An Application of the Erdös/Stone Theorem

Tom Trotter

September 13, 2001

Shift Pairs and Shift Graphs

Remark. *When listing the elements of a finite set of integers, we will always list them in increasing order.*

Definition. *An ordered pair (A, B) of k-element subsets of $\{1, 2, \ldots, n\}$ is a (k, n)- shift pair when there is a subset $\{i_1, i_2, \ldots, i_{k+1}\} \subseteq S$ so that $A = \{i_1, i_2, \ldots, i_k\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}\}$.*

Definition. *When $1 \leq k < n$, the (k, n)-shift graph $\mathbf{S}(k, n)$ is the graph whose vertex set is the set of all k-element subsets of $\{1, 2, \ldots, n\}$ with a k-element set A adjacent to a k–element set B in $\mathbf{S}(k, n)$ exactly when (A, B) is a (k, n)–shift pair.*

Historically, the graphs $\mathbf{S}(2, n)$ have been called *shift graphs*, and $\mathbf{S}(3, n)$ *double shift graphs*.

The Chromatic Number of Shift Graphs

Remark. *$\mathbf{S}(1,n)$ is a complete graph on n vertices, so $\chi\big(\mathbf{S}(1,n)\big) = n$.*

The next theorem is part of the folklore of the subject.

Theorem. *For all $n \geq 2$,*

$$\chi\big(\mathbf{S}(2,n)\big) = \lceil \lg n \rceil.$$

Question. *How hard would it be to compute the chromatic number of the double shift graph $\mathbf{S}(3,7000)$?*

Dedekind's Problem

Definition. *For an integer* t*, let* $A(t)$ *count the number of antichains in the lattice of all subsets of* $\{1, 2, \ldots, t\}$.

Note: In the preceding definition, we count the empty antichain.

Remark. *There is a natural correspondence between antichains and down sets.*

Dedekind Numbers

$$A(1) = 3$$
$$A(2) = 6$$
$$A(3) = 20$$
$$A(4) = 168$$
$$A(5) = 7781$$
$$A(6) = 7828354$$
$$A(7) = 2414682040998$$
$$A(8) = 56130437228687557907788$$

Remark. *Perhaps, the calculation of $A(10)$ is beyond reach.*

Shift Graphs and Dedekind Numbers

Theorem. [Trotter, 1984] *For every integer $n \geq 3$, the chromatic number of the double shift graph $\mathbf{S}(3,n)$ is the least t for which $A(t) \geq n$.*

Remark. *The chromatic number of the double shift graph $\mathbf{S}(3,7000)$ is 5.*

Dimension of Graphs

Definition. *Let $G = (V, E)$ be a graph. A family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear orders on V is a realizer of G if*

$(*)$ *For every edge S and every $x \in V - S$, there exists L_i so that x is larger than all elements of S in L_i.*

Definition. *The dimension of G, denoted $\dim(G)$, is the least t for which G has a realizer $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of size t.*

The Dimension of Complete Graphs

Question. *How hard would it be to compute the dimension of the complete graph* $\mathbf{K}_{1000}$*?*

HM Down Sets

Definition. *A down set D in the lattice of subsets of $\{1, 2, \ldots, t\}$ is HM if $S \cup T \neq \{1, 2 \ldots, t\}$ for all $S, T \in D$.*

Definition. *For an integer t, let $\mathrm{HM}(t)$ count the number of HM down sets in the lattice of all subsets of $\{1, 2, \ldots, t\}$.*

Remark. *The HM numbers have several other interpretations. For example, they are also:*

1. *The number of maximal intersecting families of subsets of $\{1, 2, \ldots, n\}$.*

2. *The number of self-dual monotone boolean functions.*

HM Numbers

$$\mathrm{HM}(1) = 2$$
$$\mathrm{HM}(2) = 4$$
$$\mathrm{HM}(3) = 12$$
$$\mathrm{HM}(4) = 81$$
$$\mathrm{HM}(5) = 2646$$
$$\mathrm{HM}(6) = 1422564$$
$$\mathrm{HM}(7) = 229809982112$$

Remark. *Probably* $\mathrm{HM}(8)$ *and* $\mathrm{HM}(9)$ *can also be found, but perhaps not* $\mathrm{HM}(10)$.

HM Numbers and Dimension

Theorem. [Hoşten and Morris, 1998] *For each integer $n \geq 2$, the dimension of the complete graph K_n is the least t for which $\mathrm{HM}(t-1) \geq n$.*

Remark. *As a consequence,*

$$\dim(K_{1000}) = 6.$$

Remark. *In fact,*

$$\dim(K_{2646}) = 6 \quad \textit{and} \quad \dim(K_{2647}) = 7.$$

An Extremal Problem for Graphs

Remark.

1. $\dim(G) \leq 2$ *if and only if* G *is a disjoint union of caterpillars.*

2. $\dim(G) \leq 3$ *if and only if* G *is planar.*

Problem. [Agnarsson, 1997] *Find the maximum number* $M(n,k)$ *of edges in a graph* G *on* n *nodes with* $\dim(G) \leq k$*.*

Proposition.

1. $M(n,2) = n - 1$.

2. $M(n,3) = 3n - 6$.

Larger Values of k

Remark.

1. *$M(n,4) = \binom{n}{2}$ when $n \le 12$.*

2. *$M(n,4) < \binom{n}{2}$ when $n \ge 13$.*

Although it may be possible to find an exact formula for $M(n,4)$ when n is large, we are more concerned with asymptotic values.

Remark. *For every $k \ge 4$, there exists a constant μ_k so that*

$$\lim_{n\to\infty} \frac{M(n,k)}{n^2} = \mu_k.$$

Proposition. *The sequence μ_k is increasing and converges to $1/2$.*

$k = 4$: The First Interesting Case

It is relatively easy to see that the dimension of a graph is bounded as a function of its chromatic number. Here is one important special case.

Proposition. *If $\chi(G) \leq 4$, then $\dim(G) \leq 4$.*

Proof. Let $V = V_1 \cup V_2 \cup V_3 \cup V_4$ and let L be any linear order on V. Then set:

$$L_1 = L(V_1) < L(V_2) < L(V_3) < L(V_4);$$
$$L_2 = L(V_4) < L(V_3) < L(V_2) < L(V_1);$$
$$L_3 = L^d(V_3) < L^d(V_4) < L^d(V_1) < L^d(V_2);$$
$$L_4 = L^d(V_2) < L^d(V_1) < L^d(V_4) < L^d(V_3).$$

It is straightforward to verify that these linear orders form a realizer of G.

Turán's Theorem

Definition. *Let $T(n,k)$ denoted the balanced complete k-partite graph on n nodes, and let $t(n,k)$ denote the number of edges in $T(n,k)$.*

Theorem. [Turán, 1941]
The maximum number of edges in a graph on n nodes which does not contain a complete subraph on $k+1$ nodes is $t(n,k)$.

Remark.

$$\tau_k = \lim_{n\to\infty} \frac{t(n,k)}{n^2} = \frac{1}{2} - \frac{1}{2k}$$

Remark.

$$\mu_4 \geq \frac{3}{8}.$$

The Erdős/Stone Theorem

Theorem. [Erdős and Stone, 1946]
Let k be an integer with $k \geq 3$ and let $\epsilon > 0$. Then let G be a graph with $\chi(G) \leq k$. Then there exists an integer n_0 so that if $n > n_0$ and H is any graph on n vertices with more than $(\tau_k + \epsilon)n^2$ edges, then H contains G as a subgraph.

.

Theorem. [Agnarsson, Felsner and Trotter, 1998]
For sufficiently large p, the Turán graph $T(p, 5)$ has dimension 5.

Theorem. [Agnarsson, Felsner and Trotter, 1998]

$$\mu_4 = \frac{3}{8}.$$

Larger Values of k

For $k = 5$, we can only show:

Theorem. [Agnarsson, Felsner and Trotter, 1998]

$$\frac{24}{50} \le \mu_5 \le \frac{40}{81}.$$

And for larger k, the estimates fall back to those for the dimension of the complete graph.

Theorem. [Agnarsson, Felsner and Trotter, 1998]

$$\frac{1}{1/2 - \mu_k} = \lg \lg k + \big(1/2 + o(1)\big) \lg \lg \lg k.$$

Ramsey Theory for Probability Spaces
A Motivating Problem

If we flip a coin repeatedly and let E_i be the event that the i^{th} toss is heads, then for all $i < j$,

$$\mathrm{Prob}[E_i\overline{E_j}] = \frac{1}{4}.$$

Question. *Can we do better? Does there exist an $\epsilon > 0$ so that we can have arbitrarily long sequences of events from any probability space for which*

$$\mathrm{Prob}[E_i\overline{E_j}] > \frac{1}{4} + \epsilon$$

for all $i < j$.

The Answer is NO!

Although we can do slightly better, for example by conditioning on $n/2$ of the tosses being heads, in the limit we can not do better than $1/4$.

Theorem. [Trotter and Winkler, 1998]
For every $\epsilon > 0$, there exists n_0 so that if $n > n_0$ and $E_1, E_2, \ldots, E_n$ is any sequence of events in a probability space, there exists $i < j$ for which

$$\mathrm{Prob}[E_i\overline{E_j}] < \frac{1}{4} + \epsilon$$

There are several nice proofs of this result, using ramsey theory, expectation and linear programming.

A Generalization to Shift Graphs

Definition. *Suppose we have a space in which there is an event for every k-element subset of $\{1, 2, \ldots, n\}$. Then we can find the minimum value of*

$$\mathrm{Prob}[A\overline{B}]$$

over all (k, n)-shift pairs (A, B), and let $\lambda(k, n)$ denote the maximum value of this minimum, taken over all probability spaces. Finally, let

$$\lambda_k = \lim_{n \to \infty} \lambda(k, n).$$

The Values $k = 1$ and $k = 2$

We have already seen that $\lambda_1 = 1/4$. Here's why $\lambda_2 \geq 1/3$. Take a random linear order on $\{1, 2, \ldots, n\}$. Then let A be a 2-element subset, say $A = \{i_1, i_2\}$. Let A correspond to the event that $i_1 < i_2$ in the random linear order. For every shift pair (A, B) with $A \cup B = \{i_1, i_2, i_3\}$, we see that $A\overline{B}$ means that i_2 is larger than both i_1 and i_3 in the random linear order. This happens with probability $1/3$.

On the other hand, this simple example is also asymptotically best possible.

Theorem. [Trotter and Winkler, 1998]

$$\lambda_2 = \frac{1}{3}$$

Larger Values of k

Theorem. [Trotter and Winkler, 1998]

1. $\lambda_{k+1} > \lambda_k$.

2. $\lim_{k\to\infty} \lambda_k = 1/2$.

3. $\lambda_k \geq 1/2 - 1/(2k+2)$.

4. $\lambda_k \leq 1/2 - 1/(4k-2)$, *when* $k \geq 2$.

We believe that $\lambda_3 = 3/8$ and $\lambda_4 = 2/5$. Originally, we thought that $\lambda_k = 1/2 - 1/(2k+2)$ for all k. If this were true, then we would also have $\lambda_5 = 5/12$. However, we have since been able to show that $\lambda_5 > 27/64$.

Fractional Dimension of Posets

Fractional dimension is just the linear programming relaxation of dimension, an integer valued parameter. More formally:

Definition. *Let P be a poset and let $\mathcal{F} = \{M_1, \ldots, M_t\}$ be a multiset of linear extensions of P. $\mathcal{F}$ is a k-fold realizer of P if for each incomparable pair (x, y), there are at least k linear extensions in $\mathcal{F}$ which reverse the pair (x, y), i.e.,*

$$|\{i : 1 \leq i \leq t, x > y \text{ in } M_i\}| \geq k.$$

The fractional dimension of P, denoted by $\mathrm{fdim}(P)$, is then defined as the least real number $q \geq 1$ for which there exists a k–fold realizer $\mathcal{F} = \{M_1, \ldots, M_t\}$ of P so that $k/t \geq 1/q$.

Remark. *For every poset P,*

$$\mathrm{fdim}(P) \leq \dim(P).$$

The Dimension of Posets of Bounded Degree

Definition. *Let $f(k)$ denote the largest integer t for which there exists a poset P with $\Delta(P) = k$ and $\dim(P) = t$.*

Theorem. [Rödl and Trotter, 1983]
$f(k) \leq 2k^2 + 2$.

Using the Lovász Local Lemma and other probabilistic methods, we have:

Theorem. [Füredi and Kahn, 1984]
$f(k) = O(k \log^2 k)$.

Applying correlation techniques to random posets of height two, we have:

Theorem. [Erdős, Kierstead and Trotter, 1991]
$f(k) = \Omega(k \log k)$.

The Fractional Dimension of Posets of Bounded Degree

Brightwell and Scheinerman proved that if P is a poset and $\Delta(P) = k$, then $\mathrm{fdim}(P) \leq k + 2$. They conjectured that this inequality could be improved to $\mathrm{fdim}(P) \leq k + 1$. Their conjecture was correct and the proof yielded a much stronger conclusion, a result with much the same flavor as Brooks' theorem for graphs.

Theorem. [Felsner and Trotter, 1992] *Let k be a positive integer, and let P be any poset with $\Delta(P) = k$. Then $\mathrm{fdim}(P) \leq k + 1$. Furthermore, if $k \geq 2$, then $\mathrm{fdim}(P) < k + 1$ unless one of the components of P is isomorphic to $\mathbf{S}_{k+1}$, the standard example of a poset of dimension $k + 1$.*

The Dimension of Interval Orders

In general, large height is not a prerequisite for large dimension. For example, consider the standard examples. The situation is completely different for interval orders.

Definition. *For an integer n, let $i(n)$ denote the largest integer t for which there exists an interval order of height n and dimension t.*

Using connections with shift graphs, we have:

Theorem. [Füredi, Hajnal, Rödl and Trotter, 1984]

$$i(n) = \lg\lg n + \bigl(1/2 + o(1)\bigr)\lg\lg\lg n.$$

The Fractional Dimension of Interval Orders

Interval orders enjoy many special properties. Here is an example.

Lemma. *Let A be a subset of a interval order P. Then there exists a linear extension L with $a > b$ in L whenever $a \in A$, $b \in P - A$ and $a \| b$ in P.*

Here is an immediate consequence:

Corollary. [Brightwell and Scheinerman, 1992] *If P is an interval order, then $\mathrm{fdim}(P) < 4$.*

Proof. Choose a subset A at random and apply the preceding lemma. For every distinct pair x, y, the probability that x belongs to A but y does not is $1/4$. This show $\mathrm{fdim}(P) \leq 4$. If we condition on $A \neq \emptyset$, then we get $\mathrm{fdim}(P) < 4$.

The Inequality is Best Possible

Brightwell and Scheinerman conjectured that their upper bound on the fractional dimension of interval orders was best possible—even though they did not know of any example for which the parameter was more than 2.2.

However, using the techniques they developed to investigate ramsey theoretic properties of probability spaces, the conjecture was settled in the affirmative.

Theorem. [Trotter and Winkler, 1998]
For every $\epsilon > 0$, there exists an interval order P with

$$4 - \epsilon \quad < \quad \mathrm{fdim}(P).$$

More on the Dimension of Interval Orders

Remark. *If an interval order has large dimension, then it has both large width and large height.*

Theorem. [Kierstead and Trotter, 1997] *For every interval order P, there exists an integer $t(P)$ so that if Q is any interval order with $\dim(P) > t(P)$, then Q contains a subposet isomorphic to P.*

Remark. *The proof depends on connections with the chromatic number of circle graphs (the intersection graphs of chords of a circle).*

Remark. *In fact, if $|P| = n$, then $t(P) \leq 10n$. It might be true that $t(P) = o(n)$. From below, we know that $t(P) = \Omega(\log \log n)$.*

Dimension and Chromatic Number

There are many important connections between dimension and chromatic number. Here is just one example.

Definition. *Let $G = (V, E)$ be a graph on n vertices. The adjacency poset of G, denoted A_G, is the poset whose point set consists of $A \cup B$ where $A = \{x' : x \in V\}$ is the set of minimal elements, $B = \{x'' : x \in V\}$ is the set of maximal elements, and $x' < y''$ if and only if xy is an edge in G. Note that $x' \| x''$ for all x.*

Proposition. *For every graph G, the following inequalities hold:*

1. $\dim(A_g) \geq \chi(G)$.

2. $\mathrm{girth}(A_g) > \mathrm{girth}(G)$.

Theorem. [Felsner and Trotter, 1998] *If G is planar, then $\dim(A_G) \leq 10$.*

Dimension and Chromatic Number (2)

There is a natural interpretation of dimension in terms of chromatic number.

Definition. *A linear extension L reverses the incomparable pair (x, y) if $x > y$ in L.*

Proposition. *A family $\mathcal{R}$ of linear extensions is a realizer of P if and only if for every incomparable pair (x, y), there is some $L \in \mathcal{R}$ which reverses (x, y).*

Definition. *Given a poset P, define a hypergraph H_P as follows. The vertex set of H_P is the set of incomparable pairs. A set S of incomparable pairs is an edge if and only if there is no linear extension reversing all the pairs in S, but there is one which reverses all the pairs in any proper subset of S. H_P is the hypergraph of incomparable pairs.*

Proposition. *For every poset P, $\dim(P) = \chi(H_P)$.*

Dimension and Chromatic Number (3)

Definition. *The graph of incomparable pairs, denoted G_P, is just the ordinary graph determined by the edges in H_P of size 2.*

Proposition. *For every poset P, $\dim(P) = \chi(H_P) \geq \chi(G_P)$.*

The following result is somwhat more difficult than it appears. Its proof relies on characterization theorems for comparability graphs.

Proposition. *For every poset P, if $\chi(P) = 2$, then $\dim(P) = \chi(H_P) = 2$.*

Dimension and Chromatic Number (4)

However, when the dimension of P is larger than 2, it may happen that $\chi(H_P) > \chi(G_P)$. In fact, I offer the following conjecture:

Conjecture. *For every $t \geq 3$, there exists a poset P for which*

1. $\chi(G_P) = 3$.

2. $\dim(P) = \chi(H_P) = t$.