Some Explicit Biorthogonal Polynomials

D. S. Lubinsky and H. Stahl

Abstract. Let $\alpha > 0$ and $\psi(x) = x^{\alpha}$. Let $S_{n,\alpha}$ be a polynomial of degree n determined by the biorthogonality conditions

$$\int_0^1 S_{n,\alpha} \psi^j = 0, \qquad j = 0, 1, \ldots, n-1.$$

We explicitly determine $S_{n,\alpha}$ and discuss some other properties, including their zero distribution. We also discuss their relation to the Sidi polynomials.

§1. Introduction and Results

Let $\psi : (0,1) \to \mathbb{R}$ be a strictly increasing continuous function. Then provided $\psi^j \in L_1[0,1]$ for all $j \geq 0$, we may uniquely determine a monic polynomial P_n of degree n by the biorthogonality conditions

$$\int_0^1 P_n(x) \psi(x)^j \, dx = \begin{cases} 0, & j = 0, 1, 2, \ldots, n-1, \\ I_n \neq 0, & j = n \end{cases}.$$

Biorthogonal polynomials of a more general form have been studied in several contexts - see [2]. The special form we consider here, arose for

$$\psi(x) = \log x,$$

in problems of quadrature and convergence acceleration [4], [5], [7], [8]. Then

$$P_n(x) = S_{n,0}(x) := \sum_{j=0}^{n} \binom{n}{j} (j+1)^n (-x)^j, \tag{1}$$

are the *Sidi polynomials*, and one may represent them as a contour integral. Using steepest descent, the strong asymptotics of P_n, and their zero

xxx and xxx (eds.), pp. 1–4.

ISBN 0-9728482-x-x

distribution, were established in [5]. Asymptotics for more general polynomials of this type were analyzed by Elbert [3]. The zero distribution of more general biorthogonal polynomials has been investigated in [6].

In this paper, we derive explicit expressions for $P_n(x)$ when $\psi(x) = x^\alpha$, any $\alpha > 0$. We deduce that as $\alpha \to 0$, our polynomials reduce to the Sidi polynomials, and also give a contour integral representation. This will allow steepest descent analysis of these polynomials. Our analysis is based on a Rodrigues type approach.

Our main result is:

Theorem 1. *Let $\alpha > 0, n \geq 1$ and*

$$S_{n,\alpha}(x) = \sum_{j=0}^{n} \binom{n}{j} \left[\prod_{k=0}^{n-1} \left(k + \frac{j+1}{\alpha} \right) \right] (-x)^j. \tag{2}$$

(a) Rodrigues Type Formula

$$S_{n,\alpha}\left(u^{1/\alpha}\right) = u^{1-1/\alpha} \left(\frac{d}{du} \right)^n \left[u^{n-1+1/\alpha} \left(1 - u^{1/\alpha}\right)^n \right]. \tag{3}$$

(b) Biorthogonality Relation
For $0 \leq j < n$,

$$\int_0^1 S_{n,\alpha}(x)\, x^{\alpha j} dx = 0. \tag{4}$$

(c) Confluence Relation

$$\lim_{\alpha \to 0+} \alpha^n S_{n,\alpha}(x) = S_{n,0}(x). \tag{5}$$

(d) Contour Integral Representation
Let $z \in \mathbb{C} \backslash (-\infty, 0]$ and Γ be a simple closed contour in $\mathbb{C} \backslash (-\infty, 0]$ enclosing z. Then

$$S_{n,\alpha}\left(z^{1/\alpha}\right) = \frac{n! z^{1-1/\alpha}}{2\pi i} \int_\Gamma \frac{u^{-1+1/\alpha}}{u - z} \left[\frac{u\left(1 - u^{1/\alpha}\right)}{u - z} \right]^n du. \tag{6}$$

Here the branch of $z^{1/\alpha}$ and $u^{1/\alpha}$is the principal one.

In the case $\alpha = 1$, $S_{n,\alpha}$ is the classical Legendre polynomial for $[0, 1]$, and the Rodrigues formula (3) is the classical one,

$$S_{n,1}(u) = \left(\frac{d}{du} \right)^n [u(1-u)]^n$$

apart from normalization. By contrast, the Sidi polynomials admit the Rodrigues type formula

$$S_{n,0}\left(e^{u}\right)=e^{-u}\left(\frac{d}{du}\right)^{n}\left[e^{u}\left(1-e^{u}\right)\right]^{n}.$$

The contour integral representation should allow one to apply steepest descent, yielding strong asymptotics for $S_{n,\alpha}$. One consequence of such asymptotics is the asymptotic zero distribution, which is studied for more general ψ in [6]. In this note, we show how the explicit form of the coefficients in $S_{n,\alpha}$ can be combined with results of VanAssche, Fano and Ortolani [1] to deduce information on their zero distribution. To apply those results, we consider the monic polynomials

$$\begin{aligned} S_{n,\alpha}^{*}(x) &= S_{n,\alpha}(-x)\Big/\left[\prod_{k=0}^{n-1}\left(k+\frac{n+1}{\alpha}\right)\right] \\ &= \sum_{j=0}^{n} a_{j,n}x^{n-j}, \end{aligned} \tag{7}$$

where for $0\le j\le n$,

$$a_{j,n}=\binom{n}{j}\left[\prod_{k=0}^{n-1}\left(\frac{k+\frac{n-j+1}{\alpha}}{k+\frac{n+1}{\alpha}}\right)\right]. \tag{8}$$

The biorthogonality relation (4) implies that $S_{n,\alpha}$ has n simple zeros in $(0,1)$ and hence $S_{n,\alpha}^{*}$ has n simple zeros in $(-1,0)$. Thus

$$S_{n,\alpha}^{*}(x)=\prod_{j=1}^{n}\left(x-x_{jn}\right),$$

where

$$-1<x_{nn}<x_{n-1,n}<\cdots<x_{1n}<0.$$

Form the zero counting function

$$\mu_{n}(x)=\frac{1}{n}\#\left\{j: x_{jn}\in(-1,x)\right\},$$

for $x\in(-1,0)$. For orthogonal polynomials, there is a vast array of results on zero distribution - see for example [9]. For the biorthogonal polynomials $S_{n,\alpha}$, results in [6] show that these zero counting measures converge weakly as $n\to\infty$ to an absolutely continuous distribution. That is, there exists an increasing absolutely continuous function $F:[-1,0]\to\mathbb{R}$ such that

$$\lim_{n\to\infty}\int_{-1}^{0} f\,d\mu_{n}=\int_{-1}^{0} f\,dF=\int_{-1}^{0} f\,F', \tag{9}$$

for all continuous $f : [-1, 0] \to \mathbb{R}$. The results of VanAssche, Fano and Ortolani [1] then give a representation for F'. This involves the Hilbert transform

$$H[g](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{g(t)}{t-x} dt,$$

where PV stands for principal value and the integral converges a.e. if $g \in L_1(\mathbb{R})$. We prove:

Theorem 2. *Let $\alpha > 0$ and*

$$f(x) = (1-x)^{1+\frac{1}{\alpha}} x^{-1} (\alpha + 1 - x)^{-\frac{1}{\alpha}}, \qquad x \in (0,1). \tag{10}$$

(a) If $n \to \infty$ and $j \to \infty$ in such a way that $j/n \to x \in (0,1)$, then

$$\frac{a_{j,n}}{a_{j-1,n}} \to f(x).$$

(b) Assume that the zero counting measures $\{\mu_n\}$ converge weakly to an absolutely continuous function F, as in (9). Then

$$F'(x) = -\frac{1}{\pi x} H\left[f^{[-1]}\right](x), \qquad x \in (-1,0), \tag{11}$$

where $f^{[-1]}$ denotes the inverse of the strictly decreasing function f.

In the special case $\alpha = 1$, that is the Legendre case,

$$f(x) = \frac{(1-x)^2}{x(2-x)} = \frac{(1-x)^2}{1-(1-x)^2},$$

which (comfortingly) is the form obtained in [1, p. 1609]. There

$$f^{[-1]}(x) = 1 - \frac{x}{\sqrt{x^2+x}}, \qquad x \in (0,\infty)$$

and F is the familiar arcsine distribution,

$$F'(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1+x)}}, \qquad x \in (-1,0).$$

We prove the results in the next section.

§2. Proofs

Proof of Theorem 1(a)
We see that

$$\begin{aligned}
&\left(\frac{d}{du}\right)^n \left[u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n\right] \\
&= \left(\frac{d}{du}\right)^n \left[u^{n-1+1/\alpha}\sum_{j=0}^{n}\binom{n}{j}(-1)^j u^{j/\alpha}\right] \\
&= \sum_{j=0}^{n}\binom{n}{j}(-1)^j \left(n-1+\frac{j+1}{\alpha}\right)\left(n-2+\frac{j+1}{\alpha}\right)\cdots \\
&\qquad \left(\frac{j+1}{\alpha}\right) u^{-1+(j+1)/\alpha} \\
&= u^{-1+1/\alpha} S_{n,\alpha}\left(u^{1/\alpha}\right),
\end{aligned}$$

by (2). □

Proof of Theorem 1(b)
We use the Rodrigues formula from (a). Note that $u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n$ has a zero of multiplicity n at 1, and a zero of multiplicity $n-1+1/\alpha$ at 0. Then if $n>k$ and $n-1+1/\alpha>k\geq 0$, $\left(\frac{d}{du}\right)^k\left[u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n\right]$ has a zero of multiplicity $n-k$ at 1 and multiplicity $n-k-1+1/\alpha$ at 0. Let $0\leq j<n$, and

$$\begin{aligned}
I_j &:= \int_0^1 S_{n,\alpha}(x)\, x^{\alpha j} dx \\
&= \int_0^1 S_{n,\alpha}\left(u^{1/\alpha}\right) u^{j+1/\alpha-1} du \\
&= \int_0^1 u^j \left(\frac{d}{du}\right)^n \left[u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n\right] du.
\end{aligned}$$

Integrating by parts j times, (with trivial modifications if $j=0$) and using the order of the zeros, gives

$$\begin{aligned}
I_j &= -j\int_0^1 u^{j-1}\left(\frac{d}{du}\right)^{n-1}\left[u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n\right] du \\
&= (-1)^2 j(j-1)\int_0^1 u^{j-2}\left(\frac{d}{du}\right)^{n-2}\left[u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n\right] du \\
&= \cdots \\
&= (-1)^j j!\int_0^1 \left(\frac{d}{du}\right)^{n-j}\left[u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n\right] du = 0,
\end{aligned}$$

since $n-j-1<n-1+1/\alpha$. $\square$

Proof of Theorem 1(c), (d)
The confluence relation follows from (1) and (2). The contour integral error formula is an immediate consequence of Cauchy's integral formula for derivatives and the Rodrigues formula (3). $\square$

Proof of Theorem 2(a)
A little manipulation of (8) gives

$$\frac{a_{j,n}}{a_{j-1,n}}=\frac{n-j+1}{j}\prod_{k=0}^{n-1}\left(1-\frac{1}{\alpha k+n-j+2}\right).$$

Then for large j,n such that $j/n\to x\in(0,1)$

$$\begin{aligned}\log\frac{a_{j,n}}{a_{j-1,n}}&=\log\frac{1-j/n}{j/n}-\sum_{k=0}^{n-1}\frac{1}{\alpha k+n-j+2}+O\left(\frac{1}{n}\right)\\&=\log\frac{1-j/n}{j/n}-\int_0^{n-1}\frac{1}{\alpha y+n-j+2}dy+O\left(\frac{1}{n}\right)\\&=\log\frac{1-j/n}{j/n}-\frac{1}{\alpha}\log\left(\frac{\alpha\left(n-1\right)+n-j+2}{n-j+2}\right)+O\left(\frac{1}{n}\right)\\&\to\log\frac{1-x}{x}-\frac{1}{\alpha}\log\left(\frac{\alpha+1-x}{1-x}\right)\\&=\log f\left(x\right).\end{aligned}$$

$\square$

Proof of Theorem 2(b)
Note that for $x\in(-1,0)$,

$$\begin{aligned}\frac{f'\left(x\right)}{f\left(x\right)}&=\frac{d}{dx}\log f\left(x\right)\\&=-\left[\frac{1}{x}+\frac{1+\frac{1}{\alpha}}{1-x}-\frac{\frac{1}{\alpha}}{1+\alpha-x}\right]\\&\leq-\left[\frac{1}{x}+\frac{1}{1-x}\right]<0,\end{aligned}$$

so f is a strictly decreasing function that maps $(0,1)$ onto $(0,\infty)$. Thus $f^{[-1]}$ is well defined in $(0,\infty)$ - and continuously differentiable. By Theorem 1 of [1, p. 1598],

$$\begin{aligned}f^{[-1]}\left(x\right)&=-\int_{-1}^{0}\frac{y}{x-y}dF\left(y\right)\\&=H\left[g\right]\left(x\right),\end{aligned}$$

where

$$g(y) = \pi \chi_{[-1,0]}(y)\, y F'(y)$$

and χ denotes a characteristic function. By the invertibility relation

$$H[H[g]] = -g,$$

we then obtain

$$g = -H\left[f^{[-1]}\right],$$

and then (11) follows. □

§3. References

1. W. Van Assche, G. Fano and F. Ortolani, Asymptotic behavior of the coefficients of some sequences of polynomials, SIAM J. Math. Anal. **18** (1987), 1597–1615.
2. C. Brezinski, *Biorthogonality and its Applications to Numerical Analysis*, Marcel Dekker, New York, 1992.
3. C. Elbert, Strong asymptotics of the generating polynomials of the Stirling numbers of the second kind, J. Approx. Theory **109** (2001), 198–217.
4. W. Ford and A. Sidi, An algorithm for a generalization of the Richardson extrapolation process, SIAM J. Numer. Anal. **24** (1987), 1212–1232.
5. D. S. Lubinsky and A. Sidi, Strong asymptotics for polynomials biorthogonal to powers of $\log x$, Analysis **14** (1994), 341–379.
6. D. S. Lubinsky and H. Stahl, Zero distribution of biorthogonal polynomials, in preparation.
7. A. Sidi, Numerical quadrature and non-linear sequence transformations: Unified rules for efficient computation of integrals with algebraic and logarithmic endpoint singularities, Math. Comp. **35** (1980), 851–874.
8. A. Sidi and D. S. Lubinsky, On the zeros of some polynomials that arise in numerical quadrature and convergence acceleration, SIAM J. Numer. Anal. **20** (1983), 589–598.
9. H. Stahl and V. Totik, *General Orthogonal Polynomials*, Cambridge University Press, Cambridge, 1992.

D. S. Lubinsky
School of Mathematics
Georgia Institute of Technology

Atlanta, GA 30332-0160 USA
lubinsky@math.gatech.edu

H. Stahl
Fachbereich Mathematik TFH Berlin
Luxemburgerstrasse 10
13353 Berlin Germany
HerbertStahl@aol.com