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The Abundancy index of divisors of odd perfect numbers – Part III

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Abstract: Dris gave numerical bounds for the sum of the abundancy indices of q^k and n^2, where $q^k n^2$ is an odd perfect number, in his master's thesis. In this note, we show that improving the limits for this sum is equivalent to obtaining nontrivial bounds for the Euler prime q.
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1 Introduction

Let $\sigma(N)$ denote the *sum of the divisors* of the natural number N. If $\sigma(N) = 2N$ and N is odd, then N is called an odd perfect number. Denote the *abundancy index* I of N as $I(N) = \sigma(N)/N$.

Euler proved that every odd perfect number N has to have the form $N = q^k n^2$ where q is prime (called the *Euler prime* of N) satisfying $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$.

Dris proved the following lemmas in [2]:

Lemma 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then we have the bounds*

$$L(q) < I(q^k) + I(n^2) \leq U(q),$$

where

$$L(q) = \frac{3q^2 - 4q + 2}{q(q-1)} = 3 - \frac{q-2}{q(q-1)}$$

and

$$U(q) = \frac{3q^2 + 2q + 1}{q(q+1)} = 3 - \frac{q-1}{q(q+1)}.$$

Lemma 1.2. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then we have the numerical bounds*

$$\frac{57}{20} < I(q^k) + I(n^2) < 3,$$

with the further result that they are best-possible.

In this note, we will prove the following results:

Theorem 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $q \leq 97$ if and only if*

$$I(q^k) + I(n^2) \leq \frac{299}{100}.$$

Theorem 1.2. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $q > 5$ if and only if*

$$I(q^k) + I(n^2) > \frac{43}{15}.$$

All of the proofs given in this note are elementary.

2 The proofs of Lemma 1.1 and Lemma 1.2

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

First, since N is perfect and I is (weakly) multiplicative, we have the following equation and inequalities:

$$I(N) = I(q^k n^2) = I(q^k)I(n^2) = 2 \Longleftrightarrow I(n^2) = \frac{2}{I(q^k)}$$

$$\frac{q+1}{q} = I(q) \leq I(q^k) = \frac{\sigma(q^k)}{q^k} = \frac{1+q+\ldots+q^k}{q^k} = \frac{q^{k+1}-1}{q^k(q-1)} < \frac{q^{k+1}}{q^k(q-1)} = \frac{q}{q-1}$$

$$\frac{2(q-1)}{q} < I(n^2) = \frac{2}{I(q^k)} \leq \frac{2q}{q+1}.$$

Notice that $I(q^k) < q/(q-1) \leq 5/4$ since q prime with $q \equiv 1 \pmod 4$ implies that $q \geq 5$, so that we have $I(n^2) = 2/I(q^k) > 8/5 > 5/4 > I(q^k)$.

Here is another way to prove $I(q^k) < I(n^2)$. So from before, we have $I(q^k) < \frac{q}{q-1}$ and $\frac{2(q-1)}{q} < I(n^2)$. Since q is an integer,

$$\frac{q}{q-1} \neq \frac{2(q-1)}{q},$$

because otherwise

$$\left(\frac{q}{q-1}\right)^2 = 2$$

which implies that

$$\frac{q}{q-1} = \sqrt{2}$$

contradicting $q/(q-1)$ is rational.

Now, we want to prove that $I(q^k) < I(n^2)$. It suffices to show that

$$\frac{q}{q-1} < \frac{2(q-1)}{q}.$$

Suppose to the contrary that

$$\frac{2(q-1)}{q} < \frac{q}{q-1}.$$

This implies that

$$2(q-1)^2 < q^2$$

which further means that $2q^2 - 4q + 2 < q^2$ or $q^2 - 4q + 2 < 0$. Since $q \geq 5$, $q^2 - 4q + 2 = q(q-4) + 2 \geq 5(5-4) + 2 = 7$, which is a contradiction. This then gives an alternative (albeit longer) proof for $I(q^k) < I(n^2)$.

To summarize what we have so far:

$$1 < \frac{q+1}{q} \leq I(q^k) < \frac{q}{q-1} < \frac{5}{4} < \frac{8}{5} < \frac{2(q-1)}{q} < I(n^2) \leq \frac{2q}{q+1} < 2.$$

Note that, when $k = 1$, we have the slightly stronger bounds:

$$I(q^k) = I(q) = \frac{q+1}{q} = 1 + \frac{1}{q} \leq \frac{6}{5} < \frac{5}{3} \leq I(n^2) = \frac{2}{I(q^k)} = \frac{2}{I(q)}.$$

We want to show first that

$$L(q) < I(q^k) + I(n^2) \leq U(q),$$

where

$$L(q) = \frac{3q^2 - 4q + 2}{q(q-1)} = 3 - \frac{q-2}{q(q-1)}$$

and

$$U(q) = \frac{3q^2 + 2q + 1}{q(q+1)} = 3 - \frac{q-1}{q(q+1)}.$$

Consider the product

$$\left(I(q^k) - \frac{q}{q-1}\right) \cdot \left(I(n^2) - \frac{q}{q-1}\right).$$

This product is negative since $I(q^k) < q/(q-1) < I(n^2)$. Therefore,

$$2 + \left(\frac{q}{q-1}\right)^2 = I(q^k)I(n^2) + \left(\frac{q}{q-1}\right)^2 < \frac{q}{q-1} \cdot \left(I(q^k) + I(n^2)\right)$$

from which we obtain

$$\frac{2(q-1)}{q} + \frac{q}{q-1} < I(q^k) + I(n^2).$$

Finally, we have

$$L(q) = \frac{2(q-1)}{q} + \frac{q}{q-1} = \frac{3q^2 - 4q + 2}{q(q-1)} = 3 - \frac{q-2}{q(q-1)},$$

as desired.

Next, consider the product

$$\left(I(q^k) - \frac{q+1}{q}\right) \cdot \left(I(n^2) - \frac{q+1}{q}\right).$$

This product is nonnegative since $(q+1)/q \le I(q^k) < I(n^2)$. Consequently, we have

$$2 + \left(\frac{q+1}{q}\right)^2 = I(q^k)I(n^2) + \left(\frac{q+1}{q}\right)^2 \ge \frac{q+1}{q} \cdot \left(I(q^k) + I(n^2)\right)$$

from which we obtain

$$\frac{2q}{q+1} + \frac{q+1}{q} \ge I(q^k) + I(n^2).$$

Finally, we have

$$U(q) = \frac{2q}{q+1} + \frac{q+1}{q} = \frac{3q^2+2q+1}{q(q+1)} = 3 - \frac{q-1}{q(q+1)},$$

as desired.

This proves Lemma 1.1.

(We follow the discussion in [1] for our proof of Lemma 1.2 here.)

By using a method similar to that used to prove Lemma 1.1, we can show that

$$\frac{57}{20} < I(q^k) + I(n^2) < 3.$$

We now prove that these bounds are best-possible. It suffices to get the minimum value for $L(q)$ and the maximum value for $U(q)$ in the interval $[5, \infty)$, or if either one cannot be obtained, the greatest lower bound for $L(q)$ and the least upper bound for $U(q)$ for the same interval would likewise be useful for our purposes here.

From basic calculus, we get the first derivatives of $L(q)$ and $U(q)$ and determine their signs in the interval $[5, \infty)$:

$$L'(q) = \frac{q(q-4)+2}{q^2(q-1)^2} > 0$$

and

$$U'(q) = \frac{q(q-2)-1}{q^2(q+1)^2} > 0$$

which means that $L(q), U(q)$ are increasing functions of q on the interval $[5, \infty)$. Hence, $L(q)$ attains its minimum value on that interval at $L(5) = 57/20$, while $U(q)$ has no maximum value on the same interval, but has a least upper bound of

$$\lim_{q\to\infty} U(q) = 3.$$

This confirms our earlier findings that

$$\frac{57}{20} < I(q^k) + I(n^2) < 3$$

with the further result that such bounds are best-possible.

This proves Lemma 1.2.

Note that when $k = 1$ we have the slightly stronger bound $43/15 \le I(q^k) + I(n^2)$.

3 The proofs of Theorem 1.1 and Theorem 1.2

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

First, we want to show that $q \leq 97$ if and only if

$$I(q^k) + I(n^2) \leq \frac{299}{100}.$$

Suppose that $L(q) < I(q^k) + I(n^2) \leq \frac{299}{100}$. This implies that

$$L(q) = 3 - \frac{q-2}{q(q-1)} < \frac{299}{100}$$

$$\frac{1}{100} < \frac{q-2}{q(q-1)}$$

Since $q \geq 5$, we have

$$q^2 - q < 100q - 200$$

$$q^2 - 101q + 200 < 0.$$

This implies that

$$q < \frac{101 + \sqrt{9401}}{2} \approx 98.9794$$

from which we obtain $q \leq 97$ (since q is prime and $q \equiv 1 \pmod 4$).

Does the converse hold? That is, if $N = q^k n^2$ is an odd perfect number with Euler prime q, does $q \leq 97$ imply that $I(q^k) + I(n^2) \leq \frac{299}{100}$?

To this end, assume that $q \leq 97$, and suppose to the contrary that

$$\frac{299}{100} < I(q^k) + I(n^2) \leq U(q).$$

This means that

$$U(q) = 3 - \frac{q-1}{q(q+1)} > \frac{299}{100},$$

so that we obtain

$$\frac{1}{100} > \frac{q-1}{q(q+1)}.$$

Since $q \geq 5$, we get

$$q^2 + q > 100q - 100$$

$$q^2 - 99q + 100 > 0.$$

This implies that

$$q > \frac{99 + \sqrt{9401}}{2} \approx 97.9794$$

contradicting $q \leq 97$.

This proves Theorem 1.1.

We now show that $q > 5$ if and only if

$$I(q^k) + I(n^2) > \frac{43}{15}.$$

Suppose that $U(q) \geq I(q^k) + I(n^2) > 43/15$. This means that

$$U(q) = 3 - \frac{q-1}{q(q+1)} > \frac{43}{15},$$

which implies that

$$\frac{2}{15} > \frac{q-1}{q(q+1)}.$$

Since $q \geq 5$, we have

$$\begin{aligned} 2q^2 + 2q &> 15q - 15 \\ 2q^2 - 13q + 15 &> 0 \\ (2q-3)(q-5) &> 0, \end{aligned}$$

from which we obtain $q > 5$.

Now assume that $q > 5$. We want to prove that $I(q^k) + I(n^2) > 43/15$. Suppose to the contrary that $I(q^k) + I(n^2) \leq 43/15$.

Since q is prime and $q \equiv 1 \pmod 4$, this implies that $q \geq 13$. This implies that we have the bounds

$$I(q^k) < \frac{q}{q-1} \leq \frac{13}{12} < \frac{24}{13} < I(n^2),$$

from which we obtain

$$\left(I(q^k) - \frac{13}{12}\right) \cdot \left(I(n^2) - \frac{13}{12}\right) < 0$$

$$2 + \left(\frac{13}{12}\right)^2 = I(q^k)I(n^2) + \left(\frac{13}{12}\right)^2 < \frac{13}{12} \cdot \left(I(q^k) + I(n^2)\right)$$

$$\frac{24}{13} + \frac{13}{12} < I(q^k) + I(n^2)$$

$$I(q^k) + I(n^2) > \frac{457}{156} \approx 2.929487$$

contradicting

$$I(q^k) + I(n^2) \leq \frac{43}{15} = 2.8\overline{666}.$$

This proves Theorem 1.2.

4 Generalization of Theorem 1.1

It is possible to prove the following generalization of Theorem 1.1:

Theorem 4.1. *Let $N = q^k n^2$ be an odd perfect number with Euler prime q, and let ϵ satisfy $0 < \epsilon < 3 - 2\sqrt{2}$. Then $I(q^k) + I(n^2) \leq 3 - \epsilon$ if and only if*

$$q < \frac{\epsilon + 1 + \sqrt{\epsilon^2 - 6\epsilon + 1}}{2\epsilon}.$$

Here, we only prove one direction of Theorem 4.1. We show that if $I(q^k) + I(n^2) \leq 3 - \epsilon$, then

$$q < \frac{\epsilon + 1 + \sqrt{\epsilon^2 - 6\epsilon + 1}}{2\epsilon}.$$

Since

$$3 - \frac{q-2}{q(q-1)} = L(q) < I(q^k) + I(n^2) \leq 3 - \epsilon,$$

then we have

$$0 < \epsilon < \frac{q-2}{q(q-1)}$$

from which we get

$$\epsilon q^2 - \epsilon q < q - 2$$

$$\epsilon q^2 - q(\epsilon + 1) + 2 < 0.$$

Finally we obtain

$$q < \frac{\epsilon + 1 + \sqrt{\epsilon^2 - 6\epsilon + 1}}{2\epsilon}.$$

Notice how Theorem 1.1 becomes a special case of Theorem 4.1 for $\epsilon = \frac{1}{100}$.

5 Concluding remarks

The results in this paper were motivated by the remarks of Joshua Zelinsky on a post of the author in the Math Forum at Drexel in 2005. Showing the truth of the improvements outlined in this paper remains an open problem.

Observe that, from Lemma 1.1, when $L(x)$ and $U(x)$ are viewed as functions on the domain $D = \mathbb{R} \setminus \{-1, 0, 1\}$, then

$$L(x+1) = U(x)$$

and

$$U(2) = U(3) = L(3) = \frac{17}{6} < \frac{57}{20}.$$

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