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FOURIER SERIES OF A CLASS OF ETA QUOTIENTS

KENNETH S. WILLIAMS
Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
Carleton University, Ottawa, ON, K1S 5B6 Canada
kwilliam@connect.carleton.ca

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The sum of divisors function $\sigma(m)$ is defined by

$$\sigma(m) = \begin{cases} \sum_{\substack{d \in \mathbb{N} \\ d \mid m}} d & \text{if } m \in \mathbb{N}, \\ 0 & \text{if } m \in \mathbb{Q}, \quad m \notin \mathbb{N}. \end{cases}$$

Let $\mathcal{H}$ denote the upper half of the complex plane. Let $\eta(z)$ $(z \in \mathcal{H})$ be the Dedekind eta function. A class $\mathcal{C}$ of eta quotients is given for which the Fourier series of each member of $\mathcal{C}$ can be given explicitly. One example is

$$\frac{\eta^2(2z)\eta^4(4z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^4(12z)} = 1 + \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}, \quad z \in \mathcal{H},$$

where

$$c(n) = 2\sigma(n) - 3\sigma(n/2) + 4\sigma(n/4) + 9\sigma(n/6) - 36\sigma(n/12), \quad n \in \mathbb{N}.$$

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1. Introduction

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on $\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by the product formula

$$\eta(z) := e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}). \tag{1.1}$$

The product formula (1.1) ensures that $\eta(z)$ does not vanish on $\mathcal{H}$. An eta quotient is a finite product $g(z) := \prod_{k=1}^{N} \eta^{b_k}(m_k z)$, where N is a positive integer, $m_1, \ldots, m_N$ are positive integers with $m_1 < m_2 < \cdots < m_N$, and $b_1, \ldots, b_N$ are non-zero integers. Since $\eta(z) \neq 0$ for $z \in \mathcal{H}$ the eta quotient $g(z)$ is holomorphic on $\mathcal{H}$

regardless of the choice of the exponents $b_1, \ldots, b_N$. The level of the eta quotient $g(z)$ is the positive integer

$$L := \mathrm{LCM}(m_1, \ldots, m_N). \tag{1.2}$$

Each m_k divides L and we can write the eta quotient as

$$g(z) = \prod_{\substack{m \in \mathbb{N} \\ m \mid L}} \eta^{a_m}(mz), \tag{1.3}$$

where some of the integers a_m may be 0. The eta quotient $g(z)$ has a transformation formula like that of a modular form of weight W given by

$$W := \frac{1}{2} \sum_{\substack{m \in \mathbb{N} \\ m \mid L}} a_m, \tag{1.4}$$

with respect to some multiplier system, on the congruence subgroup

$$\Gamma_0(L) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, \ ad - bc = 1, \ c \equiv 0 \pmod{L} \right\}.$$

Let

$$\frac{1}{24} \sum_{\substack{m \in \mathbb{N} \\ m \mid L}} m a_m = \frac{A}{B}, \tag{1.5}$$

where $A \in \mathbb{Z}$, $B \in \mathbb{N}$, $\mathrm{GCD}(A, B) = 1$ and $B \mid 24$. Then $g(z)$ has a Fourier expansion of the form

$$g(z) = \sum_{\substack{n \in \mathbb{Z} \\ n \geq A \\ n \equiv A \pmod{B}}} c(n) e^{2\pi i n z/B}, \quad c(A) = 1, \tag{1.6}$$

where the $c(n)$ are integers. An important continuing area of research is the explicit determination of the coefficients $c(n)$ in the Fourier series expansion (1.6) of particular eta quotients. Many papers have obtained such expansions, see, for example, [3–6, 8–10]. Recently a very comprehensive book devoted to determining such expansions has been published [7]. We just give two such examples. The first example [6, Theorem, p. 147] is an eta quotient of weight 3/2 on $\Gamma_0(2)$, namely,

$$\frac{\eta^5(2z)}{\eta^2(z)} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{n}{3}\right) n e^{2\pi i n^2 z/3}, \tag{1.7}$$

where the Legendre symbol $\left(\frac{n}{3}\right) = 0, 1, -1$ according as $n \equiv 0, 1, 2 \pmod{3}$, respectively. In this example $A = 1$, $B = 3$ and for $m \in \mathbb{N}$

$$c(m) = \begin{cases} (-1)^{n-1} \left(\dfrac{n}{3}\right) & \text{if } m = n^2, \quad n \in \mathbb{N}, \\ 0 & \text{if } m \neq n^2, \quad n \in \mathbb{N}. \end{cases}$$

The second example [6, p. 153; 7, p. 146] is an eta quotient of weight 2 and level 2, namely,

$$\frac{\eta^8(2z)}{\eta^4(z)} = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n) e^{\pi i n z}, \tag{1.8}$$

where the sum of divisors function $\sigma(n)$ was defined in the abstract. In this example $A = 1$ and $B = 2$.

In this paper we prove a theorem which gives the Fourier series expansions of many eta quotients. This theorem is proved in Sec. 2 using results deduced from the theory of theta series [1, 2]. It is convenient to define for a non-negative integer a

$$\delta(a) := \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a \geq 1. \end{cases} \tag{1.9}$$

Theorem 1.1. *Let a, b, c, d, e be non-negative integers satisfying*

$$a + b + c + d + e \leq 4. \tag{1.10}$$

Define integers A_0, A_1, A_2, A_3, A_4 by

$$x^a(1-x)^b(1+x)^c(1+2x)^d(2+x)^e = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4. \tag{1.11}$$

The integer A_0 is given by

$$A_0 = \delta(a)2^{a+e}. \tag{1.12}$$

Next define integers $a_1, a_2, a_3, a_4, a_6, a_{12}$ by

$$\begin{aligned} a_1 &:= -a + 2b - 2c - 4d - e + 4, \\ a_2 &:= 3a + b + 3c + 10d + e - 10, \\ a_3 &:= 3a + 2b + 6c + 4d + 3e - 12, \\ a_4 &:= -2a - b - c - 4d + 2e + 4, \\ a_6 &:= -9a - 7b - 9c - 10d - 7e + 30, \\ a_{12} &:= 6a + 3b + 3c + 4d + 2e - 12, \end{aligned} \tag{1.13}$$

so that

$$a_1 + a_2 + a_3 + a_4 + a_6 + a_{12} = 4 \tag{1.14}$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 12a_{12} = 24a. \tag{1.15}$$

Finally, define rational numbers $b_1, b_2, b_3, b_4, b_6, b_{12}$ *by*

$$\begin{aligned} b_1 &:= (-4A_0 + 2A_1)/2^{a+e}, \\ b_2 &:= (24A_0 - 12A_1 + 4A_2)/2^{a+e}, \\ b_3 &:= (-4A_0 + 6A_1 - 8A_2 + 8A_3)/2^{a+e}, \\ b_4 &:= (-16A_0 + 8A_1 - 8A_3 + 16A_4)/2^{a+e}, \\ b_6 &:= (-8A_0 + 4A_1 + 4A_2 - 8A_3 - 32A_4)/2^{a+e}, \\ b_{12} &:= (-16A_0 - 8A_1 + 8A_3 + 16A_4)/2^{a+e}, \end{aligned} \tag{1.16}$$

so that

$$b_1 + b_2 + b_3 + b_4 + b_6 + b_{12} = -24\delta(a). \tag{1.17}$$

Then, for $z \in \mathcal{H}$, *we have*

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z) = \delta(a) + \sum_{n=1}^{\infty} c(n)e^{2\pi inz}, \tag{1.18}$$

where for $n \in \mathbb{N}$

$$\begin{aligned} c(n) = {} & b_1\sigma(n) + b_2\sigma(n/2) + b_3\sigma(n/3) + b_4\sigma(n/4) \\ & + b_6\sigma(n/6) + b_{12}\sigma(n/12). \end{aligned} \tag{1.19}$$

Table 1 lists the eta quotient identities resulting from this theorem. Two examples are discussed in Sec. 3. Some concluding remarks are given in Sec. 4.

2. Proof of Theorem 1.1

Taking $x = 0$ in (1.11) we obtain

$$A_0 = 0^a 2^e = 0^a 2^a 2^e = \delta(a) 2^{a+e},$$

which is (1.12). Equations (1.14) and (1.15) follow from (1.13) and equation (1.17) follows from (1.16) and (1.12).

For $q \in \mathbb{C}$ with $|q| < 1$ we define the theta series $\varphi(q)$ by

$$\varphi(q) = \sum_{x \in \mathbb{Z}} q^{x^2}. \tag{2.1}$$

Then we define

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \quad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.2}$$

Using results from the theory of theta functions it was shown in [1, Theorem 1.1, p. 280] that

$$\begin{aligned} k^2 &= 1 + \sum_{n=1}^{\infty}(-4\sigma(n) + 24\sigma(n/2) - 4\sigma(n/3) - 16\sigma(n/4) \\ &\quad - 8\sigma(n/6) - 16\sigma(n/12))q^n, \end{aligned} \tag{2.3}$$

$$\begin{aligned} pk^2 &= \sum_{n=1}^{\infty}(2\sigma(n) - 12\sigma(n/2) + 6\sigma(n/3) + 8\sigma(n/4) \\ &\quad + 4\sigma(n/6) - 8\sigma(n/12))q^n, \end{aligned} \tag{2.4}$$

$$p^2k^2 = \sum_{n=1}^{\infty}(4\sigma(n/2) - 8\sigma(n/3) + 4\sigma(n/6))q^n, \tag{2.5}$$

$$p^3k^2 = \sum_{n=1}^{\infty}(8\sigma(n/3) - 8\sigma(n/4) - 8\sigma(n/6) + 8\sigma(n/12))q^n, \tag{2.6}$$

$$p^4k^2 = \sum_{n=1}^{\infty}(16\sigma(n/4) - 32\sigma(n/6) + 16\sigma(n/12))q^n. \tag{2.7}$$

In [2, pp. 48–49] each of $\prod_{n=1}^{\infty}(1-q^{rn}) \quad (r \in \{1,2,3,4,6,12\})$ was expressed in terms of p and k, namely

$$\prod_{n=1}^{\infty}(1-q^n) = q^{-1/24}2^{-1/6}p^{1/24}(1-p)^{1/2}(1+p)^{1/6}(1+2p)^{1/8}(2+p)^{1/8}k^{1/2},$$

$$\prod_{n=1}^{\infty}(1-q^{2n}) = q^{-1/12}2^{-1/3}p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/4}k^{1/2},$$

$$\prod_{n=1}^{\infty}(1-q^{3n}) = q^{-1/8}2^{-1/6}p^{1/8}(1-p)^{1/6}(1+p)^{1/2}(1+2p)^{1/24}(2+p)^{1/24}k^{1/2},$$

$$\prod_{n=1}^{\infty}(1-q^{4n}) = q^{-1/6}2^{-2/3}p^{1/6}(1-p)^{1/8}(1+p)^{1/24}(1+2p)^{1/8}(2+p)^{1/2}k^{1/2},$$

$$\prod_{n=1}^{\infty}(1-q^{6n}) = q^{-1/4}2^{-1/3}p^{1/4}(1-p)^{1/12}(1+p)^{1/4}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2},$$

$$\prod_{n=1}^{\infty}(1-q^{12n}) = q^{-1/2}2^{-2/3}p^{1/2}(1-p)^{1/24}(1+p)^{1/8}(1+2p)^{1/24}(2+p)^{1/6}k^{1/2}.$$

Solving these six equations for the six quantities p, $1-p$, $1+p$, $1+2p$, $2+p$ and k, we obtain

$$p = 2q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^3(1-q^{3n})^3(1-q^{12n})^6}{(1-q^n)(1-q^{4n})^2(1-q^{6n})^9}, \tag{2.8}$$

$$1-p=\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^{2n})(1-q^{3n})^2(1-q^{12n})^3}{(1-q^{4n})(1-q^{6n})^7}, \tag{2.9}$$

$$1+p=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^3(1-q^{3n})^6(1-q^{12n})^3}{(1-q^n)^2(1-q^{4n})(1-q^{6n})^9}, \tag{2.10}$$

$$1+2p=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{10}(1-q^{3n})^4(1-q^{12n})^4}{(1-q^n)^4(1-q^{4n})^4(1-q^{6n})^{10}}, \tag{2.11}$$

$$2+p=2\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^2(1-q^{12n})^2}{(1-q^n)(1-q^{6n})^7}, \tag{2.12}$$

$$k=\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^{4n})^2(1-q^{6n})^{15}}{(1-q^{2n})^5(1-q^{3n})^6(1-q^{12n})^6}. \tag{2.13}$$

Taking $x=p$ in (1.11), and then multiplying the resulting equation by k^2, we obtain

$$\begin{aligned} &p^a(1-p)^b(1+p)^c(1+2p)^d(2+p)^e k^2 \\ &\quad = A_0k^2+A_1pk^2+A_2p^2k^2+A_3p^3k^2+A_4p^4k^2. \end{aligned} \tag{2.14}$$

By (2.8)–(2.13) and (1.13) the left-hand side of (2.14) is

$$2^{a+e}q^a\prod_{n=1}^{\infty}(1-q^n)^{a_1}(1-q^{2n})^{a_2}(1-q^{3n})^{a_3}(1-q^{4n})^{a_4}(1-q^{6n})^{a_6}(1-q^{12n})^{a_{12}}$$

and the right-hand side of (2.14) is by (2.3)–(2.7) and (1.16)

$$\begin{aligned} A_0+2^{a+e}\sum_{n=1}^{\infty}&(b_1\sigma(n)+b_2\sigma(n/2)+b_3\sigma(n/3)+b_4\sigma(n/4) \\ &+b_6\sigma(n/6)+b_{12}\sigma(n/12))q^n. \end{aligned}$$

Equating the left and right-hand sides of (2.14), and then dividing by 2^{a+e}, we obtain

$$\begin{aligned} &q^a\prod_{n=1}^{\infty}(1-q^n)^{a_1}(1-q^{2n})^{a_2}(1-q^{3n})^{a_3}(1-q^{4n})^{a_4}(1-q^{6n})^{a_6}(1-q^{12n})^{a_{12}} \\ &\quad =\delta(a)+\sum_{n=1}^{\infty}(b_1\sigma(n)+b_2\sigma(n/2)+b_3\sigma(n/3)+b_4\sigma(n/4) \\ &\qquad +b_6\sigma(n/6)+b_{12}\sigma(n/12))q^n, \end{aligned} \tag{2.15}$$

on appealing to (1.12). Now choose $q=e^{2\pi iz}$ with $z\in\mathcal{H}$ so that $|q|<1$. Using

$$\prod_{n=1}^{\infty}(1-q^{kn})=\prod_{n=1}^{\infty}(1-e^{2\pi iknz})=e^{-\pi ikz/12}\eta(kz), \quad k\in\{1,2,3,4,6,12\}, \tag{2.16}$$

in (2.15), and appealing to (1.15), we obtain (1.18). □

An easy calculation shows that there are 126 5-tuples (a, b, c, d, e) satisfying (1.10). Hence the theorem yields the Fourier series of 126 eta quotients. A computer program was written to loop through the non-negative integers a, b, c, d, e satisfying $a + b + c + d + e \leq 4$ and then to reorder the resulting 13-tuples $(a, a_1, a_2, a_3, a_4, a_6, a_{12}, b_1, b_2, b_3, b_4, b_6, b_{12})$ with entries from left to right in increasing order of size. Table 1 of eta quotient identities was obtained.

The weight W of each eta quotient in Table 1 is by (1.4) and (1.14)

$$W = \frac{1}{2}(a_1 + a_2 + a_3 + a_4 + a_6 + a_{12}) = 2.$$

The level L of each eta quotient is given in Table 2.

By (1.5) and (1.15) the order of the cusp at ∞ is $A/B = (a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 12a_{12})/24 = a$, so that $A = a$ and $B = 1$. A computer program was run using the criterion given in [7, Corollary 2.3, p. 37] to check that all the eta quotients in Table 1 are non-cuspidal.

Table 1. Values of $a, a_1, a_2, a_3, a_4, a_6, a_{12}, b_1, b_2, b_3, b_4, b_6, b_{12}$ for which $\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)$ $\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z) = \delta(a) + \sum_{n=1}^{\infty}(b_1\sigma(n) + b_2\sigma(n/2) + b_3\sigma(n/3) + b_4\sigma(n/4) + b_6\sigma(n/6)$ $+ b_{12}\sigma(n/12))e^{2\pi i n z}$.

No.	a	a_1	a_2	a_3	a_4	a_6	a_{12}	b_1	b_2	b_3	b_4	b_6	b_{12}
1	0	−12	30	4	−12	−10	4	12	24	108	48	−648	432
2	0	−10	23	6	−9	−9	3	10	12	54	8	−324	216
3	0	−9	21	3	−6	−7	2	9	6	27	−12	−162	108
4	0	−8	16	8	−6	−8	2	8	4	24	0	−156	96
5	0	−8	20	0	−8	0	0	8	0	0	−32	0	0
6	0	−7	14	5	−3	−6	1	7	0	9	−4	−72	36
7	0	−6	9	10	−3	−7	1	6	0	10	0	−72	32
8	0	−6	12	2	0	−4	0	6	−3	0	0	−27	0
9	0	−6	13	2	−5	1	−1	6	−4	−6	−8	12	−24
10	0	−6	21	2	−9	−7	3	6	−12	−54	−72	324	−216
11	0	−5	7	7	0	−5	0	5	−2	3	0	−30	0
12	0	−5	11	−1	−2	3	−2	5	−6	−9	4	18	−36
13	0	−4	2	12	0	−6	0	4	0	4	0	−32	0
14	0	−4	5	4	3	−3	−1	4	−3	0	2	−9	−18
15	0	−4	6	4	−2	2	−2	4	−4	−4	0	12	−32
16	0	−4	10	−4	−4	10	−4	4	−8	−12	16	24	−48
17	0	−4	14	4	−6	−6	2	4	−12	−36	−16	180	−144
18	0	−3	0	9	3	−4	−1	3	0	1	0	−12	−16
19	0	−3	3	1	6	−1	−2	3	−3	0	3	0	−27
20	0	−3	4	1	1	4	−3	3	−4	−3	4	12	−36
21	0	−3	12	1	−3	−4	1	3	−12	−27	12	108	−108
22	0	−2	−2	6	6	−2	−2	2	1	0	0	−3	−24
23	0	−2	−1	6	1	3	−3	2	0	−2	0	8	−32
24	0	−2	2	−2	4	6	−4	2	−3	0	4	9	−36
25	0	−2	3	−2	−1	11	−5	2	−4	−2	8	12	−40
26	0	−2	7	6	−3	−5	1	2	−8	−18	0	96	−96
27	0	−2	11	−2	−5	3	−1	2	−12	−18	40	36	−72
28	0	−1	−4	3	9	0	−3	1	3	0	−1	0	−27

Table 1. (*Continued*)

No.	a	a_1	a_2	a_3	a_4	a_6	a_{12}	b_1	b_2	b_3	b_4	b_6	b_{12}
29	0	−1	−3	3	4	5	−4	1	2	−1	0	6	−32
30	0	−1	1	−5	2	13	−6	1	−2	3	4	6	−36
31	0	−1	5	3	0	−3	0	1	−6	−9	8	54	−72
32	0	0	−6	0	12	2	−4	0	6	0	−3	0	−27
33	0	0	−5	0	7	7	−5	0	5	0	−2	3	−30
34	0	0	−4	0	2	12	−6	0	4	0	0	4	−32
35	0	0	0	−8	0	20	−8	0	0	8	0	0	−32
36	0	0	0	8	0	−4	0	0	0	−8	0	48	−64
37	0	0	3	0	3	−1	−1	0	−3	0	6	27	−54
38	0	0	4	0	−2	4	−2	0	−4	0	16	12	−48
39	0	0	12	0	−6	−4	2	0	−12	0	96	−108	0
40	0	1	−7	−3	10	9	−6	−1	9	0	−5	0	−27
41	0	1	−6	−3	5	14	−7	−1	8	1	−4	0	−28
42	0	1	−2	5	3	−2	−1	−1	4	−3	0	24	−48
43	0	1	2	−3	1	6	−3	−1	0	9	4	0	−36
44	0	2	−8	−6	8	16	−8	−2	13	0	−8	−3	−24
45	0	2	−7	−6	3	21	−9	−2	12	2	−8	−4	−24
46	0	2	−4	2	6	0	−2	−2	9	0	−4	9	−36
47	0	2	−3	2	1	5	−3	−2	8	2	0	0	−32
48	0	2	1	−6	−1	13	−5	−2	4	18	−8	−12	−24
49	0	2	5	2	−3	−3	1	−2	0	18	32	−72	0
50	0	3	−9	−9	6	23	−10	−3	18	−1	−12	−6	−20
51	0	3	−6	−1	9	2	−3	−3	15	0	−9	0	−27
52	0	3	−5	−1	4	7	−4	−3	14	3	−8	−6	−24
53	0	3	3	−1	0	−1	0	−3	6	27	0	−54	0
54	0	4	−10	−12	4	30	−12	−4	24	−4	−16	−8	−16
55	0	4	−7	−4	7	9	−5	−4	21	0	−14	−9	−18
56	0	4	−6	−4	2	14	−6	−4	20	4	−16	−12	−16
57	0	4	−2	4	0	−2	0	−4	16	12	0	−48	0
58	0	4	2	−4	−2	6	−2	−4	12	36	−32	−36	0
59	0	5	−8	−7	5	16	−7	−5	28	−3	−20	−12	−12
60	0	5	−4	1	3	0	−1	−5	24	9	−16	−36	0
61	0	6	−9	−10	3	23	−9	−6	36	−10	−24	−12	−8
62	0	6	−6	−2	6	2	−2	−6	33	0	−24	−27	0
63	0	6	−5	−2	1	7	−3	−6	32	6	−32	−24	0
64	0	6	3	−2	−3	−1	1	−6	24	54	−96	0	0
65	0	7	−7	−5	4	9	−4	−7	42	−9	−32	−18	0
66	0	8	−8	−8	2	16	−6	−8	52	−24	−32	−12	0
67	0	8	−4	0	0	0	0	−8	48	0	−64	0	0
68	0	9	−6	−3	3	2	−1	−9	60	−27	−48	0	0
69	0	10	−7	−6	1	9	−3	−10	72	−54	−32	0	0
70	0	12	−6	−4	0	2	0	−12	96	−108	0	0	0
71	1	−9	23	3	−10	−9	6	1	6	27	20	−162	108
72	1	−7	16	5	−7	−8	5	1	4	15	4	−84	60
73	1	−6	14	2	−4	−6	4	1	3	9	−4	−45	36
74	1	−5	9	7	−4	−7	4	1	2	7	0	−42	32
75	1	−5	13	−1	−6	1	2	1	2	3	−12	−6	12
76	1	−4	7	4	−1	−5	3	1	1	3	−2	−21	18
77	1	−3	2	9	−1	−6	3	1	0	3	0	−20	16
78	1	−3	5	1	2	−3	2	1	0	0	−1	−9	9

Table 1. (*Continued*)

No.	a	a_1	a_2	a_3	a_4	a_6	a_{12}	b_1	b_2	b_3	b_4	b_6	b_{12}
79	1	−3	6	1	−3	2	1	1	0	−1	−4	0	4
80	1	−3	14	1	−7	−6	5	1	0	−9	−28	72	−36
81	1	−2	0	6	2	−4	2	1	−1	1	0	−9	8
82	1	−2	4	−2	0	4	0	1	−1	−3	0	3	0
83	1	−1	−2	3	5	−2	1	1	−2	0	1	−3	3
84	1	−1	−1	3	0	3	0	1	−2	−1	0	2	0
85	1	−1	3	−5	−2	11	−2	1	−2	−5	4	6	−4
86	1	−1	7	3	−4	−5	4	1	−2	−9	−8	42	−24
87	1	0	−4	0	8	0	0	1	−3	0	2	0	0
88	1	0	−3	0	3	5	−1	1	−3	−1	2	3	−2
89	1	0	5	0	−1	−3	3	1	−3	−9	2	27	−18
90	1	1	−5	−3	6	7	−2	1	−4	0	3	3	−3
91	1	1	−4	−3	1	12	−3	1	−4	−1	4	4	−4
92	1	1	0	5	−1	−4	3	1	−4	−5	0	24	−16
93	1	1	4	−3	−3	4	1	1	−4	−9	12	12	−12
94	1	2	−6	−6	4	14	−4	1	−5	1	4	3	−4
95	1	2	−2	2	2	−2	2	1	−5	−3	4	15	−12
96	1	3	−7	−9	2	21	−6	1	−6	3	4	2	−4
97	1	3	−4	−1	5	0	1	1	−6	0	5	9	−9
98	1	3	−3	−1	0	5	0	1	−6	−1	8	6	−8
99	1	3	5	−1	−4	−3	4	1	−6	−9	32	−18	0
100	1	4	−5	−4	3	7	−1	1	−7	3	6	3	−6
101	1	5	−6	−7	1	14	−3	1	−8	7	4	0	−4
102	1	5	−2	1	−1	−2	3	1	−8	3	16	−12	0
103	1	6	−4	−2	2	0	2	1	−9	9	8	−9	0
104	1	7	−5	−5	0	7	0	1	−10	15	0	−6	0
105	1	9	−4	−3	−1	0	3	1	−12	27	−16	0	0
106	2	−6	16	2	−8	−8	8	0	1	6	8	−39	24
107	2	−4	9	4	−5	−7	7	0	1	4	2	−21	14
108	2	−3	7	1	−2	−5	6	0	1	3	−1	−12	9
109	2	−2	2	6	−2	−6	6	0	1	2	0	−11	8
110	2	−2	6	−2	−4	2	4	0	1	2	−4	−3	4
111	2	−1	0	3	1	−4	5	0	1	1	−1	−6	5
112	2	0	−2	0	4	−2	4	0	1	0	−1	−3	3
113	2	0	−1	0	−1	3	3	0	1	0	−2	−1	2
114	2	0	7	0	−5	−5	7	0	1	0	−10	15	−6
115	2	1	−3	−3	2	5	2	0	1	−1	−1	0	1
116	2	2	−4	−6	0	12	0	0	1	−2	0	1	0
117	2	2	0	2	−2	−4	6	0	1	−2	−4	9	−4
118	2	3	−2	−1	1	−2	5	0	1	−3	−1	6	−3
119	2	4	−3	−4	−1	5	3	0	1	−4	2	3	−2
120	2	6	−2	−2	−2	−2	6	0	1	−6	8	−3	0
121	3	−3	9	1	−6	−7	10	0	0	1	3	−9	5
122	3	−1	2	3	−3	−6	9	0	0	1	1	−5	3
123	3	0	0	0	0	−4	8	0	0	1	0	−3	2
124	3	1	−1	−3	−2	3	6	0	0	1	−1	−1	1
125	3	3	0	−1	−3	−4	9	0	0	1	−3	3	−1
126	4	0	2	0	−4	−6	12	0	0	0	1	−2	1

Table 2. Values of level L of eta quotient (1.18).

L	Cases
2	67
4	5, 87
6	8,11,13,31,36,53,57,70,82,84,98,104,116
12	Otherwise

3. Two Eta Quotient Identities

Identity 87 of Table 1 asserts that

$$\frac{\eta^8(4z)}{\eta^4(2z)} = \sum_{n=1}^{\infty}(\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4))e^{2\pi inz}.$$

As

$$\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4) = \begin{cases} \sigma(n) & \text{if } n \equiv 1 \pmod 2, \\ 0 & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

we have

$$\frac{\eta^8(4z)}{\eta^4(2z)} = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n)e^{2\pi inz}.$$

Replacing z by $z/2$, we deduce

$$\frac{\eta^8(2z)}{\eta^4(z)} = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n)e^{\pi inz},$$

which is (1.8).

Identity 24 of Table 1 gives the Fourier series stated in the abstract. The eta quotient

$$\frac{\eta^2(2z)\eta^4(4z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^4(12z)}$$

is of weight 2 on $\Gamma_0(12)$.

4. Concluding Remarks

We conclude with a few remarks.

Remark 1. For any rational numbers c_1, c_2, c_3, c_4, c_6 and c_{12}, we have

$$\sum_{n=1}^{\infty}(c_1\sigma(n) + c_2\sigma(n/2) + c_3\sigma(n/3) + c_4\sigma(n/4) + c_6\sigma(n/6) + c_{12}\sigma(n/12))q^n$$
$$= c_1q + (3c_1 + c_2)q^2 + (4c_1 + c_3)q^3 + (7c_1 + 3c_2 + c_4)q^4 + 6c_1q^5 + \cdots .$$

By (2.8) and (2.13) we have

$$p^5k^2 = 32q^5 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5(1-q^{3n})^3(1-q^{12n})^{18}}{(1-q^n)(1-q^{4n})^6(1-q^{6n})^{15}} = 32q^5 + \cdots .$$

Thus, examining the coefficients of q and q^5, we see that

$$p^5k^2 \neq \sum_{n=1}^{\infty}(c_1\sigma(n) + c_2\sigma(n/2) + c_3\sigma(n/3) + c_4\sigma(n/4) + c_6\sigma(n/6) + c_{12}\sigma(n/12))q^n$$

for any rational numbers c_1, c_2, c_3, c_4, c_6 and c_{12}. Thus we cannot use a quintic polynomial in (1.11) (see also (2.14)). Hence we cannot extend the range in (1.10) from $a+b+c+d+e \leq 4$ to $a+b+c+d+e \leq 5$.

Remark 2. We observe that a few of the formulae in Table 1 are related. For example identity 67 is

$$\frac{\eta^8(z)}{\eta^4(2z)} = 1 + \sum_{n=1}^{\infty}(-8\sigma(n) + 48\sigma(n/2) - 64\sigma(n/4))e^{2\pi inz}$$

and identity 36 is

$$\frac{\eta^8(3z)}{\eta^4(6z)} = 1 + \sum_{n=1}^{\infty}(-8\sigma(n/3) + 48\sigma(n/6) - 64\sigma(n/12))e^{2\pi inz}.$$

Clearly these are related by the transformation $z \to 3z$.

Remark 3. The values of a, b, c, d and e are recoverable from the values of a_1, a_2, a_3, a_4, a_6 and a_{12} in Table 1 by means of the formulae:

$$\begin{aligned}
a &= \frac{1}{24}a_1 + \frac{1}{12}a_2 + \frac{1}{8}a_3 + \frac{1}{6}a_4 + \frac{1}{4}a_6 + \frac{1}{2}a_{12},\\
b &= \frac{1}{4}a_1 - \frac{1}{12}a_3 - \frac{1}{8}a_4 - \frac{1}{6}a_6 - \frac{5}{24}a_{12} + 1,\\
c &= \frac{1}{12}a_1 + \frac{5}{12}a_3 - \frac{1}{24}a_4 + \frac{1}{6}a_6 + \frac{1}{24}a_{12} + \frac{1}{3},\\
d &= \frac{1}{8}a_2 - \frac{1}{12}a_3 - \frac{1}{24}a_6 - \frac{1}{12}a_{12} + \frac{1}{2},\\
e &= -\frac{1}{4}a_1 - \frac{1}{8}a_2 - \frac{1}{3}a_3 + \frac{1}{8}a_4 - \frac{7}{24}a_6 - \frac{5}{24}a_{12} + \frac{3}{2}.
\end{aligned}$$

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