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Pell–Padovan-circulant sequences and their applications

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Abstract: This paper develops properties of recurrence sequences defined from circulant matrices obtained from the characteristic polynomial of the Pell–Padovan sequence. The study of these sequences modulo m yields cyclic groups and semigroups from the generating matrices. Finally, we obtain the lengths of the periods of the extended sequences in the extended triangle groups $E(2, n, 2)$, $E(2, 2, n)$ and $E(n, 2, 2)$ for $n \geq 3$ as applications of the results obtained.
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1 Introduction

The Pell–Padovan sequence $\{P(n)\}$ is defined [24,25] by a third-order recurrence equation:

$$P(n+3) = 2P(n+1) + P(n) \tag{1.1}$$

for $n \geq 0$, where $P(0) = P(1) = P(2) = 1$. The characteristic polynomial of the sequence is then

$$f(x) = x^3 - 2x - 1.$$

The circulant (or Toeplitz) matrix $C_n = \left[c_{ij}\right]_{n\times n}$ associated with the numbers $c_0, c_1, \cdots, c_{n-1}$ is defined as follows [8]:

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & \cdots & c_3 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

so that the $(n-1)$th degree polynomial

$$P(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$$

is called the associated polynomial of the circulant matrix C_n [cf. 3,4,16,21,23].

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding *k* terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [17], Kalman developed a number of closed-form formulas for this generalized sequence by the companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied by many authors [18, 26, 27, 29, 30, 31, 32, 34]. In Section 2, we define the generalized Pell–Padovan-circulant sequence and the Pell–Padovan-circulant sequences of the first, second and fourth kind such that these sequences are obtained from the circulant matrix C_4^P which are defined by using the characteristic polynomial of the Pell–Padovan sequence. Then we develop some their miscellaneous properties.

In [9, 10, 11, 12, 13, 20], the authors derived the cyclic groups and the semigroups via some special matrices. In Section 3, we consider the cyclic groups and semigroups which are generated by the multiplicative orders of the circulant matrix C_4^P and the generating matrices of the Pell–Padovan-circulant sequences of the first, second and fourth kind when read modulo *m*. Also, we study Pell–Padovan-circulant sequences and the Pell–Padovan-circulant sequences of the first, second and fourth kind modulo m and then we develop relationships among the orders of the cyclic groups obtained and the periods of these sequences.

The study of recurrence sequences in groups began with the earlier work of Wall [33] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [2, 5, 9, 12, 13, 14, 15, 19, 20, 22, 28]. In Section 4, we define the Pell–Padovan-circulant orbit by means of the elements of the groups which have two or more generators, and then we examine this sequence in finite groups. Furthermore, we examine the behaviours of the lengths of the periods of the Pell–Padovan-circulant orbits of the extended triangle groups $E(2, n, 2)$, $E(2, 2, n)$ and $E(n, 2, 2)$ for $n \geq 3$.

2 The Pell–Padovan-circulant sequences

The circulant matrix C_4^P for the polynomial $f(x)$ is as follows:

$$C_4^P = \begin{bmatrix} -1 & 1 & 0 & -2 \\ -2 & -1 & 1 & 0 \\ 0 & -2 & -1 & 1 \\ 1 & 0 & -2 & -1 \end{bmatrix}.$$

We now define new sequences, the generalized Pell–Padovan-circulant sequences, by the recurrence relations of orders 4, 5 and 7 (in which the second and the third are essentially the same):

$$P_n^c = \begin{cases} -2P_{n-1}^c + P_{n-3}^c - P_{n-4}^c, & n \equiv 1 \bmod 4, \\ P_{n-3}^c - P_{n-4}^c - 2P_{n-5}^c, & n \equiv 2 \bmod 4, \\ P_{n-3}^c - P_{n-4}^c - 2P_{n-5}^c, & n \equiv 3 \bmod 4, \\ -P_{n-4}^c - 2P_{n-5}^c + P_{n-7}^c, & n \equiv 0 \bmod 4 \end{cases} \quad \text{for } n > 4, \tag{2.1}$$

where $P_1^c = P_2^c = P_3^c = 0$ and $P_4^c = 1$. It can be readily established by mathematical induction that for $n \geq 0$,

$$\left(C_4^P\right)^n = \begin{bmatrix} P_{4n+4}^c & P_{4n+3}^c & P_{4n+2}^c & P_{4n+1}^c \\ P_{4n+1}^c & P_{4n+4}^c & P_{4n+3}^c & P_{4n+2}^c \\ P_{4n+2}^c & P_{4n+1}^c & P_{4n+4}^c & P_{4n+3}^c \\ P_{4n+3}^c & P_{4n+2}^c & P_{4n+1}^c & P_{4n+4}^c \end{bmatrix}, \tag{2.2}$$

from which it is clear that $\det C_4^P = 0$. From (2.1) we define the Pell–Padovan-circulant sequences of the first, second and fourth kind respectively by:

$$P_n^1 = -2P_{n-1}^1 + P_{n-3}^1 - P_{n-4}^1 \text{ for } n > 4 \text{ where } P_1^1 = P_2^1 = P_3^1 = 0 \text{ and } P_4^1 = 1, \tag{2.3}$$

$$P_n^2 = P_{n-3}^2 - P_{n-4}^2 - 2P_{n-5}^2 \text{ for } n > 5 \text{ where } P_1^2 = \cdots = P_4^2 = 0 \text{ and } P_5^2 = 1 \tag{2.4}$$

$$P_n^4 = -P_{n-4}^4 - 2P_{n-5}^4 + P_{n-7}^4 \text{ for } n > 7 \text{ where } P_1^4 = \cdots = P_6^4 = 0 \text{ and } P_7^4 = 1 \qquad (2.5)$$

The generating functions of the Pell–Padovan-circulant sequences of the first, second and fourth kind are then:

$$f^{(1)}(x) = \frac{x^3}{x^4 - x^3 + 2x + 1},$$

$$f^{(2)}(x) = \frac{x^4}{2x^5 + x^4 - x^3 + 1}$$

and

$$f^{(4)}(x) = \frac{x^6}{-x^7 + 2x^5 + x^4 + 1}.$$

By (2.3), (2.4) and (2.5), we can write the following companion matrices:

$$M_P^{(1)} = \begin{bmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$M_P^{(2)} = \begin{bmatrix} 0 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M_P^{(4)} = \begin{bmatrix} 0 & 0 & 0 & -1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

and we call the matrices $M_P^{(1)}$, $M_P^{(2)}$ and $M_P^{(4)}$ Pell–Padovan-circulant matrices of the first, second and fourth kind. Again by an inductive argument, we may write

$$\left(M_P^{(1)}\right)^n = \begin{bmatrix} P_{n+4}^1 & P_{n+2}^1 - P_{n+1}^1 & P_{n+3}^1 - P_{n+2}^1 & -P_{n+3}^1 \\ P_{n+3}^1 & P_{n+1}^1 - P_n^1 & P_{n+2}^1 - P_{n+1}^1 & -P_{n+2}^1 \\ P_{n+2}^1 & P_n^1 - P_{n-1}^1 & P_{n+1}^1 - P_n^1 & -P_{n+1}^1 \\ P_{n+1}^1 & P_{n-1}^1 - P_{n-2}^1 & P_n^1 - P_{n-1}^1 & -P_n^1 \end{bmatrix}, \qquad (2.6)$$

$$
\left(M_P^{(2)}\right)^n = \begin{bmatrix} P_{n+5}^2 & P_{n+6}^2 & P_{n+7}^2 & P_{n+8}^2 - P_{n+5}^2 & -2P_{n+4}^2 \\ P_{n+4}^2 & P_{n+5}^2 & P_{n+6}^2 & P_{n+7}^2 - P_{n+4}^2 & -2P_{n+3}^2 \\ P_{n+3}^2 & P_{n+4}^2 & P_{n+5}^2 & P_{n+6}^2 - P_{n+3}^2 & -2P_{n+2}^2 \\ P_{n+2}^2 & P_{n+3}^2 & P_{n+4}^2 & P_{n+5}^2 - P_{n+2}^2 & -2P_{n+1}^2 \\ P_{n+1}^2 & P_{n+2}^2 & P_{n+3}^2 & P_{n+4}^2 - P_{n+1}^2 & -2P_{n}^2 \end{bmatrix} \tag{2.7}
$$

and

$$
\left(M_P^{(4)}\right)^n = \begin{bmatrix} P_{n+7}^4 & P_{n+8}^4 & P_{n+9}^4 & P_{n+10}^4 & -2P_{n+6}^4 + P_{n+4}^4 & P_{n+5}^4 & P_{n+6}^4 \\ P_{n+6}^4 & P_{n+7}^4 & P_{n+8}^4 & P_{n+9}^4 & -2P_{n+5}^4 + P_{n+3}^4 & P_{n+4}^4 & P_{n+5}^4 \\ P_{n+5}^4 & P_{n+6}^4 & P_{n+7}^4 & P_{n+8}^4 & -2P_{n+4}^4 + P_{n+2}^4 & P_{n+3}^4 & P_{n+4}^4 \\ P_{n+4}^4 & P_{n+5}^4 & P_{n+6}^4 & P_{n+7}^4 & -2P_{n+3}^4 + P_{n+1}^4 & P_{n+2}^4 & P_{n+3}^4 \\ P_{n+3}^4 & P_{n+4}^4 & P_{n+5}^4 & P_{n+6}^4 & -2P_{n+2}^4 + P_{n}^4 & P_{n+1}^4 & P_{n+2}^4 \\ P_{n+2}^4 & P_{n+3}^4 & P_{n+4}^4 & P_{n+5}^4 & -2P_{n+1}^4 + P_{n-1}^4 & P_{n}^4 & P_{n+1}^4 \\ P_{n+1}^4 & P_{n+2}^4 & P_{n+3}^4 & P_{n+4}^4 & -2P_{n}^4 + P_{n-2}^4 & P_{n-1}^4 & P_{n}^4 \end{bmatrix}. \tag{2.8}
$$

Note that $\det M_P^{(1)} = \det M_P^{(4)} = 1$ and $\det M_P^{(2)} = -2$. It is well-known that the Simson formula for a recurrence sequence can be obtained from the determinant of its generating matrix, so that the Simpon formula for the generalized Pell–Padovan-circulant sequences is as follows:

$$
\begin{gathered}
\left(\left(P_{4n+2}^C - P_{4n+4}^C\right)\left(P_{4n+2}^C + P_{4n+4}^C\right) - \left(P_{4n+1}^C - P_{4n+3}^C\right)\left(P_{4n+1}^C + P_{4n+3}^C\right)\right)\cdot \\
\cdot\left(\left(P_{4n+2}^C - P_{4n+4}^C\right)\left(P_{4n+2}^C + P_{4n+4}^C\right) + \left(P_{4n+1}^C - P_{4n+3}^C\right)\left(P_{4n+1}^C + P_{4n+3}^C\right)\right) + \\
+4P_{4n+2}^C P_{4n+4}^C \left(\left(P_{4n+1}^C\right)^2 + \left(P_{4n+3}^C\right)^2\right) - 4P_{4n+1}^C P_{4n+3}^C \left(\left(P_{4n+2}^C\right)^2 + \left(P_{4n+4}^C\right)^2\right) = 0
\end{gathered}
$$

It is easy to see that the characteristic equations of the Pell–Padovan-circulant sequences of the first, second and fourth kind do not have multiple roots; that is, each of the eingenvalues of the matrices $M_P^{(1)}$, $M_P^{(2)}$ and $M_P^{(4)}$ are distinct.

Let $\left\{\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \alpha_4^{(1)}\right\}$, $\left\{\alpha_1^{(2)}, \alpha_2^{(2)}, \alpha_3^{(2)}, \alpha_4^{(2)}, \alpha_5^{(2)}\right\}$ and $\left\{\alpha_1^{(4)}, \alpha_2^{(4)}, \alpha_3^{(4)}, \alpha_4^{(4)}, \alpha_5^{(4)}, \alpha_6^{(4)}, \alpha_7^{(4)}\right\}$ be the sets of the eingevalues of the matrices $M_P^{(1)}$, $M_P^{(2)}$ and $M_P^{(4)}$, respectively and let $V^{(k)}$ be a $(k+3)\times(k+3)$ Vandermonde matrix as follows:

$$
V^{(k)} = \begin{bmatrix} \left(\alpha_1^{(k)}\right)^{k+2} & \left(\alpha_2^{(k)}\right)^{k+2} & \cdots & \left(\alpha_{k+2}^{(k)}\right)^{k+2} & \left(\alpha_{k+3}^{(k)}\right)^{k+2} \\ \left(\alpha_1^{(k)}\right)^{k+1} & \left(\alpha_2^{(k)}\right)^{k+1} & \ldots & \left(\alpha_{k+2}^{(k)}\right)^{k+1} & \left(\alpha_{k+3}^{(k)}\right)^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{(k)} & \alpha_2^{(k)} & \cdots & \alpha_{k+2}^{(k)} & \alpha_{k+3}^{(k)} \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix},
$$

where $k = 1, 2, 4$. Suppose now that

$$W_k^i = \begin{bmatrix} \left(\alpha_1^{(k)}\right)^{n+k+3-i} \\ \left(\alpha_2^{(k)}\right)^{n+k+3-i} \\ \vdots \\ \left(\alpha_{k+3}^{(k)}\right)^{n+k+3-i} \end{bmatrix}$$

and $V_j^{(k,i)}$ is a $(k+3)\times(k+3)$ matrix obtained from $V^{(k)}$ by replacing the jth column of $V^{(k)}$ by W_k^i. This yields the Binet-type formulas for the Pell–Padovan-circulant sequences of the first, second and fourth kind, namely:

Theorem 2.1. Let P_n^k be the nth term of the sequence of the kth kind for $k = 1, 2, 4$. Then

$$m_{ij}^{(k,n)} = \frac{\det V_j^{(k,i)}}{\det V^{(k)}}$$

where $\left(M_P^{(k)}\right)^n = \left[m_{ij}^{(k,n)}\right]$ such that $k = 1, 2, 4$.

Proof. Since the eigenvalues of the matrix $M_P^{(k)}$ are distinct, the matrix $M_P^{(k)}$ is diagonalizable. Let

$$D^{(1)} = \operatorname{diag}\left(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \alpha_4^{(1)}\right),$$

$$D^{(2)} = \operatorname{diag}\left(\alpha_1^{(2)}, \alpha_2^{(2)}, \alpha_3^{(2)}, \alpha_4^{(2)}, \alpha_5^{(2)}\right)$$

and

$$D^{(4)} = \operatorname{diag}\left(\alpha_1^{(4)}, \alpha_2^{(4)}, \alpha_3^{(4)}, \alpha_4^{(4)}, \alpha_5^{(4)}, \alpha_6^{(4)}, \alpha_7^{(4)}\right),$$

then it is readily seen that $M_P^{(k)}V^{(k)} = V^{(k)}D^{(k)}$. Since the matrix $V^{(k)}$ is invertible,

$$\left(V^{(k)}\right)^{-1} M_P^{(k)} V^{(k)} = D^{(k)}.$$

Thus, the matrix $M_P^{(k)}$ is similar to $D^{(k)}$. So we get for $n \geq 1$ that

$$\left(M_P^{(k)}\right)^n V^{(k)} = V^{(k)}\left(D^{(k)}\right)^n .$$

Then we can write the following linear system of equations for $n \geq 1$:

$$\begin{aligned} m_{i1}^{(k,n)}\left(\alpha_1^{(k)}\right)^{k+2} + m_{i2}^{(k,n)}\left(\alpha_1^{(k)}\right)^{k+1} + \cdots + m_{ik+3}^{(k,n)} &= \left(\alpha_1^{(k)}\right)^{n+k+3-i} \\ m_{i1}^{(k,n)}\left(\alpha_2^{(k)}\right)^{k+2} + m_{i2}^{(k,n)}\left(\alpha_2^{(k)}\right)^{k+1} + \cdots + m_{ik+3}^{(k,n)} &= \left(\alpha_2^{(k)}\right)^{n+k+3-i} \\ &\vdots \\ m_{i1}^{(k,n)}\left(\alpha_{k+3}^{(k)}\right)^{k+2} + m_{i2}^{(k,n)}\left(\alpha_{k+3}^{(k)}\right)^{k+1} + \cdots + m_{ik+3}^{(k,n)} &= \left(\alpha_{k+3}^{(k)}\right)^{n+k+3-i} . \end{aligned}$$

from which we obtain

$$m_{ij}^{(k,n)} = \frac{\det V_j^{(k,i)}}{\det V^{(k)}} \text{ for } k = 1, 2, 4 \text{ and } i, j = 1, 2, \ldots, k+3. \qquad \square$$

3 The Pell–Padovan-circulant sequences modulo m

For given a matrix $A = \left[a_{ij}\right]$ with m_{ij} integers, $A\left(\operatorname{mod} m\right)$ means that each element of A is reduced modulo m, that is, $A\left(\operatorname{mod} m\right) = \left(a_{ij}\left(\operatorname{mod} m\right)\right)$. Let us consider the set $\left\langle A\right\rangle_m = \left\{A^i\left(\operatorname{mod} m\right) \middle| i \geq 0\right\}$. If $\gcd\left(m, \det A\right) = 1$, then the set $\left\langle A\right\rangle_m$ is a cyclic group; if $\gcd\left(m, \det A\right) \neq 1$, then the set $\left\langle A\right\rangle_m$ is a semigroup. Let the notation $\left|\left\langle A\right\rangle_m\right|$ denote the order of the set $\left\langle A\right\rangle_m$.

Since $\det C_4^P = 0$, it is clear that the set $\left\langle C_4^P\right\rangle_m$ is a semigroup group for every positive integer m. Similarly, since $\det M_P^{(1)} = \det M_P^{(4)} = 1$, $\left\langle M_P^{(1)}\right\rangle_m$ and $\left\langle M_P^{(4)}\right\rangle_m$ are cyclic groups for every positive integer m. Moreover $\left\langle M_P^{(2)}\right\rangle_m$ is a cyclic group if m is an odd integer and $\left\langle M_P^{(2)}\right\rangle_m$ is a semigroup if m is an even integer.

We next consider the orders of the semigroups and the cyclic groups generated by the matrices C_4^P, $M_P^{(1)}$, $M_P^{(2)}$ and $M_P^{(4)}$.

Theorem 3.1. Let λ be a prime and let $\left\langle G\right\rangle_{\lambda^n}$ be any of the sets of $\left\langle C_4^P\right\rangle_{\lambda^n}$, $\left\langle M_P^{(1)}\right\rangle_{\lambda^n}$, $\left\langle M_P^{(2)}\right\rangle_{\lambda^n}$ and $\left\langle M_P^{(4)}\right\rangle_{\lambda^n}$ such that $n \in N$. If u is the largest positive integer such that $\left|\left\langle G\right\rangle_\lambda\right| = \left|\left\langle G\right\rangle_{\lambda^u}\right|$, then $\left|\left\langle G\right\rangle_{\lambda^v}\right| = \lambda^{v-u} \cdot \left|\left\langle G\right\rangle_\lambda\right|$ for every $v \geq u$. In particular, if $\left|\left\langle G\right\rangle_\lambda\right| \neq \left|\left\langle G\right\rangle_{\lambda^2}\right|$, then $\left|\left\langle G\right\rangle_{\lambda^v}\right| = \lambda^{v-1} \cdot \left|\left\langle G\right\rangle_\lambda\right|$ for every $v \geq 2$.

Proof. We consider the semigroup $\left\langle C_4^P\right\rangle_{\lambda^n}$. Suppose that a is a positive integer and $\left|\left\langle C_4^P\right\rangle_{\lambda^n}\right|$ is denoted by $h\left(\lambda^n\right)$. If $\left(C_4^P\right)^{h\left(\lambda^{a+1}\right)} \equiv I\left(\operatorname{mod} \lambda^{a+1}\right)$, then $\left(C_4^P\right)^{h\left(\lambda^{a+1}\right)} \equiv I\left(\operatorname{mod} \lambda^{a}\right)$ where I is a 4×4 identity matrix and sot $h\left(\lambda^a\right) \mid h\left(\lambda^{a+1}\right)$. Furthermore, if we denote

$$\left(C_4^P\right)^{h\left(\lambda^a\right)} = I + \left(c_{ij}^{(a)} \cdot \lambda^a\right),$$

then by the binomial expansion, we obtain

$$\left(C_4^P\right)^{h\left(\lambda^a\right)\cdot\lambda} = \left(I + \left(c_{ij}^{(a)} \cdot \lambda^a\right)\right)^\lambda = \sum_{i=0}^{\lambda} \binom{\lambda}{i} \left(c_{ij}^{(a)} \cdot \lambda^a\right)^i \equiv I\left(\operatorname{mod} \lambda^{a+1}\right),$$

which yields that $h\left(\lambda^{a+1}\right) \mid h\left(\lambda^a\right)\cdot\lambda$. Thus, $h\left(\lambda^{a+1}\right) = h\left(\lambda^a\right)$ or $h\left(\lambda^{a+1}\right) = h\left(\lambda^a\right)\cdot\lambda$.
It is clear that $h\left(\lambda^{a+1}\right) = h\left(\lambda^a\right)\cdot\lambda$ holds if and only if there exists an integer $c_{ij}^{(a)}$ which is not divisible by λ. Since u is the largest positive integer such that $h\left(\lambda\right) = h\left(\lambda^u\right)$, we have

$h\left(\lambda^{u}\right) \neq h\left(\lambda^{u+1}\right)$. Then, there exists an integer $c_{ij}^{(u+1)}$ which is not divisible by λ. So we find that $h\left(\lambda^{u+1}\right) \neq h\left(\lambda^{u+2}\right)$. To complete the proof we use an inductive method on u.
There are similar proofs for the sets $\left\langle M_P^{(1)}\right\rangle_{\lambda^n}$, $\left\langle M_P^{(2)}\right\rangle_{\lambda^n}$ and $\left\langle M_P^{(4)}\right\rangle_{\lambda^n}$. □

Theorem 3.2. Let m be an positive integer and let $\langle G\rangle_m$ be any of the sets of $\left\langle C_4^P\right\rangle_m$, $\left\langle M_P^{(1)}\right\rangle_m$, $\left\langle M_P^{(2)}\right\rangle_m$ and $\left\langle M_P^{(4)}\right\rangle_m$. Suppose that $m=\prod_{i=1}^{k}\lambda_i^{e_i},(k\geq 1)$ where λ_i's are distinct primes. Then $\left|\langle G\rangle_m\right| = \operatorname{lcm}\left[\langle G\rangle_{\lambda_1^{e_1}},\langle G\rangle_{\lambda_2^{e_2}},\ldots,\langle G\rangle_{\lambda_k^{e_k}}\right]$.

Proof. Let us consider the cyclic group $\left\langle M_P^{(1)}\right\rangle_m$. Suppose that $\left|\left\langle M_P^{(1)}\right\rangle_{\lambda_i^{e_i}}\right| = \beta_i$ for $1\leq i\leq k$ and let $\left|\left\langle M_P^{(1)}\right\rangle_m\right| = \beta$. Then by (2.6), we obtain in turn

$$\begin{aligned}
P_{\beta_i+4}^1 &\equiv 1 \bmod p_i^{e_i},\\
P_{\beta_i+3}^1 &\equiv 0 \bmod p_i^{e_i},\\
P_{\beta_i+2}^1 &\equiv 0 \bmod p_i^{e_i},\\
P_{\beta_i+1}^1 &\equiv 0 \bmod p_i^{e_i},\\
P_{\beta_i}^1 &\equiv -1 \bmod p_i^{e_i}
\end{aligned}$$

and

$$\begin{aligned}
P_{\beta+4}^1 &\equiv 1 \bmod m,\\
P_{\beta+3}^1 &\equiv 0 \bmod m,\\
P_{\beta+2}^1 &\equiv 0 \bmod m,\\
P_{\beta+1}^1 &\equiv 0 \bmod m,\\
P_{\beta}^1 &\equiv -1 \bmod m.
\end{aligned}$$

This implies that $P_{\beta+u}^1 = \lambda\cdot P_{\beta_i+u}^1$ for $0\leq u\leq 4$ and $\lambda\in N$ that is, $\left(M_P^{(1)}\right)^{\beta}$ is the form $t\cdot\left(M_P^{(1)}\right)^{\beta_i}$ for all values of i. Thus it is verified that

$$\left|\left\langle M_P^{(1)}\right\rangle_m\right| = \operatorname{lcm}\left[\left\langle M_P^{(1)}\right\rangle_{p_1^{e_1}},\left\langle M_P^{(1)}\right\rangle_{p_2^{e_2}},\ldots,\left\langle M_P^{(1)}\right\rangle_{p_k^{e_k}}\right].$$

There are similar proofs for the sets $\left\langle C_4^P\right\rangle_m$, $\left\langle M_P^{(2)}\right\rangle_m$ and $\left\langle M_P^{(4)}\right\rangle_m$. □

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period *k* if the first *k* elements in the sequence form a repeating subsequence.

If we reduce the generalized Pell–Padovan-circulant sequences and the Pell–Padovan-circulant sequences of the first, second and fourth kind by a modulus *m*, then we get the repeating sequences, respectively denoted by

$$\left\{P_n^c(m)\right\}=\left\{P_1^c(m), P_2^c(m), P_3^c(m), \ldots, P_i^c(m), \ldots\right\}$$

and

$$\left\{P_n^k(m)\right\}=\left\{P_1^k(m), P_2^k(m), P_3^k(m), \ldots, P_i^k(m), \ldots\right\},$$

where $P_i^c(m)=P_i^c(\bmod m)$, $P_i^k(m)=P_i^k(\bmod m)$ and $k=1,2,4$. They have the same recurrence relations as in (2.1), (2.3), (2.4) and (2.5), respectively.

Theorem 3.3. The cases of the sequences $\left\{P_n^c\right\}$, $\left\{P_n^1\right\}$, $\left\{P_n^2\right\}$ and $\left\{P_n^4\right\}$ (modulo *m*) are:

- The sequence $\left\{P_n^c(m)\right\}$ is periodic for every positive integer *m*.
- The sequences $\left\{P_n^1(m)\right\}$ and $\left\{P_n^4(m)\right\}$ are simply periodic for every positive integer m.
- The sequence $\left\{P_n^2(m)\right\}$ is periodic for every positive integer m. In particular, if $\gcd(2,m)=1$, then the sequence $\left\{P_n^2(m)\right\}$ will be simply periodic.

Proof. Let us consider the Pell–Padovan-circulant sequence of the fourth kind $\left\{P_n^4(m)\right\}$ as an example. Let $S=\left\{(s_1,s_2,s_3,s_4,s_5,s_6,s_7) \mid 0\le s_i\le m-1\right\}$, then $|S|=m^7$. Since there are m^7 distinct 7-tuples of elements of $\mathbb{Z}_m$, at least one of the 7-tuples appears twice in the sequence $\left\{P_n^4(m)\right\}$. Thus, the subsequence following this 7-tuple repeats; that is the sequence is periodic. Let $P_{i+7}^4(m)\equiv P_{j+7}^4(m)$, $P_{i+6}^4(m)\equiv P_{j+6}^4(m), \ldots,$ $P_{i+1}^4(m)\equiv P_{j+1}^4(m)$ and $i>j$, then $i\equiv j \bmod 7$. From the definition, we can easily derive that

$$P_i^4(m)\equiv P_j^4(m),\ P_{i-1}^4(m)\equiv P_{j-1}^4(m),\ \ldots,\ P_{i-j+2}^4(m)\equiv P_2^4(m),\ P_{i-j+1}^4(m)\equiv P_1^4(m).$$

So we get that the sequence is a simply periodic.

There are similar proofs for the sequences $\left\{P_n^c(m)\right\}$, $\left\{P_n^1(m)\right\}$ and $\left\{P_n^2(m)\right\}$. □

We next denote the periods of the sequences $\left\{P_n^c(m)\right\}$, $\left\{P_n^1(m)\right\}$, $\left\{P_n^2(m)\right\}$ and $\left\{P_n^4(m)\right\}$ by $l_P^c(m)$, $l_P^1(m)$, $l_P^2(m)$ and $l_P^4(m)$, respectively, and we present the relationships among the periods $l_P^1(m)$, $l_P^2(m)$, $l_P^4(m)$ and the orders $\left|\left\langle M_P^{(1)}\right\rangle_m\right|$, $\left|\left\langle M_P^{(2)}\right\rangle_m\right|$, $\left|\left\langle M_P^{(4)}\right\rangle_m\right|$, respectively, in the following result.

Corollary 3.1. If λ is a prime such that $\lambda\ne 2$, then the period $l_P^2(\lambda)$ is equal to the order of the cyclic group $\left\langle M_P^{(2)}\right\rangle_\lambda$. Also, $l_P^1(\lambda)=\left|\left\langle M_P^{(1)}\right\rangle_\lambda\right|$ and $l_P^4(\lambda)=\left|\left\langle M_P^{(4)}\right\rangle_\lambda\right|$ for every prime λ.

Proof. This follows directly from (2,6), (2.7) and (2.8). □

Let λ be a prime and let

$$A_1(\lambda)=\{x^n \pmod{\lambda}: n\in \mathbb{Z},\ x^4=-2x^3+x-1\}$$

and

$$A_4(\lambda)=\{x^n \pmod{\lambda}: n\in \mathbb{Z},\ x^7=-x^3-2x^2+1\}$$

such that $u\geq 1$. Then, it is clear that the sets $A_1(\lambda)$ and $A_4(\lambda)$ are cyclic groups.

Now we can give relationships among the characteristic equations of the Pell–Padovan-circulant sequences of the first and fourth kind and the periods $l_P^1(m)$ and $l_P^4(m)$.

Corollary 3.2. Let λ be a prime. Then, the cyclic groups $A_1(\lambda)$ and $A_4(\lambda)$ are isomorphic to the cyclic groups $\langle M_P^{(1)}\rangle_\lambda$ and $\langle M_P^{(4)}\rangle_\lambda$, respectively. □

4 The Pell–Padovan-circulant sequences in groups

In this section, we define the Pell–Padovan-circulant orbit by means of the elements of the groups which have two or more generators, and then we examine this sequence in finite groups. Finally, we obtain the lengths of the periods of the Pell–Padovan-circulant orbits of the extended triangle groups $E(2, n, 2)$, $E(2, 2, n)$ and $E(n, 2, 2)$ for $n\geq 3$ as applications of the results obtained.

Let G be a finite j-generator group and let X be the subset of $\underbrace{G\times G\times G\times\cdots\times G}_{j}$ such that $(x_1,x_2,\ldots,x_j)\in X$ if, and only if, G is generated by $x_1,x_2,\ldots,x_j$. We call $(x_1,x_2,\ldots,x_j)$ a generating j-tuple for G.

Definition 4.1. Let $G=\langle X\rangle$ be a finitely generated group such that $X=\{x_1,x_2,\ldots,x_j\}$. Then we denote the Pell–Padovan-circulant orbit by means of:

$$x^1_n=(x^1_{n-4})^{-1}(x^1_{n-3})(x^1_{n-1})^{-2}$$

for $n\geq 5$, with initial conditions

$$\begin{cases} x^1_1=(x_1)^{-1},\ x^1_2=x_2,\ x^1_3=x_3,\ x^1_4=x_4 & \text{if } j=4,\\ x^1_1=(x_1)^{-1},\ x^1_2=(x_1)^{-1},\ x^1_3=x_2,\ x^1_4=x_3 & \text{if } j=3,\\ x^1_1=(x_1)^{-1},\ x^1_2=(x_1)^{-1},\ x^1_3=(x_1)^{-1},\ x^1_4=x_2 & \text{if } j=2,\end{cases}$$

For a j-tuple $(x_1,x_2,\ldots,x_j)\in X$, the Pell–Padovan-circulant orbit is denoted by $P^c_{(x_1,x_2,\ldots,x_j)}(G)$.

Theorem 4.1. The Pell–Padovan-circulant orbit of a finite group is simply periodic.

Proof. Suppose that n is the order of G. Since there n^4 distinct 4-tuples of elements of G, at least one of the 4-tuples appears twice in the sequence $P^c_{(x_1,x_2,\ldots,x_j)}(G)$. Thus, consider the subsequence following this 4-tuple. Because of the repetition, the sequence is periodic. Since the orbit $P^c_{(x_1,x_2,\ldots,x_j)}(G)$ is periodic, there exist natural numbers u and v, with $u \geq v$, such that

$$x^1_{u+1} = x^1_{v+1},\ x^1_{u+2} = x^1_{v+2},\ x^1_{u+3} = x^1_{v+3} \text{ and } x^1_{u+4} = x^1_{v+4}.$$

By the definition of the sequence $P^c_{(x_1,x_2,\ldots,x_j)}(G)$, we know that

$$\left(x^1_{n-4}\right) = \left(x^1_{n-3}\right)\left(x^1_{n-1}\right)^{-2}\left(x^1_{n}\right)^{-1}.$$

Therefore, we obtain $x^1_u = x^1_v$, and hence,

$$x^1_{u-(v-1)} = x^1_{v-(v-1)} = x^1_1,\ x^1_{u-(v-2)} = x^1_{v-(v-2)} = x^1_2,\ x^1_{u-(v-3)} = x^1_{v-(v-3)} = x^1_3$$

and

$$x^1_{u-(v-4)} = x^1_{v-(v-4)} = x^1_4,$$

which implies that $P^c_{(x_1,x_2,\ldots,x_j)}(G)$ is a simply periodic sequence. □

Let $LP^c_{(x_1,x_2,\ldots,x_j)}(G)$ denote the length of the period of the orbit $P^c_{(x_1,x_2,\ldots,x_j)}(G)$. From the definition of the orbit $P^c_{(x_1,x_2,\ldots,x_j)}(G)$ it is clear that the period of this sequence in a finite group depend on the chosen generating set and the order in which the assignments of $x_1, x_2, \ldots, x_j$ are made.

The triangle group (polyhedral group) (p, q, r) for p, q, $r > 1$, is defined by the presentation

$$\left\langle x, y, z : x^p = y^q = z^r = xyz = e \right\rangle$$

or

$$\left\langle x, y : x^p = y^q = (xy)^r = e \right\rangle.$$

The triangle group (p,q,r) is finite if and only if the number

$$\mu = pqr\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right) = qr + rp + pq - pqr$$

is positive. Its order is $2pqr/\mu$.

Using Tietze transformations we may show that $(p,q,r) \cong (q,r,p) \cong (r,p,q)$ [6, pp.67–68]. The extended triangle group $E(p, q, r)$ for $p, q, r > 1$ is defined by the presentation

$$\left\langle x, y, z : x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = e \right\rangle.$$

The triangle groups $(p,\ q,\ r)$ are index two subgroups of extended triangle groups $E(p,\ q,\ r)$ [1, 5, 7] and the extended triangle groups are a very important class of groups closely linked to the automorphism groups of regular maps [6].

We now address the periods of the Pell–Padovan-circulant orbits of the extended triangle groups $E(2, n, 2)$, $E(2, 2, n)$ and $E(n, 2, 2)$ for $n \geq 3$.

Theorem 4.2. For $n \geq 3$,

$$LP^{c}_{(x,y,z)}\left(E\left(2,n,2\right)\right) = LP^{c}_{(x,y,z)}\left(E\left(2,2,n\right)\right) = \begin{cases} \dfrac{15n}{2} & \text{if } n \equiv 0 \bmod 4, \\ 15n & \text{if } n \equiv 2 \bmod 4, \\ 30n & \text{otherwise.} \end{cases}$$

Proof. We prove this by direct calculation. Note that $l_P^1(2) = 15$. The orbit $P^{c}_{(x,y,z)}\left(E\left(2,n,2\right)\right)$ is

$$\begin{aligned} &x,\ x,\ y,\ z,\ e,\ xy,\ yz,\ yzy,\ xy,\ xz,\ y,\ yzx,\ zy,\ xyz,\ yzyx,\ x(zy)^2,\ x(yz)^6, \\ &y(zy)^{12},\ y(zy)^3,\ (yz)^8,\ xy(zy)^{22},\ (yz)^9,\ y(zy)^{13},\ xy(zy)^{14},\ xy(zy)^{13}, \\ &y(zy)^4,\ xzy,\ (yz)^3,\ x(yz)^3,\ xy(zy)^{11},\ x(yz)^4,\ x(zy)^8,\ z(yz)^7,\ y(zy)^7,\ \ldots. \end{aligned}$$

Using the above, the sequence becomes:

$$\begin{aligned} &x_1 = x,\ x_2 = x,\ x_3 = y,\ x_4 = z, \ldots, \\ &x_{31} = x(yz)^4,\ x_{32} = x(zy)^8,\ x_{33} = z(yz)^7,\ x_{34} = y(zy)^7, \ldots, \\ &x_{30i+1} = x(yz)^{\beta_1 \cdot 4i},\ x_{30i+2} = x(zy)^{\beta_2 \cdot 4i},\ x_{30i+3} = z(yz)^{\beta_3 \cdot 4i-1},\ x_{30i+4} = y(zy)^{\beta_2 \cdot 4i-1}, \ldots, \end{aligned}$$

where $\beta_1, \beta_2, \beta_3$ and β_4 are positive odd numbers such that $\gcd(\beta_1, \beta_2, \beta_3, \beta_4) = 1$. So we need the smallest integer i such that $4i = n \cdot k$ for $k \in N$.

- If $n \equiv 0 \bmod 4$, then $i = \dfrac{n}{4}$, and we obtain $LP^{c}_{(x,y,z)}\left(E\left(2,n,2\right)\right) = 30 \cdot \dfrac{n}{4} = \dfrac{15n}{2}$.
- If $n \equiv 2 \bmod 4$, then $i = \dfrac{n}{2}$, and we obtain $LP^{c}_{(x,y,z)}\left(E\left(2,n,2\right)\right) = 30 \cdot \dfrac{n}{2} = 15n$.
- If $n \equiv 1 \bmod 4$ or $n \equiv 3 \bmod 4$, then $i = n$, and we obtain $LP^{c}_{(x,y,z)}\left(E\left(2,n,2\right)\right) = 30n$.

There is a similar proof for the orbit $P^{c}_{(x,y,z)}\left(E\left(2,2,n\right)\right)$. □

Theorem 4.3. For $n \geq 3$,

$$LP^{c}_{(x,y,z)}\left(E\left(n,2,2\right)\right) = \begin{cases} \dfrac{15n}{2} & \text{if } n \text{ is even,} \\ 15n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This is similar to the proof of Theorem 4.2. □

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