Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **4** (2010), 119–135. doi:10.2298/AADM1000008B

ON THE NUMBER OF CERTAIN TYPES OF STRONGLY RESTRICTED PERMUTATIONS

Vladimir Baltić

Let p be a permutation of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. We introduce techniques for counting $N(n; k, r, I)$, the number of Lehmer's strongly restricted permutations of $\mathbb{N}_n$ satisfying the conditions $-k \leq p(i) - i \leq r$ (for arbitrary natural numbers k and r) and $p(i) - i \notin I$ (for some set I). We show that $N(n; 1, r, \emptyset)$ is the Fibonacci $(r+1)$-step number.

1. INTRODUCTION

A class of permutations in which the positions of the marks after the permutation are restricted can be specified by an $n \times n$ $(0,1)$-matrix $A = (a_{ij})$ in which:

$$a_{ij} = \begin{cases} 1, & \text{if the mark } j \text{ is permitted to occupy the } i\text{-th place;} \\ 0, & \text{otherwise.} \end{cases}$$

Now, we will mention a well known fact about the number of restricted permutations. For completeness of this paper, we include a proof.

Theorem 1. *The number of restricted permutations is given by the permanent function* [**16**] *of a square matrix A:*

$$\operatorname{per} A = \sum_{p \in S_n} a_{1p(1)} a_{2p(2)} \cdots a_{np(n)},$$

where p runs through the set S_n of all permutations of $\mathbb{N}_n$.

2000 Mathematics Subject Classification. 05A15, 05A05, 11B37, 15A15.
Keywords and Phrases. Restricted permutations, exact enumeration, recurrences, permanents.

Proof. In this summation, only the products corresponding to the permutations p that satisfy all restrictions have value 1; the remaining values are 0. Hence, the number of restricted permutations equals the permanent of the associated matrix A. $\square$

In *strongly restricted permutations* [**10**], the number $r_i = \sum_{j=1}^{n} a_{ij}$ is uniformly small, i.e., $r_i \leq K$ $(i = 1, 2, \ldots, n)$, where K is an integer independent of n. In *weakly restricted permutations* [**10**], $n - r_i$ is uniformly small.

The most widely known problems from the class of weakly restricted permutations are "Le Problème des Rencontres" (derangements) and "Le Problème des ménages" (some of historical notes about them are taken from [**2**]).

Example 1. The famous problem of coincidences (matches, "Problème des Rencontres") was initially treated in the particular case of 13 cards by Pierre R. Montmort (1708) and Johann Bernoulli (1714). The statement of this problem is:

Find the number of permutations without fixed points.

The permutation without fixed points is usually called the *derangement*.

The associated matrix is $A = J_n - I_n$, where I_n denotes the unit matrix, and J_n has all elements equal to 1.

Abraham de Moivre (1718) examined the general case of n cards by using the inclusion and exclusion principle. The number D_n of derangements of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$ equals to:

$$D_n = n! \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

There are two well known recurrence relations for number of derangements:

$$D_n = n \cdot D_{n-1} + (-1)^n, \qquad D_0 = 1$$

and Leonard Euler's (1751) recurrence relation:

$$D_n = (n-1) \cdot (D_{n-1} + D_{n-2}), \qquad D_0 = 1, \quad D_1 = 0.$$

From these recurrences we can calculate number D_n of derangements:

n	0	1	2	3	4	5	6	7	8	9	10	...
D_n	1	0	1	2	9	44	265	1 854	14 833	133 496	1 334 961	...

This is the sequence A000166 at [**22**].

W. A. Whitworth (1867), in his book on the combinatorics of the games of chance, studied the matching problem and contributed to its popularization. The similarity of recurrence relations for D_n to the corresponding recurrence relations for the factorials,

$$n! = n(n-1)!, \qquad 0! = 1,$$

$$n! = (n-1)\ \ (n-1)! + (n-2)!\ \ , \qquad 0! = 1, \quad 1! = 1,$$

led Whitworth to call the number D_n *n-subfactorial*.

Example 2. The ménages problem is a classical enumeration problem formulated and solved by François Edouard Anatole Lucas (1891); before that under a different formulation this problem had been examined by Arthur Cayley (1878) and T. Muir (1878). It is asking the following:

> What is the number of different seatings of n married couples (ménages) around a circular table so that men and women alternate and no man is next to his wife?

We can assume that the wives are seated first (that could be done on $2 \cdot n!$ ways). Let us now number:

(a) the n women from 1 to n in the ordinary direction (counterclockwise) starting from any one of them,

(b) the n empty seats from 1 to n in the ordinary direction starting from the seat that is to the left of the woman with the number 1 and,

(c) the n men by assigning to every man the number of his wife.

In this way the enumeration of the different ways of placing the n men on the n empty seats between women, so that no man is seated next to his wife, is reduced to the enumeration of the number M_n of permutations $(p_1, p_2, \ldots, p_n)$ of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$ that satisfy the restrictions:

$$p_i \neq i, \quad p_i \neq i+1 \quad \text{for } i = 1, 2, \ldots, n-1, \qquad \text{and} \qquad p_n \neq n, \quad p_n \neq 1.$$

The numbers M_n are called *reduced ménages numbers*. The expression for the reduced ménages numbers is given by:

$$M_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!$$

and the proof of this fact (by the inclusion and exclusion principle) found in [**2**, Example 4.5]. That proof was presented by J. Touchard (1934, 1953).

From this formula we can calculate the reduced ménages numbers:

n	2	3	4	5	6	7	8	9	10	...
M_n	0	1	2	13	80	579	4 738	43 387	439 792	...

This is the sequence A000179 at [**22**].

The total number of different seatings of n married couples around a circular table so that men and women alternate and no man is next to his wife equals $2n! \cdot M_n$. This is the sequence A059375 at [**22**].

This section we will end with some historical data.

Irving Kaplansky gave a new approach and some generalizations of these two problems [**4**], [**5**]. A general method of enumeration of permutations with restricted positions was developed by Irving Kaplansky and John Riordan in a series of papers (they developed the theory of “Rook polynomials” [**6**], [**18**]. Noah S. Mendelsohn [**14**,**15**] studied related types of weakly restricted permutations and some particular types of strongly restricted permutations. He used the techniques of the difference operator and converting recurrences for operator polynomials into asymptotic series. M. René Lagrange [**9**] analyzed the particular types of the strongly restricted permutations satisfying the condition $|p(i) - i| \leq d$, where d is 1, 2, or 3. Richard P. Stanley [**19**, Examples 4.7.7, 4.7.15 and 4.7.16] explored the same types with the “Transfer-matrix Method” and the technique he named the “Factorization in Free Monoids”. His work was outstanding because he involved generating functions and developed a new technique which can be applied in several small cases.

Derrick Henry Lehmer [**10**] gave the following classification of some sets of strongly restricted permutations:

- $R_1^{(k)}$—no mark shall move more than k places left or right,
- $R_2^{(k)}$—when the marks are deployed in a circle, no mark shall move more than k positions clockwise or counterclockwise,
- $R_3^{(k)}$—when in a circle, each mark shall move clockwise only, but not more than k places,
- $R_4^{(k)}$—when deployed on a line, the mark n goes to the first place and all other marks move right not more than k places,
- $R_5^{(k)}$—no mark shall move more than k places left or right, but each mark must move.

Moreover, he noticed that $\left|R_2^{(k)}\right|$ is equal to $\left|R_3^{(2k+1)}\right|$. He also described six techniques for enumerating some particular cases of strongly restricted permutations.

We consider a generalization of the Lehmer's strongly restricted permutations deployed on a line, namely $R_1^{(k)}$ and $R_5^{(k)}$ ($R_4^{(k)}$ is a special case of this generalization). Furthermore, our technique (with slight modifications that result in a larger system of recurrence equations) can enumerate Lehmer's strongly restricted permutations deployed on a circle, $R_3^{(k)}$. Since Lehmer [**10**] showed that $\left|R_2^{(k)}\right| = \left|R_3^{(2k+1)}\right|$, it means that our technique can handle *all* five types of Lehmer's permutations. For the number of Lehmer's permutations $\left|R_3^{(k)}\right|$ the following is known: Richard P. Stanley [**19**, Example 4.7.7] explored the type $k = 2$ with the “Transfer-matrix Method”, Vladimir Baltić [**1**] used finite state automata for the type $k = 2$, and a group of mathematicians [**11**] explored the type $k = 3$ by expanding permanents.

Now, we will introduce the N notation, which we use in the rest of paper. Let $N(n; k, r, I)$ be the number of the strongly restricted permutations satisfying the conditions $-k \le p(i) - i \le r$ and $p(i) - i \notin I$, and let $N(n; k, r)$ be $N(n; k, r, \emptyset)$. By definition, $\left|R_1^{(k)}\right| = N(n; k, k)$ and $\left|R_5^{(k)}\right| = N(n; k, k, \{0\})$.

We introduce general techniques for evaluating $N(n; k, r)$ (Section 2) and $N(n; k, r, I)$ (Section 4), thus the number of restricted permutations of types $R_1^{(k)}$ and $R_5^{(k)}$. These are generalizations of Lehmer's type $R_1^{(k)}$. For the Lehmer's $\left|R_1^{(k)}\right|$, symmetric cases $k = r$ with $k = 1, 2$, there are known results [**9**,**10**,**19**,**20**,**21**]. Alois Panholzer [**17**] and Torleiv Kløve [**7**,**8**] made progress in symmetric cases (Panholzer used finite state automata, while Kløve used modified "Transfer-matrix Method" based on expanding permanent) and they found the asymptotic expansion and gave the bounds for the denominator of corresponding generating functions. We pursue the more general, asymmetric cases and we end with asymmetric cases with more forbidden positions. We illustrate our techniques on several cases. Specifically, we show that $N(n; 1, r)$ is the Fibonacci $(r + 1)$-step number (Section 5). We further give a bijection between the permutations satisfying the condition $-1 \le p(i) - i \le r$ and the permutations $R_4^{(k)}$ (Theorem 3). Our techniques are illustrated below by several examples. We show that computing the number of restricted permutations using our techniques is computationally much more efficient than expanding the permanent per A (Section 6). Using a program that implements our technique, we have contributed over sixty sequences to the Sloane's online encyclopedia of integer sequences [**22**].

2. COUNTING $N(n; k, r)$

We present a general technique for counting $N(n; k, r)$, the number of permutations satisfying the condition $-k \le p(i) - i \le r$ for all $i \in \mathbb{N}_n$, where $k \le r < n$. Our technique proceeds in five steps:

1. Create $\mathcal{C}$, a set of all $k+1$-element combinations of the set $\mathbb{N}_{k+r+1}$ containing element $k + r + 1$.

2. Introduce an integer sequence $a_C(n)$ for each combination $C \in \mathcal{C}$.

3. Apply the mapping φ (defined below) to each combination.

4. Create a system of linear recurrence equations:
$$a_C(n) = \sum_{C' \in \varphi(C)} a_{C'}(n-1).$$

5. Solve the system to obtain $N(n; k, r) = a_{(r+1,r+2,\ldots,r+k+1)}(n)$.

We next describe these steps in detail and then prove that $N(n; k, r)$ is indeed equal to $a_{(r+1,r+2,\ldots,r+k+1)}(n)$.

Let $\mathcal{C}$ denote a set of all combinations with $k + 1$ elements of the set $\mathbb{N}_{k+r+1}$, which contain $k + r + 1$. We represent these combinations as strictly increasing ordered $(k+1)$-tuples. For example, all such combinations with 3 elements of the set $\mathbb{N}_5 = \{1,2,3,4,5\}$ are represented (in reverse lexicographic order) by:

$$(3,4,5),\quad (2,4,5),\quad (2,3,5),\quad (1,4,5),\quad (1,3,5),\quad (1,2,5).$$

In examples we will use easier notation:

$$345,\qquad 245,\qquad 235,\qquad 145,\qquad 135,\qquad 125.$$

We split the set $\mathcal{C}$ in two disjoint sets

$$\mathcal{C}_\infty = \big\{\mathcal{C} \in \mathcal{C} \mid \infty \in \mathcal{C}\big\} \qquad \text{and} \qquad \mathcal{C}_\in = \big\{\mathcal{C} \in \mathcal{C} \mid \infty \notin \mathcal{C}\big\}.$$

We introduce the mapping

$$\varphi(C) = \begin{cases} \varphi_1(C), & C \in \mathcal{C}_1 \\ \varphi_2(C), & C \in \mathcal{C}_2 \end{cases},$$

defined by $\varphi_1 : \mathcal{C}_1 \to \mathcal{P}(\mathcal{C})$ (if $1 \in C$) and $\varphi_2 : \mathcal{C}_2 \to \mathcal{P}(\mathcal{C})$ (if $1 \notin C$) defined by

$$\varphi_1\big((1, c_2, \dots, c_k, c_{k+1})\big) = \big\{(c_2 - 1, c_3 - 1, \dots, c_k - 1, c_{k+1} - 1, k + r + 1)\big\},$$

$$\varphi_2\big((c_1, c_2, \dots, c_k, c_{k+1})\big) = \big\{C_1, C_2, \dots, C_k, C_{k+1}\big\},$$

where we get $C_i \in \mathcal{C}$ from $C = (c_1, c_2, \dots, c_k, c_{k+1})$ by deleting c_i, decreasing all other coordinates by 1, shifting all coordinates with bigger index to one place left and putting $k + r + 1$ at the end:

$$C_i = (c_1 - 1, \dots, c_{i-1} - 1, c_{i+1} - 1, \dots, c_{k+1} - 1, k + r + 1).$$

For example,

$$\varphi_1\big((1,3,5)\big) = \{(2,4,5)\} \quad \text{and} \quad \varphi_2\big((2,4,5)\big) = \big\{(3,4,5), (1,4,5), (1,3,5)\big\}.$$

Create a system of linear recurrence equations:

$$a_C(n+1) = \sum_{C' \in \varphi(C)} a_{C'}(n).$$

We use the mappings φ_1 and φ_2 to find a system of $\binom{k+r}{k}$ linear recurrence equations (for each combination we get one equation): from $\varphi_1(C) = \{C'\}$ we find the linear recurrence equation

$$a_C(n+1) = a_{C'}(n)$$

and from $\varphi_2(C) = \{C_1, C_2, \dots, C_{k+1}\}$ we find linear recurrence equation

$$a_C(n+1) = a_{C_1}(n) + a_{C_2}(n) + \dots + a_{C_{k+1}}(n).$$

This system can be easily solved, for example using the standard method based on generating functions. We will prove that $N(n;k,r) = a_{(r+1,r+2,\ldots,r+k+1)}(n)$. Thus, from the matrix of this system, S, we can find $N(n;k,r)$ as the element in the first row and the first column of the matrix S^n, i.e., the number of the closed paths in the digraph G whose adjacency matrix is S (this observation is important because we can apply the Transfer matrix method to the matrix S). We apply this observation to determine the computational complexity of our technique (Section 6).

Theorem 2. $N(n;k,r) = a_{(r+1,r+2,\ldots,r+k+1)}(n)$.

Proof. We first introduce a set of matrices $\mathcal{M}$ that correspond to the sequences $a_C(n)$. We then show that each matrix corresponds to a specific combination.

Let $\mathcal{M}$ denote the set of $n \times n$ matrices $M = (m_{ij})$ satisfying the following conditions:

1) the first $k+1$ rows start with ones and end with zeros: for $i = 1,\ldots,k+1$, $m_{ij} = 1$ for $j = 1,\ldots,d_i$ and $m_{ij} = 0$ for $j > d_i$, where $d_i \geq 1$;

2) $d_{k+1} = k+r+1$;

3) if $1 \leq i < i' \leq k+1$ then $d_i < d_{i'}$;

4) elements in the last $n-(k+1)$ rows satisfy: $m_{ij} = 1$ for $-k \leq i-j \leq r$ and $m_{ij} = 0$ otherwise.

We define the mapping $f : \mathcal{M} \to \mathcal{C}$ with $f(M) = (d_1, d_2, \ldots, d_{k+1})$. The function f is obviously a bijection.

We associate an $n \times n$ matrix $A = (a_{ij})$ defined by:

$$a_{ij} = \begin{cases} 1, & \text{if} \quad -k \leq j-i \leq r, \\ 0, & \text{otherwise} \end{cases}$$

with the strongly restricted permutations satisfying $-k \leq p(i)-i \leq r$. As stated in the introduction, $N(n;k,r) = \operatorname{per} A$. Notice that $A \in \mathcal{M}$ with $d_i = r+i$, where $1 \leq i \leq k+1$, and thus the combination corresponding to A is $(r+1, r+2, \ldots, r+k+1)$.

We next observe that the recurrence equations from step 4 above correspond to the expansion of the permanent of matrices from $\mathcal{M}$ by the first row (φ_1) or by the first column (φ_2). This observation leads to the main conclusion:

$$N(n;k,r) = \operatorname{per} A = a_{(r+1,r+2,\ldots,r+k+1)}(n). \qquad \square$$

Example 3. $\underline{k=r=2}$: The permutations of the set $\mathbb{N}_n$ satisfying the condition $-2 \leq p(i)-i \leq 2$. It is usually referred to as a permutation of length n within distance 2. In this case we have $k+r+1 = 5$.

$$\mathcal{C} = \{345, 245, 235, 145, 135, 125\}.$$

$$\begin{aligned}
\varphi_2(345) &= \{345, 245, 235\},\\
\varphi_2(245) &= \{345, 145, 135\},\\
\varphi_2(235) &= \{245, 145, 125\},\\
\varphi_1(145) &= \{345\},\\
\varphi_1(135) &= \{245\},\\
\varphi_1(125) &= \{145\},
\end{aligned}$$

from which we get the system of linear recurrence equations:

$$\begin{aligned}
a_{345}(n+1) &= a_{345}(n) + a_{245}(n) + a_{235}(n),\\
a_{245}(n+1) &= a_{345}(n) + a_{145}(n) + a_{135}(n),\\
a_{235}(n+1) &= a_{245}(n) + a_{145}(n) + a_{125}(n),\\
a_{145}(n+1) &= a_{345}(n),\\
a_{135}(n+1) &= a_{245}(n),\\
a_{125}(n+1) &= a_{145}(n),
\end{aligned}$$

with the initial conditions $a_{345}(0) = 1$, $a_{245}(0) = 0$, $a_{235}(0) = 0$, $a_{145}(0) = 0$, $a_{135}(0) = 0$, and $a_{125}(0) = 0$. If we substitute $a_{345}(n) = a_n$, $a_{245}(n) = b_n$, $a_{235}(n) = c_n$, $a_{145}(n) = d_n$, $a_{135}(n) = e_n$, and $a_{125}(n) = f_n$ we have a simpler form:

$$\begin{aligned}
a_{n+1} &= a_n + b_n + c_n,\\
b_{n+1} &= a_n + d_n + e_n,\\
c_{n+1} &= b_n + d_n + f_n,\\
d_{n+1} &= a_n,\\
e_{n+1} &= b_n,\\
f_{n+1} &= d_n.
\end{aligned}$$

The initial conditions are $a_0 = 1$, $b_0 = c_0 = d_0 = e_0 = f_0 = 0$.

One of the reasons for making the previous substitution is the transition to generating functions: for a sequence which is denoted by a lower case letter we will denote the corresponding generating function by the same upper case letter ($a_n \leftrightarrow A(z)$, $b_n \leftrightarrow B(z)$, and so on). We find the following system:

$$\begin{aligned}
\frac{A(z)-1}{z} &= A(z) + B(z) + C(z),\\
\frac{B(z)}{z} &= A(z) + D(z) + E(z),\\
\frac{C(z)}{z} &= B(z) + D(z) + F(z),\\
\frac{D(z)}{z} &= A(z),\\
\frac{E(z)}{z} &= B(z),\\
\frac{F(z)}{z} &= D(z).
\end{aligned}$$

This is the system of linear equations variables are $A(z), B(z), \ldots, F(z)$ and part of its solution that we are interested in is:

$$A(z) = \frac{1-z}{1-2z-2z^3+z^5}\,.$$

From the denominator of this generating function $1-2z-2z^3+z^5$, we can find the linear recurrence equation $a_n - 2a_{n-1} - 2a_{n-3} + a_{n-5} = 0$, i.e.

$$a_n = 2a_{n-1} + 2a_{n-3} - a_{n-5}.$$

The number of permutations, a_n, satisfying the condition $|p(i) - i| \leq 2$, for all $i \in \mathbb{N}_n$ is determined by its generating function $A(z)$:

n	0	1	2	3	4	5	6	7	8	9	10	...
a_n	1	1	2	6	14	31	73	172	400	932	2 177	...

This is the sequence A002524 at [**22**].

Remark. In [**20**, Problem 12.17], a simpler system is given:

$$\begin{aligned}
a_n &= a_{n-1} + b_{n-1} + c_{n-1},\\
b_n &= a_{n-1} + b_{n-1},\\
c_n &= b_{n-1} + d_{n-1},\\
d_n &= a_{n-1} + e_{n-1},\\
e_n &= a_{n-1}.
\end{aligned}$$

Our technique for generating a system of the recurrence equations is general and it does not give an optimum system (with minimal number of equations).

Example 4. From the system of the recurrence equations:

$$\begin{aligned}
a_{n+1} &= a_n + b_n + c_n,\\
b_{n+1} &= a_n + d_n + e_n,\\
c_{n+1} &= b_n + d_n + f_n,\\
d_{n+1} &= a_n,\\
e_{n+1} &= b_n,\\
f_{n+1} &= d_n,
\end{aligned}$$

we find the matrix of this system of the recurrence equations:

$$S = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The initial conditions are $a_0 = 1$, $b_0 = c_0 = d_0 = e_0 = f_0 = 0$. Thus, we have:

$$\begin{bmatrix} a_n\\ b_n\\ c_n\\ d_n\\ e_n\\ f_n \end{bmatrix} = S^n \cdot \begin{bmatrix} 1\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}.$$

From this equation we observe that the number of permutations, a_n, satisfying the condition $|p(i)-i| \le 2$ is equal to the element at the position $(1,1)$ in the matrix S^n. This is another way of evaluating the number a_n.

EXAMPLE 5. $\underline{k=2,\ r=3}$: The permutations of the set $\mathbb{N}_n$ satisfying the condition $-2 \le p(i)-i \le 3$. $\overline{k+r+1=6}$.

$$\mathcal{C} = \{456, 356, 346, 256, 246, 236, 156, 146, 136, 126\}.$$

$$\varphi_2(456)=\{456,356,346\},\quad \varphi_2(356)=\{456,256,246\},\quad \varphi_2(346)=\{356,256,236\}$$
$$\varphi_2(256)=\{456,156,146\},\quad \varphi_2(246)=\{356,156,136\},\quad \varphi_2(236)=\{256,156,126\}$$
$$\varphi_1(156)=\{456\},\quad \varphi_1(146)=\{356\},\quad \varphi_1(136)=\{256\},\quad \varphi_1(126)=\{156\},$$

from which we get the system of linear recurrence equations:

$$\begin{array}{ll}
a_{456}(n+1)=a_{456}(n)+a_{356}(n)+a_{346}(n), & \\
a_{356}(n+1)=a_{456}(n)+a_{256}(n)+a_{246}(n), & a_{156}(n+1)=a_{456}(n), \\
a_{346}(n+1)=a_{356}(n)+a_{256}(n)+a_{236}(n), & a_{146}(n+1)=a_{356}(n), \\
a_{256}(n+1)=a_{456}(n)+a_{156}(n)+a_{146}(n), & a_{136}(n+1)=a_{256}(n), \\
a_{246}(n+1)=a_{356}(n)+a_{156}(n)+a_{136}(n), & a_{126}(n+1)=a_{156}(n), \\
a_{236}(n+1)=a_{256}(n)+a_{156}(n)+a_{126}(n), &
\end{array}$$

with the initial conditions $a_{456}(0)=1$ and $a_C(0)=0$ for $C \ne 456$.

The number of permutations, $a_{456}(n)$, satisfying the condition $-2 \le p(i)-i \le 3$, for all $i \in \mathbb{N}_n$ is determined by its generating function:

$$A(z) = \frac{1-z^2-z^3-z^5}{1-2z^2-3z^3-4z^4-5z^5-z^6+2z^7+z^8+z^9+z^{10}}$$

n	0	1	2	3	4	5	6	7	8	9	10	...
$a_{456}(n)$	1	1	2	6	18	46	115	301	792	2 068	5 380	...

This is the sequence A072827 at [**22**].

EXAMPLE 6. $\underline{k=2,\ r=4}$: The permutations of the set $\mathbb{N}_n$ satisfying the condition $-2 \le p(i)-i \le 4$. $\overline{k+r+1=7}$.

$$\mathcal{C} = \{567, 467, 457, 367, 357, 347, 267, 257, 247, 237, 167, 157, 147, 137, 127\}.$$

Now, we will skip calculations and give only the final results. The number of permutations, $a_{567}(n)$, satisfying the condition $-2 \le p(i)-i \le 3$, for all $i \in \mathbb{N}_n$ is determined by its generating function:

$$A(z) = \frac{1-z^2-2z^3-2z^4-2z^6+z^7+z^9}{1-z-2z^2-4z^3-6z^4-10z^5-12z^6+4z^7+6z^8+6z^9+2z^{11}+2z^{12}-z^{14}-z^{15}}$$

n	0	1	2	3	4	5	6	7	8	9	10	...
$a_{567}(n)$	1	1	2	6	18	54	146	391	1081	3 004	8 320	...

This is the sequence A072850 at [**22**].

Example 7. $k = 3$, $r = 3$: The permutations of the set $\mathbb{N}_n$ satisfying the condition $-3 \leq p(i) - i \leq 3$. It is usually referred to as permutations of length n within distance 3. $k + r + 1 = 7$.

$$\begin{aligned}\mathcal{C} = \{&4567, 3567, 3467, 3457, 2567, 2467, 2457, 2367, 2357, 2347,\\ &1567, 1467, 1457, 1367, 1357, 1347, 1267, 1257, 1247, 1237\}.\end{aligned}$$

$$\begin{aligned}
\varphi_2(4567) &= \{4567, 3567, 3467, 3457\} & \varphi_1(1567) &= \{4567\},\\
\varphi_2(3567) &= \{4567, 2567, 2467, 2457\} & \varphi_1(1467) &= \{3567\},\\
\varphi_2(3467) &= \{3567, 2567, 2367, 2357\} & \varphi_1(1457) &= \{3467\},\\
\varphi_2(3457) &= \{3567, 2467, 2367, 2347\} & \varphi_1(1367) &= \{2567\},\\
\varphi_2(2567) &= \{4567, 1567, 1467, 1457\} & \varphi_1(1357) &= \{2467\},\\
\varphi_2(2467) &= \{3567, 1567, 1367, 1357\} & \varphi_1(1347) &= \{2367\},\\
\varphi_2(2457) &= \{3467, 1467, 1367, 1347\} & \varphi_1(1267) &= \{1567\},\\
\varphi_2(2367) &= \{2567, 1567, 1267, 1257\} & \varphi_1(1257) &= \{1467\},\\
\varphi_2(2357) &= \{2467, 1467, 1267, 1247\} & \varphi_1(1247) &= \{1367\},\\
\varphi_2(2347) &= \{2367, 1367, 1267, 1237\} & \varphi_1(1237) &= \{1267\},
\end{aligned}$$

from which we get the system of $\binom{k+r}{k} = \binom{6}{3} = 20$ linear recurrence equations.

The number of permutations, $a_{456}(n)$, satisfying $-2 \leq p(i) - i \leq 3$, for all $i \in \mathbb{N}_n$ is determined by its generating function:

$$A(z) = \frac{1 - z - 2z^2 - 2z^4 + z^7 + z^8}{1 - 2z - 2z^2 - 10z^4 - 8z^5 + 2z^6 + 16z^7 + 10z^8 + 2z^9 - 4z^{10} - 2z^{11} - 2z^{13} - z^{14}}$$

n	0	1	2	3	4	5	6	7	8	9	10	...
$a_{456}(n)$	1	1	2	6	24	78	230	675	2 069	6 404	19 708	...

This is the sequence A002526 at [**22**].

3. REMARKS ON OUR TECHNIQUE AND KNOWN RESULTS

Next, we discuss how our general technique encompasses some previous results. Mendelsohn [**15**] and Lehmer [**10**] got systems of recurrence equations while expanded the corresponding permanents in their specific examples. However, they did not describe a general technique for obtaining the recurrence equations; Lehmer wrote “Each of these permanents in turn can be so expanded and the process continued until no ‘new’ matrices occur” [**10**], whereas Mendelsohn wrote “For fixed $r \geq 5$, the calculation of the difference equations becomes impracticable, and even if these equations were attained, they would be so complicated that it seems unlikely they would be of any use for obtaining explicit formulae” [**15**] (r is the number of ones in the row of permanent).

Richard P. Stanley developed the general technique ("Transfer-matrix Method"), but its applications are limited. He wrote [**19**, Example 4.7.16] "To use the transfer-matrix method would be quite unwieldy, but the factorization method is very elegant". In this example it is not so hard to find a sequence of oriented graphs which generate such permutations, but already for $k = 2$ and $r = 3$ it is hard to find all generators and even harder to do the following calculations. If we know the generating function

$$A(z) = \frac{1 - z^2 - z^3 - z^5}{1 - 2z^2 - 3z^3 - 4z^4 - 5z^5 - z^6 + 2z^7 + z^8 + z^9 + z^{10}}$$

(we determined the generating function $A(z)$ on the basis of our technique), which corresponds to $N(n; 2, 3)$, we can predict this behavior. The nominator of this generating function $1 - z^2 - z^3 - z^5$ could not occur as the denominator of the sum of a simple geometric progression. Also referring to the "Factorization in Free Monoids", [**12**], Stanley wrote "while this method has limited application, when it does work it is extremely elegant and simple" [**19**, page 247].

In contrast, we give an explicit technique for creating a system of the recurrence equations based on a simple mapping φ from combinations of $\mathbb{N}_{k+r+1}$.

Now, we will give some crucial differences between our technique and the "Transfer-matrix Method". The "Transfer-matrix Method" deals with the determinants (i.e. characteristic polynomial of some adjacency matrix of a digraph), while our technique deals with expansions of the permanents. The "Transfer-matrix Method" starts with a digraph and calculates its characteristic polynomial, while our technique starts with a $(0, 1)$-matrix A corresponding to the restricted permutations, then by mappings applied to the combinations we reach the system of linear recurrence equations and from that system we can find the generating function which determinates the sequence of numbers of restricted permutations. The digraph determined by the system of linear recurrence equations is simpler then one at the beginning of the "Transfer-matrix Method" (it has less vertices).

In the next section we are going further by giving generalizations of restricted permutations from the previous section.

4. COUNTING $N(n; k, r, I)$

Here we will analyze the permutations satisfying the conditions $-k \le p(i) - i \le r$ and $p(i) - i \notin I$ for all $i \in \mathbb{N}_n$, where $k \le r < n$, and I is a fixed subset of the set $\{-k+1, -k+2, \ldots, r-1\}$. Assume that I contains x elements, $|I| = x$. In the generalization of permutations analyzed in the previous section (also, this is the generalization of Lehmer's type $R_5^{(k)}$), almost the same reasoning has been applied. Again, we will pursue with the asymmetric cases and the cases where more numbers are forbidden than in the ordinary derangements. We will associate case $k = 1$ with the number of compositions of n into elements of a finite set $P = \mathbb{N}_{k+r} \setminus (r+1-I)$, where $\alpha \pm I$ denotes the set $\alpha \pm I = \{\alpha \pm i \mid i \in I\}$.

Let $N(n;k,r,I)$ denote the number of the strongly restricted permutations satisfying the conditions $-k \le p(i)-i \le r$ and $p(i)-i \notin I$. We associate the $n \times n$ matrix $A=(a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if} \quad -k \le j-i \le r, j-i \notin I \\ 0, & \text{otherwise} \end{cases}$$

with these permutations. We have $N(n;k,r,I) = \operatorname{per} A$. For computing this permanent we will separate the set $\mathcal{C}$ into two disjoint sets $\mathcal{C}_1 = \{C \in \mathcal{C} \mid 1 \in C\}$ and $\mathcal{C}_\in = \{\mathcal{C} \in \mathcal{C} \mid \infty \notin \mathcal{C}\}$, but we will also separate the set $\mathcal{C}_2$ into $x+1$ disjoint sets $C_2^m = \{C \in \mathcal{C}_2 \mid m \text{ elements of } C \text{ are in } r+1-I\}$, $(m=0,1,\dots,x)$. We introduce the following mappings $\varphi_1 : \mathcal{C}_1 \to \mathcal{P}(\mathcal{C})$ and $\varphi_2^m : \mathcal{C}_2^m \to \mathcal{P}(\mathcal{C})$, $(m=0,1,\dots,x)$, defined by

$$\varphi_1\big((1,c_2,\dots,c_k,c_{k+1})\big) = \big\{(c_2-1,c_3-1,\dots,c_k-1,c_{k+1}-1,k+r+1)\big\},$$

$$\varphi_2^m\big((c_1,c_2,\dots,c_k,c_{k+1})\big) = \big\{D'_1,D'_2,\dots,D'_{k+1-m}\big\},$$

where we get $\big(D'_1,D'_2,\dots,D'_{k+1-m}\big)$ from $\varphi_2(C) = \{D_1,D_2,\dots,D_k,D_{k+1}\}$ $(C=(c_1,c_2,\dots,c_k,c_{k+1})$ and φ_2 is the mapping introduced in Section 2) when we delete all combinations D_y corresponding to elements c_y which satisfy the condition $c_y \in r+1-I$. Again, we use these mappings to find a system of $\binom{k+r}{k}$ linear recurrence equations (one equation per combination): if we have $\varphi_1(C) = \{D\}$ then we have the linear recurrence equation

$$a_C(n+1) = a_D(n)$$

and if we have $\varphi_2^m(C) = \{D'_1,D'_2,\dots,D'_{k+1-m}\}$ then we have the linear recurrence equation

$$a_C(n+1) = a_{D'_1}(n) + a_{D'_2}(n) + \dots + a_{D'_{k+1-m}}(n).$$

These recurrence equations correspond to expansions of the permanents of matrices from $\mathcal{M}$ by the first row (in a case of φ_1) or by the first column (in cases of all of φ_2^m; note that when we skip an element c_y, it corresponds to a zero element in the first column). We are able to get a linear recurrence equation and a generating function for $N(n;k,r,I)$ using this system.

Example 8. $\underline{k=2,\ r=3,\ I=\{-1,2\}}$: The permutations of the set $\mathbb{N}_n$, satisfying the conditions $-2 \le p(i)-i \le 3$ and $p(i)-i \ne -1,2$. Then, set $I=\{-1,2\}$, which implies $(r+1-I) = \{2,5\}$. All combinations with $k+1=3$ element of set $\mathbb{N}_{k+r+1} = \mathbb{N}_6 = \{1,2,3,4,5,6\}$ containing 6 are:

$$\mathcal{C} = \{456, 356, 346, 256, 246, 236, 156, 146, 136, 126\}.$$

$$\begin{aligned} \mathcal{C}_1 &= \{156, 146, 136, 126\}, \\ \mathcal{C}_2^0 &= \{346\}, \\ \mathcal{C}_2^1 &= \{456, 356, 246, 236\}, \\ \mathcal{C}_2^2 &= \{256\}. \end{aligned}$$

$$\begin{aligned}
\varphi_2^0(346) &= \{356, 256, 236\}; && \\
\varphi_2^1(456) &= \{456, 346\}, & \varphi_2^1(356) &= \{456, 246\}, \\
\varphi_2^1(246) &= \{156, 136\}, & \varphi_2^1(236) &= \{156, 126\}; \\
\varphi_2^2(256) &= \{146\}; && \\
\varphi_1(156) &= \{456\}, & \varphi_1(146) &= \{356\}, \\
\varphi_1(136) &= \{256\}, & \varphi_1(126) &= \{156\}.
\end{aligned}$$

We get the system of the linear recurrence equations:

$$\begin{aligned}
a_{456}(n+1) &= a_{456}(n) + a_{346}(n) \\
a_{356}(n+1) &= a_{456}(n) + a_{246}(n) \\
a_{346}(n+1) &= a_{356}(n) + a_{256}(n) + a_{236}(n) \\
a_{256}(n+1) &= a_{146}(n) \\
a_{246}(n+1) &= a_{156}(n) + a_{136}(n) \\
a_{236}(n+1) &= a_{156}(n) + a_{126}(n) \\
a_{156}(n+1) &= a_{456}(n) \\
a_{146}(n+1) &= a_{356}(n) \\
a_{136}(n+1) &= a_{256}(n) \\
a_{126}(n+1) &= a_{156}(n),
\end{aligned}$$

with the initial conditions $a_{456}(0) = 1$ and $a_C(0) = 0$ for $C \neq 456$. From this system we find the generating function:

$$A(z) = \frac{1 - z^5}{1 - z - z^3 - z^4 - 4z^5 + z^6 - z^7 + z^9 + z^{10}}.$$

The number of permutations, $a_{456}(n)$, satisfying $-2 \leq p(i) - i \leq 3$ and $p(i) - i \neq -1, 2$, for all $i \in \mathbb{N}_n$ is determined by its generating function $A(z)$:

n	0	1	2	3	4	5	6	7	8	9	10	...
$a_{456}(n)$	1	1	1	2	4	9	15	25	46	84	156	...

This is the sequence A080004 at [**22**].

5. COUNTING $R_4^{(k)}$

EXAMPLE 9. $\underline{k=1}$: All combinations with $k + 1 = 2$ elements of the set $\mathbb{N}_{r+2} = \{1, 2, \ldots, r + 2\}$ with $r + 2$ are: $(r + 1, r + 2), (r, r + 2), \ldots, (1, r + 2)$.

$$\begin{aligned}
\varphi_2\ (r+1, r+2) &= (r+1, r+2), (r, r+2) \\
\varphi_2\ (r, r+2) &= (r+1, r+2), (r-1, r+2) \\
&\vdots \\
\varphi_2\ (2, r+2) &= (r+1, r+2), (1, r+2)
\end{aligned}$$

and

$$\varphi_1 \ (1, r+2) \ = \ (r+1, r+2)$$

from which we get a system of the linear recurrence equations:

$$\begin{aligned}
a_{(r+1,r+2)}(n+1) &= a_{(r+1,r+2)}(n) + a_{(r,r+2)}(n) \\
a_{(r,r+2)}(n+1) &= a_{(r+1,r+2)}(n) + a_{(r-1,r+2)}(n) \\
&\vdots \\
a_{(2,r+2)}(n+1) &= a_{(r+1,r+2)}(n) + a_{(1,r+2)}(n) \\
a_{(1,r+2)}(n+1) &= a_{(r+1,r+2)}(n)
\end{aligned}$$

If we substitute $a_{(r+1,r+2)}(n)$ with a_n, we get Fibonacci $(r+1)$-step numbers (see [**23**]; in [**13**] they are called $(r+1)$-Fibonacci numbers):

$$a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-r-1}$$

with the initial conditions $a_0 = 1$, $a_i = 2^{i-1}$ for $i = 1, \ldots, r$ (for $r = 1$ we get Fibonacci numbers, for $r = 2$ Tribonacci numbers and for $r = 3$ Tetranacci numbers and so on).

The generating function is $A(z) = \dfrac{1}{1 - z - z^2 - \cdots - z^{r+1}}$.

Theorem 3. *The number of permutations of type* $(R_4^{(r+1)}, n)$, *of the set* $\mathbb{N}_n$ (*mark* n *goes to the first place and all other marks move right not more than* $r+1$ *places; see* [**10**]) *is equal to the number of the permutations satisfying the condition* $-1 \le p(i) - i \le r$, *for all* $i \in \mathbb{N}_{n-1}$. *That number is the Fibonacci* $(r+1)$*-step number.*

Proof. Let us denote a set of all permutations satisfying $-1 \le p(i) - i \le r$, for all $i \in \mathbb{N}_{n-1}$ with $(R_1^{(1,r)}, n-1)$. We have the bijection

$$\Phi : \big(R_4^{(r+1)}, n\big) \to \big(R_1^{(1,r)}, n-1\big)$$

given with

$$\Phi\big((n, p_2, p_3, \ldots, p_n)\big) = (p_2, p_3, \ldots, p_n).$$

In Example 9 we proved that the number of the permutations satisfying the condition $-1 \le p(i) - i \le r$ is the Fibonacci $(r+1)$-step number. □

6. COMPUTATIONAL COMPLEXITY

Next, we compare the computational complexity of our technique and some straightforward techniques for computing the number of restricted permutations. We can compute the number of restricted permutations of length n by generating all $n!$ permutations and counting those that satisfy the required conditions. We can also compute the number of restricted permutations by expanding the permanent per A according to the definition. Both of these techniques require $O(n!)$ operations. In contrast, we can compute the number of restricted permutations as the element

in the first row and the first column of the matrix S^n, where S is the matrix of the associated system of linear recurrence equations. S^n can be computed with repeated squaring [**3**] in $O(\log_2 n)$ operations. Hence, our technique evaluates the number of restricted permutations more efficiently than the straightforward techniques of filtering permutations or expanding the permanent per A.
Stanley's transfer-matrix method has the same computational complexity, $O(\log_2 n)$. Our technique goes further because for small values of k and r the order of the matrix S is less than the order of adjacency matrix A from [**19**, Example 4.7.7].

All the generating functions that we derive using our technique are rational. We have a system of $\binom{k+r}{k}$ linear recurrence equations which leads us to the upper bound for the degree d of the denominator polynomial: $d \leq \binom{k+r}{k}$. It is sufficient to compute a finite number of values, concretely $\binom{k+r}{k}$ of them, to find the generating function.

7. CONCLUDING REMARKS

We have developed a technique for generating a system of the linear recurrence equations that enumerate the Lehmer's strongly restricted permutations deployed on a line. We expect that our technique can be applied to other enumeration problems and may have applications in the theory of rook and hit polynomials. Using a program that implements our technique, we have contributed over sixty sequences to the Sloane's online encyclopedia of integer sequences [**22**]:

A072827,A072850,A072852–A072856,A079955–A080014.

We have also provided additional comments for the following sequences:

A000073,A000078,A001591,A001592

Acknowledgements. We are grateful to Darko Marinov and Dragan Stevanović for very valuable discussions. We would also like to thank Vladeta Jovović, Petar Marković, Marija Kolundžija, Vladimir Božin and Magda von Burg for comments on an earlier version of this paper. We thank the anonymous referees for a careful reading of the paper and several helpful suggestions.

REFERENCES

1. V. Baltić: *Applications of the finite state automata in the enumerative combinatorics.* Proceedings of XXXVI Symposium on Operational Research, Ivanjica (2009), 155–158.
2. C. A. Charalampides: *Enumerative Combinatorics.* Chapman & Hall/CRC, 2002.
3. T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein: *Introduction to Algorithms.* The MIT Press, 2001.
4. I. Kaplansky: *On a generalization of the "problème des rencontres".* Amer. Math. Monthly, **46** (1939), 159–161.

5. I. Kaplansky: *Solution of the "problème des ménages".* Bull. Amer. Math. Soc., **49** (1943), 784–785.
6. I. Kaplansky, J. Riordan: *The problem of the rooks and its applications.* Duke Math. J., **13** (1946), 259–268.
7. T. Kløve: *Spheres of permutations under the infinity norm - Permutations with limited displacement.* Reports in Informatics, Dept. of Informatics, Univ. Bergen, Report no. 376, Online: http://www.ii.uib.no/publikasjoner/texrap/pdf/2008-376.pdf
8. T. Kløve: *Generating functions for the number of permutations with limited displacement.* The Electronic Journal of Combinatorics, **16** (2009), #R104.
9. M. R. Lagrange: *Quelques résultats dans la métrique des permutations.* Annales Scientifiques de l'École Normale Supérieure, Paris, **79**(3) (1962), 199–241.
10. D. H. Lehmer: *Permutations with strongly restricted displacements.* Combinatorial Theory and its appl II (Proc. Colloq., Balatonfured, 1969), North-Holland, Amsterdam, (1970), 755–770.
11. L. Li, T. Tomoda, S. Midorikawa, T. Horibata: *A Counting Formula for the Number of Permutations with Constraints.* ZAMM – Z. angew. Math. Mech. **76** (1996) S3, 495–496.
12. M. Lothaire: *Algebraic Combinatorics on Words.* Cambridge University Press, 2002.
13. W. C. Lynch: *The t-Fibonacci numbers and polyphase sorting.* The Fibonacci Quarterly, **8** (1970), 6–22.
14. N. S. Mendelsohn: *The asymptotic series for a certain class of permutation problems.* Canadian J. Math, **8** (1956), 234–244.
15. N. S. Mendelsohn: *Permutations with confined displacements.* Canadian Math. Bulletin, **4** (1961), 29–38.
16. H. Minc: *Permanents.* Addison–Wesley Publishing Company, 1978.
17. A. Panholzer: *On generalized Fibonacci Permutations.* Journal of Information & Optimization Sciences, **24** (2003), 591–610.
18. J. Riordan: *An Introduction to Combinatorial Analysis.* John Wiley & Sons, Inc, 1958.
19. R. P. Stanley: *Enumerative Combinatorics, Vol. 1.* Wadsworth, 1999.
20. I. Tomescu: *Problems in combinatorics and graph theory.* Wiley–Interscience Series in Discrete Mathematics, New York, 1985. p 60, 292–295.
21. K-W. Yang: *q–determinants and permutations.* The Fibonacci Quarterly, **29** (1991), 160–163.
22. N. J. A. Sloane: *Sloane's Encyclopedia of Integer Sequences.* http://www.research.att.com/~njas/sequences/index.html
23. E. W. Weisstein: *The World of Mathematics.* http://mathworld.wolfram.com/Fibonaccin-StepNumber.html

University of Belgrade,
Faculty of Organizational Sciences,
Jove Ilića 154,
11000 Belgrade,
Serbia
E-mail: baltic@matf.bg.ac.rs

(Received June 26, 2009)
(Revised October 7, 2009)