Triangulations Of Point Sets

Applications, Structures, Algorithms

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Motivation: Triangulations in Mathematics

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When solving a difficult problem it is a natural idea to decompose complicated objects into smaller, easy-to-handle pieces. In this book we study such decompositions: *triangulations of point sets.* We will look at triangulations from many different points of view. We explore their combinatorial and geometric properties as well as some algorithmic issues arising along the way.

This first chapter is designed to informally introduce the fundamental notions to come in later chapters. We provide motivating examples to convince the reader that triangulations are rather useful and that they appear in many areas of mathematics. The reader can skip most of this chapter safely: essentially only the first two pages of the chapter are needed later on. The sections in this chapter present examples which are not meant to be read in any particular order. The examples also provide non-discrete-geometers (e.g., algebraic geometers, computer scientists, linear programming enthusiasts, etc.) that wish to learn about triangulations for their research a door connecting our book to their topic.

Without more delay we begin. A *point configuration*[1] is a finite collection of points $A = \{a_1, \dots, a_n\}$ in Euclidean space $\mathbb{R}^d$.

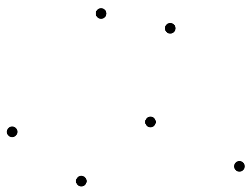


Figure 1.1: A point configuration

The *convex hull* of A is by definition the intersection of all convex sets containing the points in A. We denote it by $\operatorname{conv}(A)$.

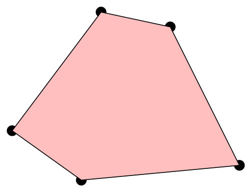


Figure 1.2: Its convex hull

A k-*simplex* is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$ (clearly $d \geq k$). Simplices are the simplest of polyhedra: points, segments, triangles, tetrahe-

[1]The word *configuration* is used to distinguish this from a *set* of points: in a subset of $\mathbb{R}^d$ there can be no multiple points, whereas in a configuration we in principle allow more than one point with the same coordinate.

dra, etc. A j-face of a k-simplex is the convex hull of $j+1$ of its vertices and thus in particular a j-simplex itself. We say that the empty set is a (-1)-dimensional face common to all simplices, so that every k-simplex has exactly $\binom{k+1}{j+1}$ j-faces for $j = -1, 0, 1, \ldots, k$. A simplex of A is a simplex whose vertices are taken from A.

Here is the main actor of this play:

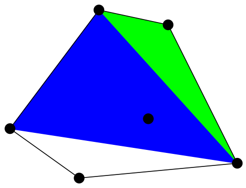


Figure 1.3: The union is not the whole convex hull

Definition 1.0.1. Given a point configuration A in $\mathbb{R}^d$, we say that a *triangulation* of A is a finite collection T of d-simplices of A that satisfies two properties:

(UP) The union of all these simplices equals $\mathrm{conv}(A)$. (*Union Property*.)

(IP) Any pair of these simplices intersects in a common face (possibly empty). (*Intersection Property*.)

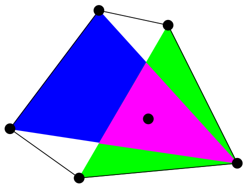


Figure 1.4: The intersection is not okay

As a particular case, by a triangulation of a convex polytope P we mean a triangulation of the point configuration given by the vertices of P.

In Figure 1.6 we show some examples of possible pathologies that may occur and prevent a triangulation.

Let us emphasize two features that distinguish our definition of other definitions of the word triangulation that people sometimes use:

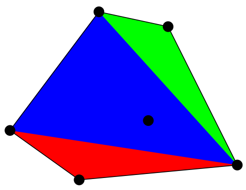


Figure 1.5: Union and intersections okay: a triangulation

1. With very few exceptions, we will fix in advance the set of vertices that can be used, and it is a finite set. In particular, the number of triangulations of a point configuration or a polytope is always finite. This does not happen in classic geometry or in some applications, where one is free to decide which points to use as vertices.

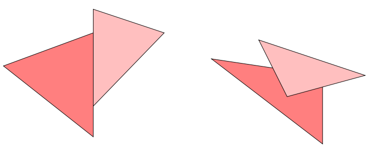


Figure 1.6: Simplices not intersecting in a common face (possibly empty) are forbidden in a triangulation.

2. We do not insist that all the vertices of A need to be used as vertices in a triangulation. For example, if our point set consists of points in $\mathbb{R}$, then there is one triangulation with only one simplex (the whole segment $\mathrm{conv}(A)$) and two vertices (the two convex hull extremes of the line segment $\mathrm{conv}(A)$), regardless of how many points we may have in A. This differs from the standard use of triangulations in

Computational Geometry, where one usually wants all the points to be used.

The first of these two features gives our setting a strong combinatorial flavor. Actually, to describe a particular triangulation we will normally number the points of A from 1 to n and give the list of vertex sets of the d-simplices in the triangulation. For example, a pentagon has five triangulations which we would normally write as:

$$\{\,\{1,2,3\},\{1,3,4\},\{1,4,5\}\,\}, \qquad \{\,\{1,2,5\},\{2,4,5\},\{2,3,5\}\,\},$$

$$\{\,\{1,2,3\},\{1,3,5\},\{3,4,5\}\,\}, \qquad \{\,\{1,4,5\},\{1,2,4\},\{2,3,4\}\,\},$$

$$\text{and} \qquad \{\,\{1,2,5\},\{2,3,5\},\{3,4,5\}\,\}.$$

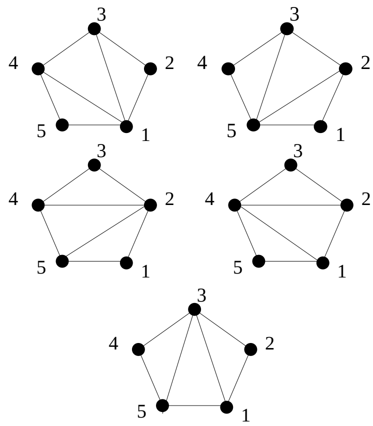


Figure 1.7: The five triangulations of a pentagon

We will even abbreviate $\{1,2,3\}$ as 123 and so on, whenever this creates no confusion.

Why should anyone care about studying triangulations of point sets? It is our intention to illustrate, with some examples, how several of the fundamental defining properties of triangulations draw themselves into topics that, at first sight, seemed far apart from the geometry of triangulations.

1.1 Combinatorics and triangulations

It is well-known that polyhedra can be quite useful when dealing with combinatorial problems. In this section we show two examples of combinatorial identities that have interpretations in terms of triangulations. Let us start with what is possibly the simplest example of the structures studied in this book: the set of triangulations of a convex polygon. Let P be a convex polygon with n vertices, numbered from 1 to n in counterclockwise order. The first observation is that the number of triangulations does not depend on the coordinates of the vertices. Indeed, a triangulation will be given by any $n-3$ diagonals not crossing one another, and two diagonals cross if and only if they involve four vertices in an alternating way. That is to say, if $1 \le i < j < k < l \le n$, then the only two diagonals involving these four points and crossing each other are ik and jl.

In particular, the number of triangulations of a convex n-gon is a certain number depending only on n and that we will denote t_n. The first instances are easy to compute: $t_3 = 1$, $t_4 = 2$, $t_5 = 5$ (see Figure 1.7) and $t_6 = 14$ (see Figure 1.16).

Proposition 1.1.1. *Setting* $t_2 = 1$ *by convention, the sequence of numbers* $t_2, t_3, t_4, \dots$ *satisfies the following recurrence relation:*

$$t_n = t_2 t_{n-1} + t_3 t_{n-2} + \cdots + t_{n-1} t_2.$$

Proof. In every triangulation of P the edge $\{1, n\}$ is a side of exactly one of the triangles, say $\{1, k, n\}$. The total number of triangulations, then, is the sum of the triangulations using the triangle $\{1, k, n\}$ for k ranging from 2 to $n-1$.

For a fixed k, since the complement of the triangle $\{1, k, n\}$ are two polygons S_1 and S_2 with k and $n-k-1$ vertices, respectively. The polygons S_1 and S_2 can be triangulated separately. Hence, the number of triangulations using that triangle equals $t_k t_{n-k-1}$. (Of course, we admit S_1 or S_2 being a single edge, or a "2-gon", if $k = 2$ or $k = n-1$. We take $t_2 = 1$ because this makes $t_2 t_{n-1}$ equal to t_{n-1}).

The recurrence formula is obtained adding this for $k = 2, \dots, n-1$. □

The recurrence formula in the statement allows to easily compute some more terms in the sequence, for example $t_7 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$ and $t_7 = 1 \cdot 42 + 1 \cdot 14 + 2 \cdot 5 + 5 \cdot 2 + 14 \cdot 1 + 42 \cdot 1 = 132$. A closed formula for t_i can be deduced from the recurrence with the method of *generating functions* (see Exercise 1.5). But here we use a more direct approach to find it.

Theorem 1.1.2. *The number* t_n *of triangulations of a convex* n*-gon equals*

$$\frac{1}{n-1}\binom{2n-4}{n-2},$$

Proof. As before, we assume the vertices of the n-gon labeled from 1 to n in clockwise order. Denote by $\mathcal{T}_n$ the set of all triangulations of an n-gon, and by t_n their number. We are going to set up a simple surjective map f from $\mathcal{T}_{n+1}$ onto $\mathcal{T}_n$. A triangulation in $\mathcal{T}_{n+1}$ is mapped to the triangulation in $\mathcal{T}_n$ obtained by contracting the boundary edge $\{1, n+1\}$ (see Figure 3.1).

Our crucial observation is that the number of triangulations in $\mathcal{T}_{n+1}$ mapped to a certain triangulation T in $\mathcal{T}_n$ equals the number of edges incident to vertex 1 in T (the "degree" of vertex 1 in T). This is true because to "reverse" the map f we must choose one edge incident to 1 and "double it" to become a triangle incident to edge $\{1, n+1\}$. (For example, in Figure 1.9 one has to "double" edge $\{1,4\}$.) This implies that

$$t_{n+1} = \sum_{T \in \mathcal{T}_n} \deg_1(T).$$

By cyclic symmetry of the n-gon, this same formula must hold for any other vertex of it. Hence:

$$n t_{n+1} = \sum_{i=1}^{n} \sum_{T \in \mathcal{T}_n} \deg_i(T) = \sum_{T \in \mathcal{T}_n} \sum_{i=1}^{n} \deg_i(T).$$

But it turns out that the sum $\sum_{i=1}^{n} \deg_i(T)$ is independent of T: it equals twice the number of edges of T, that is, $2(2n-3)$. Hence:

$$t_{n+1} = \frac{2(2n-3)}{n} t_n, \quad \text{or} \quad t_n = \frac{2(2n-5)}{n-1} t_{n-1},.$$

From this we conclude that:

$$t_n = \frac{2^{n-2}(2n-5)(2n-7)\cdots 3\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2} =$$

$$\frac{(2n-4)!}{(n-1)!(n-2)!} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

□

The sequence of numbers we have just found is known as the *Catalan numbers* (see Definition 1.1.4 below), and is one of the most important "non-trivial" number sequences in combinatorics, perhaps comparable to the well-known Fibonacci sequence. We remark that asymptotically these numbers asymptotically grow (up to a

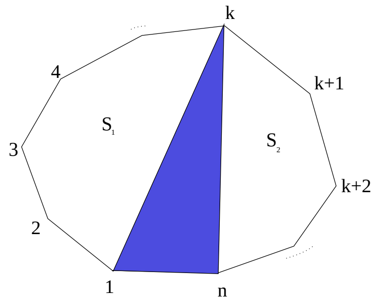


Figure 1.8: Setting up another recursion for $R(n)$.

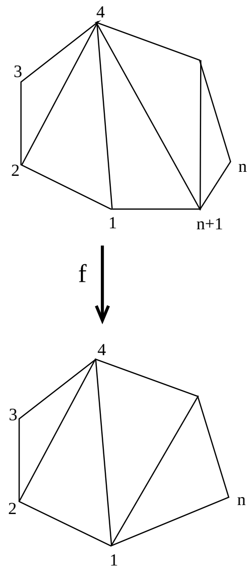


Figure 1.9: The contracting map.

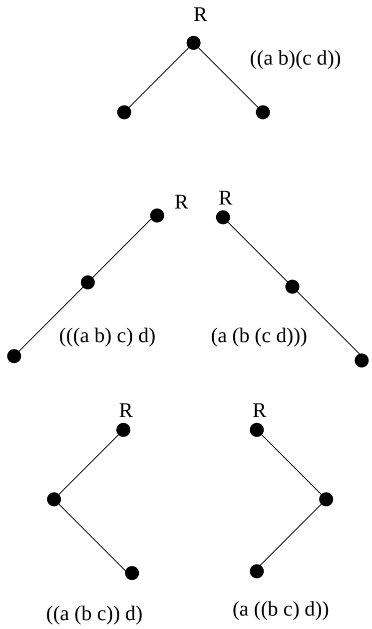


Figure 1.10: The five binary trees in 3 nodes, with their associated parenthesizations.

constant factor) like $4^n n^{-3/2}$. This may be seen using Stirling's approximation of the factorial. As an example of the ubiquity of Catalan numbers, the following statement lists other four combinatorial structures whose cardinality is given by the Catalan sequence. Exercise 6.19 in [83, p. 219] contains 61 additional such examples:

Theorem 1.1.3. *There are as many triangulations of a convex polygon with $n+2$ vertices as:*

(i) Binary trees with n nodes (and hence $n-1$ edges).

(ii) Parenthesizations of the product of $n+1$ factors. That is to say, ways of placing $n-2$ pairs of parentheses in order to perform the product .

(iii) Sequences of length $2n$ consisting of n plus signs and n minus signs, with the property that in every initial segment of the sequence there are at least as many pluses as minuses.

(iv) Monotone paths in the integer grid, going from $(0,0)$ to (n,n) by steps of length 1 in the positive directions of the axes, and not going above the diagonal.

One interesting feature of the equivalence to sign sequences is that it immediately shows that the number of triangulations of an n-gon is bounded above by 2^{2n-4}. Of course, that is also clear from Theorem 1.1.2, but its proof needed some work. Moreover, the equivalence explicitly tells us how to write a given triangulation as a binary number of length $2n-4$.

Before proving the theorem let us define binary trees, which the reader may not be familiar with. A *tree* is a connected simple graph with no cycles [13]. We are interested in rooted trees, i.e., trees with a special distinguished node, called the *root*. In rooted trees, one can direct the edges naturally along the unique paths from each node to the root node. This establishes a hierarchy among the nodes: node v_1 becomes a *child of node* v_2 if they are adjacent and the edge joining them is directed from v_1 to v_2 (v_2 is the *parent of* v_1). Rooted trees are normally drawn with the root on top and with parents above their children.

A *binary tree* is a rooted tree in which each edge is marked as a *right* or *left* edge of its parent and each vertex has at most one right child and at most one left child (in particular, each vertex has either $0, 1$ or 2 children). In Figure 1.10 we show the five different binary trees on three nodes. As it is customary, a left child is drawn toward the left-down direction and a right child is drawn toward the right-down direction. Binary trees are very useful combinatorial structures due to applications in data structures and design of algorithms (see for example [54]).

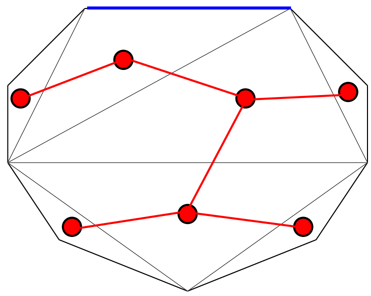
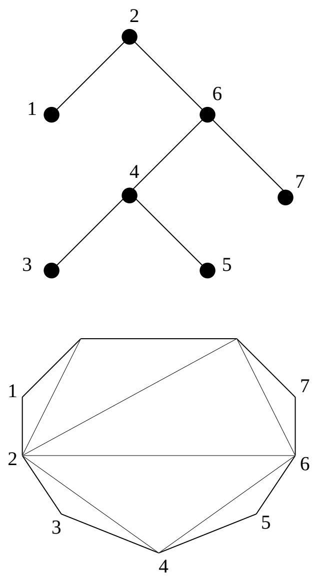


Figure 1.11: A binary tree dual of a triangulation.

Proof of Theorem 1.1.3.

1. From triangulations to binary trees: Let us see how to build up a binary tree from a given triangulation of the $n+2$-gon. As usual, we assume vertices of the polygon labeled from 1 to $n+2$. We call the edge $\{1, n+2\}$ the *reference edge* of the polygon. We draw a node of the binary tree inside each of the n triangles of the triangulation, and join nodes of adjacent triangles by an edge. We declare the root of the tree to be the node of the unique triangle that contains the reference edge. The three sides of each triangle become clearly identified as a "parent edge" (the one towards the root), a "right edge" (the next one in the clockwise direction) and a "left edge" (the third one). In particular, every node in the tree has a parent (unless it is the root node) and its children are labeled as right or left depending on whether the corresponding edge in the triangle is the right or the left one. See Figure 1.11.

To show that this construction is indeed a bijection it suffices to show that it can be reversed: starting with a binary tree, draw a triangle for the root and call its edges "parent", "right" and "left" appropriately. Then glue triangles to its right and left edge for the right and left children of the root, if they exist. Recursively continue with grand children and all the other descendants (great-grand-children, etc) and, after you have finished, number the vertices of the $n+2$-gon starting and ending with the end-points of the parent edge of the root



$\begin{array}{cccccccccccccc} + & - & + & + & - & + & + & - & - & + & + & - & - & - \\ 1 & 1 & 2 & 3 & 3 & 4 & 5 & 5 & 4 & 6 & 7 & 7 & 6 & 2 \end{array}$

Figure 1.12: A binary tree with its symmetric order traversal, its associated triangulation, and its associated sequence of signs.

triangle.

There is actually a nice correspondence between the n nodes in the binary tree and the n vertices of the $n+2$-gon out of the reference edge, exhibited in Figure 1.12. The following way in which the nodes of the tree are numbered is called *symmetric order traversal*: Starting with the root node, if a node has at least one child, then process recursively its left subtree first, then traverse the node itself, then process recursively its right subtree, and finally return to its parent. We give numbers 1 to n to the n nodes as we visit them during the traversal. If a node has no children then simply visit it by assigning to it the next available number from 1 to n, and return to its parent. We will use the symmetric order traversal a bit later.

2. From binary trees to parenthesizations: The bijection between binary trees with n nodes and parenthesizations of products of $n+1$ factors was displayed in Figure 1.10: we place a pair of parentheses for each node of the binary tree, starting with the root parentheses which enclose the whole product and inserting inner parentheses for children with the following rule: if the right/left child of a given parent node has k descendants the corresponding parentheses will enclose the $k+1$ leftmost/rightmost factors within the ones enclosed by the parent parentheses. Alternatively, one can start drawing parentheses for the leafs of the tree (leaving placeholders for the two variables they contain, which we cannot still identify) and add greater and greater parentheses for their parents, inserting right or left factors (placeholders) depending on the type of edge leading to the parent. We leave it to the reader to convince him or herself that this is indeed a bijection.

3. From binary trees to sign sequences: Clearly, there is going to be a plus and a minus sign corresponding to each node in the tree, and the plus sign will appear before the minus to guarantee that every initial segment has at least as many pluses as minuses. The way to construct the sequence is: Go along the tree in the symmetric order traversal presented above. When visiting a node, first process its left subtree, then place the plus sign for this node, then visit the right subtree, then place the minus sign. Figure 1.12 shows an example where, to make things clear, each plus or minus is labeled by its corresponding vertex of the tree.

In the exercises you will see how to construct the sequences of signs directly from the triangulation.

4. From sign sequences to monotone paths: Figure 1.13 shows the monotone paths under consideration and, at the same time, their bijection to sign sequences. Essentially, plus signs correspond to steps to the right and minus signs to steps upwards. The condition that the monotone paths do not cross the diagonal is exactly equivalent to saying that every initial segment has at least as many plus signs as minus signs.

□

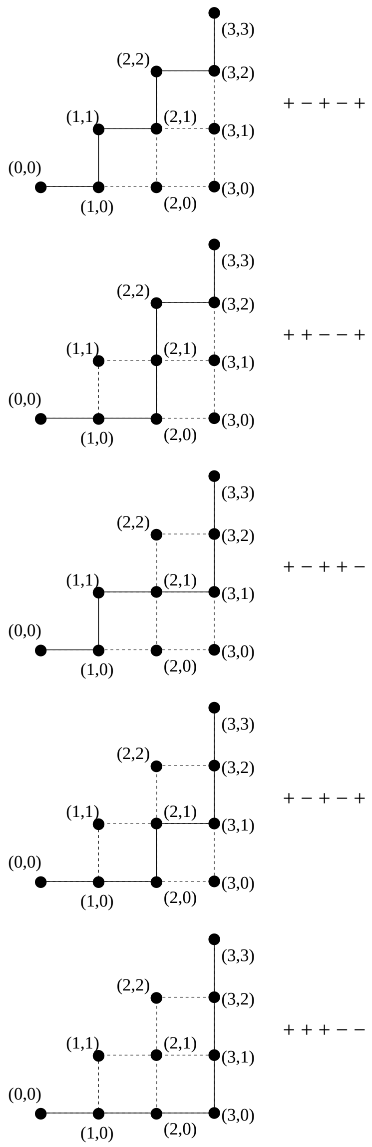


Figure 1.13: The five monotone paths/sign sequences, of length six.

Definition 1.1.4. The n-th Catalan number, where $n = 0, 1, 2, \ldots$ is the number C_n defined by the following recurrence formula:

$$C_0 = 0, \qquad C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}, \quad \forall k > 0. \tag{1.1}$$

Equivalently, it is the number of triangulations of the convex $n+2$-gon, which equals

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.2}$$

Theorem 1.1.3 can be read as saying that the five combinatorial structures described there are just different formulations of one and the same structure, that we can call the “Catalan structure”. Having the different formulations, besides its mathematical appeal, has practical sequences: properties which are obvious in one formulation may be invisible in others, and the many appearances of the structure provide additional insight and more tools to attack it.

As an example, our proof of Theorem 1.1.2 is heavily based on the cyclic symmetry of the convex n-gon, while none of the other four structures of Theorem 1.1.3 have a cyclic symmetry at all.

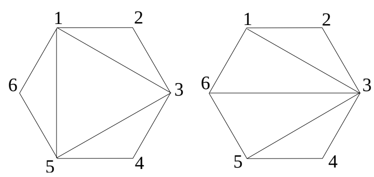


Figure 1.14: A diagonal flip in the quadrilateral 1356.

Once we know the cardinality of the set T_n of triangulations of the convex n-gon, let us see that it is "more than a set". That is, that there are natural relations between triangulations. Every internal edge in a triangulation is a diagonal of a convex quadrilateral formed by two adjacent triangles. One can change this diagonal to the opposite one and get a triangulation which is as similar as possible to the initial one. This operation is called a "diagonal flip". Figure 1.14 shows a flip between two triangulations of a hexagon.

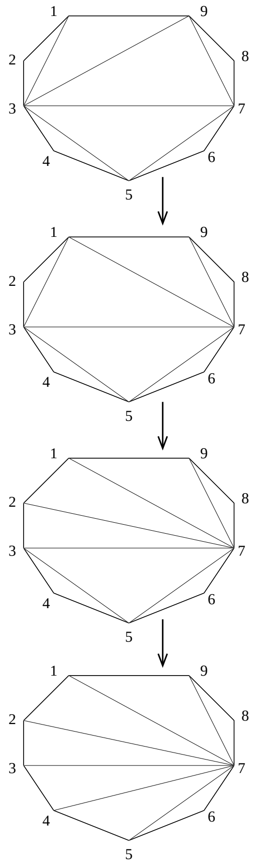


Figure 1.15: Flipping towards the 7th standard triangulation.

We can thus consider the set of triangulations of the n-gon as the vertices of a graph, whose edges are diagonal flips. This graph is called the flip-graph of triangulations of the n-gon. Some straightforward properties of the graph are:

1. It is regular of degree $n-3$ (that is to say, every triangulation has exactly $n-3$ flips). This is so because there is one flip associated to each internal diagonal.

2. It is connected. To prove this, let us pick any particular vertex, say the i-th one, and consider the unique triangulation in which all the triangles are incident to i. We call this the i-th *standard triangulation* of the n-gon. An example is in the right part of Figure 1.14. In any triangulation other than this one there is always at least one flip which increases the degree of vertex i: just flip the diagonal jk for any triangle ijk with j and k not consecutive vertices of the n-gon. This shows that every triangulation can be transformed into the standard one by a sequence of at most $n-3$ flips.

Other not-so-easy properties of the graph are that it is Hamiltonian [66] and that it is the graph of a convex and simple polytope of dimension $n-3$ called the *associahedron* [57] (see also [101, Chapter 0]). The associahedron is a particular case of the *secondary polytope* or "polytope of triangulations and flips", which exists for any finite point set in any dimension. Flips and secondary polytopes are introduced in chapters 2 and 4, and are among the central topics in this book. See Figure 1.16 for a picture of the graph of flips on triangulations of a hexagon (well, we have forgotten to draw one edge. Can you find it?). You should try to verify in the picture all the properties of the graph mentioned so far.

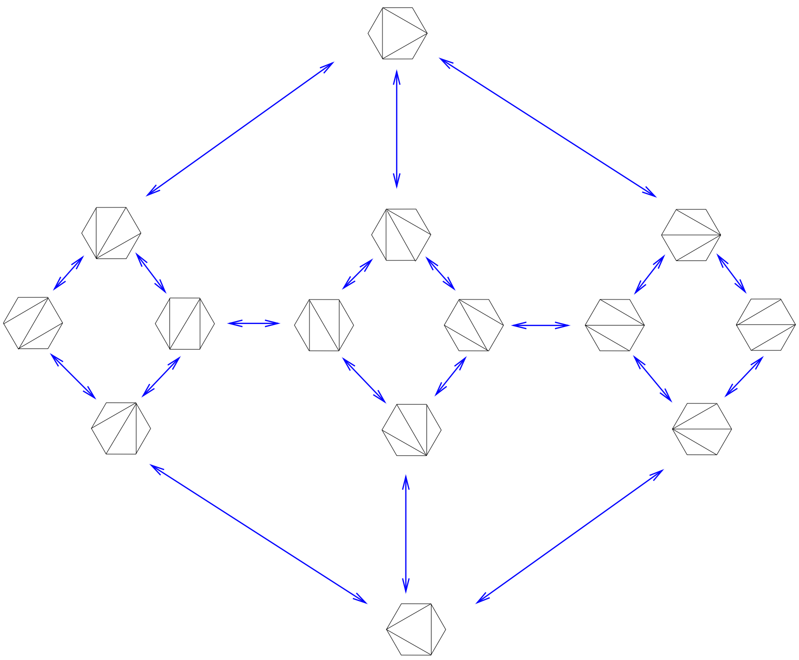


Figure 1.16: The Graph of flips for a hexagon, with one edge missing.

You may be wondering whether the graph of flips is meaningful in the other formulations of the Catalan structure that we have mentioned. The answer is yes and no. For example, the monotone path formulation possesses its own natural notion of flip, (move the path along a single square of the grid) but these flips are certainly not equivalent to the flips in triangulations: In Figure 1.13 you can see paths with one, two, and three flips.

In the context of binary trees, however, diagonal flips can be described easily: They arise as the so-called "rotation of an edge". If an edge connects a node X to its right child Y, let P, A, B and C denote the parent subtree of X, left subtree of X, left subtree of Y and right subtree of Y, respectively. The rotation changes this to the binary tree in which X is a left child of Y and P, A,

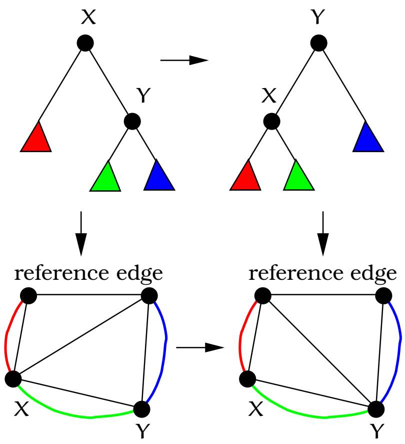


Figure 1.17: The diagonal flip corresponding to a rotation

B and C are, respectively, the parent subtree of Y, left subtree of X, right subtree of X and right subtree of Y. A rotation and its correspondence to a flip in triangulations is depicted in Figure 1.17. In the context of parenthesizations a flip is given by a single application of the associative law $x(yz) \mapsto (xy)z$.

But how many flips does it take to move from one triangulation to another? Remember that the distance between two nodes in a connected graph is the minimal number of edges needed to go from one node to the other, and that the diameter of the graph is the maximum distance between nodes. It is interesting to say something about the diameter of the graph of flips:

Proposition 1.1.5. *Let D_n be the diameter of the graph of flips between triangulations of the convex n-gon. Then:*

(i) $D_n \leq 2n - 10 - 12/n$ *for every n (in particular, it is bounded by $2n - 10$ for every $n \geq 12$).*

(ii) $D_n + 1 \leq D_{n+1} \leq D_n + 3$ *for every n.*

Proof. Part (i) can be proved by slightly refining the argument that proved connectedness. Let T and T' be two triangulations, and let d_j and d_j' denote the degrees of the vertex j in T and T' respectively, for each $j = 1, \ldots, n$. What we have shown is that for every $i = 1, \ldots, n$ there is a path from T to T' consisting of $2n - 2 - d_i - d_i'$ flips: just start flipping from T and T' in a way that always increases the degree of the i-th vertex. Now we wonder what is the minimum length of these n paths we constructed, but since this is a difficult question we look at the *average* length, which is:

$$\frac{1}{n}\sum_{i=1}^{n}(2n - 2 - d_i - d_i') = 2n - 2 - \frac{1}{n}\left(\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} d_i'\right) \tag{1.3}$$

$$= 2n - 2 - \frac{8n - 12}{n} \tag{1.4}$$

$$= 2n - 10 - \frac{12}{n}. \tag{1.5}$$

In Equation (1.4) we have used that $\sum d_i$ equals twice the number $2n-3$ of edges, a property already used in the proof of Theorem 1.1.2.

Part (ii) is left as an exercise. For the left inequality use the contraction map of Theorem 1.1.2. For the other one, use the arguments of part 1 but with "anti-standard" triangulations, that is to say, triangulations with no internal edge at the given vertex i. $\square$

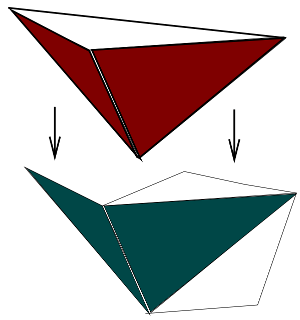


Figure 1.18: A diagonal flip viewed as glueing a tetrahedron to a surface.

Part (ii) says that the bound in Part (i) is not too bad, but the following statement says more; it gives the exact diameter for almost all values of n:

Theorem 1.1.6 (Sleator, Tarjan, Thurston). *The diameter of the graph of flips of an n-gon is $2n-10$ for all sufficiently large values of n.*

As far as we know there is no purely combinatorial proof of the lower bound implicit in this theorem. The proof contained in [80] is far from elementary and we will avoid all the details. Still we will sketch the main idea. We wish to give a lower estimate on how many flips are necessary to move from one triangulation T to another T'. For this, we construct a three dimensional abstract simplicial complex homeomorphic to a ball S whose boundary is obtained by glueing together the two triangulations T and T'.

Glue the two triangulations T and T' of an n-gon along their boundary edges, abstractly this yields a triangulation of a sphere $S = T \cup T'$. When T_1 and T_2 are joined by a sequence of flips, we think of a flip as a tetrahedron inside this ball, whose "lower" and "upper" facets project to the quadrilateral in the flip (see Figure 1.18)

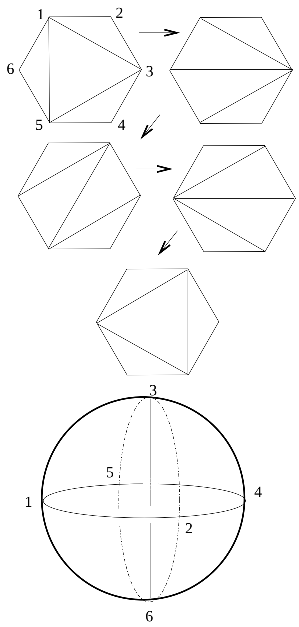


Figure 1.19: Four flips in a hexagon and the corresponding triangulated 3-ball

The flipping process slowly "fills-in" the whole sphere with "abstract" tetrahedra until we get a fully triangulated ball P. Figure 1.19 shows an example where four flips give a triangulated ball with the four tetrahedra $\{1,3,5,6\}$, $\{1,2,3,6\}$, $\{3,4,5,6\}$, $\{2,3,4,6\}$. This interpretation of flips is also popular in the context of Delaunay triangulations [31].

The next nice idea is to embed what was the abstract simplicial ball P into hyperbolic space getting a hyperbolic polyhedron. Thus we are allowed to talk about the hyperbolic volume of each of these simplices. What happens then is that the number of flips, which equals the number of hyperbolic tetrahedra used, is bounded below by the volume of P divided by the maximum volume of one of these tetrahedra. Thus if we want to find a better lower bound we look among n-vertex hyperbolic polyhedra with very large volumes and the paper [80] deals extensively on how to do this. We must remark that the same idea was used by W. Smith to give the best lower bound known for the size of smallest triangulations of combinatorial n-dimensional cubes [81].

Triangulating and computing volumes are intimately related activities. Every convex polytope can be triangulated (why?), and thus the sum of the volumes of simplices of any triangulation of the polytope P equals the volume of P. To compute the volume of P one simply needs to count with a triangulation and have a formula for the volume of a simplex. The volume formula of an simplex in Euclidean space is just a determinant (it should be said that, unlike Euclidean space, the calculation of the volume of a simplex in hyperbolic space is much more complicated). Volume computations are useful throughout mathematics. For example, the calculation of volumes of non-Euclidean convex polytopes has become of interest in topology. The reason is that every hyperbolic manifold can be obtained by identifying the faces of a convex polytope in hyperbolic space and its volume is a topological invariant (when the dimension is greater or equal to three). The volume has been used in the classification of hyperbolic manifolds (see [71] for references). The computation of volumes of polytopes in Euclidean space gets also used in algebra [7, 34, 87]. But most important for us here are the fascinating connections to combinatorics [82], here we show how the computation of volume is equivalent to counting linear extensions of posets, and that the linear extensions are simplices on a triangulation! This was first observed by R. Stanley in [85]:

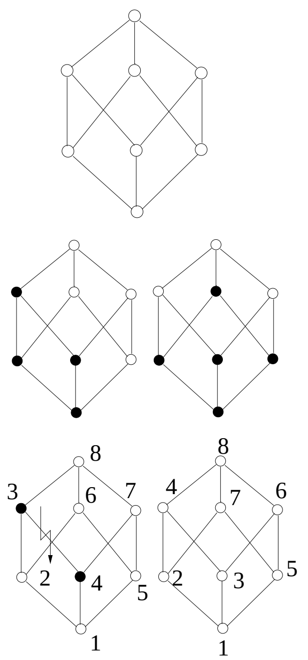


Figure 1.20: A poset P with two order ideals and two bijections $P \to \{1, \dots, 8\}$: only one of them is a linear extension

Definition 1.1.7. We define:

(i) A partially ordered set (or *poset*) is a finite set P with an ordering $<$ that is reflexive, antisymmetric, and transitive.

(ii) A *linear extension* of a poset on n vertices is a bijection λ from the set of vertices of P to $\{1, \ldots, n\}$ such that $\lambda(x) < \lambda(y)$ whenever $x < y$ in P.

(iii) An *order ideal* of a poset is a subset of the poset P such that if $x \in I$ and $y < x$ then $y \in I$.

As it is typical, a poset is represented by a graph, its *Hasse diagram*. Without going into more details we recommend the reader Chapter 3 of [84] for a thorough discussion of posets. Here we simply show in Figure 1.20 the Hasse diagram for the poset of subsets of the set $\{1, 2, 3\}$ ordered by containment, as well as two of its order ideals (elements labeled by the letter o) and two of its linear orderings. Given a poset P with elements $x_1, \ldots, x_n$ one can define the *order polytope* $O(P)$ in $\mathbb{R}^n$ (see [85]) by the following linear constraints:

$$O(P) = \{X \in \mathbb{R}^n | 1 \geq X_i \geq 0,\ X_i > X_j \text{ if } x_i > x_j \text{ in } P\}.$$

Theorem 1.1.8. *The following hold for the order polytope $O(P)$ of a poset P:*

(i) The vertices of the order polytope $O(P)$ are in bijection with the order ideals of the poset P.

(ii) The number of distinct linear extensions of the poset P equals the number of simplices in a maximal size triangulation of the order polytope $O(P)$.

Remark 1.1.9. For a poset with n elements, $O(P)$ is always contained in the n-cube. If P has no relations at all then $O(P)$ is the whole n-cube. In particular, the previous theorem yields a triangulation of the n-cube. See Figures 1.21 to 1.25 for all order polytopes of posets with three elements. One order polytope is hidden because it is asked in Exercise 1.12.

Proof. We claim first that the coordinates of the vertices of $O(P)$ are the incidence vectors of order ideals of the poset P. To see this first note that all the vertices of the order polytope must be 0-1 coordinates: Let $x \in O(P)$

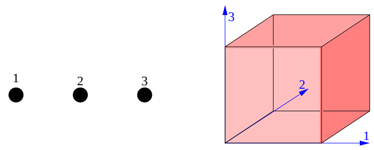


Figure 1.21: The order polytope of an antichain with three elements is a cube

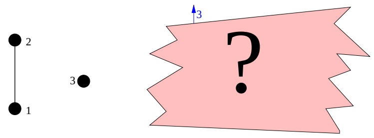


Figure 1.22: The order polytope for three elements with one relation is asked for in Exercise 1.12

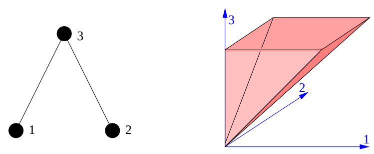


Figure 1.23: The order polytope of this is a pyramid upside-down

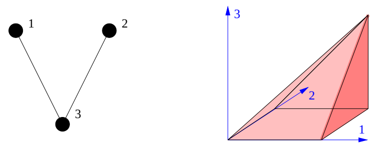


Figure 1.24: The order polytope of this is a pyramid

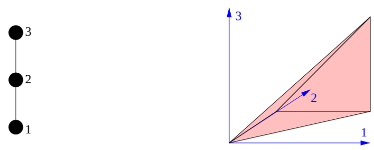


Figure 1.25: The order polytope of a chain of three elements is a simplex

have some entry not zero or one. If $\alpha = max\{x_j | 0 < x_j < 1\}$, then by replacing all coordinates $x_j = \alpha$ by $\alpha + \epsilon$ or $\alpha - \epsilon$ we get again a point of $O(P)$. Thus x cannot be a vertex of $O(P)$. The correspondence of vertices and order ideals is the following

$$x \in \text{vert}(O(P)) \to S_x = \{x_j | X_j = 0\}$$

We should verify that S_x is an ideal: Suppose $X_k < X_j$ with $X_j \in S_x$, then clearly by definition of $O(P)$ we have $X_k = 0$ too! Different vertices have different support so they give different ordered ideals. Now given an ordered ideal, we construct a 0-1 vector by its support. The inequalities are satisfied inside $O(P)$.

Any of the $n!$ total orderings $x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}$ defines a simplex $\{X \in [0,1]^n | X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n)}\}$ inside the unit n-cube. The cube is partitioned into these $n!$ simplices (no two intersect in the interior, the union of all of those equals the cube). It is important to notice that all these simplices have the same volume $1/n!$. Now, given a poset P all its linear extensions correspond to simplices that form a triangulation of $O(P)$. This is because $O(P)$ contains one such simplex iff the ordering where it came from is a linear extension and all of $O(P)$ is covered. The number of simplices in the triangulation equals the number of linear extensions for the poset P. Because the simplices cannot have smaller volume, we have a maximal triangulation of $O(P)$. □

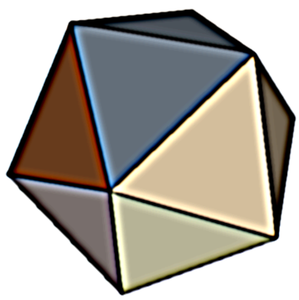


Figure 1.26: The volume of this icosahedron . . .

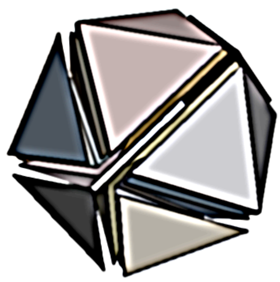


Figure 1.27: . . . can be computed by triangulating it . . .

Figure 1.28: . . . and summing up the volumes of the simplices

Brightwell and Winkler [14] proved that enumerating the linear extensions of a finite poset is a #P-complete problem (besides, the enumeration of their vertices is also a hard problem [69]). Therefore, from the previous theorem we get the following:

Corollary 1.1.10. *Given a d-dimensional polytope P represented by its facets it is #P-hard to compute its volume.*

Roughly speaking this means that computing the volume is equivalent to a number of other computational problems that are already known to be "extremely hard" in a well-defined way, the class of #P-hard problems, and if it were the case that computing the volume admits a fast solution all the other members of the same family of problems would have a fast solution too. This indicates, it is more likely we will never find a fast algorithm. The #P-hard class contains counting problems such as “How many hamiltonian cycles are there in a finite graph?”, “How many matchings are there in bipartite graph?”, “How many different 3-colorings are possible of a planar graph?”, and many more [33]).

Computing the volume of a polytope of arbitrary dimension presented by its vertices is also hard [25, 47]. We even know that it is hard to compute the volume of *zonotopes* [27]. (Zonotopes are centrally symmetric polytopes that arise as projections of cubes or, equivalently, as Minkowski sums of line segments [101, Chapter 7].) We refer the reader to the nice paper [15] for a survey and evaluation of practical methods to compute the volume of a convex polyhedron.

How about the problem of approximating the volume? It is possible to have fast randomized approximation [26] although for general convex sets the situation is much worse: It was proved by Elekes (see [29]) that the volume of the convex hull of any m points of an n-dimensional ball with volume V is at most $Vm/2^n$. This implies that no polynomial-time algorithm can compute the volume of convex sets (given by an oracle) with less than exponential relative error.

1.2 Optimization and Triangulations

Polyhedra are useful tools in discrete and combinatorial optimization [77]. They are at the foundation of linear programming which is a widespread method in optimization. We wish to outline a beautiful relationship between triangulations and parametric optimization problems.

Let A be a $d \times n$ matrix. We will sometimes think of A as a set of its column vectors. In particular, $\mathrm{cone}(A)$ denotes the real cone generated by the nonnegative linear combination of the columns of the matrix A. A *cone subdivision* of $\mathrm{cone}(A)$ is a finite collection of subcones $\mathrm{cone}(A_\sigma)$, where A_σ is a submatrix of A, such that the

$$\begin{aligned} &\min c(x)\\ &\text{subject to}\\ &\qquad Ax = b\\ &\qquad x \geq 0 \end{aligned}$$

Figure 1.29: A linear program with $A \in \mathbb{R}^{d\times n}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}^d$

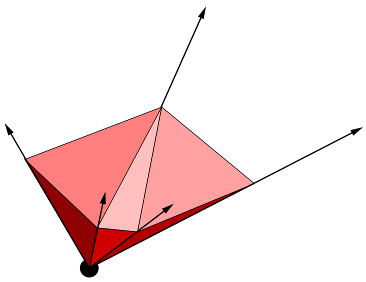


Figure 1.30: A triangulation of a pointed cone, cut off by an affine hyperplane; the section of the cone looks like a triangulation of a point configuration

intersection of any pair of subcones in Γ is again a subcone in Γ and the union of all the subcones is $\mathrm{cone}(A)$. By a *cone triangulation* we mean a cone subdivision all of whose cones are simplicial cones, that is, all the submatrices are square matrices. Note that this definition is essentially our original definition of triangulation but using vectors (the columns of A) instead of points.

Fix a matrix A as above, and for each cost vector $c \in \mathbb{R}^n$ and for each right-hand-side vector $b \in \mathrm{cone}(A) \subset \mathbb{R}^d$ consider the linear programming problem $LP_{A,c}(b) := \min\{c(x) | Ax = b,\ x \geq 0\}$.

Our main focus will be the study of the parametric family of linear programs $LP_{A,c} = \{LP_{A,c}(b) | b \in \mathrm{cone}(A)\}$.

For simplicity we will assume for the rest of this section that $\{x \in \mathbb{R}^n | Ax = 0\} \cap \{x \in \mathbb{R}^n | x \geq 0\} = \{0\}$. This assumption makes $LP_{A,c}$ a family of bounded linear programs (the minimum exists in all cases). The main intuition is that $\mathrm{cone}(A)$ will be triangulated by each choice of cost vector c: A *cone subdivision* of $\mathrm{cone}(A)$ is a finite collection of subcones $\mathrm{cone}(A_\sigma)$, where A_σ is a submatrix of A, such that the intersection of any pair of subcones in Γ is again a subcone in Γ and the union of all the subcones is $\mathrm{cone}(A)$. By a *cone triangulation* we mean a cone subdivision all of whose cones are simplicial cones. Note that this definition is essentially our original definition of triangulation but using vectors (the columns of A) instead of points.

What happens to the linear program as the right-hand side b changes? What are the optimal solutions for each b? The main intuition is that $\mathrm{cone}(A)$ will be divided into regions consisting of "equivalent" linear programs. And it turns out that the subdivision is a cone subdivision and, if b is generic, a triangulation. This was first observed by Walkup and Wets [97]. The geometric theory of parametric linear programs has recently been extended to parametric integer programs (see [89]).

Theorem 1.2.1 (Walkup-Wets). *Let* $LP_{A,c}(b)$ *denote the linear program*

$$\min\{cx : Ax = b, x \geq 0\},$$

where c *and* A *are fixed. Then, if* $\mathrm{cone}(A)$ *is the cone* *spanned by the nonnegative linear combinations of the columns of* A*:*

(i) $LP_{A,c}(b)$ *is feasible if and only if* b *lies in* $\mathrm{cone}(A)$.

(ii) $LP_{A,c}(b)$ *is bounded for all* $b \in \mathrm{cone}(A)$ *and all* c *if and only if* $\ker(A) \cap \mathbb{R}^n_+ = \{0\}$

(iii) If $LP_{A,c}(b)$ *is bounded, then there exists a triangulation* T *of* $\mathrm{cone}(A)$ *such that the* d*-dimensional simplices of the triangulation are cones whose rays form an optimal basis for all* b *inside the simplex.*

For simplicity we will assume for the rest of this section that $\{x \in R^n : Ax = 0\} \cap \{x \in R^n | x \geq 0\} = \{0\}$. As the theorem says, this assumption makes every $LP_{A,c}(b)$ a bounded linear program (the minimum exists in all cases).

Here is an example. Consider the parametric linear programming problems $LP_{A,c}(b)$ with

$$c = (0, -2, -2, 0, -1, 0), \qquad \text{(cost)}$$

$$A = \begin{bmatrix} 4 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 4 \end{bmatrix}, \qquad \text{(coefficients)}$$

$$b = (b_1, b_2, b_3). \qquad \text{(righthand side)}$$

According with the theorem we can find a triangulation of $\mathrm{cone}(A)$ associated to c and its simplicial cones are useful for finding optimal solutions. We can visualize the triangulation of $\mathrm{cone}(A)$ for the above example by taking a 2-dimensional slice of the 3-dimensional $\mathrm{cone}(A)$. The triangulation for the cost vector is shown in Figure 1.31.

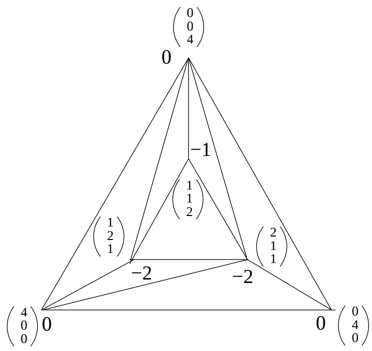


Figure 1.31: The cone triangulation associated with the cost vector c. This shows a two-dimensional slice of the cone.

In the proof of Theorem 1.2.1 we need to use a basic fact from the theory of linear programming (see [77, Section 7.9]):

Lemma 1.2.2 (Complementary slackness). *Let* A *be a matrix,* b *and* c *the right-hand-side and cost vectors of* $LP_{A,c}(b) = \min\{cx | Ax = bx \geq 0\}$*. There is an associated dual problem and the duality equation:*

$$\max\{yb | yA \leq c\} = \min\{cx | Ax = bx \geq 0\}. \qquad (1.6)$$

Then if both optima are finite and x^* *and* y^* *are feasible solutions, then the following conditions are equivalent:*

(i) x^* *and* y^* *are optimum solutions of their problems.*

(ii) If a component of x^* *is positive, the corresponding inequality in* $yA \leq c$ *is satisfied by* y^* *with equality, i.e.,* $x^*(c - y^*A) = 0$.

In other words, the minimum value of $LP_{A,c}(b)$ *is attained by at a vector* x^* *if and only if there exists a* y *such that* $yA_j \leq c_j$ *for all* $j = 1, \ldots, n$ *and for all indices either* $x^*_j = 0$ *or* $yAj = c_j$.

Proof of Theorem 1.2.1. Part (i) is easy. Part (ii): If we had a non-zero z vector in the intersection of $\ker(A) \cap \mathbb{R}^n_+$ we could use it to decrease indefinitly the value of cx with any c with all negative entries, just add large multiples of z. If $LP_{A,c}(b)$ is unbounded for some choice of b, c, there is an extreme ray d as part of the polyhedron $\{x | Ax = bx \geq 0\}$ and the vector direction d gives the desired vector [77]. For Part (iii), for each cost vector c consider the polyhedral subdivision T_c given by the cones $\text{cone}(\{A_{i_1}, \ldots, A_{i_j}\})$ where A_i denotes a column of A and there exist a vector y such that $yA_{i_k} = c_{i_k}$ for all $i_k \in \{i_1, \ldots, i_j\}$. This means in particular that $yA_j < c_j$ when A_j is not a ray of $\text{cone}(\{A_{i_1}, \ldots, A_{i_j}\})$.

If T_c is not already a triangulation we can take a perturbation of c that will indeed be one. We are sure of this because the condition $yA_{i_k} = c_{i_k}$ is typically satified simultaneously by at most rank of A many A_{i_k} (the more equations in a linear system, the less likely it is to have a solution). Now finally, take any d-dimensional simplicial cone $\text{cone}(\{A_{i_1}, \ldots, A_{i_d}\})$ by the complementary slackness theorem, the columns of A which are rays of the cone are indeed a basis that supports an optimal solution. □

We emphasize that not all cone triangulations come from the use of some cost vector! Triangulations that arise from a cost vector are called *regular* and will be studied in Chapter 4. They actually are the triangulations that appear as vertices of the secondary polytope of A (to be studied in Chapter 4).

Let us now see another subdivision of $\mathrm{cone}(A)$ related to the family of linear programs. For each b, we have a polytope $P_b := \{x|Ax = b, x \geq 0\}$. We say that P_b and P'_b are *normally equivalent* if their *normal fans* coincide. The normal fan of a polytope $P \subset \mathbb{R}^d$ is the decomposition of $\mathbb{R}^d$ (now regarded as the space of linear functionals on P) into functionals that select the same face of P. Being normally equivalent means in particular that the polytopes look combinatorially the same (same face lattice) but, more strongly, corresponding facets are parallel [101].

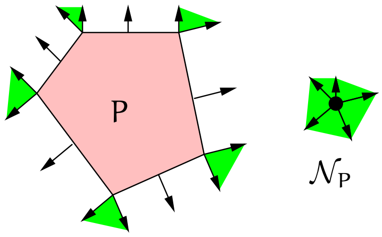


Figure 1.32: The normal fan of a polygon

Now, the notion of normally equivalence creates an equivalence relation on the right-hand-side vectors b. We can say that right-hand-side vectors b, b' inside $\mathrm{cone}(A)$, are equivalent if P_b and $P_{b'}$ are normally equivalent. And this provides us now with yet another partition of $\mathrm{cone}(A)$ into polyhedral cones. This partition is *not* a cone subdivision in the sense defined above. Among other things, some new vertices are introduced.

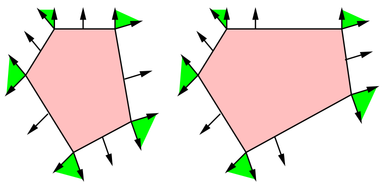


Figure 1.33: Normally equivalent polygons

We call the cells in this partition *chambers*. They are the maximal cells in the common polyhedral refinement of *all* triangulations of $\mathrm{cone}(A)$. In other words, a chamber γ is the intersection of a finite collection Γ_γ of simplicial subcones with the properties: (a) each $\sigma \in \Gamma_\gamma$ is generated by $\mathrm{rank}(A)$ many linearly independent columns of A, (b) the intersection $\cap_{\sigma\in\Gamma_\gamma} \sigma$ has non-empty interior, and (c) Γ_γ is maximal with respect to property (b).

Figure 1.34 shows the polyhedral complex that we obtain by "overlapping" of all the cost-vector-induced triangulations. We show them all for our running example in the upper part of 1.35.

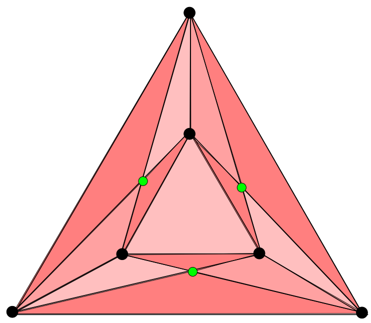


Figure 1.34: The chamber complex of $\mathrm{cone}(A)$ for the example. As before, for ease of drawing, we show a 2-dimensional slice

First we allowed the right-hand-side vector b to move with c fixed to discover a triangulation; then we let b be fixed to discover that as we change c all the b is contained in a chamber of $\mathrm{cone}(A)$ and that chambers are indeed what results from overlapping all triangulations of $\mathrm{cone}(A)$. Now there is a final surprise: If we let also the cost vectors c and the right hand side b to be an arbitrary vector inside $\mathrm{cone}(A)$ we witness different triangulations for cone(A) appear. Even more surprinsingly,

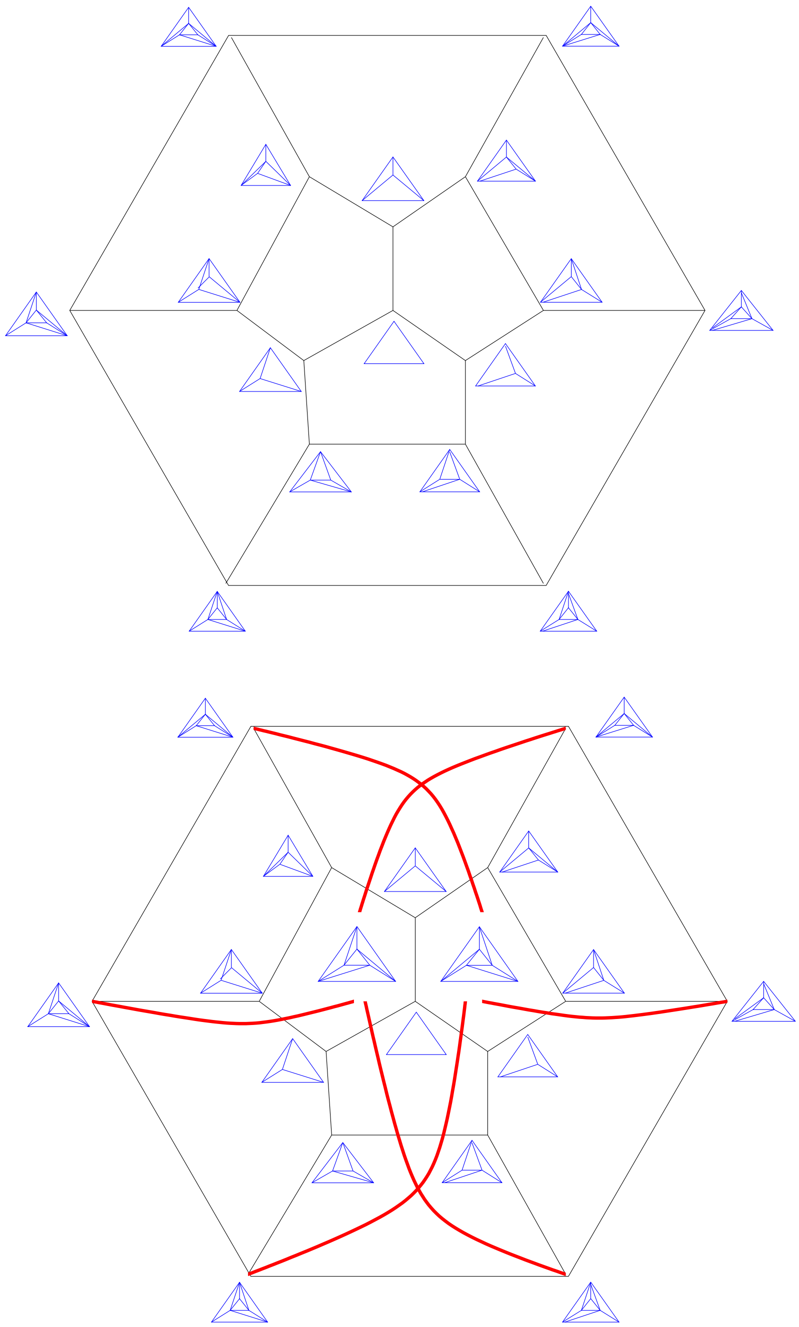


Figure 1.35: The flip graph of all regular triangulations (top) and all triangulations (bottom) of the point configuration in Figure 1.31.

the triangulations we get with the variation of the cost vector are "connected" to one another in a rather nice way. They have in fact the structure of a polytope. We show in Figure 1.35 the vertices and edges of the secondary polytope of the point configuration in Figure 1.31.

Take c and c' two generic cost vectors then $LP_{A,c}(b)$ and $LP_{A,c'}(b)$ have the same set of optimal solutions for every value of b if and only if c and c' define the same cone triangulation. Define an equivalence relation among the cost vectors (vectors in $\mathbb{R}^n$): We say c is a equivalent to c' if they define the same triangulation. This equivalence relation decomposes $\mathbb{R}^n$ into finitely many equivalence classes, each of them is a convex polyhedral cone. The collections of all such cones covers $\mathbb{R}^n$ and receives the name of the *secondary fan*. Gel'fand, Kapranov and Zelevinksy demonstrated that this fan is actually the normal fan of the *secondary polytope* of A via cost variations.

The connection of triangulations of point sets with linear optimization problems does not end here. One can consider a similar study of parametric integer programming problems. There are several methods to attack such problems [77] but a new algebraic approach, presented in [18] and extended in [90] provides a nice connection of the theory of Gröbner bases of toric ideals to our setting.

Let us now turn attention to another topic in optimization where triangulations play an important role: fixed points of continuous maps. In Game theory and Economics the notion of equilibrium is very important [76]. Mathematically an equilibrium is a fixed point of a continuous mapping. And this is not the only reason why finding a fixed point is an issue of practical importance. A wide variety of algorithms have been proposed and there is an extensive literature in the mathematical programming community. One of the most famous theorems about fixed points is due to the Dutch mathematician L. E. J. Brouwer:

Theorem 1.2.3 (Brouwer). *If C is a topological d-dimensional ball and $f : C \mapsto C$ is a continuous function, then f has a fixed point, namely, there is a point x^* in C with $f(x^*) = x^*$.*

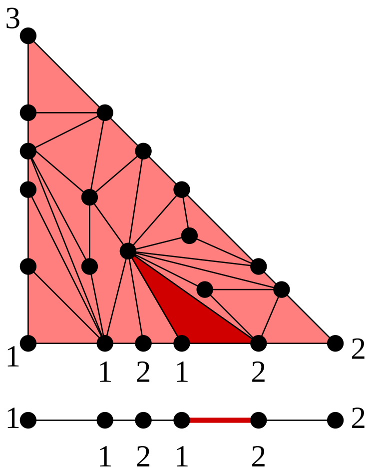


Figure 1.36: In order to find a fully labeled simplex, one can start with a fully labeled simplex of one dimension less in the boundary; then one dives into the big simplex until one finds a fully labeled simplex in the triangulation; Exercise 1.15 is about making this rigorous

Recall that a homeomorphism is a one-to-one and onto continuous function whose inverse is also continuous. A topological d-ball is the image of the standard unit ball $B^d = \{x \in \mathbb{R}^d | \sum_i x_i^2 \leq 1\}$ under a homeomorphism. A simplex is our favorite example of a topological ball. Brouwer's original proof says nothing about how to find the fixed point or a good approximation to a fixed point, not even in the case when C is a simplex. In the case of a simplex (see exercises for extension) Brouwer's theorem may be demonstrated via a combinatorial result about labeling triangulations due to Sperner:

Lemma 1.2.4 (Sperner). *Let A be a point configuration whose convex hull is a d-dimensional simplex Δ and T a triangulation of A. There are $d+1$ facets $\Delta_1, \ldots, \Delta_{d+1}$ in the simplex Δ. Label all the vertices of T using the numbers $1, 2, \ldots, d+1$ in such a way that no vertex that lies on the facet Δ_i receives the label i. Then there is a simplex in T whose vertices carry all the different $d+1$ labels.*

A rather easy proof can be derived by induction on the dimension of the simplex (see exercises). Curiously, Sperner's lemma has a simple generalization to labelings of triangulations of arbitrary polytopes (see Figure 1.37 for an example and [22] for details concerning the following theorem):

Theorem 1.2.5 (Polytopal Sperner Lemma). *Let T be a triangulation of a d-dimensional polytope P using n vertices $V_1, V_2, \ldots, V_n$ (in the boundary or the interior of the polytope). Label the vertices of T by $1, 2, \ldots, n$ in such a way that a vertex of T belonging to the interior of a face F of P can only be labeled by j if V_j is a vertex of F. In this way there are as many labels being used as vertices on the convex hull of P. Then there are at least $n-d$ full dimensional simplices of T, each labeled with $d+1$ different labels.*

Now, how can one use Sperner's lemma, or its generalization, to prove Brouwer fixed-point theorem for simplices? Triangulate the simplex and apply a labeling suitable for a particular continuous function f: Associate to a vertex of the triangulation a the label i if the

ith-barycentric coordinate of a is smaller or equal than the ith-barycentric coordinate of $f(a)$. There will be at least one such index for each vertex unless the vertex is a fixed point (because the barycentric coordinates add up to one) but if there are several, simply make an arbitrary choice among them. Now, a simplex of T can be found so that for each of the $d+1$ vertices a the corresponding barycentric coordinates of a are *not* increased by f.

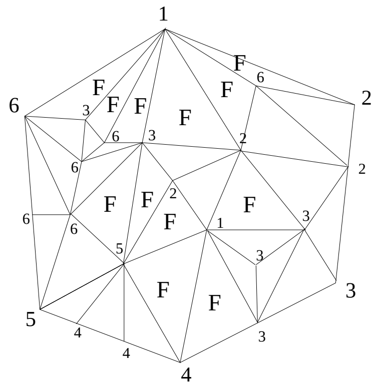


Figure 1.37: A Sperner labeling of a triangulated hexagon, with many extra vertices, and its fully labeled cells.

Finally re-triangulate adding more and more points in such a way that the maximum diameter of the simplices appearing in the triangulation goes to zero (would any triangulation do?). At each step we can find a fully labeled simplex, the barycenters of all such simplices will produce an infinite sequence of points that must converge to a point x^*. Since the map f is continous, ith-barycentric coordinate of x^* is smaller or equal than the ith-barycentric coordinate of $f(x^*)$ (the difference is smaller than any positive epsilon) and therefore is a fixed point of the map.

There is a practical difficulty on using Sperner's lemma to explicitly find an approximation to the fixed point. First, finite versions of this method do not in general find a point arbitrarily close to a fixed point but rather a point that is arbitrarily close to being a fixed point, i.e., whose image is arbitrarily close to itself. Moreover, the number of vertices necessary to refine the successive triangulations may be very large and moreover there is no clear procedure to find the special simplices that receive all the labelings. Today there is a large set of triangulation-based techniques to compute fixed points (see [91]). The development of triangulation-based algorithms is still active and has brought new interesting questions [41, 92].

All such algorithms use an essential property of triangulations: If T is a triangulation of a point set in $\mathbb{R}^d$ and τ is a $(d-1)$-simplex that is a face of a simplex of T, then either (1) τ belongs to the boundary of $\mathrm{conv}(A)$ or (2) τ is a face of precisely two simplices in T. You can easily verify that this is true from the Definition 1.0.1. This simple property makes triangulations useful because one can iteratively search for a "fully labeled" simplex of the triangulation by moving to an adjacent simplex. Even non-triangulation-based algorithms use a similar pivoting property [75, 76].

Computational experience with different fixed-point algorithms has shown a considerable sensitivity to the triangulation used. Is there a theoretical measure that can predict the relative efficiency? When we want to find the "approximate fixed point", the general principle is to move from simplex to adjacent simplex until we reach a fully labeled simplex. Hence, a rough measure of efficiency of a triangulation would be the number of simplices used. This has brought attention to the problem of finding triangulations of point sets that use the fewest simplices.

For example, take the vertices of a regular cube. Figure 1.38 shows all triangulations modulo

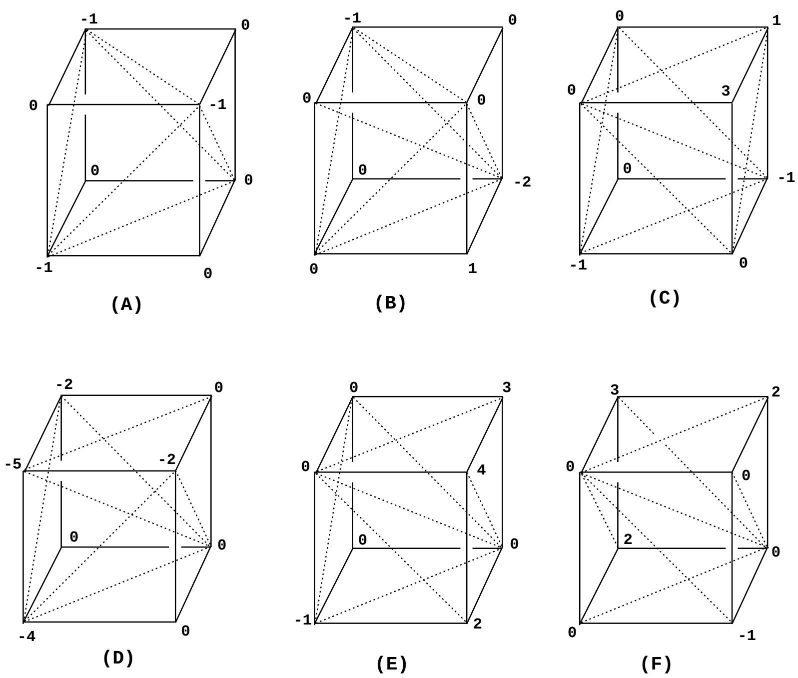


Figure 1.38: The six non-isomorphic triangulations of a regular cube.

symmetries. If you consider them as cone triangulations of the 4-dimensional cone over the 3-cube, they are all regular. The picture shows cost vectors producing them. We see in the figure that the smallest triangulation has 5 tetrahedra. It is a famous open problem to determine the size of smallest triangulation of the regular d-cube. So far the answer is only known up to $d = 7$ (and it is 1493) [41]. We will discuss more about this topic later on in Chapter 9.

1.3 Algebra and Triangulations

Consider the system of polynomial equations

$$ax + by + c = 0 \quad \text{and} \quad dx^3y^3 + ex^3 + fy^3 + g = 0.$$

The coefficients a, b, c, d, e, f, g are non-zero complex parameters. (This particular system is a counterexample to an important conjecture, see [59].) The question that we would like to ask is how many roots over $\mathbb{C}^2$ for the system should one expect as the parameters change? We are looking for bounds that will be valid for "almost all" values. We certainly know of this kind of bounds, for instance, the famous Bezout's theorem! We now recall two versions. The second version is very specific for discussing real solutions and will be used later on, the first version is more useful for us now. The books [96] and [19] have nice expositions about this theorem.

Theorem 1.3.1 (Bézout). *There are the following bounds on the roots of a sparse system of polynomial equations:*

- Affine complex version: *Let* $f_1(x, y) = 0$ *and* $f_2(x, y) = 0$ *be a system of two polynomial equations in two unknowns. If it has only finitely many common complex roots* $(x, y) \in \mathbb{C}^2$*, then the number of those roots is at most* $\deg(f_1)\deg(f_2)$*.*

- Projective smooth real curve version: *If* C_{f_1}, C_{f_2} *are two non-singular real projective curves, given by homogeneous polynomials* f_1 *and* f_2 *respectively, and the intersection* $C_{f_1} \cap C_{f_2}$ *is a finite set of points, then its cardinality is at most* $\deg(f_1)\deg(f_2)$*. If in addition* f_1 *and* f_2 *intersect transversaly then* $|C_{f_1} \cap C_{f_2}| \equiv \deg(f_1)\deg(f_2)$ *(modulo 2).*

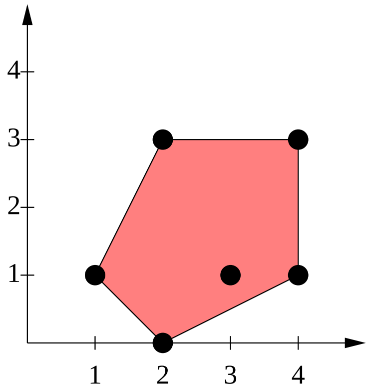


Figure 1.39: The Newton polytope for the polynomial $x^2 + xy + x^3y + x^4y + x^2y^3 + x^4y^3$

In our particular example the bound is six roots. In most instances the Bezout bound will not be too tight (though, we will later prove that in this example it is sharp) because Bezout's bound counts solutions at infinity as well. Our example is a sparse system, missing many of the terms of degree six that can be formed with two variables. We would like to have a bound that reflects somehow the "shape" of the system and possibly a method that finds the solution without changing its "shape" like in the case of Gröbner bases techniques. We must then define what we mean by the shape of a system:

Definition 1.3.2. The *support* of a polynomial $f(x_1, \ldots, x_n)$ is the set of monomials that appear with non-zero coefficient. Each monomial is regarded as an *exponent vector* in $\mathbb{N}^n$, i.e., its coordinates are the exponents of the n variables. The *Newton polytope* of f, denoted by $N(f)$, is the convex hull of the exponent vectors of the monomials in the support of f.

In this way the Newton polytope of the polynomials presented at the beginning are a triangle and a rectangle. Note that in some situations the vertices of the Newton polytope may not equal the support of the polynomial. The *Minkowski sum* of two convex polytopes P and Q, denoted $P + Q$ is the convex polytope $\{p + q | p \in P, \ q \in Q\}$. Note that the vertices (respectively faces) of $P + Q$ are sums of vertices (faces) of P and Q. Note that the Minkowski sum of the Newton polytopes of two polynomials f and g equals the Newton polytope $N(fg)$. We can see an example of Minkowski sum in figure 1.40.

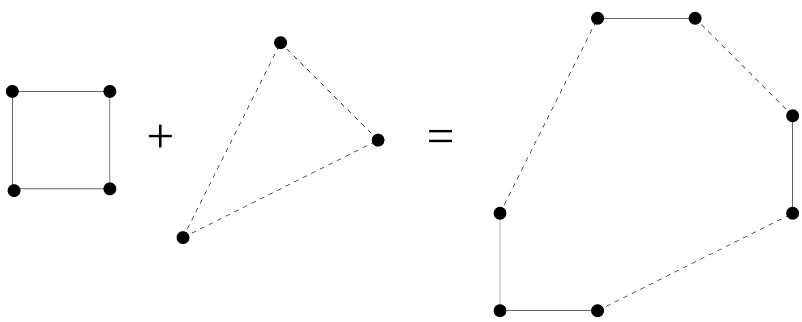


Figure 1.40: Minkowski sum of a triangle and a rectangle.

Definition 1.3.3. Given d polytopes $Q_1, Q_2, \ldots, Q_d$ in $\mathbb{R}^d$, their *mixed volume*, $\mu(Q_1, Q_2, \ldots, Q_d)$ equals the absolute value of the following alternating sum of ordinary volumes

$$\sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \operatorname{vol}(\sum_{j \in I} Q_j).$$

This is a concrete way of defining a certain real number, but what does it mean? and how does one compute its value? This has to do with the following very intuitive construction. Let $A_1, A_2, \ldots, A_d$ be a collection of lattice point configurations, $A_i \subset \mathbb{Z}^d$ (you can think of them as the support exponent vectors for certain polynomials). Denote by Q_i the convex hull of A_i. Now let us perform the following construction:

1. Choose random values $w_i(a)$ for each of the points $a \in A_i$ (random values work fine, see discussion in [40]). Do this for each set A_i. Consider the polytope
$$\bar{Q}_i = \operatorname{conv}(\{(a, w_i(a)) | a \in A_i\}).$$
Note that $\bar{Q}_i$ is a polytope in $\mathbb{R}^{d+1}$.
2. Compute the lower convex hull $\bar{\Delta}$ of the Minkowski sum $\bar{Q}_1 + \cdots + \bar{Q}_d$. The facets of $\bar{\Delta}$ are of the form $\bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_d$ where $\bar{F}_i$ is a face of $\bar{Q}_i$ and $\sum_{i=1}^{d} \dim(\bar{F}_i) = d$. We say one such facet is *mixed* if $\dim(\bar{F}_i) = 1$ for all d.

Theorem 1.3.4. *The image of the polyhedral complex $\bar{\Delta}$ under the projection that forgets the last coordinate of every point is a polyhedral subdivision of the Minkowski sum $\sum_{i=1}^{d} Q_i$. The mixed volume $\mu(Q_1, \ldots, Q_d)$ equals the sum of the volumes of the mixed cells induced by the projection of $\bar{\Delta}$.*

In Figure **??** we show an example of two mixed subdivision of the Minkowski sum of the Newton polytopes of the system of polynomials $ax + by + c = 0$ and $dx^3y^3 + ex^3 + fy^3 + g = 0$. Each is obtained as the Minkowski sum Mixed subdivision obtained by specified height vectors; mixed cells are drawn grey again fig:mixed-lifting of two polytopes, one is a lifting of a square (in fact a tetrahedron) and the other a lifted triangle. To compute the mixed volume we must simply identify the mixed cells (in this case we show them highlighted on the picture).

The mixed volume is equal to six in this particular example. Note that as we change the heights or lifting values we use for the points, we obtain different subdivisions. We can find the equivalence classes of the lifting vectors that induced the same mixed subdivisions. We obtain a collection of polyhedral cones partitioning real space. It follows from the theory of fiber polytopes, developed by L. Billera and B. Sturmfels [10], that this is the normal fan of a polytope. We show a diagram of the polytope for our example in Figure 1.41. Readers that carefully read the previous sections will feel a sense of dejà-vu, especially comparing this figure to Figures 1.16 and 1.35. Indeed, the polytope in question is the same as the secondary polytope of a certain three-dimensional point set, and this is not a coincidence as we will see in Chapter10.

Now we are ready to state the main result of this part (see [7] and [40] for proofs of the theorem as well as the closely related papers [56] and [51]).

Theorem 1.3.5 (D.N. Bernstein, 1976). *Given* d *subsets* $A_1, \dots, A_d$ *of* $\mathbb{Z}^d$ *and* $Q_i = \mathrm{conv}(A_i)$, *consider the sparse polynomial system of equations*

$$\sum_{a \in A_1} c_{1,a} t_1^{a_1} t_2^{a_2} \dots t_d^{a_d} = 0, \tag{1.7}$$

$$\sum_{a \in A_2} c_{2,a} t_1^{a_1} t_2^{a_2} \dots t_d^{a_d} = 0, \tag{1.8}$$

$$\vdots$$

$$\sum_{a \in A_d} c_{d,a} t_1^{a_1} t_2^{a_2} \dots t_d^{a_d} = 0. \tag{1.9}$$

For almost all choices of coefficients $(c_{i,a})$, *the number of roots in* $(\mathbb{C}^*)^d$ *equals the mixed volume* $\mu(Q_1, \dots, Q_d)$.

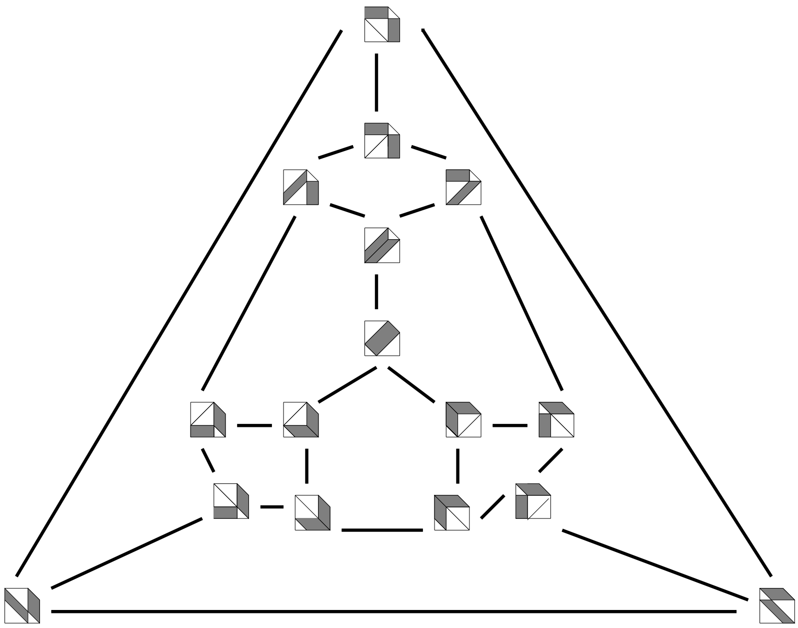


Figure 1.41: All mixed subdivisions of a Minkowski sum; the gray cells are mixed cells; what the edges in this picture mean will be discussed in Section 10.1

As a more striking application of Bernstein's theorem consider the system of equations $ax^3y^2 + bx + cy^2 + d = 0$ and $exy^4 + fx^3 + gy = 0$. The Bézout bound estimates 25 complex roots. The number of roots in the torus $(\mathbb{C}^*)^2$ using mixed volumes equals 18. Using Gröbner bases one can see that 18 is the actual number of roots in $\mathbb{C}^2$.

Note that the number of solutions in the torus $(\mathbb{C}^*)^2$ of a system B, obtained from a system A by multiplying by a fixed monomial, is the same as the torus solutions for A, but the total number of affine solutions may change! Bernstein's theorem applies only to solutions on $(\mathbb{C}^*)^d$ but recently there has been work on trying to extend it for counting affine solutions [40], [73]. For sparse systems there exist explicit homotopies for finding all roots. This is a combination of numerical and combinatorial methods that has been explored in [40], [93].

How many real roots are there for a sparse system? One would like to generalize the case of one univariate polynomials where Descartes rule of signs [12] indicates the number of real solutions for a univariate polynomial, is bounded by the just the number of terms. Can something similar be done for multivariate systems? It is still unknown. On a positive note Khovanskii [52] made a major breakthrough when he provided a bound that did not depend on the degrees of the equations. On the negative side Kushnirenko had conjectured that, if $f_1 = f_2 = \cdots = f_k = 0$ are k polynomial equations in k variables, and m_i is the number of terms of f_i, the number of nondegenerate isolated positive roots of this system is at most

$$(m_1 - 1)(m_2 - 1)\ldots(m_k - 1).$$

Bertrand Haas found a counterexample that consists of two polynomials in two variables with three terms having five roots (instead of the conjectured four). And this can be easily generalized to more variables [39].

Figure 1.42: David Hilbert around the time he proposed 23 open problems

In preparation for our next algebraic topic it is very relevant to mention the work of Sturmfels who gave lower bounds on the number of real roots for sparse systems of equations from studying the signs of the coefficients and marking with them the mixed cells of the mixed subdivisions [86]. This is a generalization of Viro's method for complete intersections. In this way one can construct zero dimensional polynomial systems that have an "easy to count" number of real roots. The roots are in fact cells of a mixed subdivision of the type we saw before. It was proposed by Itenberg and Roy [44] that this construction could provide a combinatorial bound for the number of real solutions of a polynomial system with fixed Newton polytopes. Unfortunately this was disproved by Li and Wang [59]. For a nice introduction to the topic of solving systems of polynomial equations we highly recommend the book [88]

Now we move to an application in algebraic geometry. The study of the topology of smooth real algebraic curves has a long history (perharps the earliest result is the well-known theorem of projective geometry, due to Poncelet, which says that any pair of smooth conics are equivalent under projective transformations). Informally, it deals with the following question: What are the possible topological types of smooth real curves, with a given degree? Hilbert popularized this question by including a version of it in his famous collection of problems, proposed in 1900. He asked about the classification of curves of degree six and surfaces of degree four. Both cases were solved by 1977, but only the curves of degree seven have been classified since then, and hardly anything is known for real smooth hypersurfaces of arbitrary degree.

For the classification, two types of results are needed. One the one hand, it is necessary to describe "prohibitions" or obstructions that narrow down the possible topological types. On the other hand, the interested researcher must construct hypersurfaces for the topological types allowed by the obstructions. This part of the book considers in detail this last aspect of the problem: The construction of hypersurfaces with prescribed topology. We will focus on the work of Oleg Viro [94, 95] who developed a very succesful combinatorial technique in the 1980's. This technique is based on the triangulations of points sets associated with the possible monomials of a polynomial function. For simplicity, the discussion will be done for the case of plane curves but the theory works in arbitrary dimension.

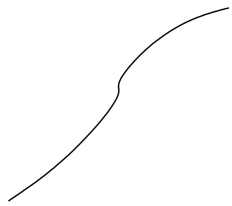


Figure 1.43: A pseudoline. Its complement has one component, homeomorphic to an open circle. The picture only shows the "affine part"; you have to think the two ends as meeting at infinity.

Let us see what the problem actually looks like. Let $f(x, y, z) = \sum_{i+j+k=d} a_{i,j,k} x^i y^k z^k$ be a smooth homogeneous polynomial of degree d. The solution set of f is the collection $C_f = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = 0\}$. The solution set is a closed subvariety of dimension one inside $\mathbb{R}P^2$. Since it is smooth, each component is homeomorphic to a circle. But there are two topologically different (i.e., "non-isotopic") ways to embed a circle in $\mathbb{R}P^2$: It can be isotopic to a conic, or to a line. The two cases are called, respectively, an *oval* and a *pseudoline*. The main difference is that the complement of a pseudoline has only one connected component (for example, the "line at infinity" is a pseudoline whose complement is the affine plane) while the complement of an oval has two connected components: one homeomorphic to an open circle, which we call the interior of the oval, and one homeomorphic to a Möbius band. See Figures 1.43 and 1.44. The two cases are also distinguishable by the double cover of $\mathbb{R}P^2$ by the two dimensional sphere S^2. An oval is covered by two ovals in the sphere, and a pseudoline

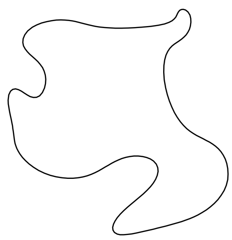


Figure 1.44: An oval. Its interior is a (topological) circle and its exterior is a Möbius band.

by only one, which wraps around twice.

Ovals can be nested, that is to say, one contained in the interior of the other, or not. A nest of ovals (of depth k) consists of k ovals each nested in the previous one. Also relevant to the question we want to study is that an oval meets any generic line (or pseudoline, for that matter) in an even number of points, while two generic pseudolines meet always in an odd number of points. In particular:

Corollary 1.3.6. *The following hold for real plane curves:*

(i) A smooth plane curve of degree d has a pseudoline if and only if d is odd, and in this case it only has one. In particular, a curve of odd degree is never empty.

(ii) If a curve of degree d has two nests, of depths i and j, then $i+j \leq d/2$. Here i and j are allowed to be zero.

(iii) If the ovals of the curve of degree d are distributed into at most five nests, then there are at most d ovals in the curve.

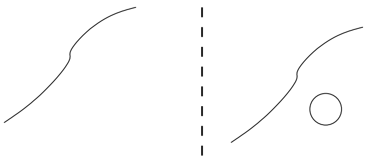


Figure 1.45: Two configurations are possible in degree 3

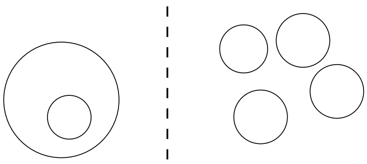


Figure 1.46: Six configurations are possible in degree 4. Only the two maximal ones are shown.

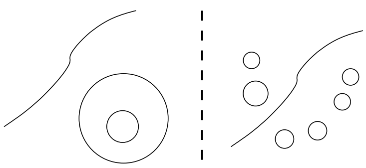


Figure 1.47: Eight configurations are possible in degree 5. Only the two maximal ones are shown.

Proof. For the first part, take a line l cutting the whole curve transversally. The cardinality of $l \cap C_f$ is congruent to d modulo 2, by Bezout's Theorem. And two pseudolines cannot appear because they would produce at least a singular point. Similarly for the second part we can find a line that passes through the center of the innermost ovals in the two nests. For the last assertion, we consider the conic that passes through points inside of the innermost ovals in the five (or less) nests. $\square$

This is a result of the type we have called "prohibitions". The first part is specially important. It says that the classification of curves of a certain degree consists just in telling how many different ovals can appear and what are the possible nesting structures. In other words, we want to know which partially order sets (posets) can arise in this way from a smooth curve of given degree. The equivalence class of curves that induce the same poset is an isotopy class.

A second prohibition is Harnack's Theorem, known well before Hilbert:

Theorem 1.3.7 (Harnack). *The number of connected components of a nonsingular algebraic curve* f *of degree* d *is at most* $(d^2 - 3d + 4)/2$*, or equivalently one plus the genus of the Riemann surface associated with* f*.*

This theorem already allows to easily give the classification up to dimension five. Observe that, by Part (ii) of Corollary 1.3.6, nesting appears only in dimensions four and higher. Moreover, in dimensions four and five only a single nest of two ovals is allowed, and if the nest appears then no more ovals can be present. This gives only the following possibilities, displayed in Figures 1.45, 1.46 and 1.47.

1. In degree one and two, we can have only one connected component and it is a pseudoline or oval depending on the degree (well, we knew this already, didn't we?) In degree one we have a line, and in dimension two we have a conic, which in the projective plane is indeed an oval.

2. In degree three, there is a pseudoline together with zero or one ovals.

3. In degree four, there is either a nest of two ovals or a number of unnested ovals ranging from 0 to 4 (only the maximal case is shown in Figure 1.46).

4. In degree five, there is a pseudoline together with either a nest of two ovals or a number of unnested ovals ranging from 0 to 6 (again, only the maximal case is shown).

As it turns out, all these non-prohibited configurations can (easily) be constructed algebraically. But in degree six things start to be more complicated. By Corollary 1.3.6, you cannot get two nests of depth at least two, and if there is a nest of depth three then there is nothing else. Hence, the possibilities are:

- A single nest with three ovals.
- A number of zero to eleven unnested ovals.
- An oval having i ovals inside (unnested to one another) and j ovals outside, with $i + j \leq 10$.

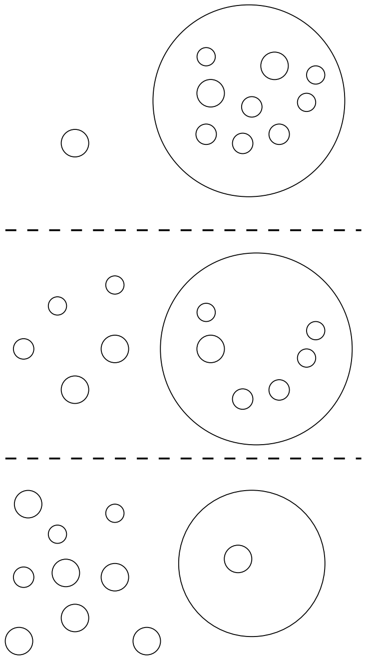


Figure 1.48: The three curves of degree six with eleven ovals.

But it was soon noticed that there are some extra restrictions. Petrovskii proved in the 1930's the following one, which in degree six prohibits the appearance of 11 unnested ovals; at least one nesting must occur.

Theorem 1.3.8 (Petrovskii). *Let* po *and* io *be the numbers of even and odd ovals (that is to say, ovals nested in an even or odd number of other ovals, respectively) of a nonsingular curve of even degree* $d = 2k$*. Then:*

$$-3/2(k^2 - k) \leq \mathrm{po} - \mathrm{io} \leq 3/2(k^2 - k) + 1.$$

Even more restrictions were found later (most notably by Rokhlin and Arnold) which excluded other cases. In the end, combining constructions and restrictions, Gudkov finished in the 1970's the classification of curves of degree six [36]: there are 56 distinct types (see Exercise 1.20). As an example, if we restrict our attention to the maximal case of 11 ovals in degree 6, there are only the three configurations of Figure 1.48. Two of these three curves were already constructed in the times of Hilbert. Gudkov's achievement was the construction of the third one.

A big boost to the "constructions" part of the problem was given by Viro around 1990 (incidentally, five years after solving himself the degree seven case). And his construction uses triangulations in an essential way! Consider the point configuration

$$A_d = \{(i, j) | i + j \leq d,\ i, j \in \mathbb{Z},\ i, j \geq 0\}.$$

We denote the convex hull of A_d by Q_d. (How many points do you have in A_d?). Take a triangulation T of A_d. At this moment we will not assume any special property for T, only later we will state what is needed of T. Consider also a sign function $\sigma : A_d \to \{+, 0, -\}$. Denote by $\mathbb{RP}^2_{++} = \{(x : y : z) | x \geq 0,\ y \geq 0,\ z \geq 0\}$. The real projective plane.

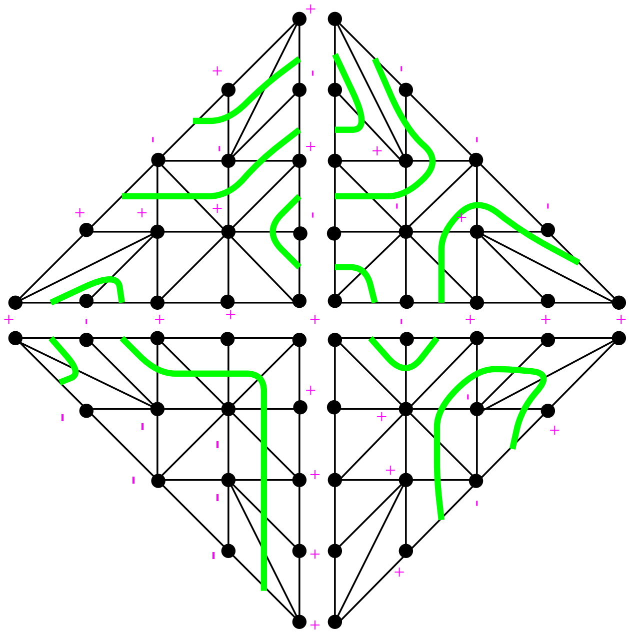


Figure 1.49: The graph $G_{T,\sigma}$ of degree 4.

Naturally we can divide the real projective plane into 4 components $\mathbb{R}P^2_{++}$, $\mathbb{R}P^2_{+-}$, $\mathbb{R}P^2_{-+}$ and $\mathbb{R}P^2_{--}$. Take 3 copies of the triangulation T, those obtained by reflecting the T over the coordinate axis. The projective plane is obtained by glueing the four triangles so that opposite sides are glued in opposite direction. We can also extend the sign function σ modified according to which orthant we are working on. In this way $\sigma(i,j) = \sigma(i,j)(-1)^i$ for $\mathbb{R}P^2_{+-}$, $\sigma(i,j) = \sigma(i,j)(-1)^j$ for $\mathbb{R}P^2_{-+}$, $\sigma(i,j) = \sigma(i,j)(-1)^{i+j}$ for $\mathbb{R}P^2_{--}$. We are going to define a graph, $G_{T,\sigma}$, whose vertices are the midpoints of those edges of T whose endpoints have distinct signs under σ. Two of these nodes will be connected if and only if they lie in a common triangle of T. The graph is embedded in the combinatorial model of the projective plane. Figure 1.49 shows an example.

Let $f_t(x,y) = \sum c_{i,j}x^iy^jt^{w(i,j)}$ where $\text{sign}(c_{i,j}) = \sigma(i,j)$. Note that for each value of t we get a curve C_{f_t} in the projective plane. We are now ready to state the main theorem of Viro:

Theorem 1.3.9. *Under the assumption that the trian-*

gulation T *is regular, for* t *large enough,* f_t *is an affine equation for a smooth plane algebraic curve and there exists a homeomorphism between* $\mathbb{R}P^2$ *and its combinatorial model which maps the curve* C_{f_t} *into the graph* $G_{T,\sigma}$.

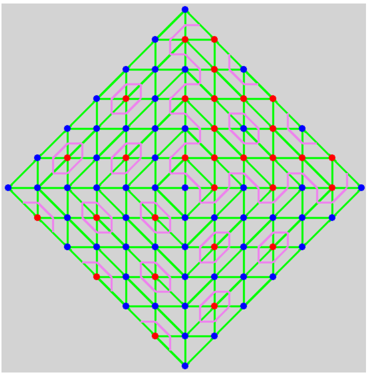


Figure 1.50: A curve of degree 6 constructed using Viro's method

What does it mean to be a regular triangulation? Roughly speaking a triangulation is regular when it is isomorphic to the lower hull of a convex polytope. More precisely, the triangulation is regular if we can lift each of the vertices a_i of T to height w_i and obtain a convex polyhedral surface combinatorially equivalent to T. Therefore a regular triangulation T is given by generic lifting vectors $w : A_d \to Z$. Our running example is in fact a good piecewise linear picture of a certain smooth curve of degree four. Perhaps the reader should play with this construction.

The construction is powerful enough to obtain all the maximal curves of degree six. Figure 1.50 shows one.

We finish with one of the most successful applications of Viro's construction. Virginia Ragsdale conjectured in her 1906-paper [70] that the following inequalities would be satisfied by all non-singular curves of degree $2k$ (the second equation is in fact a correction by O. Viro to the original statement of Ragsdale). Here, po and io are as in Theorem 1.3.8:

$$po \leq 3/2(k^2 - k) + 1 \qquad io \leq 3/2(k^2 - k) + 1$$

She obtained this conjecture from her extensive analysis of Hilbert's and Harnack's results. This condition was first proved to be false by Ilia Itenberg in 1993 (see [43]). Itenberg used Viro's construction to show that there are examples of curves with $(3k^2 - 3k + 2)/2 + h(k)$ even ovals, where $h(k)$ is a quadratic function of k. We present in figure 1.51 the case for $k = 5$.

Itenberg's construction gives $\frac{13}{32}d^2 \pm O(d)$ positive ovals. Bertrand Haas improved the result to $\frac{10}{24}d^2 \pm O(d)$ [37]. Of course, an upper bound of $\frac{7}{16}d^2 \pm O(d)$ can be derived from Rokhlin's theorem. This shows the power of Viro's construction. The construction of Viro in principle is purely combinatorial and depends on a triangulated point set. The algorithms can be carried out even for non-regular triangulations. It has to be emphasized, though, that only regular triangulations are guaranteed to produce algebraic curves. For non-regular triangulations, the construction provides a curve embedded in the projective plane but there is no evidence that these curves are always algebraic. However, up to now no counter-example is known. Several properties of algebraic curves seem to be valid too for curves arising from arbitrary triangulations [65, 38].

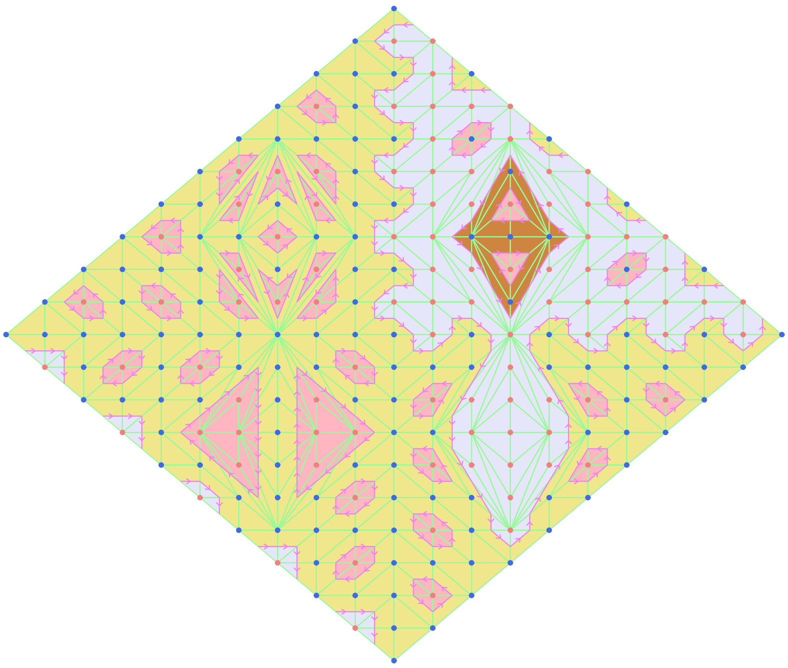


Figure 1.51: Itenberg's counterexample to Ragsdale's conjecture (degree 10).

For example, using non-regular triangulations Francisco Santos has constructed curves with $\frac{17}{40}d^2 \pm O(d)$ positive ovals, which would be the "best" counterexamples to Ragsdale's conjecture. Figure 1.52 shows a look of Santos's triangulation.

Viro's construction generalizes naturally for the construction of real smooth hypersurfaces of higher dimension. From a signed triangulation of the simplex with lattice points $(i_1, \ldots, i_s)$ with $i_1+i_2+\cdots+i_s \leq d$ one can recover a piecewise linear hypersurface of degree d in $\mathbb{RP}^s$. In particular, Viro's construction has been used to investigate smooth real algebraic surfaces embedded in $\mathbb{R}P^3$. Little is known in terms of theorems that restrict the possible topologies [49, 98]. At present the highest degree completely understood is degree four. Kharlamov was able to classify real smooth surfaces of degree four [48] (no more than ten components).

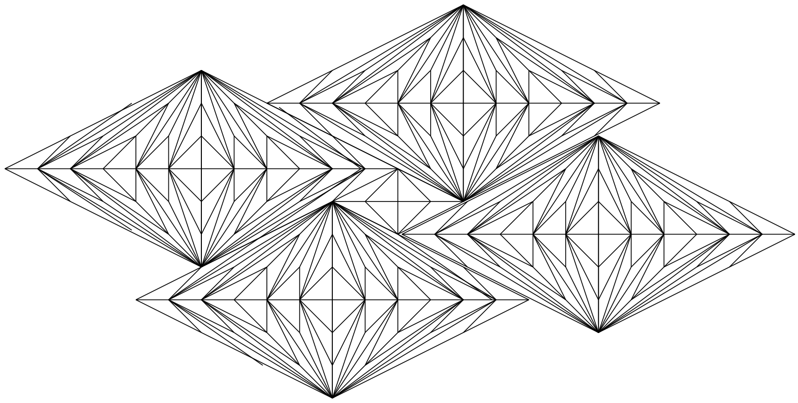


Figure 1.52: A portion of Santos's non-regular triangulation related to Ragsdale conjecture.

Hardly anything is known for the case of smooth surfaces of degree 5 and higher, not even an upper bound on the number of components is fully understood. Itenberg and Kharlamov [50] constructed a new degree five surface in RP^3 with 22 components (three away from the bound provided by Smith and Comessatti inequalities [98]). They used a hybrid technique that takes some elements from T-surfaces. Itenberg [42] and Bihan [8] have constructed counterexamples to Viro's conjecture as well. Viro's conjecture stated that for smooth compact complex hypersurface X of degree d its real part RX satisfies $\dim H_1(RX, Z/2) \leq 2/3d^3 - 2d^2 + 7/3d$. Itenberg's counterexamples to Ragsdale conjecture provided in fact counterexamples to Viro's conjecture. Here we present a picture of Bihan's counterexample (in degree 8) that is not constructed that way.

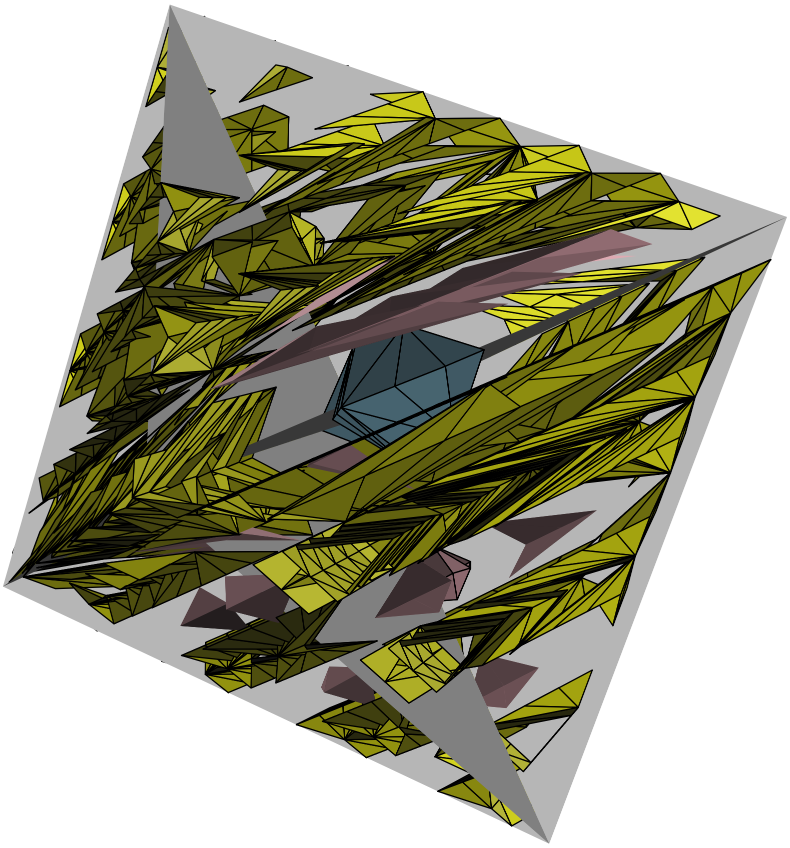


Figure 1.53: A view of Bihan's counterexample to Viro's conjecture (degree 8).

Some more exciting results with Viro's combinatorial construction include the construction of singular curves with prescribed collection of singularities [79], and a remarkable new connection to dynamical systems with the construction of planar polynomial vector fields with large number of limit cycles [45].

1.4 The Rest of this Book

Now the reader has a minimal familiarity with the objects to be studied and why triangulations are relevant in mathematics. For a given point set A there are many questions that one can ask about their triangulations. Here is a sample of general issues that will be of interest for us in the rest of the book:

1. *Count* the *number* of different triangulations.

2. *Decide* whether there is a triangulation with *property* X.
3. *Find* an *"optimal"* such triangulation.
4. *Study* the algebraic or topological *structure* of the set of all triangulations, or special subsets.

Exercises

Exercise 1.1. Identify how the five binary trees in 3 nodes (Figure 1.10) biject to the five triangulations of a pentagon (Figure 1.7, but you better turn the figure upside-down and check that rotations correspond to diagonal flips.

Exercise 1.2. Prove that any point set with four elements in the plane, not all in a line, has exactly two triangulations.

Exercise 1.3. Take a regular tetrahedron, can you triangulate it, with the help of extra interior points, in such a way that only regular tetrahedra appear inside? (Hint: find the dihedral angle between adjacent facets of a tetrahedron).

Exercise 1.4. Prove that the graph of flips of a 6-gon is Hamiltonian. (This holds in general for any n-gon, see [66].)

Exercise 1.5. (Catalan numbers via generating functions.) Find formula (1.2) for the Catalan number C_n from the recurrence relation (1.1) of Definition 1.1.4. In other words, prove Theorem 1.1.2 from Proposition 1.1.1.

For this, call $F(x)$ the *generating function* of the sequence C_i, that is to say the series

$$F(x) = \sum_{i=0}^{\infty} C_i x^i.$$

1. Prove that $F^2(x) = \sum_{i=0}^{\infty} C_{i+1} x^i = \frac{F(x)-1}{x}$.
2. Deduce that

$$F(x) = \frac{1 + \sqrt{1-4x}}{2x}.$$

Hint: solve for F in $xF^2 = F - 1$, and discard the solution which diverges when $x \to 0^+$; our function must have $\lim\limits_{x \to 0^+} F(x) = 1$.

3. Recall Newton's binomial theorem:

$$(1+z)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} z^k$$

where the fractional binomial is defined as:

$$\binom{1/2}{k} = \frac{[1/2(1/2-1)(1/2-2)\ldots(1/2-k+1)]}{k!}.$$

Apply it to $z = -4x$ and express the fractional binomial in terms of $\binom{2k}{k}$.

Exercise 1.6. For each of the four "Catalan structures" of Theorem 1.1.3 (other than triangulations) show that the recurrence formula (1.1) holds.

Exercise 1.7. (From triangulations to sign sequences) Show that the way we have constructed sign sequences in Theorem 1.1.3 is equivalent to the following one: Given a triangulation of the $n+2$-gon, the sequence consists $n+1$ blocks, one for each vertex of the polygon other than the first one. The $i-1$-th block ($i = 2, \ldots, n+2$) has length equal to the number of edges ji, with $j < i$ except we do not count the reference edge $\{1, n+2\}$ in the last block. Each block consists of zero or more minuses ending in a single plus, except the last block where we only put minuses. Figure 1.54 shows the construction and it also hints an alternative description of the same construction, where a sign is assigned to every edge other than the reference edge.

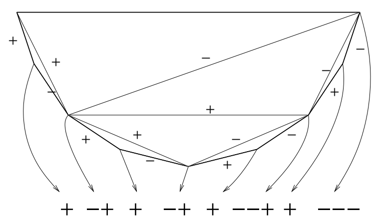


Figure 1.54: The sign sequence associated to a triangulation. Compare with Figure 1.12

Exercise 1.8. Let D_n denote the diameter of the graph of flips of the convex n-gon. Show that

$$D_n + 1 \leq D_{n+1} \leq D_n + 3,$$

for every $n \leq 3$. (This is Part (ii) of Proposition 1.1.5. See the hints given there).

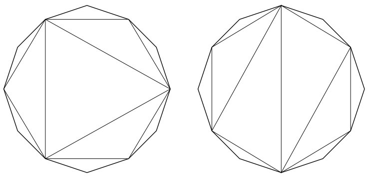


Figure 1.55: These two triangulations are at distance at least 13 in the graph of flips of the 12-gon

Exercise 1.9. Prove that the diameter of the graph of flips in a convex n-gon is at least $\lfloor \frac{3n}{2} - 5 \rfloor$ for every n. More precisely, prove that for every even n, at least that number of flips is needed to go from a triangulation with no internal edges incident to even vertices to a triangulation with no internal edges incident to odd vertices, such as the ones in the figure.

Exercise 1.10. Suppose you are given a planar polygon, but the subroutine that computes triangulations of polygons is broken (your little brother spilled coffee on the CPU!). How can you still calculate the area of the polygon without using a triangulation? (Hint: essentially you only know the boundary of the polygon right?)

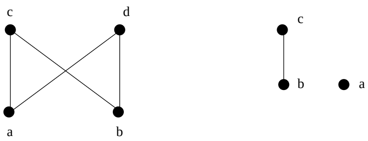


Figure 1.56: Describe the order polytopes associated to these two posets

Exercise 1.11. (Open Problem) Suppose the input polytopes are given in terms of its vertices. How hard is it to compute the size of the largest triangulation in that case? (Hint: The construction in [25] will not help you).

Exercise 1.12. Describe as completely as you can (dimension, vertices, facets, address, age, etc.) the order polytopes of the posets in the margin.

Exercise 1.13. Provide a proof of Sperner's lemma for arbitrary dimension. (Hint: Start with dimension two, then apply induction.)

Exercise 1.14. Prove that Brouwer's theorem is true for all homeomorphic balls if it is true for the simplex.

Exercise 1.15. Prove Sperner's lemma using Brouwer's theorem.

Exercise 1.16. Take a "deformed" combinatorial 3-cube, say the cartesian product of a trapezoid with a segment, and find all possible triangulations. How many are there?

Exercise 1.17. Consider the family of parametric linear programming problems $LP_{A,c}(b)$. Where A is given by the 3×6 matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

Describe the secondary polytope of variations of the cost vector c. Make a picture!

Exercise 1.18. Consider the parametric system of equations

$$a_0xy^3 + x^3 + a_1x^3 + a_2 = 0, \qquad (1.10)$$
$$b_0x^2y^2 + b_1x^2 + b_2y^2 + b_3 = 0. \qquad (1.11)$$

Determine bounds for the number of complex roots of the system using mixed subdivisions. Verify your answer using Gröbner bases.

Exercise 1.19. Use Viro's method to construct the three isotopy types of maximal curves of degree six shown in Figure 1.48. How many distinct triangulations did you need?

Exercise 1.20. Check that there are exactly 53 configurations of ovals that you can get from the three in Figure 1.48 by removing some of the ovals. (These 53, together with a single nest of three ovals and the two configurations in Figure 1.57 form the 56 possible configurations of real algebraic curves of degree six).

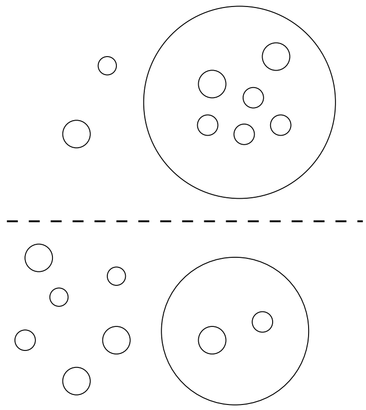


Figure 1.57: Two configurations which are possible in degree six

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