The Positive Grassmannian (from a mathematician's perspective)

Lauren K. Williams, UC Berkeley

Plan of the talk

The totally non-negative Grassmannian (also called *positive Grassmannian*) is a subset of the real Grassmannian with remarkable properties.

I will start by explaining some of the reasons why mathematicians have been interested in it.

I'll then describe how it arose naturally in a physical context – shallow water waves (via the KP hierarchy). Is this setting related to scattering amplitudes?

- Background on the positive Grassmannian
- Why do mathematician's care?
- Interactions of shallow water waves
- Using the positive Grassmannian and the KP equation to study shallow water waves
- What shallow water waves taught us (regularity ⇔ positivity; tropical curves; criterion for reduceness; nonplanar plabic graphs)

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Total positivity on the Grassmannian

The real Grassmannian and its positive and non-negative parts

The Grassmannian $Gr_{k,n}(\mathbb{R}) = \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the *Plücker coordinate* $\Delta_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The *totally positive part* of the Grassmannian $(Gr_{k,n})_{>0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where all Plucker coordinates $\Delta_I(A) > 0$.

Similarly define the TNN Grassmannian $(Gr_{k,n})_{\geq 0}$ using $\Delta_I(A) \geq 0$.

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Background on total positivity

1930's: Classical theory of *totally positive matrices*. A square matrix is *totally positive* (TP) if every minor is positive i.e. the determinant of every square sub-matrix is positive. Similarly define the *totally non-negative* (TNN) matrices.

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 4 \\ 2 & 3 & 10 \end{pmatrix}$$

1990's: Lusztig developed total positivity in Lie theory. Defined the TP and TNN parts of a reductive group, so that TP part of GL_n is totally positive matrices. Also defined TP and TNN parts of any flag variety (includes $Gr_{k,n}$).

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Background on total positivity (cont.)

1995-2000: Fomin and Zelevinsky studied Lusztig's theory.

- Sample question: "How many and which minors must we test, to determine whether a given matrix is totally positive?"
- Answer uses combinatorics of *double wiring diagrams* for longest permutation in the symmetric group.
- To answer the same question replacing "positive" with "non-negative," need to partition the space of TNN matrices into cells and answer the question separately for each cell (each cell is equi-dimensional; the biggest cell is the set of TP matrices). Cells labeled by pairs of permutations.
- This and related questions led them to discover *cluster algebras*.

Background on total positivity (cont.)

1997-2003: Rietsch and March-Rietsch studied TP parts of flag varieties.

2001-2006: Postnikov studied $(Gr_{k,n})_{\geq 0}$.

- His theory is in many ways parallel to study of totally positive matrices.
- He gave a decomposition into cells, indexed by *decorated permutations* (among other things).
- *Plabic graphs* are the analogue of double wiring diagrams, and allow one to answer the question "How many minors, and which ones, must we test to determine whether an element of the Grassmannian is totally positive?"

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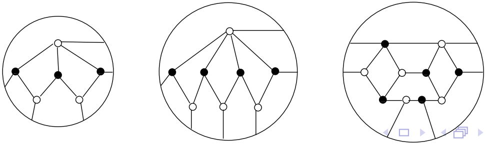

Postnikov's decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells

Recall: Elements of $(Gr_{k,n})_{\geq 0}$ are represented by full-rank $k \times n$ matrices A, with all $k \times k$ minors $\Delta_I(A)$ being non-negative.

Let $\mathcal{M} \subset \binom{[n]}{k}$. (Think of this as a collection of Plücker coordinates.)
Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid \Delta_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations on $[n]$ with k weak excedances.
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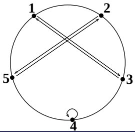

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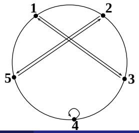

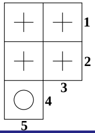

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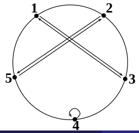

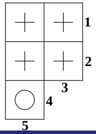

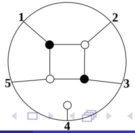

How many cells does the TNN Grassmannian have?

Let $A_{k,n}(q)$ be the polynomial in q whose q^r coefficient is the number of positroid cells in $Gr^+_{k,n}$ which have dimension r.

Theorem (W.): Let $[i] := 1 + q + q^2 + \cdots + q^{i-1}$. Then

$$A_{k,n}(q) = \sum_{i=0}^{k-1} \binom{n}{i} q^{-(k-i)^2} ([i-k]^i [k-i+1]^{n-i} - [i-k+1]^i [k-i]^{n-i}).$$

Theorem (W.): Define $E_{k,n}(q) := q^{k-n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_{k,n-i}(q)$. Then:

- $E_{k,n}(0)$ is the Narayana number $N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$
- $E_{k,n}(1)$ is the Eulerian number $E_{k,n} = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n$.

Remark: Narayana and Eulerian numbers appear in the BCFW recurrence and twistor string theory (Eulerian connection: Spradlin-Volovich).

What does the TNN Grassmannian look like?

The face poset of a cell complex

The face poset $F(K)$ of a cell complex K is the partially ordered set which specifies when one cell is contained in the closure of another.

(Postnikov) Explicit description of face poset of $(Gr_{k,n})_{\geq 0}$.

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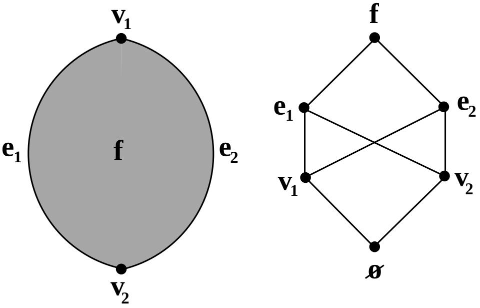

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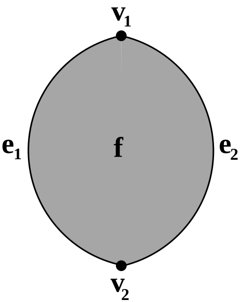

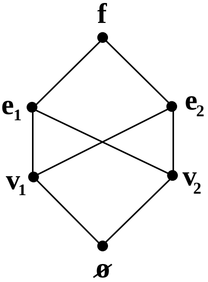

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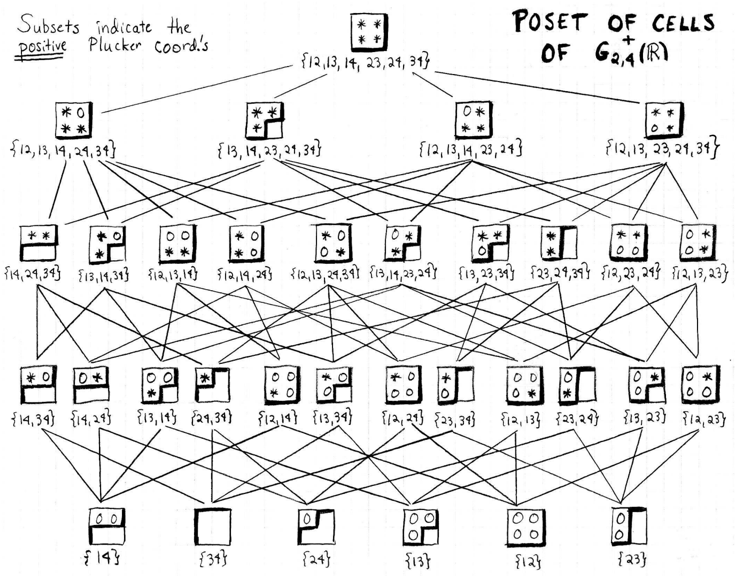

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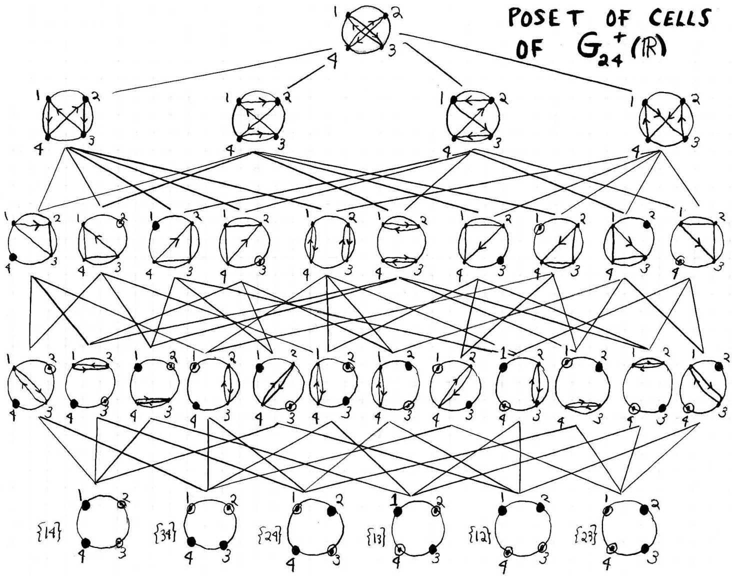

What does the positive Grassmannian look like?

Conjecture (Postnikov): The $(Gr_{k,n})_{\geq 0}$ is homeomorphic to a ball, and its cell decomposition is a regular CW complex – i.e. the closure of every cell is homeomorphic to a closed ball with boundary a sphere.

Theorem (W.): The face poset of $(Gr_{k,n})_{\geq 0}$ is the face poset of some regular CW decomposition of a ball. In particular, it is an Eulerian poset.

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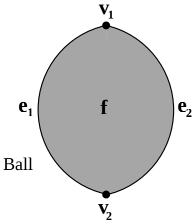

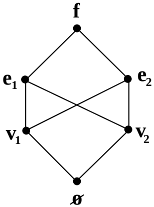

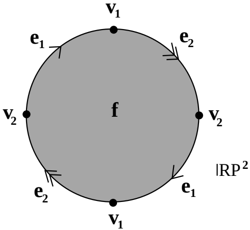

What does the positive Grassmannian look like?

Theorem (Rietsch-W.)

Postnikov's conjecture is true up to homotopy-equivalence: the closure of every cell is contractible, with boundary homotopy-equivalent to a sphere. In particular, $(Gr_{k,n})_{\geq 0}$ is contractible, with boundary homotopy-equivalent to a sphere.

Remark

All these results hold in much greater generality. Rietsch gave a cell decomposition of $(G/P)_{\geq 0}$ (1997) which coincides with Postnikov's in the case of the Grassmannian, and described its face poset.

Moreover, we showed that $(G/P)_{\geq 0}$ is contractible, with boundary homotopy-equivalent to a sphere, and the same is true for the closure of each cell.

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The interaction of shallow water waves

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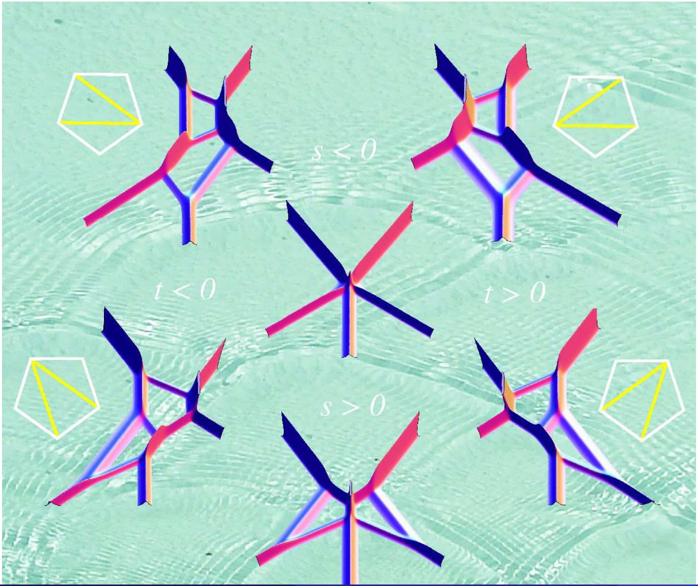
Many possible combinatorial configurations can arise!

How can we describe them?



The positive Grassmannian and shallow water waves

The key to answering the question lies in the study of the positive Grassmannian and the KP equation.

The KP equation

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\frac{\partial^2 u}{\partial y^2}=0$$

- Proposed by Kadomtsev and Petviashvili in 1970 (in relation to KdV)
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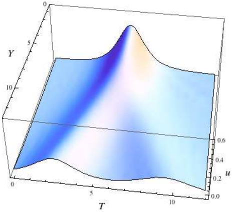
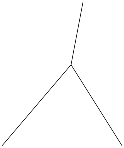
- Solutions provide a model for shallow water waves





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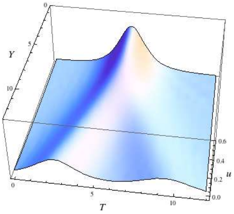

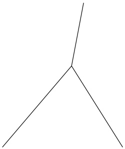

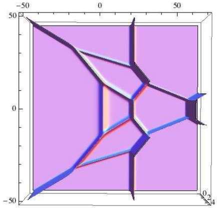

Soliton solutions to the KP equation

Recall: the Grassmannian $Gr_{k,n}(\mathbb{R}) = \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$.
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.
Given $I \in \binom{[n]}{k}$, $\Delta_I(A)$ is the minor of the I-submatrix of A.

From $A \in Gr_{k,n}(\mathbb{R})$, can construct τ_A, and then a solution u_A of the KP equation.
(cf Sato, Hirota, Satsuma, Freeman-Nimmo, ...)

The τ function τ_A

Fix real boundary data κ_j such that $\kappa_1 < \kappa_2 < \cdots < \kappa_n$.
(κ_j's control slopes of waves coming in from the disk)
Define $E_j(t_1, \ldots, t_n) := \exp(\kappa_j t_1 + \kappa_j^2 t_2 + \cdots + \kappa_j^n t_n)$.
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Set $x = t_1, y = t_2, t = t_3$ (treat other t_i's as constants). Then

$$u_A(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \tau_A(x, y, t) \quad \text{is a solution to KP.} \qquad (1)$$

Note: If all $\Delta_I(A) \geq 0$, this solution is everywhere regular. Therefore we will initially restrict attention to those $A \in (Gr_{k,n})_{\geq 0}$.

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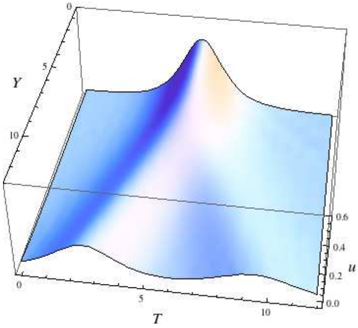
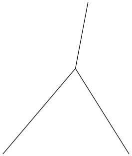
Visualing soliton solutions to the KP equation

The contour plot of $u_A(x, y, t)$

We analyze $u_A(x, y, t)$ by fixing t, and drawing its *contour plot* $\mathcal{C}_t(u_A)$ for fixed times t – this will approximate the subset of the xy plane where $u_A(x, y, t)$ takes on its maximum values.





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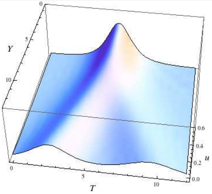
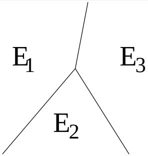
When the κ_i's are integers, $\mathcal{C}_t(u_A)$ is a *tropical curve.*





Labeling regions of the contour plot by dominant exponentials

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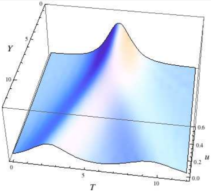

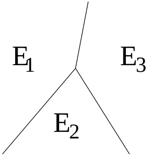

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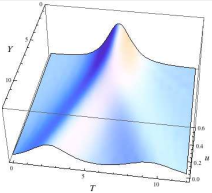

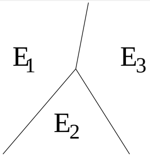

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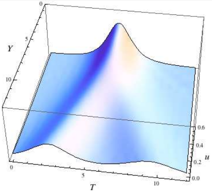

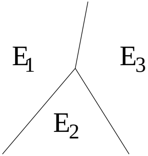

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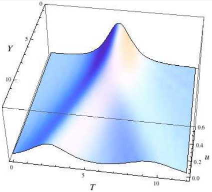

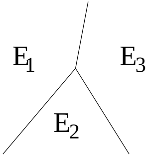

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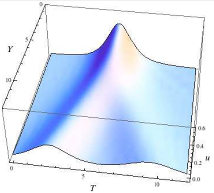

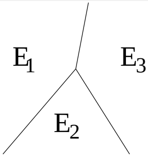

Labeling regions of the contour plot by dominant exponentials

One term E_I dominates u_A in each region of the complement of $\mathcal{C}_t(u_A)$.

Label each region by the dominant exponential.

Definition of the contour plot at fixed time t

$u_A(x,y,t)$ is defined in terms of $\tau_A(x,y,t) := \sum_{I \in \binom{[n]}{k}} \Delta_I(A) E_I(x,y,t)$.

At most points (x,y,t), $\tau_A(x,y,t)$ will be dominated by one term –
– at such points, $u_A(x,y,t) \sim 0$.

Define the *contour plot* $\mathcal{C}_t(u_A)$ to be the subset of the xy plane where two or more terms dominate $\tau_A(x,y,t)$.

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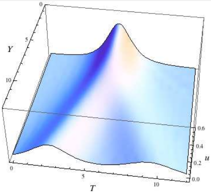

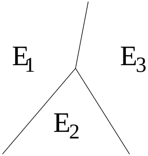

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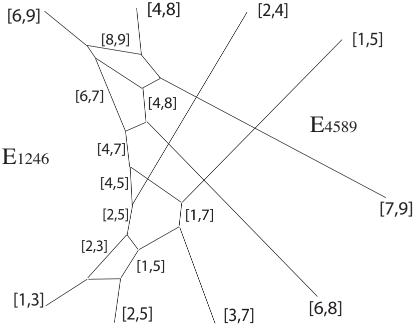
Visualizing soliton solutions to the KP equation

Generically, interactions of *line-solitons* are trivalent or are *X-crossings* (think of this as a crossing of two edges in a *non-planar* graph).



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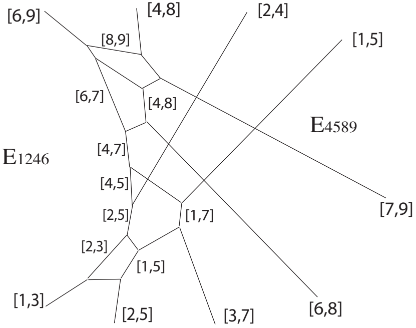

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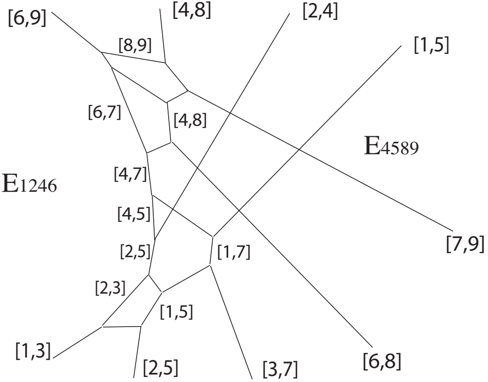
Soliton graphs

We associate a *soliton graph* $G_t(u_A)$ to a contour plot $\mathcal{C}_t(u_A)$ by: forgetting lengths and slopes of edges, and marking a trivalent vertex black or white based on whether it has a unique edge down or up.



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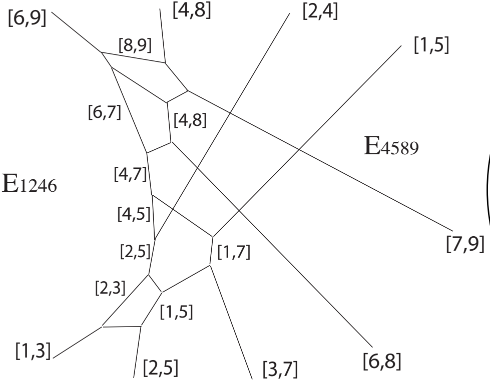

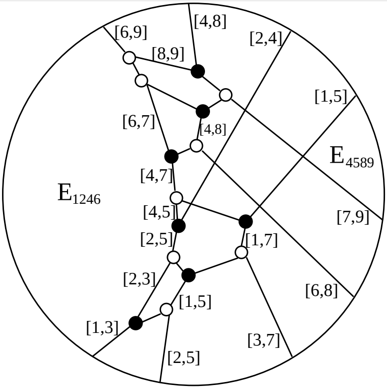

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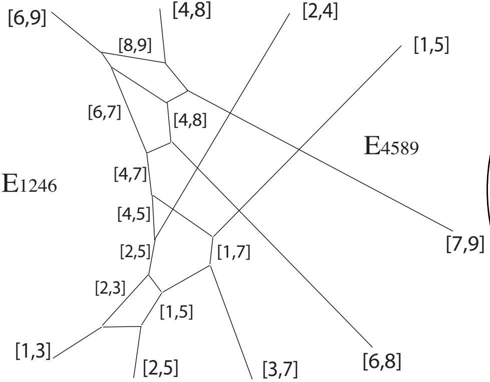

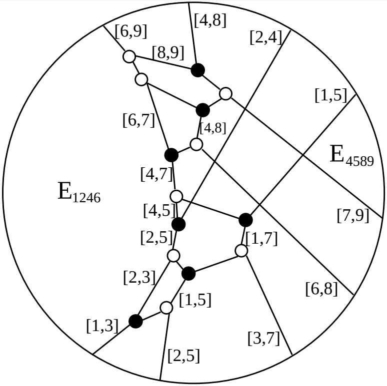

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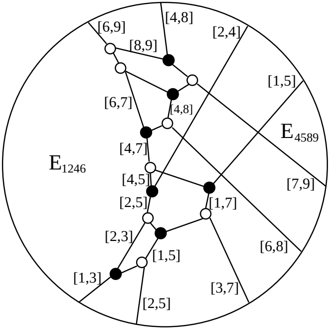
Soliton graph → generalized plabic graph



Associate a *generalized plabic graph* to each soliton graph by:

- For each unbounded line-soliton $[i, j]$ (with $i < j$) heading to $y >> 0$, label the incident bdry vertex by j.
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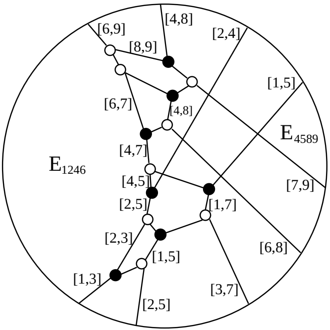

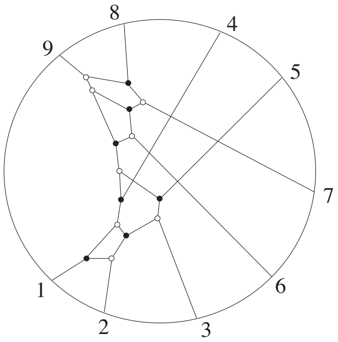

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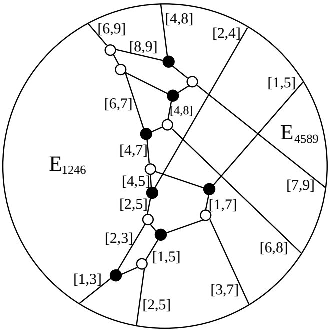

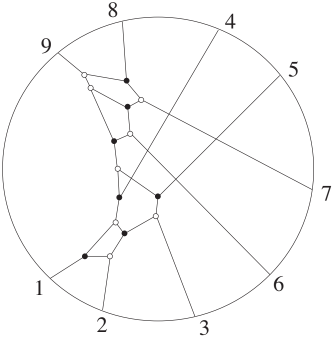

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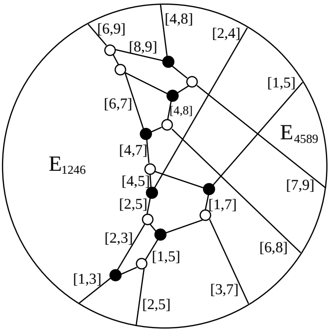

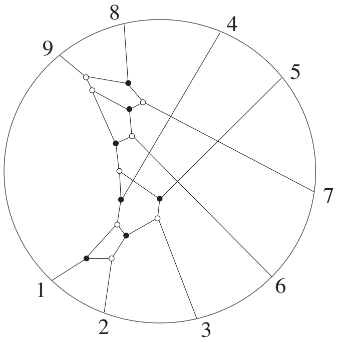

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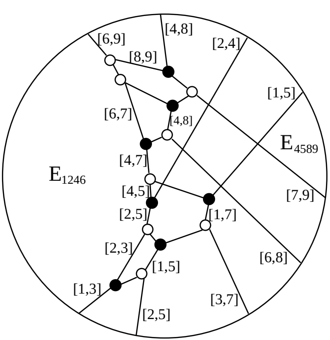

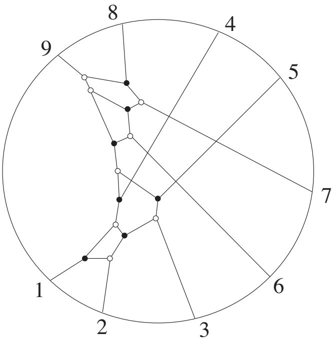

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Theorem (Kodama-W). Passing from the soliton graph to the generalized plabic graph does not lose any information!

We can reconstruct the labels by following the "rules of the road" (zig-zag paths). From the bdry vertex i, turn right at black and left at white.

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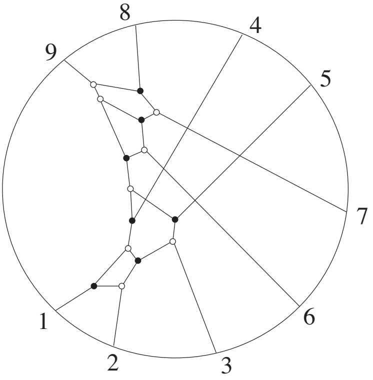

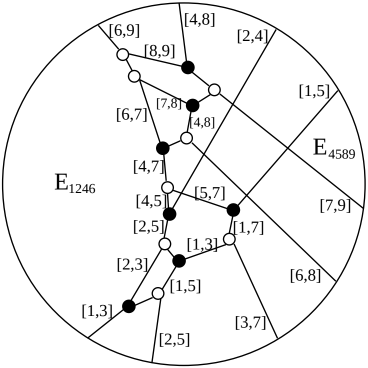

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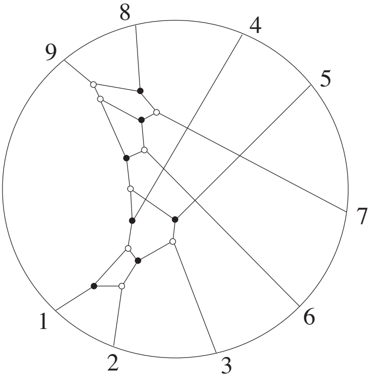

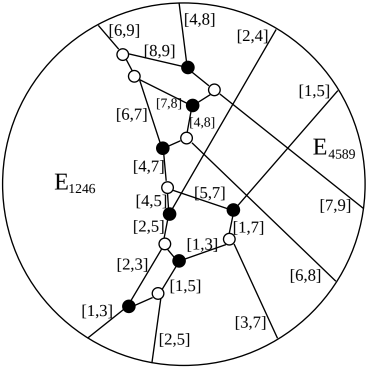

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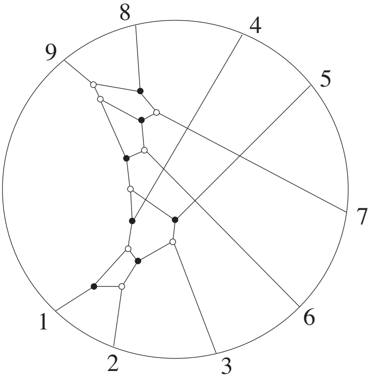

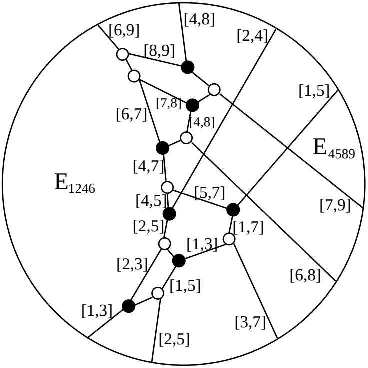

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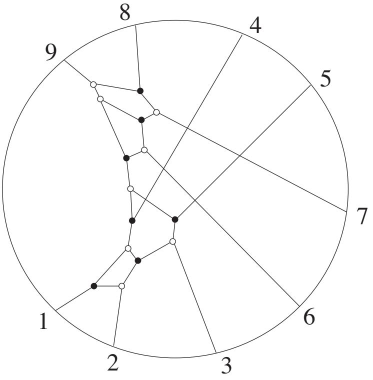

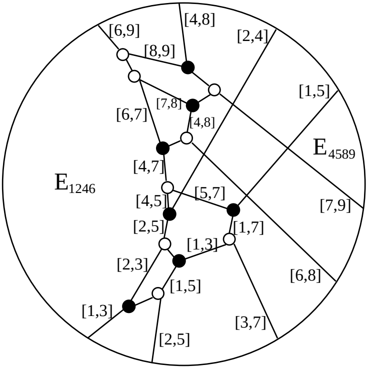

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Classification of soliton graphs for $(Gr_{2,n})_{>0}$

Theorem (K.-W.)

Up to graph-isomorphism,[a] the generic soliton graphs for $(Gr_{2,n})_{>0}$ are in bijection with triangulations of an n-gon. Therefore the number of different soliton graphs is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

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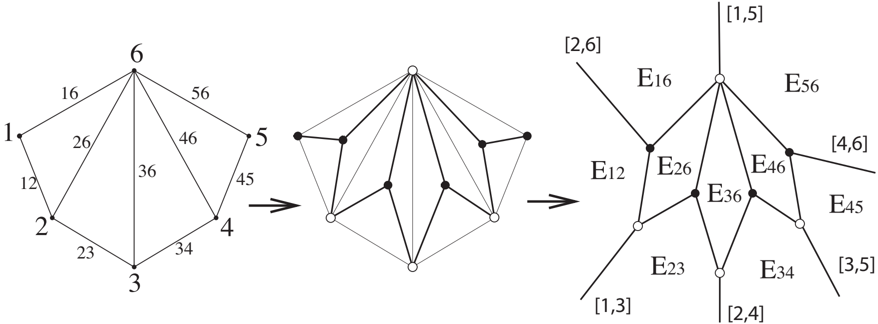

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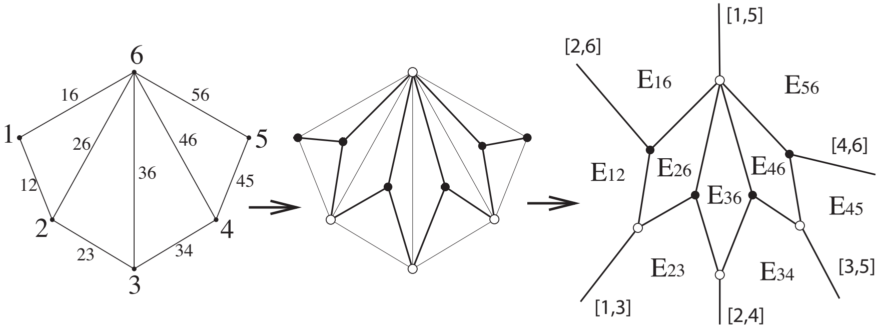
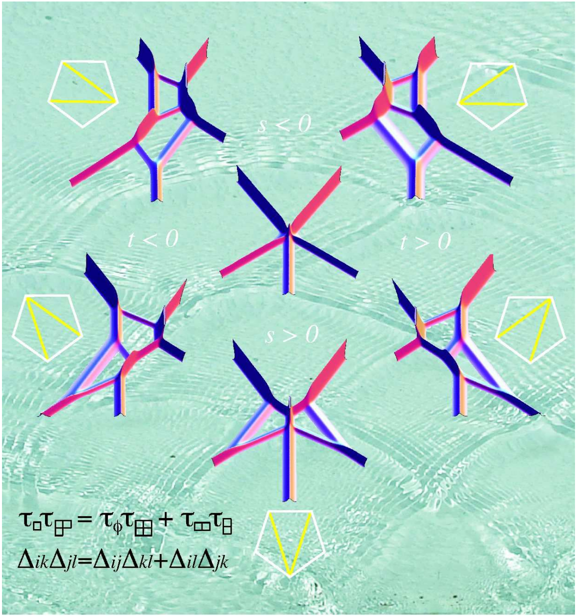


The soliton graphs for $(Gr_{2,5})_{>0}$



What about soliton graphs for $(Gr_{k,n})_{\geq 0}$, for $k > 2$?

The positroid cell decomposition

Recall that the positroid cell decomposition partitions elements of $(Gr_{k,n})_{\geq 0}$ into cells $S_{\mathcal{M}}^{tnn}$ based on which $\Delta_I(A) > 0$ and which $\Delta_I(A) = 0$.

Recall that positroid cells of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- decorated permutations π of $[n]$ with k weak excedances
- Ⅎ-diagrams L contained in a $k \times (n-k)$ rectangle

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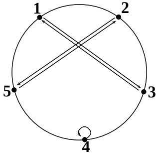

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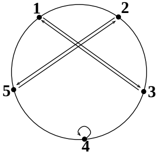

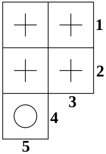

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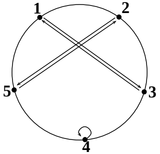
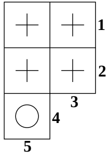
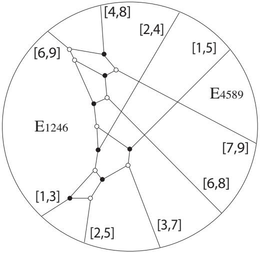




Total positivity on the Grassmannian and KP solitons



Let A be an element of a positroid cell in $(Gr_{kn})_{\geq 0}$. What can we say about the soliton graph $G_t(u_A)$?

Metatheorem

Which cell A lies in determines the asymptotics of $G_t(u_A)$ as $y \to \pm\infty$ and $t \to \pm\infty$. Use the decorated permutation and ⅃-diagram labeling the cell.

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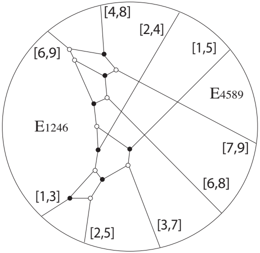

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How the positroid cell determines asymptotics at $y \to \pm\infty$

Recall: positroid cells in $(Gr_{kn})_{\geq 0} \leftrightarrow$ decorated permutations $\pi \in S_n$ with k weak excedances.

Definition

A decorated permutation π on $[n] = \{1, 2, \ldots, n\}$ is a permutation on $[n]$ in which a fixed point may have one of two colors, red or blue.

An *excedance* of π is a position i such that $\pi(i) > i$.

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Let A lie in the positroid cell $\mathcal{S}_\pi^{tnn}$ *of* $(Gr_{kn})_{\geq 0}$. *For any* t:

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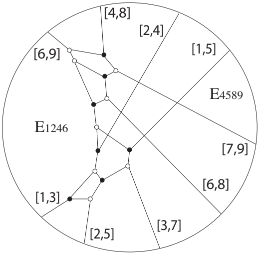

$G_t(u_A)$ where $A \in \mathcal{S}_\pi^{tnn}$ for $\pi = (5, 4, 1, 8, 2, 9, 3, 6, 7)$.

How the positroid cell determines asymptotics at $t \to -\infty$

Recall: positroid cells in $(Gr_{k,n})_{\geq 0} \leftrightarrow$ ⅃-diagrams contained in $k \times (n-k)$ rectangle

Definition

A ⅃-diagram is a filling of the boxes of a Young diagram by $+$'s and 0's such that: there is no 0 with a $+$ above it in the same column, and a $+$ to its left in the same row.

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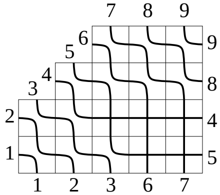

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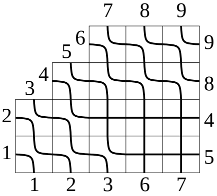

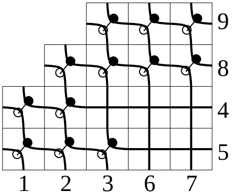

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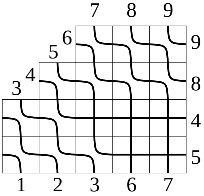

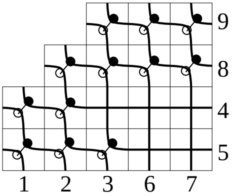

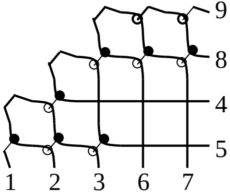

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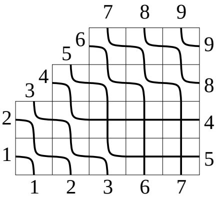

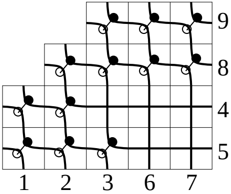

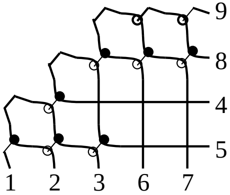

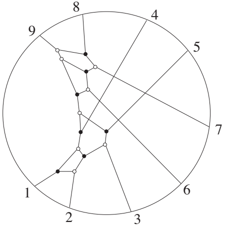

Soliton graphs and cluster algebras

Cluster algebras (Fomin and Zelevinsky)

Cluster algebras are an important class of commutative algebras; they come with distinguished generating sets called *clusters*.

Theorem (K.-W.)

Let $A \in (Gr_{k,n})_{>0}$. If $G_t(u_A)$ is *generic* (no vertices of degree > 3), then the set of dominant exponentials labeling $G_t(u_A)$ is a *cluster* for the cluster algebra associated to the Grassmannian.

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Application: solving the inverse problem for soliton graphs

Inverse problem

Given a time t together with the contour plot of a soliton solution of KP, can one reconstruct the point of $(Gr_{k,n})_{\geq 0}$ which gave rise to the solution?

Theorem (K.-W.)

1. For $t << 0$, we can solve the inverse problem, no matter what cell of $(Gr_{k,n})_{\geq 0}$ the element A came from.
2. If the contour plot is generic and came from a point of $(Gr_{k,n})_{>0}$, we can solve the inverse problem, regardless of time t.

Proof of 1: uses our description of soliton graphs at $t << 0$, and work of Kelli Talaska.

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Extending results from $(Gr_{k,n})_{\geq 0}$ to $Gr_{k,n}$.

Almost all our results can be extended to $Gr_{k,n}$, using the *Deodhar decomposition* of $Gr_{k,n}$ instead of the positroid decomposition.

Recall: If $A \in (Gr_{k,n})_{\geq 0}$, the solution $u_A(x,y,t)$ to the KP equation is regular for all times t. IS THE CONVERSE TRUE?

Theorem – the regularity problem

Choose $A \in Gr_{k,n}(\mathbb{R})$. The solution $u_A(x,y,t)$ is regular for all times t if and only if $A \in (Gr_{k,n})_{\geq 0}$.

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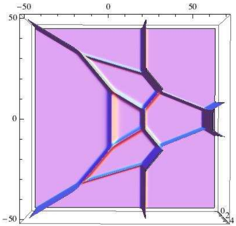

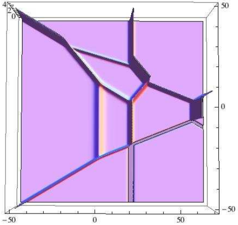

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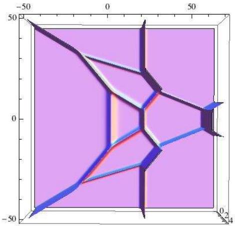

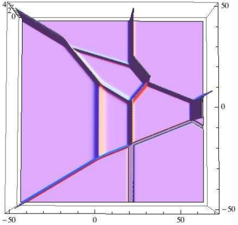

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- The subset $(Gr_{k,n})_{\geq 0}$ of $Gr_{k,n}$ has a natural physical interpretation: it picks out the set of regular solutions to the KP equation (among all those coming from the real Grassmannian $Gr_{k,n}$).
- Reduced plabic graphs can be realized as *tropical curves*. This leads to a simple and local characterization of reduced plabic graphs (K.-W.):
- Nonplanar plabic graphs arise naturally in the study of solutions of the KP equation. These also satisfy the characterization above.
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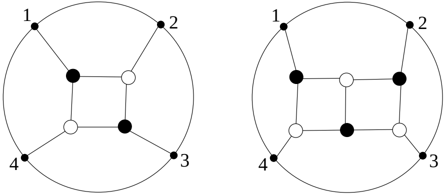

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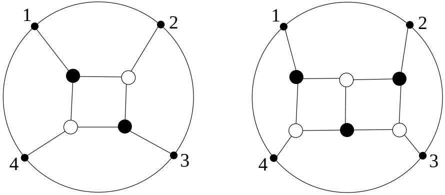

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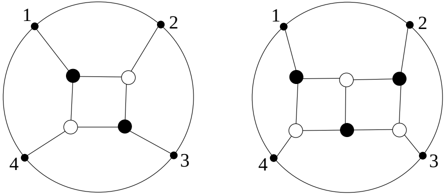

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- Free probability. *We interpret the number of positroid (respectively, connected positroids) as the moments and cumulants of a random variable. (Ardila-Rincon-W.)*.

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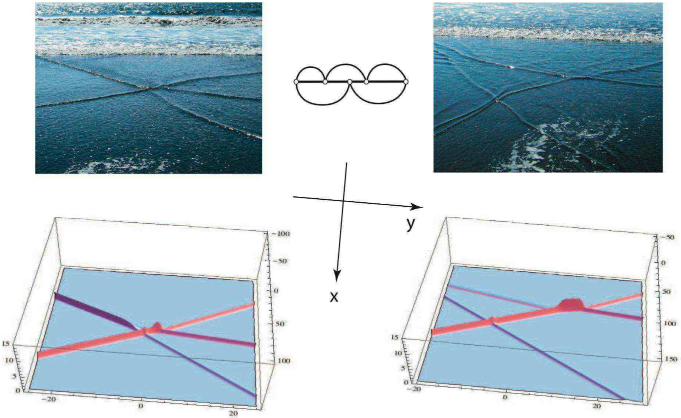
Thanks for listening! (movies?)



Why look at asymptotics as $y \to \pm\infty$ and not $x \to \pm\infty$?

The equation for a line-soliton separating dominant exponentials E_I and E_J is where $I = \{i, m_2, \ldots, m_k\}$ and $J = \{j, m_2, \ldots, m_k\}$ is

$$x + (\kappa_i + \kappa_j)y + (\kappa_i^2 + \kappa_i\kappa_j + \kappa_j^2)t = constant.$$

So we may have line-solitons parallel to the y-axis, but never to the x-axis. (κ_i's are fixed)

References

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- KP solitons and total positivity on the Grassmannian (K. + W.), to appear in Inventiones.
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