Theory) Topics

14.1 Monoidal Functors

We'll return to the situation of Section 3 and discuss what it means for a functor to preserve the monoidal structure of a monoidal category **C**.

Let **C** and **D** be monoidal categories, and $F\colon \mathbf{C} \to \mathbf{D}$ a functor. Let M be a monoid in **C**, with $M \bullet M \xrightarrow{\mu} M$, $I \xrightarrow{\iota} M$ it's multiplication and identity maps. Is $F(M)$ a monoid object in **D**?

We can apply F to both μ and ι, but this doesn't quite get us a structure of a monoid on $F(M)$. We need, in addition, maps $F(M) \bullet F(M) \to F(M \bullet M)$ and $I \to F(I)$.

$$F(M) \bullet F(M) \dashrightarrow F(M \bullet M) \xrightarrow{F(\mu)} F(M)$$

$$IF(I) \xrightarrow{F(\iota)} F(M)$$

Definition 14.1. A **(lax) monoidal functor** $(\mathbf{C}, \bullet, I) \to (\mathbf{D}, \bullet, I)$ is a triple (F, ϕ, ϕ_0) where $F\colon \mathbf{C} \to \mathbf{D}$ is a functor,

$$\phi_{X,Y}\colon F(X) \bullet F(Y) \to F(X \bullet Y)$$

a natural transformation, and

$$\phi_0\colon I \to F(I)$$

a morphism, subject to the following diagrams commuting.

$$\begin{tikzcd}
F(X) \bullet F(Y) \bullet F(Z) \arrow[r, "\phi_{X,Y} \bullet \mathrm{id}_Z"] \arrow[d, "\mathrm{id}_{F(X)} \bullet \phi_{Y,Z}"] & F(X \bullet Y) \bullet F(Z) \arrow[d, "\phi_{X \bullet Y, Z}"] \\
F(X) \bullet F(Y \bullet Z) \arrow[r, "\phi_{X, Y \bullet Z}"] & F(X \bullet Y \bullet Z)
\end{tikzcd}$$

$$\begin{tikzcd}
F(I) \bullet F(X) \arrow[r, "\phi_{I,X}"] & F(I \bullet X) \arrow[d, equal] \\
I \bullet F(X) \arrow[u, "\iota \bullet \mathrm{id}_X"] \arrow[r, equal] & F(X)
\end{tikzcd}$$

Definition 14.2. A monoidal functor (F, ϕ, ϕ_0) is **strong** if ϕ, ϕ_0 are isomorphisms, and **strict** if ϕ, ϕ_0 are identities.

Proposition 14.3. *If* (F, ϕ, ϕ_0) *are lax monoidal, and* (M, μ, ι) *is a monoid in* **C***, then* $(FM, F(\mu) \circ \phi_{M,M}, F(\iota) \circ \phi_0)$ *is a monoid in* **D**.

Exercise 14.4. Prove the proposition above.

Definition 14.5. A **colax** monoidal functor $(\mathbf{C}, \bullet, I) \to (\mathbf{D}, \bullet I)$ is a triple (F, ψ, ψ_0) where $F\colon \mathbf{C} \to \mathbf{D}$ is a functor

$$\psi_{X,Y}\colon F(X, Y) \to F(X) \bullet F(Y)$$

a natural transformation and

$$\psi_0 \colon F(I) \to I$$

a morphism, subject to the dual axioms as in Definition 14.1.

Proposition 14.6. *Colax monoidal functors preserve comonoids.*

Remark 14.7. Strong monoidal functors are both lax and colax monoidal functors.

Definition 14.8. Let $\mathbf{C}$ and $\mathbf{D}$ be braided monoidal categories with braiding β. A **bilax monoidal functor $\mathbf{C} \to \mathbf{D}$** consists of a functor $F \colon \mathbf{C} \to \mathbf{D}$ that is both lax and colax

$$\begin{tikzcd} F(X) \bullet F(Y) \arrow[r, shift left, "\phi_{X,Y}"] & F(X \bullet Y) \arrow[l, shift left, "\psi_{X,Y}"] \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, shift left, "\phi_0"] & F(I) \arrow[l, shift left, "\psi_0"] \end{tikzcd}$$

such that the following diagrams commute for any $A, B, C, D \in \mathbf{C}$.

$$\begin{tikzcd}[column sep=small] & F(A \bullet B) \bullet F(C \bullet D) \arrow[dl, "\psi_{A,B} \bullet \psi_{B,C}"'] \arrow[dr, "\psi(A \bullet B, C \bullet D)"] & \\ F(A) \bullet F(B) \bullet F(C) \bullet F(D) \arrow[d, "\mathrm{id} \bullet \beta \bullet \mathrm{id}"] & & F(A \bullet B \bullet C \bullet D) \arrow[d, "F(\mathrm{id} \bullet \beta \bullet \mathrm{id})"] \\ F(A) \bullet F(C) \bullet F(B) \bullet F(D) \arrow[dr, "\phi_{A,C} \bullet \phi_{B,C}"'] & & F(A \bullet B \bullet C \bullet D) \arrow[dl, "\psi_{A \bullet C, B \bullet D}"] \\ & F(A \bullet C) \bullet F(B \bullet D) & \end{tikzcd}$$

(and three other axioms)

Proposition 14.9. *Bilax monoidal functors preserve bimonoids.*

Definition 14.10. A bilax monoidal functor is **bistrong** if $\phi = \psi^{-1}$.

Remark 14.11. It turns out that F is bistrong if and only if

- (F, ϕ, ϕ_0) is strong,
- (F, ψ, ψ_0) is costrong,
- The diagram below commutes.

$$\begin{tikzcd} F(A) \bullet F(B) \arrow[r, "\phi_{A,B}"] \arrow[d, "\beta"] & F(A \bullet B) \arrow[r, "\psi_{A,B}"] \arrow[d, "F(\beta)"] & F(A) \bullet F(B) \arrow[d, "\beta"] \\ F(B) \bullet F(A) \arrow[r, "\phi_{B,A}"] & F(B \bullet A) \arrow[r, "\psi_{B,A}"] & F(B) \bullet F(A) \end{tikzcd}$$

14.2 Monoidal properties of the Schur Functor

Example 14.12. Consider the Schur functor. Given a species P and vector space V,

$$\mathcal{F}_P(V) = \bigoplus_{n \geq 0} V^{\otimes n} \otimes_{S_n} P[n].$$

This gives rise to a functor

$$\begin{aligned} \mathbf{Sp} &\longrightarrow \mathrm{End}(\mathbf{Vect}) \\ P &\longmapsto \mathcal{F}_P. \end{aligned}$$

The objects of $\mathrm{End}(\mathbf{Vect})$ are the functors $\mathcal{F} \colon \mathbf{Vect} \to \mathbf{Vect}$, and the morphisms are natural transformations.

$(\mathbf{Sp}, \circ)$ is a monoidal category under **substitution**:

$$(P \circ Q)[I] = \bigoplus_{X \in \Pi[I]} P[X] \otimes \bigotimes_{B \in X} Q[B].$$

$(\mathrm{End}(\mathbf{Vect}), \circ)$ is a monoidal category under composition:

$$(F \circ G)(V) = F(G(V)).$$

Fact 14.13. $\mathcal{F} \colon (\mathbf{Sp}, \circ) \to (\mathrm{End}(\mathbf{Vect}), \circ)$ *is strong monoidal.*

Definition 14.14. An **operad** is a monoid in a monoidal category. A **monad** on $\mathbf{C}$ is a monoid in the monoidal category $(\mathrm{End}(\mathbf{C}), \circ)$.

Remark 14.15. As a consequence of Fact 14.13, if P is an operad, then $\mathcal{F}_P$ is a monad on $\mathbf{Vect}$. Moreover, $\mathcal{F}_P(V)$ is the **free algebra** on V over the operad P.

Example 14.16. kL is an operad, and therefore $\mathcal{F}_{kL}(V) = T(V)$ is the free associative algebra on V. kE is an operad, and therefore $\mathcal{F}_{kE}(V) = S(V)$ is the free commutative algebra on V. So we say that kL is the **associative operad**, and kE is the **commutative operad**.

Example 14.17. Define three bilax monoidal functors $\mathcal{K}, \mathcal{K}^\vee, \overline{\mathcal{K}} \colon \mathbf{Sp} \to \mathbf{gVec}$ by

$$\begin{aligned} \mathcal{K}(P) &= (P[n])_{n \geq 0}, \\ \mathcal{K}^\vee(P) &= (P[n])_{n \geq 0}. \end{aligned}$$

Although $\mathcal{K}$ and $\mathcal{K}^v$ee are the same on objects, they will have different bilax structures.

Given species P and Q, define ϕ, ψ on components of degree n by

$$\begin{tikzcd}
\mathcal{K}(P) \bullet \mathcal{K}(Q) \arrow[r, shift left, "\phi_{P,Q}"] \arrow[d, equal] & \mathcal{K}(P \bullet Q) \arrow[l, shift left, "\psi_{P,Q}"] \arrow[d, equal] \\
\bigoplus_{i+j=n} P[i] \otimes Q[j] & \bigoplus_{S \sqcup T = [n]} P[S] \otimes Q[T] = (P \bullet Q)[n]
\end{tikzcd}$$

$$\phi\colon P[i] \otimes Q[j] \xrightarrow{\mathrm{id} \otimes \mathrm{can}_{[i+1,n]}} P[i] \otimes Q[i+1, \dots, i+j = n]$$

$$\psi\colon P[S] \otimes Q[T] \xrightarrow{\mathrm{can}_S^{-1} \otimes \mathrm{can}_T^{-1}} P[s] \otimes Q[t]$$

Likewise, define $\phi^\vee$ and $\psi^\vee$ by

$$\phi^\vee\colon P[i] \otimes Q[j] \xrightarrow{\sum \mathrm{can}_S \otimes \mathrm{can}_T} \sum_{\substack{|S|=i \\ |T|=j}} P[S] \otimes Q[T]$$

$$\psi^\vee P[S] \otimes P[T] \xrightarrow{\mathrm{id} \otimes \mathrm{can}_T^{-1}} \begin{cases} P[s] \otimes Q[t] & \text{if } S \text{ initial} \\ 0 & \text{otherwise.} \end{cases}$$

Fact 14.18. *$(\mathcal{K}, \phi, \psi)$ is a bilax monoidal functor from* **Sp** *to* **gVec***.*

Definition 14.19 (Notation)**.** If $S \subseteq \mathbb{N}$ is a finite subset, and $s = |S|$, then there is a unique order-preserving bijection $\sigma\colon [s] \to S$. Hence we have $\sigma_*\colon P[s] \to P[S]$. Call $\phi_* = \mathrm{can}_S$.

Remark 14.20. If G is a finite group and W is a G-module, then the **norm** transformation is

$$\begin{aligned} W &\longrightarrow W \\ w &\longmapsto \sum_{g \in G} g \cdot w \end{aligned}$$

Proposition 14.21. *There exists a morphism of bilax monoidal functors $\kappa\colon \mathcal{K} \to \mathcal{K}^\vee$. This is given by*

$$\begin{aligned} \kappa_P\colon \mathcal{K}(P) &\longrightarrow \mathcal{K}^\vee(P) \\ P[n] &\longrightarrow P[n] \\ x &\longmapsto \sum_{\sigma \in S_n} \sigma \cdot x \end{aligned}$$

In characteristic zero, this gives rise to a new bilax monoidal functor $\overline{\mathcal{K}}(P)$, which is the image of κ. It is given by

$$\overline{\mathcal{K}}(P) = (P[n]_{S_n})_{n \geq 0},$$

where V_G denotes the space of coinvariants of the action of G on V.

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "\kappa"] \arrow[dr] & & \mathcal{K}^\vee \\
& \overline{\mathcal{K}} \arrow[ur, hook] &
\end{tikzcd}$$

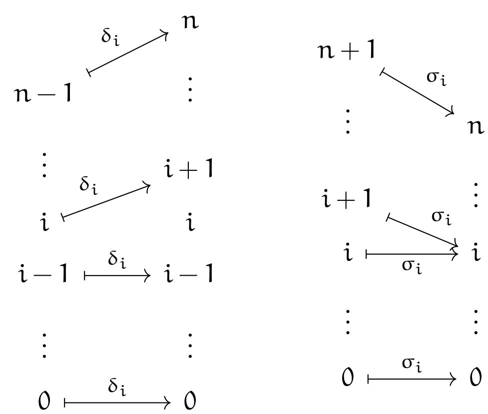
Note also that $\overline{\mathcal{K}}(P) = \mathcal{F}_P(k)$, because for any G-module W, $k \otimes_G W = W_G$.

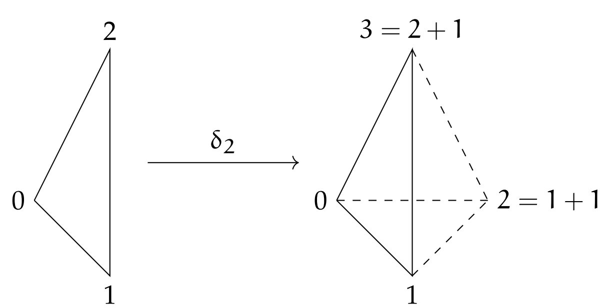
14.3 Simplicial Objects

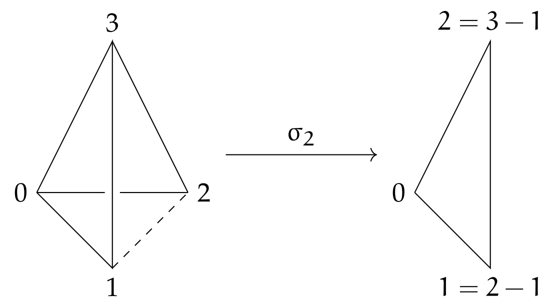
Definition 14.22. The **simplicial category** Δ has for objects the finite, nonempty, totally ordered sets, and for morphisms the order-preserving maps.

Definition 14.23. The **face map** $\delta_i \colon \{0 < 1 < \ldots < n-1\} \to \{0 < 1 < \ldots < n\}$ is the unique injective, order preserving map with image missing i.

The **degeneracy map** $\sigma_i \colon \{0 < 1 < \ldots < n+1\} \to \{0 < 1 < \ldots < n\}$ is the unique surjective, order preserving map that identifies i and $i+1$.







Definition 14.24. Let **C** be a category. A **simplicial object** in **C** is a contravariant functor $X \colon \Delta \to \mathbf{C}$. Write

$$X_n := X(\{0 < 1 < \ldots < n\})$$

for $n \geq 0$. By contravariance, we have morphisms in **C**

$$\delta_i^* \colon X_n \to X_{n-1} \text{ for } n \geq 1, 0 \leq i \leq n$$

$$\sigma_i^* \colon X_n \to X_{n+1} \text{ for } n \geq 1, 0 \leq i \leq n$$

Proposition 14.25. *Let* X *be a simplicial vector space. Define*

$$\partial_n = \sum_{i=0}^{n} (-1)^i \delta_i^* \colon X_n \to X_{n-1}$$

for all $n \geq 1$*. Then* $\delta_{n-1} \circ \delta_n = 0$ *for all* $n \geq 2$*. Thus;*

$$\cdots \xrightarrow{\partial_{n+1}} X_n \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} X_0$$

is a chain complex.

Lemma 14.26. *For* $0 \leq i < j \leq n$*,*

$$\delta_j \delta_i = \delta_i \delta_{j-1} \colon \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n+1\}$$

Proof. Both composites are injective and order-preserving, and miss i and j from the image. Hence, they are equal. $\square$

Proof of Proposition 14.25.

$$
\begin{aligned}
\partial_{n-1} \circ \partial_n &= \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} \delta_i^* \circ \delta_j^* \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} (\delta_j \circ \delta_i)^* \\
&= \sum_{i<j} (-1)^{i+j} (\delta_j \circ \delta_i)^* + \sum_{j \le i} (-1)^{i+j} (\delta_j \circ \delta_i)^* \\
&= \sum_{i<j} (-1)^{i+j} (\delta_i \circ \delta_{j-1})^* + \sum_{j \le i} (-1)^{i+j} (\delta_j \circ \delta_i)^* && \text{by Lemma 14.26} \\
&= -\sum_{h \le k} (-1)^{h+k} (\delta_h \circ \delta_k)^* + \sum_{j \le i} (-1)^{i+j} (\delta_j \circ \delta_i)^* && \text{set } h = i,\ k = j-1 \\
&= 0
\end{aligned}
$$

□

Example 14.27. Let X be a topological space. There is a simplicial set $S(X)$ such that $S_n(X) = \{f\colon \Delta^n \to X \text{ continuous}\}$, where Δ^n is the geometric n-simplex ($\dim \Delta^n = n$).

$\delta_i^*\colon S_n(X) \to S_{n-1}(X)$ is restriction to the facet of Δ^n that misses i. σ_i^* is similarly defined from σ_i. More generally, an order-preserving map $\gamma\colon \{0, 1, \dots, n\} \to \{0, 1, \dots, m\}$ gives rise to a map $\gamma\colon \Delta^n \to \Delta^m$ and then

$$
\begin{aligned}
\gamma^*\colon S_m(X) &\longmapsto S_n(X) \\
f &\longmapsto f \circ \gamma
\end{aligned}
$$

Let $C_n(X) = kS_n(X)$. Then $C(X)$ is a simplicial vector space.

By Proposition 14.25, $(C(X), \partial)$ is a chain complex, called the **singular chain complex** of X.

Example 14.28. Let G be a group. There is a simplicial set $B(G)$ such that $B_n(G) = G^n$. $\delta_i^*\colon B_n(G) \to B_{n-1}(G)$ is defined by

$$
(g_1, \dots, g_n) \xmapsto{\delta_i^*} \begin{cases} (g_2, \dots, g_n) & i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 0 < i < n, \\ (g_1, \dots, g_{n-1}) & i = n. \end{cases}
$$

$\sigma_i^*\colon B_n(G) \to B_{n+1}(G)$ is defined by

$$
(g_1, \dots, g_n) \longmapsto (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n)
$$

Let $C_n(G) = kB_n(G)$. Then $(C(G), \partial)$ is the **bar complex** of G.

Example 14.29. Let K be a simplicial complex with vertex set V (a family of subsets of V ordered under inclusion). Fix a partial order on V that induces a total order each simplex of K (for example, any total order on V will do). There is a simplicial set $S(K)$ such that

$$S_n(K) = \left\{ (v_0, \ldots, v_n) \in V^n \;\middle|\; v_0 \le v_1 \le \ldots \le v_n \text{ and } \{v_0, \ldots, v_n\} \in K \right\}$$

$$\begin{aligned} \delta_i^* \colon S_n(K) &\longrightarrow S_{n-1}(K) \\ (v_0, \ldots, v_n) &\longmapsto (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots v_n) \end{aligned}$$

$$\begin{aligned} \sigma_i^* \colon S_n(K) &\longrightarrow S_{n+1}(K) \\ (v_0, \ldots, v_n) &\longmapsto (v_0, \ldots, v_i, v_i, v_{i+1}, \ldots v_n) \end{aligned}$$

Remark 14.30. Example 14.28 can be carried out for any small category $\mathbf{C}$. The simplicial set is the **nerve** of $\mathbf{C}$.

If we think of a poset as a category, we have a commutative diagram as below.

$$\begin{tikzcd} & \text{small categories} \arrow[dr, "\text{nerve (Example 14.28)}"] & \\ \text{posets} \arrow[ur] \arrow[dr, "\text{order complex}"'] & & \text{simplicial sets} \\ & \text{ordered simplicial complex} \arrow[ur, "\text{Example 14.29}"'] & \end{tikzcd}$$

Poset homology is the chain complex given by either of the two ways around this diagram, either through nerves of small categories or through an ordered simplicial complex.

Example 14.31. Let C be any set-theoretic comonoid in the category of set species. There is a simplicial set $B(C)$ such that

$$B_n(C) = \{(I, x, S_1, \ldots, S_n) \mid I = S_1 \sqcup \ldots \sqcup S_n, x \in C[I]\}.$$

Then $\delta_i^* \colon B_n(C) \to B_{n-1}(C)$ is given by

$$(I, x, S_1, \ldots, S_n) \xmapsto{\delta_i^*} \begin{cases} (I \setminus S_1, x/_{S_1}, S_2, \ldots, S_n) & i = 0 \\ (I, x, S_1, \ldots, S_i \sqcup S_{i+1}, \ldots, S_n) & 0 < i < n \\ (I \setminus S_n, x|_{I \setminus S_n}, S_1, \ldots, S_{n-1}) & i = n. \end{cases}$$

$\sigma_i^* \colon B_n(C) \to B_{n+1}(C)$ is simpler:

$$(I, x, S_1, \ldots, S_n) \longmapsto (I, x, S_1, \ldots, S_i, \emptyset, S_{i+1}, \ldots, S_n).$$

This gives rise to a notion of homology of species.

Definition 14.32. For **C** a category, let s**C** be the category of simplicial objects in **C**.

Fact 14.33. *If* $(C, \otimes)$ *is monoidal, then* s**C** *is monoidal as well. Everything is done diagonally.*

$$\begin{tikzcd}
(X \otimes Y)_n \arrow[r, equal] \arrow[d, "\delta_i^*"] & X_n \otimes Y_n \arrow[d, "\delta_i^* \otimes \delta_i^*"] \\
(X \otimes Y)_{n-1} \arrow[r, phantom, "="] & X_{n-1} \otimes Y_{n-1}
\end{tikzcd}$$

Remark 14.34. $S\colon$ **Top** $\to$ s**Set** is strong monoidal.

Definition 14.35. Let **dgVec** be the category of chain complexes of vector spaces. The objects are pairs (V, ∂), where $V \in$ **gVec**, with $V = (V_n)_{n \geq 0}$ and also $\partial = (\partial_n \colon V_n \to V_{n-1})_{n \geq 1}$ with $\partial^2 = 0$.

dgVec is monoidal under the Cauchy product:

$$\begin{tikzcd}
(V \cdot W)_n \arrow[r, equal] \arrow[d, "\partial_n"] & \bigoplus_{i+j=n} V_i \otimes V_j \arrow[d, "\partial_n"] \\
(V \cdot W)_{n-1} \arrow[r, phantom, "\supseteq"] & V_{i-1} \otimes W_j + V_i \otimes W_{j-1}
\end{tikzcd}$$

dgVec is braided under

$$\begin{array}{ccc}
V \cdot W & \xrightarrow{\beta} & W \cdot V \\
\cup & & \cup \\
V_i \otimes W_j & \longrightarrow & W_j \otimes V_i \\
\Cup & & \Cup \\
v \otimes w & \longmapsto & (-1)^{ij} v \otimes v
\end{array}$$

Theorem 14.36. *The chain complex functor*

$$\begin{array}{ccc}
(s\mathbf{Vec}, \otimes, \beta) & \longrightarrow & (\mathbf{dgVec}, \cdot, \beta) \\
X & \longmapsto & (C(X), \delta)
\end{array}$$

is bilax monoidal, with

- $\phi\colon C(X) \cdot C(Y) \to C(X \otimes Y)$, *called the* ***Eilenberg-Zilber map*** *(degeneracies).*
- $\psi\colon C(X \otimes Y) \to C(X) \cdot C(Y)$, *called the* ***Alexander-Whitney map*** *(faces).*

14.4 The Eckmann-Hilton argument

Proposition 14.37 (Eckmann-Hilton 1960)**.** *Consider a set with two operations $+$ and $\times$, and two elements 0 and 1. Assume*

$$a + 0 = a = 0 + a$$
$$a \times 1 = a = 1 \times a$$

and moreover,

$$(a + b) \times (c + d) = (a \times c) + (b \times d).$$

Then $+$ and $\times$ coincide, $1 = 0$, and the operation is commutative.

Proof. Set $a = d = 0$, $b = c = 1 \implies 1 = 0$. Set $b = c = 0 \implies a \times d = a + d$. Finally, set $a = d = 0 \implies b \times c = c + b$. $\square$

Exercise 14.38. Deduce that the operations $+$ and $\times$ are associative.

Example 14.39. Recall that any continuous map $f\colon X \to Y$ induces a morphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ for some choice of basepoints $x_0 \in X, y_0 \in Y$. Let G be a topological group. Then in particular $\mu\colon G \times G \to G$ is continuous, so we get a morphism of groups

$$\begin{array}{rcl} \pi_1(G, e) \times \pi_1(G, e) \cong \pi_1(G \times G, (e, e)) & \xrightarrow{\mu_*} & \pi_1(G, e) \\ (a, b) & \longmapsto & a + b \end{array}$$

Let's denote the product of $\pi_1(G, e)$ by $a \cdot b$.

The constant map e is the unit for both $\cdot$ and $+$. We have that

$$(a, b) \mapsto a + b$$

$$(c, d) \mapsto c + d$$

Since μ_* is a morphism of topological groups, we get

$$(a \cdot c, b \cdot d) \mapsto (a + b) \cdot (c + d)$$

Therefore,

$$(a \cdot c) + (b \cdot d) = (a + b) \cdot (c + d).$$

Then by the Eckmann-Hilton argument, $a + b = a \cdot b$ and $\pi_1(G, e)$ is abelian.

Example 14.40. Let $I = [0, 1]$. Recall that

$$\pi_2(X, x_0) = \left\{ \text{homotopy classes of maps } f\colon I^2 \to X \text{ such that } f(\partial I^2) = x_0 \right\}$$

$\pi_2(X, x_0)$ is a group where the product of f and g is defined as the homotopy class of

$$(f \cdot g)(x,y) = \begin{cases} f(2x,y) & \text{if } 0 \le x \le \frac{1}{2} \\ g(2x-1,y) & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \qquad f \cdot g = \begin{array}{|c|c|} \hline f & g \\ \hline \end{array}$$

We may define $f + g$ by

$$(f + g)(x,y) = \begin{cases} f(x,2y-1) & \text{if } \frac{1}{2} \le y \le 1 \\ g(x,2y) & \text{if } 0 \le y \le \frac{1}{2}. \end{cases} \qquad f + g = \begin{array}{|c|} \hline f \\ \hline g \\ \hline \end{array}$$

We have that

$$(f+g)\cdot(h+k) = \begin{array}{|c|c|} \hline f & h \\ \hline g & k \\ \hline \end{array} = (f \cdot h) + (g \cdot k)$$

Here $0 = 1$ is the constant map x_0. Then by the Eckmann-Hilton argument, $\pi_2(X, x_0)$ is abelian.

14.5 2-monoidal categories

Definition 14.41. A **2-monoidal category** $(\mathbf{C}, \diamond, *, I, J)$ or a **duoidal category** consists of a category $\mathbf{C}$ with two monoidal structures: $(\mathbf{C}, \diamond, I)$ and $(\mathbf{C}, *, J)$, and a natural transformation

$$\zeta_{A,B,C,D} \colon (A * B) \diamond (C * D) \to (A \diamond C) * (B \diamond D),$$

and maps

$$\sigma \colon I \to I * I, \qquad \tau \colon J \to J \diamond J, \qquad \zeta_0 \colon I \to J.$$

Such that the following diagrams commute:

$$\begin{tikzcd}
(A * B) \diamond (C * D) \diamond (E * F) \arrow[r, "\mathrm{id} \diamond \zeta_{C,D,E,F}"] \arrow[d, "\zeta_{A,B,C,D} \diamond \mathrm{id}"] & (A * B) \diamond ((C \diamond E) * (D \diamond F)) \arrow[d, "\zeta_{A,B,C \diamond E, D \diamond F}"] \\
((A \diamond C) * (B \diamond D)) \diamond (E * F) \arrow[r, "\zeta_{A \diamond C, B \diamond D, E, F}"] & (A \diamond C \diamond E) * (B \diamond D \diamond F)
\end{tikzcd}$$

$$\begin{tikzcd}
(A * B * C) \diamond (D * E * F) \arrow[r, "\zeta_{A,B*C,D,E*F}"] \arrow[d, "\zeta_{A*B,C,D*E,F}"] & (A \diamond D) * ((B * C) \diamond (E * F)) \arrow[d, "\mathrm{id} * \zeta_{B,C,E,F}"] \\
((A * B) \diamond (D * E)) * (C \diamond F) \arrow[r, "\zeta_{A,B,D,E} * \mathrm{id}"] & (A \diamond D) * (B \diamond E) * (C \diamond F)
\end{tikzcd}$$

(some unit axioms)

(Note: we do not require any of these maps to be invertible!)

Definition 14.42. If $\zeta, \zeta_0, \sigma, \tau$ are isomorphisms, then the 2-monoidal category is called **strong**.

Example 14.43. Let G be a monoid and **C** be the category of G-graded vector spaces. Then an object looks like

$$V = (V_g)_{g \in G},$$

and there are both the **Cauchy product**

$$(V \diamond W)_g = \bigoplus_{g = xy} V_x \otimes W_y$$

and the **Hadamard product**

$$(V * W)_g = V_g \otimes W_g.$$

The map ζ is given component-wise by the map across the bottom of the following square.

$$\begin{tikzcd}
(A * B) \diamond (C * D) \arrow[rr, "\zeta"] & & (A \diamond C) * (B \diamond D) \\
\bigoplus_{g=xy} A_x \otimes B_x \otimes C_y \otimes D_y \arrow[u, hook] \arrow[r, hook] & \left(\bigoplus_{xy=g} A_x \otimes C_y \right) \otimes \left(\bigoplus_{x'y'=g} B'_x \otimes D'_y \right) \arrow[ur, hook] &
\end{tikzcd}$$

Notice that ζ is just an inclusion; there's no chance of it being invertible here.

Exercise 14.44. Adapt this to the category of species.

Example 14.45. Let P be a lattice. Let **C** be the poset associated to the poset P, with a unique map $x \to y$ when $x \le y$ in P.

Then define $x \diamond y = x \vee y$ and $x * y = x \wedge y$. Then

$$\zeta \colon (a \wedge b) \vee (c \wedge d) \to (a \vee c) \wedge (b \vee d)$$

exists if and only if $(a \wedge b) \vee (c \wedge d) \le (a \vee c) \wedge (b \vee d)$ in P. It's enough to check that $a \wedge b \le (a \vee c) \wedge (b \vee d)$, and $c \wedge d \le (a \vee c) \wedge (b \vee d)$.

For the first, it's enough to know that $a \wedge b \leq a \vee c$ and $a \wedge b \leq b \vee c$, and likewise for the second, it's enough to know that $c \wedge d \leq a \vee c$ and $c \wedge d \leq b \vee d$. All of these hold by general properties of meets and joins.

Therefore, there is such a map ζ. But again, it may not be an isomorphism!

Exercise 14.46. Generalize to any category $\mathbf{C}$ with finite products and coproducts.

Exercise 14.47. Fix a set X. A **digraph** on X is a triple (A, s, t) where A is a set and $s, t \colon A \to X$ are two maps. Given $a \in A$, we write $a \colon s(a) \to t(a)$. Let $\mathbf{C}$ be the category of digraphs on X, where morphisms $(A, s, t) \to (B, s, t)$ preserve source and target.

Define

$$A \diamond B = \{(a, b) \in A \times B \mid s(a) = t(b)\} = \left\{ * \xrightarrow{b} * \xrightarrow{a} * \right\}$$

$$A * B = \{(a, b) \in A \times B \mid s(a) = s(b), t(a) = t(b)\} = \left\{ \begin{tikzcd} * \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & * \end{tikzcd} \right\}$$

The exercise is to define ζ.

Proposition 14.48. *Let $(\mathbf{C}, \bullet, \beta)$ be a braided monoidal category. Then $\mathbf{C}$ is strong 2-monoidal with $\diamond = * = \bullet$, and*

$$\zeta = (A \bullet B) \bullet (C \bullet D) \xrightarrow{\mathrm{id} \bullet \beta \bullet \mathrm{id}} (A \bullet C) \bullet (B \bullet D)$$

Proof Sketch. The axioms for ζ follow from the axioms for β. $\square$

This is a kind of converse to an Eckmann-Hilton argument. A version of the Eckmann-Hilton argument for categories is that the converse holds.

Theorem 14.49 (Eckmann-Hilton for **Cat**, Joyal Street)**.** *Let $(\mathbf{C}, \diamond, *, I, J)$ be a strong 2-monoidal category. Then $\mathbf{C}$ satisfies $\diamond = *$, $I = J$, and the monoidal structure $(\mathbf{C}, \diamond, I) = (\mathbf{C}, *, J)$ on $\mathbf{C}$ is braided.*

14.6 Double monoids

Definition 14.50. Let $(\mathbf{C}, \diamond, *, I, J)$ be a 2-monoidal category. A **double monoid** $(A, \mu_1, \mu_2, \iota_1, \iota_2)$ consists of an object A together with two monoid structures $(A, \mu_1 \colon A \diamond A \to A, \iota_1 \colon I \to A)$ and $(A, \mu_2 \colon A * A \to A, \iota_2 \colon J \to A)$, such that the following commute.

$$\begin{tikzcd} (A * A) \diamond (A * A) \arrow[rr, "\zeta_{A,A,A,A}"] \arrow[d, "\mu_2 \diamond \mu_2"] & & (A \diamond A) * (A \diamond A) \arrow[d, "\mu_1 * \mu_1"] \\ A \diamond A \arrow[r, "\mu_1"] & A & A * A \arrow[l, "\mu_2"'] \end{tikzcd}$$

(some unit axioms)

Similarly, we may define **double comonoids**.

Definition 14.51. A **double bimonoid** consists of an object B that is both a monoid $(B, \mu\colon B \diamond B \to B, \iota\colon I \to B)$ with respect to δ, and also a comonoid $(B, \delta\colon B \to B * B, \varepsilon\colon B \to J)$ with respect to $*$, such that the following commute.

$$
\begin{tikzcd}
(B * B) \diamond (B * B) \arrow[rr, "\zeta_{B,B,B,B}"] & & (B \diamond B) * (B \diamond B) \arrow[d, "\mu * \mu"] \\
B \diamond B \arrow[u, "\Delta \diamond \Delta"] \arrow[r, "\mu"] & B \arrow[r, "\Delta"] & B * B
\end{tikzcd}
$$

(some unit axioms)

Theorem 14.52 (Categorical Eckmann-Hilton)**.** *Let $(\mathbf{C}, \bullet, \beta)$ be a braided monoidal category. View it as a strong 2-monoidal category. Let (A, μ_1, μ_2) be a double monoid. Then $\mu_1 = \mu_2$, and the monoid structure on A is commutative.*

Proof sketch. Consider

$$
\begin{tikzcd}
 & A \bullet I \bullet I \bullet A \arrow[r, "\mathrm{id}\bullet\iota_1\bullet\iota_1\bullet\mathrm{id}"] \arrow[d, "\mathrm{id}\bullet\beta\bullet\mathrm{id}=\zeta"] & A \bullet A \bullet A \bullet A \arrow[r, "\mu_1\bullet\mu_1"] \arrow[d, "\mathrm{id}\bullet\beta\bullet\mathrm{id}=\zeta"] & A \bullet A \arrow[dr, "\mu_2"] & \\
A \bullet A \arrow[ur, equal] \arrow[dr, equal] & & & & A \\
 & A \bullet I \bullet I \bullet A \arrow[r, "\mathrm{id}\bullet\iota_2\bullet\iota_2\bullet\mathrm{id}"] & A \bullet A \bullet A \bullet A \arrow[r, "\mu_2\bullet\mu_2"] & A \bullet A \arrow[ur, "\mu_1"] &
\end{tikzcd}
$$

The top is μ_1, and the bottom is μ_1. Each smaller diagram commutes by some axioms of either the monoidal structure or naturality. Therefore, $\mu_1 = \mu_2$. $\square$

Remark 14.53. The classical Eckmann-Hilton argument is the case of the above when $\mathbf{C} = \mathbf{Set}$.

Remark 14.54. To summarize, categorical Eckmann-Hilton says that double monoids in $\mathbf{C}$ are the same as commutative monoids in $\mathbf{C}$. Here are some settings where we know versions of the Eckmann-Hilton argument.

$\mathbf{C} = \mathbf{Set}$	$\mathbf{C} =$ braided monoidal category	$\mathbf{C} =$ 2-monoidal category
classical	True	False
$\mathbf{C} = \mathbf{Cat}$	$\mathbf{C} =$ braided monoidal 2-category	2-monoidal 2-category
Theorem 14.49	To be done! (By you?)	False