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On $k-$balancing numbers

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Abstract: In this work, we consider some algebraic properties of $k-$balancing numbers. We deduce some formulas for the greatest common divisor of $k-$balancing numbers, divisibility properties of $k-$balancing numbers, sums of $k-$balancing numbers and simple continued fraction expansion of $k-$balancing numbers.

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1 Introduction

Recently, Behera and Panda [1] introduced *balancing numbers* $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

for some positive integer r which is called *balancer* or *cobalancing number*. For example $6, 35, 204, 1189$ and 6930 are balancing numbers with balancers $2, 14, 84, 492$ and 2870, respectively. If n is a balancing number with balancer r, then from (1.1) one has $\frac{(n-1)n}{2} = rn + \frac{r(r+1)}{2}$ and so

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \text{ and } n = \frac{2r+1+\sqrt{8r^2+8r+1}}{2}. \tag{1.2}$$

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then $B_{n+1} = 6B_n - B_{n-1}$ and $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$, where $B_1 = 1, B_2 = 6, b_1 = 0$ and $b_2 = 2$. From (1.2), we see that B_n is a balancing number iff $8B_n^2 + 1$ is a perfect square and b_n is a cobalancing number iff $8b_n^2 + 8b_n + 1$ is a perfect square. So we set

$$C_n = \sqrt{8B_n^2 + 1} \quad \text{and} \quad c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

which are called the n^{th} *Lucas–balancing number* and n^{th} *Lucas–cobalancing number*, respectively.

Ray derived some nice results on balancing numbers and *Pell numbers* (sequence A000129 in OEIS) which are the numbers given by $P_0 = 0, P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ in his/her Phd thesis [8]. Since x is a balancing number if and only if $8x^2 + 1$ is a perfect square, we set $8x^2 + 1 = y^2 \Leftrightarrow y^2 - 8x^2 = 1$ for some $y \neq 0$, which is a Pell equation. The fundamental solution is $(y_1, x_1) = (3, 1)$. So $y_n + x_n\sqrt{8} = (3 + \sqrt{8})^n$ for $n \geq 1$ and hence $y_n - x_n\sqrt{8} = (3 - \sqrt{8})^n$. Let $\gamma = 3 + \sqrt{8}$ and $\delta = 3 - \sqrt{8}$. Then we get $x_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$ which is the Binet formula for balancing numbers, that is, $B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$. The characteristic equation for Pell numbers is $x^2 - 2x - 1 = 0$ and hence its roots are $u = 1 + \sqrt{2}$ and $v = 1 - \sqrt{2}$. Since $u^2 = \gamma$ and $v^2 = \delta$, we easily get $B_n = \frac{u^{2n} - v^{2n}}{4\sqrt{2}}$. Thus there is a correspondence between balancing numbers and Pell numbers. Further we can give the n^{th} term of balancing numbers as $B_n = \frac{P_{2n}}{2}$. Binet formulas for other balancing numbers are $b_n = \frac{u^{2n-1} - v^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{u^{2n} + v^{2n}}{2}, c_n = \frac{u^{2n-1} + v^{2n-1}}{2}$ and also $b_n = \frac{P_{2n-1} - 1}{2}, C_n = P_{2n} + P_{2n-1}$ and $c_n = P_{2n-1} + P_{2n-2}$ (for further details see [5, 6, 7]).

In [4], Liptai, Luca, Pinter and Szalay generalized the theory of balancing numbers to numbers defined as: Let $y, k, l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x such that $x \leq y - 2$ is called a $(k, l)-$power numerical center for y if $1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l$. They derived some algebraic relation on it. Later Kovacs, Liptai and Olajas [3] extended of the concept of balancing numbers to the $(a, b)-$*balancing numbers* defined as follows: Let $a > 0$ and let $b \geq 0$ be co prime integers. If for some positive integers n and r, one has

$$(a + b) + \cdots + (a(n-1) + b) = (a(n+1) + b) + \cdots + (a(n+r) + b)$$

then the number $an + b$ is called an $(a, b)-$balancing number. The sequence of $(a, b)-$balancing numbers is denoted by $B_m^{(a,b)}$ for $m = 1, 2, \cdots$.

In [2], Dash, Ota and Dash considered the $t-$balancing numbers and their properties. A positive integer n is a $t-$*balancing number* if

$$1 + 2 + \cdots + n = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t) \tag{1.3}$$

for some positive integer r which is called $t-$*balancer* or $t-$*cobalancing number*. We deduce from (1.3) that

$$\begin{aligned} r &= \frac{-(2n + 2t + 1) + \sqrt{8n^2 + 8n(1+t) + (2t+1)^2}}{2} \qquad (1.4) \\ n &= \frac{(2r - 1) + \sqrt{8r^2 + 8tr + 1}}{2}. \end{aligned}$$

We denote the n^{th} $t-$balancing number by B_n^t and denote the n^{th} $t-$cobalancing number by b_n^t. From (1.4), we see that B_n^t is a $t-$balancing number iff $8(B_n^t)^2 + 8B_n^t(1+t) + (2t+1)^2$ is a perfect square and b_n^t is a $t-$cobalancing number iff $8(b_n^t)^2 + 8tb_n^t + 1$ is a perfect square. So we set

$$C_n^t = \sqrt{8(B_n^t)^2 + 8B_n^t(1+t) + (2t+1)^2} \text{ and } c_n^t = \sqrt{8(b_n^t)^2 + 8tb_n^t + 1}$$

which are called the n^{th} *Lucas $t-$balancing number* and n^{th} *Lucas $t-$cobalancing number*, respectively. In [9], Tekcan, Tayat and Özbek considered the integer solutions of the Diophantine Equation $8x^2 - y^2 + 8x(1+t) + (2t+1)^2 = 0$ in order to determine the general terms of all $t-$balancing numbers.

2 $k-$balancing numbers

In this section, we consider the *$k-$balancing numbers*, which are the numbers defined by

$$\begin{aligned}
&B_0^k = 0, B_1^k = 1, B_{n+1}^k = 6kB_n^k - B_{n-1}^k \text{ for } n \geq 1 \\
&b_1^k = 0, b_2^k = 2, b_{n+1}^k = 6kb_n^k - b_{n-1}^k + 2 \text{ for } n \geq 2 \\
&C_0^k = 1, C_1^k = 3, C_{n+1}^k = 6kC_n^k - C_{n-1}^k \text{ for } n \geq 1 \\
&c_1^k = 1, c_2^k = 7, c_{n+1}^k = 6kc_n^k - c_{n-1}^k \text{ for } n \geq 2
\end{aligned} \tag{2.1}$$

for some positive integer $k \geq 1$. We first consider the relations.

Theorem 2.1. *Let B_n^k denote the n^{th} $k-$balancing number, let b_n^k denote the n^{th} $k-$cobalancing number, let C_n^k denote the n^{th} $k-$Lucas balancing number and let c_n^k denote the n^{th} $k-$Lucas cobalancing number. Then*

$$\begin{aligned}
C_n^k &= 3B_n^k - B_{n-1}^k \textit{ for } n \geq 1 \\
c_n^k &= 7B_{n-1}^k - B_{n-2}^k \textit{ for } n \geq 2 \\
B_n^k &= 2kC_{n-1}^k + (2k-1)B_{n-2}^k \textit{ for } n \geq 2 \\
2B_n^k &= b_{n+1}^k - b_n^k \textit{ for } n \geq 1.
\end{aligned}$$

Before prove it, we need Binet formulas.

Theorem 2.2. *Binet formulas for $k-$balancing numbers are*

$$\begin{aligned}
B_n^k &= \frac{\alpha^n - \beta^n}{2\sqrt{9k^2-1}} \\
b_n^k &= \frac{(\alpha+1)\alpha^{n-1} + (\beta+1)\beta^{n-1} - 6k - 2}{2(9k^2-1)} \\
C_n^k &= \frac{(3-\beta)\alpha^n - (3-\alpha)\beta^n}{2\sqrt{9k^2-1}} \\
c_n^k &= \frac{(7\alpha-1)\alpha^{n-2} - (7\beta-1)\beta^{n-2}}{2\sqrt{9k^2-1}}
\end{aligned}$$

for $n \geq 1$, where $\alpha = 3k + \sqrt{9k^2-1}$ and $\beta = 3k - \sqrt{9k^2-1}$.

Proof. The characteristic equation for $k-$balancing numbers is $x^2 - 6kx + 1 = 0$ and hence its roots are $\alpha = 3k + \sqrt{9k^2-1}$ and $\beta = 3k - \sqrt{9k^2-1}$. Now let $B_n^k = T_1\alpha^n + T_2\beta^n$ for some integers T_1, T_2 not both zero. Then for $n = 1$ and $n = 2$, we have $T_1\alpha + T_2\beta = 1$ and $T_1\alpha^2 + T_2\beta^2 = 6k$. So

$$T_1 = \frac{6k-\beta}{\alpha^2-\alpha\beta} = \frac{3k+\sqrt{9k^2-1}}{2(3k+\sqrt{9k^2-1})\sqrt{9k^2-1}} = \frac{1}{2\sqrt{9k^2-1}}$$

and

$$T_2 = \frac{6k-\alpha}{\beta^2-\alpha\beta} = \frac{3k-\sqrt{9k^2-1}}{-2(3k-\sqrt{9k^2-1})\sqrt{9k^2-1}} = \frac{-1}{2\sqrt{9k^2-1}}.$$

Thus $B_n^k = \frac{\alpha^n-\beta^n}{2\sqrt{9k^2-1}}$ as we claimed. The others can be proved similarly. □

Proof. **(Proof of Theorem 2.1)** Applying Binet formulas, we easily get

$$\begin{aligned} 3B_n^k - B_{n-1}^k &= 3\left(\frac{\alpha^n-\beta^n}{2\sqrt{9k^2-1}}\right) - \left(\frac{\alpha^{n-1}-\beta^{n-1}}{2\sqrt{9k^2-1}}\right) \\ &= \frac{\alpha^n(3-\alpha^{-1}) - \beta^n(3-\beta^{-1})}{2\sqrt{9k^2-1}} \\ &= \frac{\alpha^n(3-\beta) - \beta^n(3-\alpha)}{2\sqrt{9k^2-1}} \\ &= C_n^k \end{aligned}$$

since $\alpha^{-1} = \beta$ and $\beta^{-1} = \alpha$. The others can be proved similarly. □

Apart from above theorem, we can give the following result.

Theorem 2.3. *Let B_n^k denote the n^{th} $k-$balancing number, let b_n^k denote the n^{th} $k-$cobalancing number. Then the product of $(n+1)^{st}$ and $(n-1)^{st}$ terms of $k-$balancing and $k-$cobalancing numbers and adding 1 are perfect squares, that is*

$$\begin{aligned} B_{n+1}^k B_{n-1}^k + 1 &= (B_n^k)^2 \text{ for } n \geq 1 \\ b_{n+1}^k b_{n-1}^k + 1 &= (b_n^k - 1)^2 \text{ for } n \geq 2. \end{aligned}$$

Proof. Using Binet formulas, we easily get

$$\begin{aligned} B_{n+1}^k B_{n-1}^k + 1 &= \left(\frac{\alpha^{n+1}-\beta^{n+1}}{2\sqrt{9k^2-1}}\right)\left(\frac{\alpha^{n-1}-\beta^{n-1}}{2\sqrt{9k^2-1}}\right) + 1 \\ &= \frac{\alpha^{2n}+\beta^{2n} - (\alpha\beta)^n(\alpha\beta^{-1}+\beta\alpha^{-1}) + 4(9k^2-1)}{4(9k^2-1)} \\ &= \frac{\alpha^{2n}+\beta^{2n} - (36k^2-2) + 4(9k^2-1)}{4(9k^2-1)} \\ &= \frac{\alpha^{2n}+\beta^{2n} - 2(\alpha\beta)^n}{4(9k^2-1)} = \left(\frac{\alpha^n-\beta^n}{2\sqrt{9k^2-1}}\right)^2 = (B_n^k)^2 \end{aligned}$$

since $\alpha\beta = 1, \alpha\beta^{-1} + \beta\alpha^{-1} = 36k^2 - 2$. The other case can be proved similarly. □

The product of $(n+1)^{\text{st}}$ and $(n-1)^{\text{st}}$ terms of $k-$Lucas balancing and $k-$Lucas cobalancing numbers are not perfect square. Nevertheless, we can give the following theorem.

Theorem 2.4. *Let C_n^k denote the n^{th} $k-$Lucas balancing number and let c_n^k denote the n^{th} $k-$Lucas cobalancing number. Then*

$$\begin{aligned} C_{n+1}^k C_{n-1}^k &= 3\sum_{i=1}^{n-1} C_{2i+1}^k - \sum_{i=1}^{n-2} C_{2i+2}^k \\ c_{n+1}^k c_{n-1}^k &= 7\sum_{i=1}^{n-2} c_{2i+2}^k - \sum_{i=1}^{n-3} c_{2i+3}^k. \end{aligned}$$

Proof. It can be proved in the same way as Theorem 2.3 was proved. □

Let (X, Y) denote the greatest common divisors of two integers X and Y. Then we can give the following theorem for $k-$balancing numbers.

Theorem 2.5. *Any two consecutive $k-$balancing numbers are relatively prime, that is,*

$$(B_n^k, B_{n-1}^k) = 1.$$

Proof. Using the Euclidean algorithm and the recurrence relations $B_n^k = 6kB_{n-1}^k - B_{n-2}^k$ for $n \geq 2$, we deduce that

$$\begin{aligned} B_n^k &= 6kB_{n-1}^k + (-B_{n-1}^k + B_{n-1}^k) - B_{n-2}^k \\ B_{n-1}^k &= (B_{n-1}^k - B_{n-2}^k) \times 1 + B_{n-2}^k \\ (B_{n-1}^k - B_{n-2}^k) &= (6k-2)B_{n-2}^k + (B_{n-2}^k - B_{n-3}^k) \\ B_{n-2}^k &= (B_{n-2}^k - B_{n-3}^k) \times 1 + B_{n-3}^k \\ (B_{n-2}^k - B_{n-3}^k) &= (6k-2)B_{n-3}^k + (B_{n-3}^k - B_{n-4}^k) \\ &\cdots \\ B_2^k &= (6k-1)B_1^k + (B_1^k + B_0^k) \\ (6k-1) &= 1 \times (6k-1) + 0. \end{aligned}$$

Since $B_1^k = 1$ and $B_0^k = 0$, we conclude that $(B_n^k,\ B_{n-1}^k) = B_1^k = 1$. □

As in above theorem, we can give the following theorem which can be proved similarly.

Theorem 2.6. *Let B_n^k denote the n^{th} $k-$balancing number. Then*

1. *For every positive integers n and m, we have*

$$(B_m^k, B_n^k) = B_{(m,n)}^k \ \text{ and } \ (B_{mn}^k, B_{m(n\pm1)}^k) = B_m^k.$$

2. *For every positive integers $n \geq m \geq 1$, we have*

$$(B_n^k, B_{n\pm m}^k) = \begin{cases} B_m^k & \text{if } n = mt \\ 1 & \text{otherwise} \end{cases}$$

for some integer $t \geq 1$.

3. *If* $m \geq 2$ *is even, then*

$$(B^k_{nm-1}, B^k_{nm+1}) = 1$$

and if $m \geq 1$ *is odd, then*

$$(B^k_{nm-1}, B^k_{nm+1}) = \begin{cases} 1 & \text{for even } n \\ 6k & \text{for odd } n. \end{cases}$$

4. *If* $m = 2tu$ *for some positive integers* t *and* u*, then*

$$(B^k_{nm-t}, B^k_{nm+t}) = \begin{cases} \sum_{i=0}^{\frac{t-2}{2}} (-1)^i \begin{pmatrix} t-1-i \\ i \end{pmatrix} (6k)^{t-1-2i} & \text{for even } t \geq 2 \\ \sum_{i=0}^{\frac{t-1}{2}} (-1)^i \begin{pmatrix} t-1-i \\ i \end{pmatrix} (6k)^{t-1-2i} & \text{for odd } t \geq 3. \end{cases}$$

5. *Let* $p \geq 3$ *be a prime number and let* $t \geq 1$ *is an integer. Then for every positive integers* n *and* m*, we have*

$$(B^k_{nm\pm p}, B^k_n) = \begin{cases} B^k_p & \text{if } n = pt \\ 1 & \text{otherwise} \end{cases}$$

and

$$(B^k_{np}, B^k_n) = B^k_n \text{ and } (B^k_{np}, B^k_p) = B^k_p.$$

Also for every integer $n \geq 1$*, we have*

$$(B^k_n, B^k_p) = \begin{cases} \sum_{i=0}^{\frac{p-1}{2}} (-1)^i \begin{pmatrix} p-1-i \\ i \end{pmatrix} (6k)^{p-1-2i} & \text{for } n = pt \\ 1 & \text{otherwise.} \end{cases}$$

For divisibility properties, we can give the following theorem.

Theorem 2.7. *Let* B^k_n *denote the* n^{th} $k-$*balancing number. Then*

1. $B^k_m | B^k_{mn}$ *for every positive integers* n *and* m*.*

2. $B^k_m | B^k_{m(n\pm m)}$ *for every positive integers* n *and* m*.*

Before prove it, we need the following theorem.

Theorem 2.8. *Let* B^k_n *denote the* n^{th} $k-$*balancing number. Then*

1. $(B^k_{n+1} + B^k_n)(B^k_{n+1} - B^k_n) = B^k_{2n+1}$ *for every integers* $n \geq 1$*.*

2. $B^k_{n+m} = B^k_n B^k_{m+1} - B^k_{n-1} B^k_m$ *for every positive integers* n, m*.*

3. $(B^k_n + B^k_m)(B^k_n - B^k_m) = B^k_{n+m} B^k_{n-m}$ *for every integers* $n \geq m \geq 1$*.*

Proof. (1) Using Binet formulas, we easily get

$$\begin{aligned}(B^k_{n+1}+B^k_n)(B^k_{n+1}-B^k_n) &= (B^k_{n+1})^2-(B^k_n)^2 \\ &= \left(\frac{\alpha^{n+1}-\beta^{n+1}}{2\sqrt{9k^2-1}}\right)^2-\left(\frac{\alpha^{n}-\beta^{n}}{2\sqrt{9k^2-1}}\right)^2 \\ &= \frac{\alpha^{2n}(\alpha^2-1)+\beta^{2n}(\beta^2-1)}{4(9k^2-1)}.\end{aligned} \tag{2.2}$$

Here we note that $\frac{\alpha^2-1}{4(9k^2-1)}=\frac{\alpha}{2\sqrt{9k^2-1}}$ and $\frac{\beta^2-1}{4(9k^2-1)}=\frac{-\beta}{2\sqrt{9k^2-1}}$. So (2.2) becomes

$$\begin{aligned}(B^k_{n+1}+B^k_n)(B^k_{n+1}-B^k_n) &= (B^k_{n+1})^2-(B^k_n)^2 \\ &= \left(\frac{\alpha^{n+1}-\beta^{n+1}}{2\sqrt{9k^2-1}}\right)^2-\left(\frac{\alpha^{n}-\beta^{n}}{2\sqrt{9k^2-1}}\right)^2 \\ &= \frac{\alpha^{2n+1}-\beta^{2n+1}}{2\sqrt{9k^2-1}} \\ &= B^k_{2n+1}.\end{aligned}$$

The others can be proved similarly. □

Now we can prove Theorem 2.7.

Proof. **(Proof Theorem 2.7)** For $n=1$ we get $B^k_m|B^k_m$ and so it is clearly true. Now assume that $B^k_m|B^k_{mn}$ is true for n. So

$$B^k_m|B^k_{m(n+1)}.$$

We know from (2) of Theorem 2.8 that $B^k_{n+m}=B^k_nB^k_{m+1}-B^k_{n-1}B^k_m$. So

$$B^k_m|B^k_{mn+m}\Leftrightarrow B^k_m|B^k_{mn}B^k_{m+1}-B^k_{mn-1}B^k_m$$

since $B^k_{mn}B^k_{m+1}-B^k_{mn-1}B^k_m$ is divided by B^k_m because $B^k_m|B^k_{mn}$ and $B^k_m|B^k_m$ as we claimed. □

We can also give the following result which can be proved similarly.

Theorem 2.9. *Let B^k_n denote the n^{th} $k-$balancing number. Then*

1. *For every relatively prime positive integers n and m, we have*

$$B^k_mB^k_n \mid B^k_{mn}.$$

2. *For every positive integers n and m, we have*

$$B^k_mB^k_n \mid B^k_{mn}B^k_{(m,n)}.$$

3. *For every positive integers n and m, we have*

$$B^k_n|B^k_m \Leftrightarrow n|m.$$

For the other $k-$balancing numbers, we can give the following three theorems which can be proved similarly.

Theorem 2.10. *Let b_n^k denote the n^{th} $k-$cobalancing number.*

1. *Any two consecutive $k-$cobalancing numbers are not relatively prime, that is,*

$$(b_n^k, b_{n-1}^k) \neq 1.$$

In fact,

$$(b_n^k, b_{n-1}^k) = \begin{cases} b_{\frac{n+1}{2}}^k - b_{\frac{n-1}{2}}^k & \text{for odd } n \geq 3 \\ b_{\frac{n+2}{2}}^k - b_{\frac{n-2}{2}}^k & \text{for even } n \geq 4. \end{cases}$$

2. *For every positive integer $m \geq 1$, we have*

$$(b_n^k,\ b_{(n-1)^2+n(n-1)m}^k) = (b_n^k,\ b_{n(n-1)(m+1)}^k) = b_n^k$$

for $n \geq 2$ and

$$(b_n^k,\ b_{n^2+n(n-1)m}^k) = (b_n^k,\ b_{n(n-1)(m+1)+1}^k) = b_n^k$$

for $n \geq 1$.

3. *For every positive integers m, t, we have*

$$b_n^k | b_m^k \Leftrightarrow m = n + n(n-1)t \text{ or } m = n(n-1) + n(n-1)t$$

for $n \geq 2$.

4. *For every positive integer m, we have*

$$b_n^k | b_{(n-1)^2+n(n-1)m}^k,\ b_n^k |\ b_{n(n-1)(m+1)}^k \text{ for } n \geq 1$$
$$b_n^k |\ b_{n^2+n(n-1)m}^k,\ b_n^k |\ b_{n(n-1)(m+1)+1}^k \text{ for } n \geq 2.$$

Theorem 2.11. *Let C_n^k denote the n^{th} $k-$Lucas balancing number. Then*

1. *Any two consecutive $k-$Lucas balancing numbers are relatively prime, that is,*

$$(C_n^k, C_{n-1}^k) = 1.$$

2. *For every integer $m \geq 1$, we have*

$$(C_{nm}^k, C_{m(n\pm 1)}^k) = 1.$$

3. *For any two consecutive odd numbers n and m, we have*

$$(C_n^k, C_m^k) = C_{(n,m)}^k = 3$$

and for any two consecutive even numbers n and m, we have

$$(C_n^k, C_m^k) = 1 \text{ while } C_{(n,m)}^k = 18k - 1.$$

4. *If $m \geq 2$ is even, then*
$$(C_{nm}^k, C_n^k) = 1$$
for every $n \geq 1$ and if $m \geq 3$ is odd, then
$$(C_{nm}^k, C_n^k) = \begin{cases} 1 & \text{for } n \geq 2 \text{ is even} \\ 3 & \text{for } n \geq 1 \text{ is odd.} \end{cases}$$

5. *If $m \geq 2$ is even, then*
$$(C_{nm}^k, C_m^k) = \begin{cases} C_m^k & \text{for } n = 1 \\ 1 & \text{for } n \geq 2 \end{cases}$$
and if $m \geq 3$ is odd, then
$$(C_{nm}^k, C_n^k) = \begin{cases} C_m^k & \text{for } n = 1 \\ 1 & \text{for } n \geq 2 \text{ is even} \\ 3 & \text{for } n \geq 3 \text{ is odd} \end{cases}$$

6. *If $m \geq 2$ is even, then*
$$(C_n^k, C_m^k) = 1$$
for every $n \geq 1$ such that $n \neq m$, and if $m \geq 1$ is odd, then
$$(C_n^k, C_m^k) = \begin{cases} 1 & \text{for } n \geq 2 \text{ is even} \\ 3 & \text{for } n \geq 1 \text{ is odd, } n \neq m. \end{cases}$$

Theorem 2.12. *Let c_n^k denote the n^{th} $k-$Lucas cobalancing number. Then*

1. *Any consecutive two $k-$Lucas cobalancing numbers are relatively prime, that is,*
$$(c_n^k, c_{n-1}^k) = 1.$$

2. *For any two consecutive odd numbers n and m, we have*
$$(c_n^k, c_m^k) = c_{(n,m)}^k = 1$$
and for any two consecutive even numbers n and m, we have
$$(c_n^k, c_m^k) = 1 \text{ while } c_{(n,m)}^k = 7.$$

3. *For every integer $m \geq 2$, we have*
$$(c_{nm}^k, c_n^k) = 1 \text{ for every } n \geq 1$$
$$(c_{nm}^k, c_m^k) = 1 \text{ for every } n \geq 2$$
$$(c_n^k, c_m^k) = 1 \text{ for every } n \geq 1, n \neq m.$$

2.1 Sums of $k-$balancing numbers

In this subsection, we consider the sums of $k-$balancing numbers.

Theorem 2.13. *The sums of first $n-$terms of $k-$balancing numbers are*

$$\sum_{i=1}^{n} B_i^k = \frac{(6k-1)B_n^k - B_{n-1}^k - 1}{6k-2} \qquad \sum_{i=1}^{n} C_i^k = \frac{(6k-1)C_n^k - C_{n-1}^k - 2}{6k-2}$$

for $n \geq 1$ and

$$\sum_{i=2}^{n} b_i^k = \frac{(6k-1)b_n^k - b_{n-1}^k - 2n + 2}{6k-2} \qquad \sum_{i=2}^{n} c_i^k = \frac{(6k-1)c_n^k - c_{n-1}^k - 6}{6k-2}$$

for $n \geq 2$.

Proof. Since $B_n^k = 6kB_{n-1}^k - B_{n-2}^k$, we get $B_n^k + B_{n-2}^k = 6kB_{n-1}^k$ and so

$$\begin{aligned} B_2^k + B_0^k &= 6kB_1^k \\ B_3^k + B_1^k &= 6kB_2^2 \\ &\cdots \\ B_{n-1}^k + B_{n-3}^k &= 6kB_{n-2}^k \\ B_n^k + B_{n-2}^k &= 6kB_{n-1}^k. \end{aligned} \tag{2.3}$$

If we sum of both sides of (2.3), then we obtain

$$(B_0^k + B_1^k + \cdots + B_{n-2}^k) + (B_2^k + B_3^k + \cdots + B_n^k) = 6k(B_1^k + B_2^k + \cdots + B_{n-1}^k)$$

and hence $2(B_1^k + B_2^k + \cdots + B_n^k) = 6k(B_1^k + B_2^k + \cdots + B_n^k) + 1 + B_{n-1}^k + (1-6k)B_n^k$. So $(2-6k)(B_1^k + B_2^k + \cdots + B_n^k) = (1-6k)B_n^k + B_{n-1}^k + 1$. Thus

$$B_1^k + B_2^k + \cdots + B_n^k = \frac{(6k-1)B_n^k - B_{n-1}^k - 1}{6k-2}.$$

The others can be proved similarly. □

In 1876, the French mathematician François Edouard Anatole Lucas (1842–1891) discovered an explicit formula for the *Fibonacci numbers* F_n (sequence A000045 in OEIS) which are the numbers given by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, namely

$$F_n \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i},$$

and for the *Lucas numbers* L_n (sequence A000032 in OEIS) which are the numbers given by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$,

$$L_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \left[\binom{n-i}{i} + \binom{n-1-i}{i-1} \right].$$

Similarly we can give the following theorem which can be proved by induction on n.

Theorem 2.14. *Let B_n^k denote the n^{th} $k-$balancing number. Then*

$$B_n^k = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \binom{n-1-i}{i} (6k)^{n-1-2i}$$

for $n \geq 1$.

By virtue of Theorem 2.4, we can give the following result.

Theorem 2.15. *Let C_n^k denote the n^{th} $k-$Lucas balancing number and let c_n^k denote the n^{th} $k-$Lucas cobalancing number. Then*

$$C_n^k = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \left[3\binom{n-1-i}{i} (6k)^{n-1-2i} - \binom{n-2-i}{i} (6k)^{n-2-2i} \right]$$

for $n \geq 2$ and

$$c_n^k = \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} (-1)^i \left[7\binom{n-2-i}{i} (6k)^{n-2-2i} - \binom{n-3-i}{i} (6k)^{n-3-2i} \right]$$

for $n \geq 3$.

From (2.1), we note that the recurrence relation for $k-$cobalancing numbers ($b_{n+1}^k = 6kb_n^k - b_{n-1}^k + \mathbf{2}$ for $n \geq 2$) is different from the other $k-$balancing numbers (adding 2). Consequently, we can not give a formula for $k-$cobalancing numbers like in Theorems 2.14 and 2.15. But nevertheless, we can give the following result.

Theorem 2.16. *Let b_n^k denote the n^{th} $k-$cobalancing number. Then*

$$b_n^k = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 2(-1)^i \left[\binom{n-2-i}{i} (6k)^{n-2-2i} + \binom{n-3-i}{i} (6k)^{n-3-2i} \right] + b_{n-2}^k$$

for $n \geq 4$.

For the sums of $k-$balancing numbers, we can give the following results.

Theorem 2.17. *Let B_n^k denote the $n^{th}k-$balancing number. Then*

1. *the sum of $(2i-1)^{st}$ $k-$balancing numbers from 1 to n is a perfect square, that is,*

$$\sum_{i=1}^{n} B_{2i-1}^k = (B_n^k)^2.$$

2. *Also*

$$\sum_{i=1}^{2n} B_i^k = B_n^k (B_n^k + B_{n+1}^k) \qquad \sum_{i=1}^{2n+1} B_i^k = B_{n+1}^k (B_n^k + B_{n+1}^k)$$

$$\sum_{i=1}^{2n} (B_i^k + B_{i+1}^k) = B_{n+1}^k (B_{n+1}^k + 2B_n^k + B_{n-1}^k)$$

$$\sum_{i=1}^{2n+1} (B_i^k + B_{i+1}^k) = (B_n^k + B_{n+1}^k)(B_{n+1}^k + B_{n+2}^k)$$

$$\sum_{i=0}^{2n} (B_{2i+1}^k + B_{2i+2}^k) = B_{2n+1}^k (B_{2n+1}^k + B_{2n+2}^k).$$

Proof. (1) Recall that $B_n^k = \frac{\alpha^n-\beta^n}{2\sqrt{9k^2-1}}$. So

$$\begin{aligned}\sum_{i=1}^{n} B_{2i-1}^k &= B_1^k + B_3^k + B_5^k + \cdots + B_{2n-1}^k \\ &= \frac{1}{2\sqrt{9k^2-1}}[\alpha + \alpha^3 + \cdots + \alpha^{2n-1} - (\beta + \beta^3 + \cdots + \beta^{2n-1})].\end{aligned}$$

Since $\sum_{i=1}^{n}\alpha^{2i-1} = \frac{\alpha^{2n}-1}{2\sqrt{9k^2-1}}$ and $\sum_{i=1}^{n}\beta^{2i-1} = \frac{1-\beta^{2n}}{2\sqrt{9k^2-1}}$, we conclude that

$$\begin{aligned}\sum_{i=1}^{n} B_{2i-1}^k &= B_1^k + B_3^k + B_5^k + \cdots + B_{2n-1}^k \\ &= \frac{1}{2\sqrt{9k^2-1}}[\alpha + \alpha^3 + \cdots + \alpha^{2n-1} - (\beta + \beta^3 + \cdots + \beta^{2n-1})] \\ &= \frac{1}{2\sqrt{9k^2-1}}\left(\frac{\alpha^{2n}-1}{2\sqrt{9k^2-1}} + \frac{\beta^{2n}-1}{2\sqrt{9k^2-1}}\right) = \frac{\alpha^{2n}+\beta^{2n}-2}{4(9k^2-1)} = (B_n^k)^2.\end{aligned}$$

The others can be proved similarly. □

For $k-$Lucas balancing and $k-$Lucas cobalancing numbers, we can give the following result which can be proved similarly.

Theorem 2.18. *Let C_n^k denote the n^{th} $k-$Lucas balancing number and let c_n^k denote the n^{th} $k-$Lucas cobalancing number. Then*

$$\begin{aligned}\sum_{i=1}^{2n} C_i^k &= B_n^k(C_n^k + C_{n+1}^k) \qquad \sum_{i=0}^{2n}(C_{2i+1}^k + C_{2i+2}^k) = B_{2n+1}^k(C_{2n+1}^k + C_{2n+2}^k) \\ \sum_{i=1}^{n} C_{2i}^k &= B_n^k C_{n+1}^k \qquad \sum_{i=1}^{2n+1}(C_i^k + C_{i+1}^k) = (B_n^k + B_{n+1}^k)(C_{n+1}^k + C_{n+2}^k) \\ \sum_{i=1}^{2n+1} C_i^k &= C_{n+1}^k(B_n^k + B_{n+1}^k) \qquad \sum_{i=1}^{2n}(C_i^k + C_{i+1}^k) = B_n^k C_{n+1}^k(B_2^k + b_2^k)\end{aligned}$$

and

$$\sum_{i=1}^{2n} c_i^k = B_n^k(c_n^k + c_{n+1}^k)B_n^k \qquad \sum_{i=1}^{2n+1} c_i^k = c_{n+1}^k(B_n^k + B_{n+1}^k)$$

$$\sum_{i=1}^{n} c_{2i}^k = B_n^k c_{n+1}^k \qquad \sum_{i=1}^{2n}(c_i^k + c_{i+1}^k) = B_n^k c_{n+1}^k(B_2^k + b_2^k)$$

$$\sum_{i=1}^{2n+1}(c_i^k + c_{i+1}^k) = (B_n^k + B_{n+1}^k)(c_{n+1}^k + c_{n+2}^k) \qquad \sum_{i=0}^{2n}(c_{2i+1}^k + c_{2i+2}^k) = B_{2n+1}^k(c_{2n+1}^k + c_{2n+2}^k).$$

Remark 1. *Here, one may wonder are there any formulas for the sums of $k-$cobalancing numbers. No! At least one can check that the sum $\sum_{i=1}^{2n} b_i^k$ can not be written as the product of*

$k-$balancing numbers. Indeed

n	$\sum\limits_{i=1}^{2n} b_i^k$
1	2
2	$72k^2+24k+4$
3	$2592k^4+864k^3+6$
4	$93312k^6+31104k^5-5184k^4-1728k^3+144k^2+48k+8.$

2.2 Continued fraction expansion of $k-$balancing numbers

In this subsection, we consider the continued fraction expansion of $k-$balancing numbers.

Theorem 2.19. *Let B_n^k denote the n^{th} $k-$balancing number, let b_n^k denote the n^{th} $k-$cobalancing number, let C_n^k denote the n^{th} $k-$Lucas balancing number and let c_n^k denote the n^{th} $k-$Lucas cobalancing number. Then*

1.

$$\frac{B_{n+1}^k}{B_n^k}=\begin{cases}[6k;\underbrace{-6k,6k}_{\frac{n-2}{2}\text{ times}},-6k] & \text{for even } n\geq 2\\ [6k;\underbrace{-6k,6k}_{\frac{n-1}{2}\text{ times}}] & \text{for odd } n\geq 3.\end{cases}$$

2. Let n be even, say $n=2m$. Then

$$\frac{b_{n+1}^k}{b_n^k}=\begin{cases}[6k;\underbrace{-6k,6k}_{\frac{m-2}{2}\text{ times}},-6k-1] & \text{for even } m\geq 2\\ [6k;\underbrace{-6k,6k}_{\frac{m-3}{2}\text{ times}},-6k,6k+1] & \text{for odd } m\geq 3.\end{cases}$$

Let n be odd, say $n=2m+1$. Then

$$\frac{b_{n+1}^k}{b_n^k}=\begin{cases}[6k;\underbrace{-6k,6k}_{\frac{m-2}{2}\text{ times}},-6k] & \text{for even } m\geq 2\\ [6k;\underbrace{-6k,6k}_{\frac{m-3}{2}\text{ times}},-6k,6k] & \text{for odd } m\geq 3.\end{cases}$$

3.

$$\frac{C_{n+1}^k}{C_n^k}=\begin{cases}[6k;\underbrace{-6k,6k}_{\frac{n-2}{2}\text{ times}},-6k,3] & \text{for even } n\geq 2\\ [6k;\underbrace{-6k,6k}_{\frac{n-1}{2}\text{ times}},-3] & \text{for odd } n\geq 1.\end{cases}$$

4.

$$\frac{c_{n+1}^k}{c_n^k} = \begin{cases} [6k; \underbrace{-6k, 6k}_{\frac{n-2}{2} \text{ times}}, -7] & \text{for even } n \geq 2 \\ [6k; \underbrace{-6k, 6k}_{\frac{n-3}{2} \text{ times}}, -6k, 7] & \text{for odd } n \geq 3. \end{cases}$$

Proof. (1) Let n be even, say $n = 2t$ for some positive integer $t \geq 1$. For $t = 1$, we have $n = 2$ and hence

$$\frac{B_3^k}{B_2^k} = \frac{36k^2 - 1}{6k} = 6k + \frac{1}{-6k}.$$

So $\frac{B_3^k}{B_2^k} = [6k; -6k]$, that is, it is true for $n = 2$. Let us assume that it is true for $n - 2$, that is,

$$\frac{B_{n-1}^k}{B_{n-2}^k} = [6k; \underbrace{-6k, 6k}_{\frac{n-4}{2} \text{ times}}, -6k].$$

Then

$$\begin{aligned} \frac{B_{n+1}^k}{B_n^k} &= [6k; \underbrace{-6k, 6k}_{\frac{n-2}{2} \text{ times}}, -6k] = 6k + \cfrac{1}{-6k + \cfrac{1}{6k + \cfrac{1}{-6k + \cfrac{1}{\ddots \; 6k - \frac{1}{6k}}}}} \\ &= 6k + \cfrac{1}{-6k + \cfrac{1}{\frac{B_{n-1}^k}{B_{n-2}^k}}} = \frac{6kB_n^k - B_{n-1}^k}{6kB_{n-1}^k - B_{n-2}^k}. \end{aligned}$$

So it is true for every n since $B_n^k = 6kB_{n-1}^k - B_{n-2}^k$. The other cases can be proved similarly. □

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