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On the extensibility of the $D(4)$-triple $\{k-2, k+2, 4k\}$ over Gaussian integers

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Abstract: In this paper, we prove that if $\{k-2, k+2, 4k, d\}$, where $k \in \mathbb{Z}[i]$, $k \neq 0, \pm 2$, is a $D(4)$-quadruple in the ring of Gaussian integers, then $d = 4k^3 - 4k$.

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1 Introduction

Let n be a non-zero integer. A set of m distinct positive integers such that the product of any two of its elements increased by n is a perfect square is called $D(n)$-m-tuple. The most studied case is $n = 1$ in which we call $D(1)$-m-tuple simply Diophantine m-tuple. There is the folklore conjecture that there does not exist a Diophantine quintuple. This conjecture is intensively studied in recent years. To see all references the reader should consult the webpage [5]. The most important contribution was done by Dujella [7] in 2004 when he proved that there does not exist a Diophantine sextuple and that there are only finitely many Diophantine quintuples. Moreover, he also gave an upper bound for the number of Diophantine quintuples, which was recently significantly improved by various authors. The best known bound can be found in [4]. But that upper bound is still too large such that it is impossible to completely solve the problem using today's computers. Therefore, the conjecture is still open. Moreover, in the case $n = 1$, there is a stronger version of the main conjecture which says that every $D(1)$-triple can be extended to a quadruple with a larger element in a unique way. More precisely, the conjecture says that if $\{a, b, c, d\}$ is a Diophantine quadruple such that $a < b < c < d$, then $d = a+b+c+2(abc+rst)$, where r, s, and t are positive integers defined by $ab + 1 = r^2$, $ac + 1 = s^2$ and $bc + 1 = t^2$. Such Diophantine quadruples are called regular. So the conjecture implies that all Diophantine quadruples are regular. Similar results and conjectures exist when $n = -1$ and $n = 4$. In the case $n = -1$, there is a conjecture that there does not exist $D(-1)$-quadruple, while in the case $n = 4$, we have an analogous conjecture (as in the case $n = 1$) that there does not exist $D(4)$-quintuple. Those conjectures are also extensively studied recently, and for details the reader could again consult [5].

There are various generalizations of the above problems. One of them consists in taking elements of m-tuples in a commutative ring with unity. Let R be a commutative ring with unity 1. One can consider m different non-zero elements in R such that the product of any two distinct elements increased by some element of R is a perfect square in R. Recently, there are also results on the extendibility of Diophantine m-tuples in other rings such as the ring of Gaussian integers and rings of integers of quadratic fields. For example, in 2008, Franušić [8] proved that the Diophantine triple $\{k-1, k+1, 4k\}$ extends uniquely to a Diophantine quadruple in the ring $\mathbb{Z}[i]$. The first, third and the fourth author [2] proved the same result for the triple $\{k, 4k + 4, 9k + 6\}$. In this paper, we will consider a $D(4)$-tuple in the ring of Gaussian integers. Precisely, we study the $D(4)$-extensions of the triple $\{k - 2, k + 2, 4k\}$ in $\mathbb{Z}[i]$ and our main result is the following theorem.

Theorem 1.1. *Let $k \in \mathbb{Z}[i]$ and $k \neq 0, \pm 2$. If $\{k - 2, k + 2, 4k, d\}$ is a $D(4)$ quadruple in $\mathbb{Z}[i]$, then $d = 4k^3 - 4k$.*

That result was proven in the ring of integers by Fujita [9]. This is the first time that a $D(4)$-tuple is studied in the ring of Gaussian integers. The organization of this paper is as follows. In Section 2, we determine a system of Pell equations and the sequences v_m, w_n and w'_n corresponding to our problem of extendibility. Moreover, we prove some useful results that we will need to prove Theorem 1.1. We find the relation between indices m and n in the equations $v_m = w_n$ and $v_m = w'_n$ and the lower bound on $|x|$ in terms of $|k|$ in Section 3. To prove the main result for $|k| \geq 155402$, we devote Section 4 to the use of a generalization of Bennett's result (see [10]), that consists in simultaneous rational approximations of algebraic numbers that are close to 1. The study of the remaining cases is done in two steps. First, by the means of the Baker-Davenport reduction method, we complete the proof of Theorem 1.1 for $5 < |k| < 155402$ in Section 5. In the section 6, we deal with the second step that consists in looking the problem for $2 < |k| \leq 5$. In the final section, we study the cases $|k| \leq 2$. This is really a laborious work. Our proof follows mostly the methods from [8].

2 Solving a system of Pell equations

Let $k \in \mathbb{Z}[i], k \neq 0, \pm 2$. The aim of this paper is to determine all $D(4)$-quadruples of the form $\{k-2, k+2, 4k, d\}$ in $\mathbb{Z}[i]$. Thus, we have to solve the system

$$(k-2)d + 4 = x^2, \quad (k+2)d + 4 = y^2, \quad (4k)d + 4 = z^2, \tag{1}$$

in $d, x, y, z \in \mathbb{Z}[i]$. Eliminating d in (1), we obtain the following system of diophantine equations

$$(k+2)x^2 - (k-2)y^2 = 16 \tag{2}$$

$$(4k)x^2 - (k-2)z^2 = 12k + 8. \tag{3}$$

Lemma 2.1. *Let $k \in \mathbb{Z}[i]$ with $|k| > 2$. Then there exist x_0 and $y_0 \in \mathbb{Z}[i]$ such that*

(i) *(x_0, y_0) is a solution of* (2)

(ii) *the estimates*

$$|x_0|^2 \leq \frac{8\,|k-2|}{|k|-2}, \tag{4}$$

$$|y_0|^2 \leq \frac{8\,|k+2|}{|k|-2} + \frac{16}{|k-2|}. \tag{5}$$

(iii) *For each solution (x, y) of equation* (2) *there exists $m \in \mathbb{Z}$ such that*

$$x\sqrt{k+2} + y\sqrt{k-2} = (x_0\sqrt{k+2} + y_0\sqrt{k-2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m. \tag{6}$$

Proof. If (x, y) is a solution of (2), then (x_m, y_m) is obtained by

$$x_m\sqrt{k+2} + y_m\sqrt{k-2} = (x\sqrt{k+2} + y\sqrt{k-2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m \tag{7}$$

is also a solution of (2), for all $m \in \mathbb{Z}$.
Let (x_0, y_0) be an element of the sequence $(x_m, y_m)_{m\in\mathbb{Z}}$ (defined by (7)) such that the absolute value $|x_0|$ is a minimal. We put

$$\begin{aligned} x'\sqrt{k+2} + y'\sqrt{k-2} &= (x_0\sqrt{k+2} + y_0\sqrt{k-2})\left(\frac{k+\sqrt{k^2-4}}{2}\right) \\ x''\sqrt{k+2} + y''\sqrt{k-2} &= (x_0\sqrt{k+2} + y_0\sqrt{k-2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^{-1} \\ &= (x_0\sqrt{k+2} + y_0\sqrt{k-2})\left(\frac{k-\sqrt{k^2-4}}{2}\right). \end{aligned}$$

Due to minimality of $|x_0|$, we have

$$\begin{aligned} |x_0| &\leq |x'| = \left|\frac{x_0k + y_0(k-2)}{2}\right| \\ |x_0| &\leq |x"| = \left|\frac{x_0k - y_0(k-2)}{2}\right|. \end{aligned}$$

At least one of the expressions $\left|\frac{x_0k+y_0(k-2)}{2}\right|$ and $\left|\frac{x_0k-y_0(k-2)}{2}\right|$ must be greater or equal to $\left|\frac{x_0k}{2}\right|$, since $\left|\frac{x_0k+y_0(k-2)}{2}\right| + \left|\frac{x_0k-y_0(k-2)}{2}\right| \geq 2\left|\frac{x_0k}{2}\right|$. Let us assume that $\left|\frac{x_0k+y_0(k-2)}{2}\right| \geq \left|\frac{x_0k}{2}\right|$. Hence, we get

$$\left|\frac{x_0^2k}{2}\right| \leq \left|\frac{(x_0k)^2 - (y_0(k-2))^2}{4}\right|,$$

and

$$\left|\frac{x_0^2k}{2}\right| \leq \left|\frac{4x_0^2 + 16(k-2)}{4}\right|.$$

Immediately, we obtain the following estimate for $|x_0|$

$$|x_0|^2 \leq \frac{8\,|k-2|}{|k|-2}.$$

This implies that the estimate for $|y_0|$ is

$$\left|(k-2)y_0^2\right| = \left|(k+2)x_0^2 - 16\right| \leq |k+2|\,\frac{8\,|k-2|}{|k|-2} + 16.$$

It is obvious that there exist only finitely many x_0 an y_0 such that above estimates are fulfilled. Thus, the definition of x_0 implies there exist $m_0 \in \mathbb{Z}$ such that

$$x_0\sqrt{k+2} + y_0\sqrt{k-2} = (x\sqrt{k+2} + y\sqrt{k-2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^{m_0}.$$

Therefore, we obtain

$$x\sqrt{k+2} + y\sqrt{k-2} = (x_0\sqrt{k+2} + y_0\sqrt{k-2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^{-m_0}$$

□

Analogously, all solutions of (3) are given by the following lemma.

Lemma 2.2. *Let $k \in \mathbb{Z}[i]$ with $|k| > 2$. Then, there exist x_1 and $z_1 \in \mathbb{Z}$ such that*

(i) *(x_1, z_1) is a solution of* (3)

(ii) *the estimates*

$$|x_1|^2 \leq \frac{|3k+2|\,|k-2|}{|k-1|-1}, \tag{8}$$

$$|z_1|^2 \leq \frac{|4k|\,|3k+2|}{|k-1|-1} + \frac{|12k+8|}{|k-2|}. \tag{9}$$

(iii) *For each solutions* (x,z) *of equation* (3) *there exist* $n\in\mathbb{Z}$ *such that*

$$x\sqrt{4k}+z\sqrt{k-2}=(x_1\sqrt{4k}+z_1\sqrt{k-2})\left(k-1+\sqrt{k^2-2k}\right)^n. \tag{10}$$

Now, we define the following sequences

$$v_0=x_0,\ \ v_1=\frac{1}{2}\left(x_0k+y_0(k-2)\right),\ \ v_{m+2}=kv_{m+1}-v_m, \tag{11}$$

$$v'_0=x_0,\ \ v'_1=\frac{1}{2}\left(x_0k-y_0(k-2)\right),\ \ v'_{m+2}=kv'_{m+1}-v'_m, \tag{12}$$

for all $m\in\mathbb{N}$. If x is a solution of (2), then there exists a nonnegative integer m such that $x=v_m$ or $x=v'_m$.

Similarly, if x is a solution of (3), then there exists a nonnegative integer n such that $x=w_n$ or $x=w'_n$, where

$$w_0=x_1,\ \ w_1=\frac{1}{2}\left(2x_1(k-1)+z_1(k-2)\right),\ \ w_{n+2}=2(k-1)w_{n+1}-w_n, \tag{13}$$

$$w'_0=x_1,\ \ w'_1=\frac{1}{2}\left(2x_1(k-1)-z_1(k-2)\right),\ \ w'_{n+2}=2(k-1)w'_{n+1}-w'_n. \tag{14}$$

The next result will specify the values of x_0 and y_0.

Lemma 2.3. *Let* $k\in\mathbb{Z}[i]$*,* $k\neq\pm6$ *and* $|k|>5$*. Then,* $x_0=\pm2$ *and* $y_0=\pm2$ *are the only fundamental solutions of* (2) *and all solutions are represented by the sequences* $(v_m)_m$ *and* $(-v_m)_m$ *defined by*

$$v_0=2,\ \ v_1=2k-2,\ \ v_{m+2}=kv_{m+1}-v_m,\ \ m\in\mathbb{Z}. \tag{15}$$

Proof. Suppose (x_0,y_0) is a fundamental solution of (2). Then we have

$$x_0^2(k+2)-y_0^2(k-2)=16.$$

Thus, we have $x_0^2-y_0^2=\frac{16-4x_0^2}{k-2}$. Relation (1) implies $x_0^2-y_0^2=4d_0,\ x_0^2\equiv4 \pmod{k-2}$, with $d_0\in\mathbb{Z}[i]$. Thus, we have

$$|d_0|=\frac{|4-x_0^2|}{|k-2|}\leq\frac{4+|x_0^2|}{|k-2|}\leq\frac{12}{|k|-2}. \tag{16}$$

If $|k|>14$, then we have $|d_0|<1$, so $d_0=0$. Thus, we have $x_0^2-y_0^2=0$. So we deduced $x_0=\pm2$ and $y_0=\pm2$.
If $5<|k|\leq14$ and $k\neq\pm6$, using Maple, the following relations $x_0\in\mathbb{Z}[i], y_0\in\mathbb{Z}[i], x_0^2=4+d_0(k-2), y_0^2=4+d_0(k+2)$ and (16) are simultaneous satisfied if and only if $x_0=\pm2$ and $y_0=\pm2$. □

Let us mention that from this lemma, we have that for $k\neq\pm6$ and $|k|>5$, all possible d's are of the form $d=2d'$ for some $d'\in\mathbb{Z}[i]$. We can conclude that, because those fundamental solutions will generate all possible elements that extend $D(4)$-pair $\{k-2,k+2\}$, and it is easy to see that all of them will be of that form $2d'$. It furthermore implies, using the above mentioned result from [8], that for such k and $k = 2t$ for some $t \in \mathbb{Z}[i]$, the statement of the main Theorem is valid.

Before proceeding further, let us recapitulate our results. For $|k| > 5$ and $k \neq \pm 6$ the problem of solving (2) and (3) is reduced to looking for the intersections of recursive sequences, i.e. to solving the equations

$$v_m = \pm w_n, v_m = \pm w'_n, m, n \geq 0. \tag{17}$$

3 The congruence method

In this section, we will determine all fundamental solutions of equation (3) under the assumption that one of the equations in (17).

Lemma 3.1. *Let $k \in \mathbb{Z}[i]$, $|k| > 5$ and $k \neq \pm 6$. If (x_1, z_1) is a fundamental solution of* (3)*, then*

$$x_1 \pmod{k-1} \in \{\pm 2, 0\}$$

or

$$\frac{1}{2}(k-2)z_1 \pmod{k-1} \in \{\pm 2, 0\}.$$

Proof. We have

$$(v_m \pmod{k-1})_{m\geq 0} = (2, 0, -2, -2, 0, 2, 2, 0, -2, -2, 0, 2, \dots),$$

$$(w_n \pmod{k-1})_{n\geq 0} = (x_1, \frac{1}{2}(k-2)z_1, -x_1, -\frac{1}{2}(k-2)z_1, x_1, \frac{1}{2}(k-2)z_1 \dots),$$

$$(w'_n \pmod{k-1})_{n\geq 0} = (x_1, -\frac{1}{2}(k-2)z_1, -x_1, \frac{1}{2}(k-2)z_1, x_1, -\frac{1}{2}(k-2)z_1, \dots).$$

These congruence relations are obtained by induction from (15), (13), (14), respectively. The rest follows immediately from (17). □

In what follows, we will discuss all the possibilities given in Lemma 3.1.

- $x_1 \equiv \pm 2 \pmod{k-1}$**:**
  In this case, if $x_1 \neq \pm 2$, we have: $|k-1| - 2 \leq |x_1|$. From this relation and (8), we have

$$(|k| - 3)^2 \leq |x_1|^2 \leq \frac{|3k+2|\,|k-2|}{|k-1| - 1}.$$

Therefore, we obtain

$$|k|^3 - 11\,|k|^2 + 17\,|k| - 22 \leq 0.$$

If $|k| \geq 10$, then

$$|k|^3 - 11\,|k|^2 + 17\,|k| - 22 > 0.$$

So, if $|k| \geq 10$ then $x_1 = \pm 2$.

If $5 < |k| < 10$, then we put $x_1 = \pm 2 + l_0(k-1)$. Suppose that $|l_0| \geq 2$. We have $|x_1| \geq 2\,|k-1| - 2$. This relation and (8) give

$$4\,|k|^3 - 27\,|k|^2 + 44\,|k| - 36 \leq 0.$$

If $|k|>5$, then we get

$$4|k|^3-27|k|^2+44|k|-36>0.$$

So, if $5<|k|<10$ then $|l_0|\le\sqrt{2}$. In this case, $l_0=\pm1$ or $l_0=\pm i$ or $l_0=\pm(1\pm i)$.

$l_0=\pm1$ implies $x_1=\pm(k+1)$ or $x_1=\pm(k-3)$. In this case, the following relation $x_1^2\equiv4 \pmod{(k-2)}$, (8) are not simultaneously satisfied.

$l_0=\pm i$ implies $x_1^2\equiv 3\pm4i \pmod{(k-2)}$. This relation and (1) imply $1\pm4i\equiv0 \pmod{(k-2)}$. We deduce $|k|\le\sqrt{17}+2$. In this case, relation (8) is not satisfied.

$l_0=1\pm i$ and $x_1=2+l_0(k-2)$ imply $|x_1|\ge\sqrt{2}|k|-\sqrt{2}$. These relations and (8) imply $2|k|^3-11|k|^2+6|k|-8\le0$. This inequality is not satisfied if $5<|k|<10$.

$l_0=-1\pm i$ and $x_1=2+l_0(k-2)$ imply $2x^3-(7+4\sqrt{5})x^2+(8\sqrt{5}+6)x-24\le0$. This inequality is not satisfied if $|k|\ge\sqrt{41}$. If $5<|k|<\sqrt{41}$, using Maple in the various cases the relation (8) is not satisfied. If $x_1=-2+l_0(k-2)$, we have similar results.

So, if $x_1\equiv\pm2 \pmod{k-1}$ and $|k|>5$, then $x_1=\pm2$.

- $x_1\equiv0 \pmod{(k-1)}$**:**

In this case, if $x_1\ne0$, we have $|k-1|\le|x_1|$. This relation and (8) give

$$(|k|-1)^2\le|x_1|^2\le\frac{|3k+2|\,|k-2|}{|k-1|-1}.$$

Therefore, we obtain

$$|k|^3-7|k|^2+|k|-6\le0.$$

If $|k|\ge7$, then

$$|k|^3-7|k|^2+|k|-6>0.$$

So we deduce that $x_1=0$.

Suppose that $x_1=l_1(k-1)$. If $5<|k|<7$ then $|l_1|\le1$. If $|l_1|=1$, thus relations (1) or (8) are not satisfied. This implies that $l_1=0$. Consequently, if $x_1\equiv0 \pmod{(k-1)}$ then $x_1=0$.

If $x_1=0$, then relation (3) implies $-(k-2)z_1^2=12k+8$. In this case, using Maple, the equation $-(k-2)z_1^2=12k+8$ has no solution in $\mathbb{Z}[i]$ for all $k\in\mathbb{Z}[i]$ with $5<|k|<7$.

If $x_1=\pm2$, then relation (3) implies $z_1=\pm2$.

- $k=6$**:**

In this case, the fundamental solutions of (2) are $(x_0,y_0)=(\pm2,\pm2)$, $(x_0,y_0)=(0,\pm2i)$ and the fundamental solutions of (3) are $(x_1,z_1)=(\pm2,\pm2)$, $(x_1,z_1)=(\pm(2+2i),\pm(4+6i))$, $(x_1,z_1)=(\pm(2-2i),\pm(4-6i))$. In Section 2, the solutions of (2) are given by recurrence sequences (11) and (12). In these cases, we have

$$q_0=0,\ q_1=4i,\ q_{m+2}=6q_{m+1}-q_m,\ m\in\mathbb{N},$$

$$q_0'=0,\ q_1=-4i,\ q_{m+2}=6q_{m+1}-q_m,\ m\in\mathbb{N},$$

$$v_0=2,\ v_1=10,\ v_{m+2}=6v_{m+1}-v_m,\ m\in\mathbb{N}.$$

From (13) and (14), all solutions of (3) are given by

$$x_0 = 2+2i,\ x_1 = 18+22i,\ x_{n+2} = 10x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$x'_0 = 2+2i,\ x'_1 = 2-2i,\ x'_{n+2} = 10x'_{n+1} - x'_n,\ n \in \mathbb{N},$$

$$u_0 = 2-2i,\ u_1 = 18-22i,\ u_{n+2} = 10u_{n+1} - u_n,\ n \in \mathbb{N},$$

$$u'_0 = 2-2i,\ u'_1 = 2+2i,\ u'_{n+2} = 10u'_{n+1} - u'_n,\ n \in \mathbb{N},$$

$$w_0 = 2,\ w_1 = 14,\ w_{n+2} = 10w_{n+1} - w_n,\ n \in \mathbb{N},$$

$$w'_0 = 2,\ w'_1 = 6,\ w'_{n+2} = 10w'_{n+1} - w'_n,\ n \in \mathbb{N}.$$

So, we have the following relations:
a) $q_m = \pm x_n$ or $q_m = \pm x'_n$ or $q_m = \pm w_n$ or $q_m = \pm w'_n$ or $q_m = u_n$ or $q_m = u'_n$,
b) $q'_m = \pm x_n$ or $q'_m = \pm x'_n$ or $q'_m = \pm w_n$ or $q'_m = \pm w'_n$ or $q'_m = u_n$ or $q'_m = u'_n$,
c) $v_m = \pm x_n$ or $v_m = \pm x'_n$ or $v_m = \pm w_n$ or $v_m = \pm w'_n$ or $v_m = u_m$ or $v_m = u'_n$.

We see that

$$(q_m \pmod{(10)} = (0, 4i, 4i, 0, -4i, -4i, 0, 4i, 4i, ...),$$

$$(q'_m \pmod{10} = (0, -4i, -4i, 0, 4i, 4i, 0, -4i, -4i, ...),$$

$$(v_m \pmod{10} = (2, 0, -2, -2, 0, 2, 2, 0, -2, ...),$$

$$(x_n \pmod{10} = (2+2i, -2+2i, -2-2i, 2-2i, 2+2i, -2+2i, ...),$$

$$(x'_n \pmod{10} = (2+2i, 2-2i, -2-2i, -2+2i, 2+2i, 2-2i, ...),$$

$$(u_n \pmod{10} = (2-2i, -2-2i, -2+2i, 2+2i, 2-2i, -2-2i, -2+2i, ...),$$

$$(u'_n \pmod{10} = (2-2i, 2+2i, -2+2i, -2-2i, 2-2i, 2+2i, ...),$$

$$(w_n \pmod{10} = (2, 4, -2, -4, 2, 4, -2, -4, ...),$$

$$(w'_n \pmod{10} = (2, -4, -2, 4, 2, -4, -2, 4, ...).$$

By these relations, we deduce that the only following equality are possible:

$$v_m = \pm w_n,\ \text{or } v_m = \pm w'_n.$$

• The case $k = -6$ is similar to the case $k = 6$.

The above results can be summarized in the following lemma.

Lemma 3.2. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$. If at least one of the equations in* (17) *is solvable, then all fundamental solutions of equation* (3) *are $x_1 = \pm 2$, $z_1 = \pm 2$ and related sequences (w_n) and (w'_n) are given by*

$$w_0 = 2, w_1 = 3k-4, w_{n+2} = (2k-2)w_{n+1} - w_n, \tag{18}$$

$$w'_0 = 2, w'_1 = k, w'_{n+2} = (2k-2)w'_{n+1} - w'_n, \tag{19}$$

for $n \in \mathbb{N}$.

We now prove the following results:

Lemma 3.3. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$.*
- *The equation $v_m = w_{2n+1}$ has no solution.*
- *The equation $v_m = w'_{2n+1}$ has no solution.*

Proof. The result comes from the following facts:

$$(v_m \pmod{k-1})_{m\geq 0} = (2, 0, -2, -2, 0, 2, 2, 0, -2, -2, 0, 2, \dots),$$

$$(w_n \pmod{k-1})_{m\geq 0} = (2, -1, -2, 1, 2, -1, -2, 1, 2, -1, -2, \dots),$$

$$(w'_n \pmod{k-1})_{m\geq 0} = (2, 1, -2, -1, 2, 1, -2, -1, 2, 1, -2, -1, \dots).$$

□

Lemma 3.4. *The sequences $(v_m)_m$, $(w_n)_n$ and $(w'_n)_n$ defined by* (15), (18) *and* (19) *respectively, satisfy the following congruences*

$$(v_m \pmod{k})_{m\geq 0} = (2, -2, -2, 2, 2, -2, -2, 2, 2, -2, ...),$$

$$(w_n \pmod{k})_{n\geq 0} = (-1)^n(2n+2),$$

$$(w'_n \pmod{k})_{n\geq 0} = (-1)^{n+1}(2n-2).$$

Proof. This can be verified by induction. □

Lemma 3.4 implies the following lemma.

Lemma 3.5. *Let $k \in \mathbb{Z}[i], |k| > 5$ and let $x \in \mathbb{Z}[i]$ be a solution of the system of equations* (2) *and* (3). *Then, there exist $m, n \in \mathbb{N}, (2n \pm 2) \equiv \pm 2 \pmod{(k)}$, such that*

$$x = v_m = w_n \text{ or } x = v_m = -w_n \text{ or } x = v_m = w'_n \text{ or } x = v_m = -w'_n,$$

where $(v_m)_m$, $(w_n)_n$ and $(w'_n)_n$ are defined by (15), (18) *and* (19), *respectively.*

Now, observe that $v_0 = w_0 = w'_0 = 2$ and $v_2 = w'_2 = 2k^2 - 2k - 2$. So, $x = \pm 2$ and $x = \pm(2k^2 - 2k - 2)$ are solutions of the system of equations (2) and (3). The solutions $x = \pm 2$ are not interesting, because they correspond to $d = 0$. The solutions $x = \pm(2k^2 - 2k - 2)$ correspond to $d = 4k^3 - 4k$. Since we intend to prove that this is the unique nontrivial extension of the $D(4)$-triple $\{k-2, k+2, 4k\}$ in $\mathbb{Z}[i]$. We have to show that the system of equations (2), (3) has no other solution. Our next step is to determine an upper bound for all solutions of (2) and (3) that are different from the previous ones.

Lemma 3.6. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$. If $x \in \mathbb{Z}[i] \backslash \{\pm 2, \pm(2k^2 - 2k - 2)\}$ is a solution of the system of equations* (2) *and* (3), *then $2n \geq |k| - 4$.*

Proof. If $m \geq 2$, then $|w'_1| < |v_m|$. Otherwise,

$$|w_1 - v_1| \neq 0,\ |w_1 - v_3| \neq 0,\ |w_1 - v_4| \neq 0.$$

We have $|v_0| < |v_1|$. Suppose $|v_m| < |v_{m+1}|$. We have

$$|v_{m+2}| \geq |kv_{m+1}| - |v_n| = (|k| - 1)\,|v_{m+1}| + |v_{m+1}| - |v_m| > |v_{m+1}|\,.$$

$|w_1| < |v_4| < |v_5| < ....$

$$|v_1 - w_2| \neq 0,\ |v_3 - w_2| \neq 0,\ |v_4 - w_2| \neq 0,\ |v_5 - w_2| \neq 0,\ |w_2| < |v_6| < |v_7| < ...$$

So, by Lemma 3.5 if x is a solution of the system of equations (2) and (3), then we have $2n - 2 \geq 2 + |k|$ or $2n - 2 \geq -2 + |k|$ or $2n + 2 \geq 2 + |k|$ or $2n + 2 \geq -2 + |k|$. □

Lemma 3.7. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$. If $x \in \mathbb{Z}[i] \backslash \{\pm 2, \pm(2k^2 - 2k - 2)\}$ is a solution of the system of equations* (2) *and* (3) *then*

$$|x| \geq |k|\,(2\,|k| - 3)^{|k|-5}.$$

Proof. If $x \in \mathbb{Z}[i] \backslash \{\pm 2, \pm(2k^2 - 2k - 2)\}$, then $n \geq 2$. We have $|w_2| \leq |w_3|$. Now, assume that $|w_n| \leq |w_{n+1}|$. From (18), we have that

$$|w_{n+2}| \geq (2\,|k| - 3)\,|w_{n+1}| + |w_{n+1}| - |w_n| \geq (2\,|k| - 3)\,|w_{n+1}|\,.$$

So, we get $|w_{n+2}| \geq |w_{n+1}|$.

One can see that $|w_2| \geq |k|\,(2\,|k| - 3)$. Assume that $|w_n| \geq |k|\,(2\,|k| - 3)^{n-1}$. From (18), we have that

$$|w_{n+1}| \geq |k|\,(2\,|k| - 3)\,|w_n| + |w_n| - |w_{n-1}| \geq |k|\,(2\,|k| - 3)\,|w_n|\,.$$

So, we conclude that

$$|w_{n+1}| \geq |k|\,(2\,|k| - 3)^n.$$

Analogously, if $n \geq 2$, we obtain $|w'_n| \geq |k|\,(2\,|k| - 3)^{n-1}$. From Lemma 3.6, If $x \in \mathbb{Z}[i] \backslash \{\pm 2, \pm(2k^2 - 2k - 2)\}$ is a solution of the system of equations (2) and (3) then

$$|x| \geq |k|\,(2\,|k| - 3)^{|k|-5}.$$

□

4 An application of the theorem on simultaneous approximations

In this section, we prove that if the parameter $|k|$ is large enough, then $x = \pm 2$ and $x = \pm(2k^2 - 2k - 2)$ give all solutions of the system of equations (2), (3). For that reason, we apply the following generalization of Bennett's theorem [3] on simultaneous rational approximations of square roots which are close to one.

Theorem 4.1. *([10]) Let $\theta_i = \sqrt{1+\frac{a_i}{T}}, i = 1, 2$, with a_1 and a_2 pairwise distinct quadratic integers in the imaginary quadratic field K and let T be an algebraic integer of K. Further, let $M = \max\{|a_1|, |a_2|\}, |T| > M$ and*

$$l = \frac{27|T|}{64|T| - M},$$

$$L = \frac{27}{16|a_1|^2|a_2|^2|a_1-a_2|^2}(|T|-M)^2 > 1,$$

$$p = \sqrt{\frac{2|T|+3M}{2|T|-2M}},$$

$$P = 16\frac{|a_1|^2|a_2|^2|a_1-a_2|^2}{\min\{|a_1|, |a_2|, |a_1-a_2|\}^3}(2|T|+3M).$$

Then

$$\max\left(\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right) > c|q|^{-\lambda},$$

for all algebraic integers $p_1, p_2, q \in K$, where

$$\lambda = 1 + \frac{\log P}{\log L},$$

$$c^{-1} = 4pP(\max\{1, 2l\})^{\lambda-1}.$$

First, let us show the following technical lemma.

Lemma 4.1. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$ and let $(x, y, z) \in \mathbb{Z}[i]^3$ be a solution of the system of (2) and (3). Furthermore, let*

$$\theta_1^{(1)} = \pm\sqrt{1+\frac{4}{k-2}},\ \theta_1^{(2)} = -\theta_1^{(1)},$$

$$\theta_2^{(1)} = \pm\sqrt{1+\frac{2}{k-2}},\ \theta_2^{(2)} = -\theta_2^{(1)},$$

where the signs are chosen such that

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \le \left|\theta_1^{(2)} - \frac{y}{x}\right|,\ \left|\theta_2^{(1)} - \frac{z}{2x}\right| \le \left|\theta_2^{(2)} - \frac{z}{2x}\right|.$$

Then, we obtain

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \le \frac{16}{\sqrt{|k^2-4|}\,|x|^2},$$

$$\left|\theta_2^{(1)} - \frac{z}{2x}\right| \le \frac{|3k+2|}{\left|\sqrt{k(k-2)}\right||x|^2}.$$

Proof. We have

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| = \left|(\theta_1^{(1)})^2 - \frac{y^2}{x^2}\right| \cdot \left|\theta_1^{(1)} + \frac{y}{x}\right|^{-1} = \frac{16}{|k-2|\,|x|^2}\left|\theta_1^{(2)} - \frac{y}{x}\right|^{-1}.$$

Because of the assumptions on $\theta_1^{(1)}$ and $\theta_1^{(2)}$, we get

$$\left|\theta_1^{(2)} - \frac{y}{x}\right| \geq \frac{1}{2}\left(\left|\theta_1^{(2)} - \frac{y}{x}\right| + \left|\theta_1^{(1)} - \frac{y}{x}\right|\right) \geq \frac{1}{2}\left|\theta_1^{(1)} - \theta_1^{(2)}\right| = \left|\sqrt{\frac{k+2}{k-2}}\right|.$$

Hence, we see that

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \leq \frac{16}{\sqrt{|k^2-4|}\,|x|^2}.$$

Similarly, we get

$$\begin{aligned}\left|\theta_2^{(1)} - \frac{z}{2x}\right| &= \left|(\theta_2^{(1)})^2 - \frac{z^2}{(2x)^2}\right| \cdot \left|\theta_2^{(1)} + \frac{z}{2x}\right|^{-1} \\ &= \frac{|3k+2|}{|k-2|\,|x|^2}\left|\theta_2^{(2)} - \frac{z}{2x}\right|^{-1},\end{aligned}$$

and

$$\left|\theta_2^{(2)} - \frac{z}{2x}\right| \geq \left|\sqrt{\frac{k}{k-2}}\right|,$$

the other estimate is obtained. □

Now, we apply Theorem 4.1 on $\theta_1^{(1)}$ and $\theta_2^{(1)}$. In our case, we have

$$a_1 = 4,\ a_2 = 2,\ T = k-2,\ M = 4$$

and

$$l = \frac{27\,|k-2|}{64\,|k-2|-4}, L = \frac{27}{4096}\left(|k-2|-4\right)^2, p = \sqrt{\frac{2\,|k-2|+12}{2\,|k-2|-8}} = \sqrt{\frac{|k-2|+6}{|k-2|-4}},$$

$$P = 1024(|k-2|+6).$$

If $|k| \geq 155402$, then we have $L > 1$. Thus, the condition $L > 1$ of Theorem 4.1 is satisfied. So, we conclude that

$$\max\left\{\left|\theta_1^{(1)} - \frac{y}{x}\right|, \left|\theta_2^{(1)} - \frac{z}{2x}\right|\right\} > c\,|2x|^{-\lambda}, \tag{20}$$

where

$$\lambda = 1 + \frac{\log P}{\log L},\ c^{-1} = 4pP(\max\{1, 2l\})^{\lambda-1}.$$

If we assume that $|k| \geq 155402$, then $\max\{1, 2l\} = 1$ and $c^{-1} = 4pP$. Furthermore, according to Lemma 4.1, we have

$$\max\left\{\left|\theta_1^{(1)} - \frac{y}{x}\right|, \left|\theta_2^{(1)} - \frac{z}{2x}\right|\right\} \leq \frac{3.1}{|x|^2},$$

and inequality (20) implies

$$\frac{1}{4}\sqrt{\frac{|k-2|-4}{|k-2|+6}}\cdot\frac{1}{1024(|k-2|+6)}\,|2x|^{-\lambda}<\frac{3.1}{|x|^2}.$$

Hence,

$$|2x|^{2-\lambda}\le 12697.6\sqrt{\frac{|k-2|+6}{|k-2|-4}}(|k-2|+6). \tag{21}$$

If $|k|\ge 155402$, then $2-\lambda\ge 0.000014$. Thus, using the estimate for x, $|x|\ge |k|\,(2\,|k|-3)^{|k|-5}$ (from Lemma 3.7), and after taking the logarithm of (21), we obtain

$$(2-\lambda)\,(\log 2+\log|k|+(|k|-5)\log(2\,|k|-3))<\log\left(12697.6\sqrt{\frac{|k-2|+6}{|k-2|-4}}(|k-2|+6)\right) \tag{22}$$

If $|k|\ge 155402$, then we have a contradiction. Therefore, we just prove the following statement.

Theorem 4.2. *Let $k\in\mathbb{Z}[i]$ and $|k|\ge 155402$. Then, all solution of the system of equations* (2) *and* (3) *are given by*

$$x=\pm 2, y=\pm 2, z=\pm 2$$

and

$$x=\pm(2k^2-2k-2), y=\pm(2k^2+2k-2), z=\pm(4k^2-2).$$

5 Linear forms in three logarithms

In this section, we will use the already well-known method in solving the equations of the form $v_m=w_n$ or $v_n=w'_n$, that consists in searching for the intersection of binary recurrence sequences. This will be done using Baker's theory on linear forms in logarithms. It gives the upper bound of m (and n) which will later be reduced using the reduction method. For more details, one can refer to [6] (for the integer case) and [8] (for the case of Gaussian integers). We assume $5<|k|<155402$ and we will use it at several places.

Let $v_m=w_n$ or $v_m=w'_n$, for $n>2$ and $|k|>5$. Then we have

$$v_m=\frac{\sqrt{k+2}+\sqrt{k-2}}{\sqrt{k+2}}\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m+\frac{\sqrt{k+2}-\sqrt{k-2}}{\sqrt{k+2}}\left(\frac{k-\sqrt{k^2-4}}{2}\right)^m,$$

$$w_n=\frac{\sqrt{4k}+\sqrt{k-2}}{\sqrt{4k}}\left(k-1+\sqrt{k^2-2k}\right)^n+\frac{\sqrt{4k}-\sqrt{k-2}}{\sqrt{4k}}\left(k-1-\sqrt{k^2-2k}\right)^n,$$

$$w'_n=\frac{\sqrt{4k}-\sqrt{k-2}}{\sqrt{4k}}\left(k-1+\sqrt{k^2-2k}\right)^n+\frac{\sqrt{4k}+\sqrt{k-2}}{\sqrt{4k}}\left(k-1-\sqrt{k^2-2k}\right)^n.$$

Now, let us define

$$P=\frac{\sqrt{k+2}+\sqrt{k-2}}{\sqrt{k+2}}\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m,$$

$$Q=\frac{\sqrt{4k}\pm\sqrt{k-2}}{\sqrt{4k}}\left(k-1+\sqrt{k^2-2k}\right)^{2n},$$

where

$$\sqrt{\frac{k-2}{k+2}},\pm\sqrt{\frac{k-2}{4k}},\sqrt{1-\frac{4}{k^2}},\sqrt{1-\frac{1}{(k-1)^2}}$$

are chosen such that

$$Re\left(\sqrt{\frac{k-2}{k+2}}\right)\geq 0, Re\left(\pm\sqrt{\frac{k-2}{4k}}\right)\geq 0, Re\left(\sqrt{1-\frac{4}{k^2}}\right)\geq 0, Re\left(\sqrt{1-\frac{1}{(k-1)^2}}\right)\geq 0.$$

The equation $v_m=w_{2n}$ or $v_m=w'_{2n}$ implies that

$$P+\frac{4}{k+2}P^{-1}=Q+\frac{3k+2}{4k}Q^{-1}. \tag{23}$$

We have

$$|P|=\left|1+\sqrt{\frac{k-2}{k+2}}\right|\left|\frac{k}{2}\right|^m\left|1+\sqrt{1-\frac{4}{k^2}}\right|^m>\left|\frac{k}{2}\right|^2>2.5^2>6.$$

$$|Q|=\left|1\pm\sqrt{\frac{k-2}{4k}}\right||k-1|^{2n}\left|1+\sqrt{1-\frac{1}{(k-1)^2}}\right|^{2n}>|k-1|^{2n}>4^4\geq 256.$$

Furthermore, from (23) we have

$$||P|-|Q||\leq|P-Q|\leq\left|\frac{3k+2}{4k}\right||Q|^{-1}+\left|\frac{4}{k+2}\right||P|^{-1}<0.23.$$

Hence, $|P|\leq|Q|+0.23\leq 1.001\,|Q|$, which yields $|Q|^{-1}\leq 1.001\,|P|^{-1}$ and

$$\left|\frac{P-Q}{P}\right|\leq\left|\frac{3k+2}{4k}\right||Q|^{-1}|P|^{-1}+\left|\frac{4}{k+2}\right||P|^{-2}<2.19\,|P|^{-2}<0.061.$$

Finally, we get

$$\begin{aligned}\left|\log\frac{|P|}{|Q|}\right| &= \left|\log\left(1-\frac{|P|-|Q|}{|P|}\right)\right|\\ &< 2.19\,|P|^{-2}+(2.19\,|P|^{-2})^2\\ &< 0.061\cdot 2.19\cdot|P|^{-2}\\ &< \left|\frac{k}{2}\right|^{-2m}<6^{-m}.\end{aligned}$$

The above expression can be written as a linear form in three logarithms:

$$\left|m\log\left|\frac{k+\sqrt{k^2-4}}{2}\right|-2n\log\left|k-1+\sqrt{k^2-2k}\right|+\log\left|\frac{\sqrt{4k}(\sqrt{k+2}+\sqrt{k-2})}{\sqrt{k+2}(\sqrt{4k}\pm\sqrt{k-2})}\right|\right|<6^{-m}. \tag{24}$$

Lemma 5.1. *If $v_m=w_{2n}$, then $|P|\neq|Q|$ for $k\in\mathbb{Z}[i]$, $k\neq 0,\pm 2$.*

Proof. The proof is analogous as [8, Lemma 5.2]. Assume that $|P| = |Q|$. If $P = Q$, then (23) implies $3k^2 - 8k + 4 = 0$. The only solution in $\mathbb{Z}[i]$ of this equation is $k = 2$, so we conclude that $P \neq Q$. Let us denote

$$\alpha = \sqrt{\frac{k-2}{k+2}},\ \beta = \sqrt{\frac{k-2}{k}}.$$

Then, we have

$$P = a + b\alpha\, Q = c + d\beta,$$

where $a, b, c, d \in \mathbb{Q}[i]$. Furthermore, the assumption $v_m = w_n$ implies that $a = c$ because

$$v_m = a + b\alpha + a - b\alpha = 2a,$$

$$w_n = c + d\beta + c - d\beta = 2c.$$

Moreover, we have

$$|P|^2 = p + u\alpha + \overline{u\alpha} + q|\alpha|^2,$$

$$|Q|^2 = r + v\beta + \overline{v\beta} + s|\beta|^2.$$

where $p, q, r, s \in \mathbb{Q}$ and $u, v \in \mathbb{Q}[i]$. In the exactly same way as in [8] we can prove several important facts:

- The complex numbers α and β are algebraic numbers of degree 2, for $k \in \mathbb{Z}[i]$, $k \neq 0, \pm 2$.
- The basis for $\mathbb{Q}[i](\alpha, \overline{\alpha})$, considered as a vector space over $\mathbb{Q}[i]$ is $B_\alpha = \{1, \alpha, \overline{\alpha}, |\alpha|^2\}$, and the basis for $\mathbb{Q}[i](\beta, \overline{\beta})$ is $B_\beta = \{1, \beta, \overline{\beta}, |\beta|^2\}$.
- The set $B = \{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2\}$ is linearly independent.

Now we have $|P|^2 \in \mathbb{Q}[i](\alpha, \overline{\alpha})$ and $|Q|^2 \in \mathbb{Q}[i](\beta, \overline{\beta})$ and they are uniquely represented. Thus, the assumption $|P|^2 = |Q|^2$ implies $p = q = r = s = 0$ because B is linearly independent set. So we have $P = a$ and $Q = c$ which is not possible because we showed that $a = c$. $\square$

Put

$$\Lambda = m \log \left| \frac{k + \sqrt{k^2 - 4}}{2} \right| - 2n \log \left| k - 1 + \sqrt{k^2 - 2k} \right| + \log \left| \frac{\sqrt{4k}(\sqrt{k+2} + \sqrt{k-2})}{\sqrt{k+2}(\sqrt{4k} \pm \sqrt{k-2})} \right|.$$

From (24) and 5.1, if $|k| > 5$, we have $0 < |\Lambda| < 6^{-m}$.

Lemma 5.2. *Let $k \in \mathbb{Z}[i]$ such that $|k| > 5$. If $v_m = \pm w_n$ or $v_m = \pm w'_n$ then $n \leq m < 1.33n$.*

Proof. The statement is trivially satisfied if $m = n = 0$, which is the only possibility if one of the index is equal to 0. Moreover, it can be proved by induction that

$$(2\,|k| - 2)(|k| - 1)^{m-1} \leq |v_m| \leq (2\,|k| + 2)(|k| + 1)^{m-1},$$

$$(3\,|k| - 4)(2\,|k| - 3)^{n-1} \leq |w_n| \leq (3\,|k| + 4)(2\,|k| + 3)^{n-1},$$

$$|k|\,(2\,|k| - 3)^{n-1} \leq |w_n| \leq |k|\,(2\,|k| + 3)^{n-1},$$

for $n, m \geq 1$. So $v_m = w_n$ or $v_m = w'_n$ implies

$$|k|\,(2\,|k| - 3)^{n-1} \leq (2\,|k| + 2)(|k| + 1)^{m-1},$$

which yields $n < m$. On the other hand, $v_m = w_n$ or $v_m = w'_n$ also implies

$$2(|k| - 1)^m \leq (3\,|k| + 4)(2\,|k| + 3)^{n-1} < 2(2\,|k| + 3)^n,$$

if $|k| > 5$. Now, if we take the logarithm of both sides of the inequality and use the condition $|k| > 5$, we easily get $m < 1.33n$. □

We use the following theorem of Baker and Wüstholz (see [1], p.20) to obtain a upper bound for m.

Theorem 5.1. *Let Λ be a nonzero linear form in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$. Then*

$$\log \Lambda \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B,$$

where $B = \max(|b_1|, \ldots, |b_l|)$ and d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$ over the rationals.

Here

$$h'(\alpha) = \max(h(\alpha), \frac{1}{d}\,|\log \alpha|\,, \frac{1}{d}),$$

where $h(\alpha)$ denotes the standard logarithmic Weil height of α.

In our case, we have

$$\log|m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3| \geq -18 \cdot 4!3^4(32d)^5 h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B,$$

where

$$\alpha_1 = \left|\frac{k + \sqrt{k^2-4}}{2}\right|,\ \alpha_2 = \left|k - 1 + \sqrt{k^2 - 2k}\right|,\ \alpha_3 = \left|\frac{\sqrt{4k}(\sqrt{k+2} + \sqrt{k-2})}{\sqrt{k+2}(\sqrt{4k} \pm \sqrt{k-2})}\right|.$$

The minimal polynomials of α_1, α_2 are respectively

$$p_1(x) = (2x^4 - |k|^2 x^2 + 2)^2 - \left|k^2 - 4\right|^2 x^4, \tag{25}$$

$$p_2(x) = (x^4 - 2\,|k-1|^2 + 1)^2 - 4\left|k^2 - 2k\right|^2 x^4. \tag{26}$$

The roots of p_1 are

$$x_1,\ x_2 = \pm\left|\frac{k + \sqrt{k^2-4}}{2}\right| = \pm\alpha_1,\ \ x_3,\ x_4 = \pm\left|\frac{k - \sqrt{k^2-4}}{2}\right|,$$

$$x_5,\ x_6 = \pm\frac{\sqrt{|k|^2 - |k^2-4| + \sqrt{(|k|^2 - |k^2-4|)^2 - 4}}}{2},$$

$$x_7,\ x_8 = \pm \frac{\sqrt{|k|^2 - |k^2-4| - \sqrt{(|k|^2 - |k^2-4|)^2 - 4}}}{2}.$$

If $i = 1, 2$ then $|x_i| > 1$ and if $i = 3, 4, 5, 6, 7, 8$ then $|x_i| \leq 1$. Thus, if $5 < |k| < 155402$ then we have

$$h(\alpha_1) = \frac{1}{4}\log(2\left|k + \sqrt{k^2-4}\right|) < 3.34.$$

The roots of p_2 are

$$x_1,\ x_2 = \pm\left|k - 1 + \sqrt{k^2-2k}\right| = \pm\alpha_2,\quad x_3,\ x_4 = \pm\left|k - 1 - \sqrt{k^2-2k}\right|,$$

$$x_5,\ x_6 = \pm\sqrt{|k-1|^2 - |k^2-2k| + \sqrt{(|k-1|^2 - |k^2-2k|)^2 - 1}},$$

$$x_7,\ x_8 = \pm\sqrt{|k-1|^2 - |k^2-2k| - \sqrt{(|k-1|^2 - |k^2-2k|)^2 - 1}}.$$

If $i = 1, 2$, then $|x_i| > 1$ and if $i = 3, 4, 5, 6, 7, 8$, thus $|x_i| \leq 1$. So, if $5 < |k| < 155402$ then we have

$$h(\alpha_2) = \frac{1}{4}\log\left(\left|k - 1 + \sqrt{k^2-2k}\right|\right) < 3.17.$$

For the estimate of $h(\alpha_3)$, we compute the minimal polynomial using Maple. However, because we do not need this estimate to be so accurate (we will significantly improve the bound on m using the reduction method), we only estimate the leading coefficient of the minimal polynomial and the conjugates of α_3. For that we use $|k| > 5$. We have

$$h(\alpha_3) \leq \frac{1}{32}\log(|a_0|\,|\alpha'|^{32}) < 14.74,$$

as

$$|a_0| \leq (\sqrt{|k|+2}(\sqrt{4\,|k|} + \sqrt{|k|+2}))^{32} < 4.51 \cdot 10^{181},$$

$$|\alpha'| < \frac{\sqrt{4\,|k|}(2\sqrt{|k|+2})}{\sqrt{|k|-2}(\sqrt{4\,|k|} - \sqrt{|k|+2})} < 5.27.$$

Applying Baker-Wüstholz theorem with $l = 3$ and $d \leq 8 \cdot 32 \cdot 8 = 2048$, we get $-m\log 6 > -34992 \cdot 1.3 \cdot 10^{24} \cdot 3.34 \cdot 3.17 \cdot 14.74 \cdot \log 12288 \cdot \log 2m$, which yields

$$\frac{m}{\log 2m} < 3.74 \cdot 10^{31}$$

and $m < 2.91 \cdot 10^{33}$. Therefore, we have just proved that for $5 < |k| < 155402$, equation $v_m = w_n$ or $v_m = w'_n$ implies $m < 2.91 \cdot 10^{33}$. As this bound for m is very large, we have to reduce it.

Lemma 5.3. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < n\gamma - m + \mu < AB^{-n},$$

in positive integers m, n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leq n \leq M.$$

As

$$0 < m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3 < 6^{-m},$$

we apply Lemma 5.3 with

$$\gamma = \frac{\log \alpha_1}{2 \log \alpha_2}, \quad \mu = \frac{\log \alpha_3}{2 \log \alpha_2}, \quad A = \frac{1}{2 \log \alpha}, \quad B = 6, \quad M = 2.91 \cdot 10^{33}.$$

The program was developed in Mathematica running with 500 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. In a few days all the computations were done. After the first run, we obtained $M \leq 67$. For the second run, we set $M = 67$ to obtain another bound $M = 16$. The third run of the reduction method yields $n \leq m \leq 14$. In this range, we check the equations $v_m = w_n$ and $v_m = w'_n$ and see that they confirm the result in the main theorem.

6 The case $2 < |k| \leq 5$

There are some extra fundamental solutions of (2) and (3) for certain values of parameter k. Precisely, these fundamental solutions of (2) also appear (besides $x = \pm 2$):

- $x_0 = \pm(1-i)$, $y_0 = \pm(1+i)$ and $x_0 = \pm(1-3i)$, $y_0 = \pm(1+3i)$ for $k = 4i$,
- $x_0 = \pm(1+i)$, $y_0 = \pm(1-i)$ and $x_0 = \pm(1+3i)$, $y_0 = \pm(1-3i)$ for $k = -4i$,
- $x_0 = \pm 4i$, $y_0 = \pm 4(1+i)$ for $k = 1+2i$,
- $x_0 = \pm 4i$, $y_0 = \pm 4(1-i)$ for $k = 1-2i$,
- $x_0 = \pm 4i$, $y_0 = \pm 4(1+i)$ for $k = -1+2i$,
- $x_0 = \pm 4i$, $y_0 = \pm 4(1-i)$ for $k = -1-2i$,
- $x_0 = \pm(2+2i)$, $y_0 = \pm(6+2i)$ for $k = 2+i$,
- $x_0 = \pm(2-2i)$, $y_0 = \pm(6-2i)$ for $k = 2-i$
- $x_0 = \pm(6-2i)$, $y_0 = \pm(2-2i)$ for $k = -2+i$,
- $x_0 = \pm(6+2i)$, $y_0 = \pm(2+2i)$ for $k = -2-i$,
- $x_0 = \pm(2-2i)$, $y_0 = \pm 4i$ and $x_0 = 0$, $y_0 = \pm(2+2i)$ for $k = 2+2i$,
- $x_0 = \pm(2+2i)$, $y_0 = \pm 4i$ and $x_0 = 0$, $y_0 = \pm(2-2i)$ for $k = 2-2i$,
- $x_0 = \pm(2-2i)$, $y_0 = 0$ and $x_0 = \pm 4i$, $y_0 = \pm(2+2i)$ for $k = -2+2i$,
- $x_0 = \pm(2+2i)$, $y_0 = 0$ and $x_0 = \pm 4i$, $y_0 = \pm(2-2i)$ for $k = -2-2i$,
- $x_0 = \pm 6i$, $y_0 = \pm 2i$ and $x_0 = \pm 4i$, $y_0 = 0$ for $k = -3$,
- $x_0 = \pm(1-i)$, $y_0 = \pm(1-3i)$ for $k = 4$,
- $x_0 = \pm(1+3i)$, $y_0 = \pm(1+i)$ and $x_0 = \pm(1-3i)$, $y_0 = \pm(1-i)$ for $k = -4$, and for

(3), we have:

- $x_1 = \pm 4i$, $z_1 = \pm(2+8i)$ for $k = 4i$,
- $x_1 = \pm 4i$, $z_1 = \pm(2-8i)$ for $k = -4i$,
- $x_1 = \pm(1+2i)$, $z_1 = \pm 4(1+i)$ for $k = 1+2i$,
- $x_1 = \pm(1-2i)$, $z_1 = \pm 4(1-i)$ for $k = 1-2i$,
- $x_1 = \pm(1-2i)$, $z_1 = \pm 4i$ for $k = -1+2i$,

- $x_1 = \pm(1-2i)$, $z_1 = \pm 4i$ for $k = -1-2i$,
- $x_1 = \pm(2+3i)$, $z_1 = \pm(10+6i)$ and $x_1 = \pm(2+i)$, $z_1 = \pm(6+2i)$ for $k = 2+i$,
- $x_1 = \pm(2-3i)$, $z_1 = \pm(10-6i)$ and $x_1 = \pm(2-i)$, $z_1 = \pm(6-2i)$ for $k = 2+i$,
- $x_1 = \pm(2-i)$, $z_1 = \pm(2-2i)$ for $k = -2+i$,
- $x_1 = \pm(2+i)$, $z_1 = \pm(2-2i)$ for $k = -2-i$,
- $x_1 = \pm(2+2i)$, $z_1 = \pm(6+4i)$ and $x_1 = 0$, $z_1 = \pm(2+4i)$ for $k = 2+2i$,
- $x_1 = \pm(2-2i)$, $z_1 = \pm(6-4i)$ and $x_1 = 0$, $z_1 = \pm(2-4i)$ for $k = 2-2i$,
- $x_1 = \pm(2-2i)$, $z_1 = \pm(-2+4i)$ for $k = -2+2i$,
- $x_1 = \pm(2+2i)$, $z_1 = \pm(-2-4i)$ for $k = -2-2i$,
- $x_1 = \pm 3$, $z_1 = \pm 4$ for $k = -3$,
- $x_1 = \pm 3i$, $z_1 = \pm 10i$ and $x_1 = \pm i$, $z_1 = \pm 6i$ for $k = 4$,
- $x_1 = \pm 1$, $z_1 = \pm 2i$ for $k = -4$.

Each of the above cases will be treated separately.

- $k = 4i$

In Section 2, the solutions of (2) are given by recurrence sequences (11) and (12). In this cases we have

$$q_0 = 1-i,\ q_1 = -1+3i,\ q_{m+2} = 4iq_{m+1} - q_m,\ m \in \mathbb{N},$$

$$q_0' = 1-i,\ q_1 = 5+i,\ q_{m+2} = 4iq_{m+1} - q_m,\ m \in \mathbb{N},$$

$$u_0 = 1-3i,\ u_1 = -1+i,\ u_{m+2} = 4iu_{m+1} - u_m,\ m \in \mathbb{N},$$

$$u_0' = 1-3i,\ u_1' = 13+3i,\ u_{m+2}' = 4iu_{m+1}' - u_m',\ m \in \mathbb{N},$$

$$v_0 = 2,\ v_1 = -2+8i,\ v_{m+2} = 4iv_{m+1} - v_m,\ m \in \mathbb{N}.$$

The relations (13) and (14) giving all solutions of (3) become

$$x_0 = 4i,\ x_1 = -34-8i,\ x_{n+2} = 2(4i-1)x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$x_0' = 4i,\ x_1' = 2,\ x_{n+2}' = 2(4i-1)x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$w_0 = 2,\ w_1 = -4+12i,\ w_{n+2} = 2(4i-1)w_{n+1} - w_n,\ n \in \mathbb{N},$$

$$w_0' = 2,\ w_1' = 4i,\ w_{n+2}' = 2(4i-1)w_{n+1}' - w_n',\ n \in \mathbb{N}.$$

So, we have the following relations:

a): $q_m = \pm x_n$ or $q_m = \pm x_n'$ or $q_m = \pm w_n$ or $q_m = \pm w_n'$,

b): $q_m' = \pm x_n$ or $q_m' = \pm x_n'$ or $q_m' = \pm w_n$ or $q_m' = \pm w_n'$,

c): $u_m = \pm x_n$ or $u_m = \pm x_n'$ or $u_m = \pm w_n$ or $u_m = \pm w_n'$,

d): $u_m' = \pm x_n$ or $u_m' = \pm x_n'$ or $u_m' = \pm w_n$ or $u_m' = \pm w_n'$,

e): $v_m = \pm x_n$ or $v_m = \pm x_n'$ or $v_m = \pm w_n$ or $v_m = \pm w_n'$.

Thus, we get:

$$\begin{aligned}
(q_m \pmod 2) &= (1-i,-1+i,-1+i,1-i,1-i,-1+i,...),\\
(q'_m \pmod 2) &= (1-i,1+i,-1+i,-1-i,1-i,1+i,...),\\
(u_m \pmod 2) &= (1-i,-1+i,-1+i,1-i,1-i,-1+i,-1+i,...),\\
(u'_m \pmod 2) &= (1-i,1+i,-1+i,-1-i,1-i,1+i,-1+i,...),\\
(v_m \pmod 2) &= (0,0,0,0,0,0,0,0,...),\\
(x_n \pmod 2) &= (0,0,0,0,0,0,0,0,0,...),\\
(x'_n \pmod 2) &= (0,0,0,0,0,0,0,0,0,...,),\\
(w_n \pmod 2) &= (0,0,0,0,0,0,0,0,0,...),\\
(w'_n \pmod 2) &= (0,0,0,0,0,0,0,0,0,...).
\end{aligned}$$

By these relations, we deduce that the only possible equations are:

$$v_m = \pm w_n, \text{ or } v_m = \pm w'_n, \text{ or } v_m = \pm x_n, \text{ or } v_m = \pm x'_n.$$

The equations $v_m = \pm w_n$ and $v_m = \pm w'_n$ can be solved similarly as in the previous sections. Moreover, $x_n = w'_{n+1}$ and $x'_{n+1} = w_n$.

• $k = 1 + 2i$

In Section 2, the solutions of (2) are given by recurrence sequences (11) and (12). In these cases, we have

$$q_0 = 4i,\ q_1 = -10 + 4i,\ q_{m+2} = (1 + 2i)q_{m+1} - q_m,\ m \in \mathbb{N},$$

$$q'_0 = 4i,\ q'_1 = 2,\ q_{m+2} = (1 + 2i)q_{m+1} - q_m,\ m \in \mathbb{N},$$

$$v_0 = 2,\ v_1 = 4i,\ v_{m+2} = (1 + 2i)v_{m+1} - v_m,\ m \in \mathbb{N}.$$

The relations (13) and (14) that give all solutions of (3) become

$$x_0 = 1 + 2i,\ x_1 = -10 + 4i,\ x_{n+2} = 2(2i)x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$x'_0 = 1 + 2i,\ x'_1 = 2,\ x'_{n+2} = 2(2i)x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$w_0 = 2,\ w_1 = -1 + 6i,\ w_{n+2} = 2(2i)w_{n+1} - w_n,\ n \in \mathbb{N},$$

$$w'_0 = 2,\ w'_1 = 1 + 2i,\ w'_{n+2} = 2(2i)w_{n+1} - w_n,\ n \in \mathbb{N}.$$

Thus, the following cases should be analyzed:

a): $q_m = \pm x_n$ or $q_m = \pm x'_n$ or $q_m = \pm w_n$ or $q_m = \pm w'_n$,

b): $q'_m = \pm x_n$ or $q'_m = \pm x'_n$ or $q'_m = \pm w_n$ or $q'_m = \pm w'_n$,

c): $v_m = \pm x_n$ or $v_m = \pm x'_n$ or $v_m = \pm w_n$ or $v_m = \pm w'_n$.

Moreover, $q_m = v_{m+1}$, $q'_{m+2} = v_m$, $x_n = w'_{n+1}$, $x'_{n+1} = w_n$. So, the cases a), b), c) can be reduced to

$$v_m = \pm w_n \text{ or } v_m = \pm w'_n.$$

This case can be solved similarly as in the previous sections.

• $k=-2+i$

In Section 2, the solutions of (2) are given by recurrence sequences (11) and (12). In these cases, we have

$$q_0=6-2i,\ q_1=-8+10i,\ q_{m+2}=(-2+i)q_{m+1}-q_m,\ m\in\mathbb{N},$$

$$q_0'=6-2i,\ q_1'=-2,\ q_{m+2}=(-2+i)q_{m+1}-q_m,\ m\in\mathbb{N},$$

$$v_0=2,\ v_1=-6+2i,\ v_{m+2}=(-2+i)v_{m+1}-v_m,\ m\in\mathbb{N}.$$

The relations (13) and (14) that all solutions of (3) are

$$x_0=2-i,\ x_1=-8+10i,\ x_{n+2}=2(-3+i)x_{n+1}-x_n,\ n\in\mathbb{N},$$

$$x_0'=2-i,\ x_1'=-2,\ x_{n+2}'=2(-3+i)x_{n+1}-x_n,\ n\in\mathbb{N},$$

$$w_0=2,\ w_1=-10+3i,\ w_{n+2}=2(-3+i)w_{n+1}-w_n,\ n\in\mathbb{N},$$

$$w_0'=2,\ w_1'=-2+i,\ w_{n+2}'=2(-3i)w_{n+1}-w_n,\ n\in\mathbb{N}.$$

So, the following cases should be analyzed:

a): $q_m=\pm x_n$ or $q_m=\pm x_n'$ or $q_m=\pm w_n$ or $q_m=\pm w_n'$,

b): $q_m'=\pm x_n$ or $q_m'=\pm x_n'$ or $q_m'=\pm w_n$ or $q_m'=\pm w_n'$,

c): $v_m=\pm x_n$ or $v_m=\pm x_n'$ or $v_m=\pm w_n$ or $v_m=\pm w_n'$.

Moreover, $q_m=-v_{m+1}$, $q_{m+2}'=v_m$, $x_n=-w_{n+1}'$, $x_{n+1}'=-w_n$. So, the cases a), b), c) are reduced to

$$v_m=\pm w_n \text{ or } v_m=w_n'.$$

These equations can be solved similarly as in the previous sections.

• $k=-2+2i$

In the Section 2, the solutions of (2) are given by recurrence sequences (11) and (12). In this cases we have

$$q_0=2-2i,\ q_1=4i,\ q_{m+2}=(-2+2i)q_{m+1}-q_m,\ m\in\mathbb{N},$$

$$q_0'=4i,\ q_1'=-10-6i,\ q_{m+2}'=(-2+2i)q_{m+1}'-q_m',\ m\in\mathbb{N},$$

$$q_0''=4i,\ q_1''=2-2i,\ q_{m+2}''=(-2+2i)q_{m+1}''-q_m'',\ m\in\mathbb{N},$$

$$v_0=2,\ v_1=-6+4i,\ v_{m+2}=(-2+2i)v_{m+1}-v_m,\ m\in\mathbb{N}.$$

The relations (13) and (14) that give all solutions of (3) are

$$x_0=2-2i,\ x_1=-2,\ x_{n+2}=2(-3+2i)x_{n+1}-x_n,\ n\in\mathbb{N},$$

$$x_0'=2-2i,\ x_1'=-2+20i,\ x_{n+2}'=2(-3+2i)x_{n+1}-x_n,\ n\in\mathbb{N},$$

$$w_0=2,\ w_1=-10+3i,\ w_{n+2}=2(-3+i)w_{n+1}-w_n,\ n\in\mathbb{N},$$

$$w_0'=2,\ w_1'=-2+i,\ w_{n+2}'=2(-3i)w_{n+1}-w_n,\ n\in\mathbb{N}.$$

Hence, the following cases should be analyzed:

a): $q_m = \pm x_n$ or $q_m = \pm x'_n$ or $q_m = \pm w_n$ or $q_m = \pm w'_n$,
b): $q'_m = \pm x_n$ or $q'_m = \pm x'_n$ or $q'_m = \pm w_n$ or $q'_m = \pm w'_n$,
c): $q''_m = \pm x_n$ or $q''_m = \pm x'_n$ or $q''_m = \pm w_n$ or $q''_m = \pm w'_n$,
d): $v_m = \pm x_n$ or $v_m = \pm x'_n$ or $v_m = \pm w_n$ or $v_m = \pm w'_n$.

Moreover, we have $q_{m+1} = q'_m$, $q_m = q''_{m+1}$, $w_n = -x_{n+1}$, $w'_{n+1} = -x'_n$. So, the cases a), b), c), d) are reduced to

$$v_m = \pm w_n \text{ or } v_m = w'_n \text{ or } q_m = \pm w_n \text{ or } q_m = \pm w'_n.$$

These equations $v_m = \pm w_n$ or $v_m = w'_n$ can be solved similarly as in previous sections. Moreover, by the congruence method, the equations $w_n = \pm q_m$ have no solution. Now, we study the equations $q_m = \pm w'_n$. One can check that for the small values of m, n, i.e. $m, n \leq 2$ and one gets the only solutions: $q_0 = -w'_1 = 2 - 2i$, which gives $d = 2i$. Thus, we have to solve $q_m = \pm w'_n$, for $n > 2$. The same way as in Lemma 5.2, we get $n < m$. Moreover, using Baker's theory on linear forms in logarithms we get

$$0 < |m \log \alpha_1 - n \log \alpha_2 + \log \alpha_3| < 1.42^{-m},$$

where

$$\alpha_1 = \left|-1 + i + \sqrt{-2i - 1}\right|, \alpha_2 = \left|-3 + 2i + \sqrt{4 - 12i}\right|, \alpha_3 = \left|\frac{(1-i)\sqrt{-8+8i}}{\sqrt{-8+8i} - \sqrt{-4+2i}}\right|.$$

Now, we have to combine this bound with that obtained by Baker-Wüstholz theorem. Similarly as for general case, we get

$$h(\alpha_1) < 0.27, h(\alpha_2) < 0.5, h(\alpha_3) < 3.61.$$

This yields

$$\frac{m}{\log m} < 5.54 \cdot 10^{29}$$

and then $m < 4.1 \cdot 10^{31}$. The previous reduction method confirms the result.

• $k = -3$

In Section 2, the solutions of (2) are given by recurrence sequences (11) and (12). In this cases we have

$$q_0 = 6i,\ q_1 = -14i,\ q_{m+2} = (-3)q_{m+1} - q_m,\ m \in \mathbb{N},$$

$$q'_0 = 6i,\ q'_1 = -4i,\ q'_{m+2} = (-3)q'_{m+1} - q'_m,\ m \in \mathbb{N},$$

$$q''_0 = 4i,\ q''_1 = -6i,\ q''_{m+2} = (-3)q''_{m+1} - q''_m,\ m \in \mathbb{N},$$

$$v_0 = 2,\ v_1 = -8,\ v_{m+2} = (-3)v_{m+1} - v_m,\ m \in \mathbb{N}.$$

The relations (13) and (14) of all solutions of (3) are given by

$$x_0 = 3,\ x_1 = -22,\ x_{n+2} = -8x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$x'_0 = 3,\ x'_1 = -2,\ x'_{n+2} = -8x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$w_0 = 2,\ w_1 = -13,\ w_{n+2} = -8w_{n+1} - w_n,\ n \in \mathbb{N},$$

$$w'_0 = 2,\ w'_1 = -3,\ w'_{n+2} = -8w_{n+1} - w_n,\ n \in \mathbb{N}.$$

Therefore, the following cases will be analyzed:

a): $q_m = \pm x_n$ or $q_m = \pm x'_n$ or $q_m = \pm w_n$ or $q_m = \pm w'_n$,

b): $q'_m = \pm x_n$ or $q'_m = \pm x'_n$ or $q'_m = \pm w_n$ or $q'_m = \pm w'_n$,

c): $q''_m = \pm x_n$ or $q''_m = \pm x'_n$ or $q''_m = \pm w_n$ or $q''_m = \pm w'_n$,

d): $v_m = \pm x_n$ or $v_m = \pm x'_n$ or $v_m = \pm w_n$ or $v_m = \pm w'_n$.

We have $q_m \in i\mathbb{Z}$, $q'_m \in i\mathbb{Z}$, $q''_m \in i\mathbb{Z}$, $x_n \in \mathbb{Z}$, $x'_n \in \mathbb{Z}$, $w_n \in \mathbb{Z}$, $w'_n \in \mathbb{Z}$. So in the cases a), b), c) have no solution. Moreover, $x_n = -w_{n+1}$ and $x'_{n+1} = -w_n$. Consequently, the case d) can be summed to $v_m = \pm w_n$ or $v_m = w'_n$. This case can be solved similarly as in previous sections.

• $k = 4$

In Section 2, the solutions of (2) are given by recurrence sequences (11) and (12). In these cases, we have

$$q_0 = 1 - i,\ q_1 = 3 - 5i,\ q_{m+2} = 4q_{m+1} - q_m,\ m \in \mathbb{N},$$

$$q'_0 = 1 - i,\ q_1 = 1 + i,\ q_{m+2} = 4q_{m+1} - q_m,\ m \in \mathbb{N},$$

$$v_0 = 2,\ v_1 = 6,\ v_{m+2} = 4v_{m+1} - v_m,\ m \in \mathbb{N}.$$

The relations (13) and (14) of all solutions of (3) become

$$x_0 = 3i,\ x_1 = 19i,\ x_{n+2} = -6x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$x'_0 = 3i,\ x'_1 = -i,\ x'_{n+2} = -6x_{n+1} - x_n,\ n \in \mathbb{N},$$

$$u_0 = i,\ u_1 = 9i,\ u_{n+2} = -6u_{n+1} - x_n,\ n \in \mathbb{N},$$

$$u'_0 = i,\ u'_1 = -3i,\ u'_{n+2} = -6u'_{n+1} - u'_n,\ n \in \mathbb{N},$$

$$w_0 = 2,\ w_1 = 8,\ w_{n+2} = -6w_{n+1} - w_n,\ n \in \mathbb{N},$$

$$w'_0 = 2,\ w'_1 = 4,\ w'_{n+2} = -6w_{m+1} - w_m,\ n \in \mathbb{N}.$$

So, we have the following relations:

a): $q_m = \pm x_n$ or $q_m = \pm x'_n$ or $q_m = \pm w_n$ or $q_m = \pm w'_n$ or $q_m = u_n$ or $q_m = u'_n$,

b): $q'_m = \pm x_n$ or $q'_m = \pm x'_n$ or $q'_m = \pm w_n$ or $q'_m = \pm w'_n$ or $q'_m = u_n$ or $q'_m = u'_n$,

c): $v_m = \pm x_n$ or $v_m = \pm x'_n$ or $v_m = \pm w_n$ or $v_m = \pm w'_n$ or $v_m = u_n$ or $v_m = u_n$.

We have:

$$\begin{aligned}
(q_m \pmod 4) &= (1-i, -1-i, -1+i, 1+i, 1-i, ...),\\
(q'_m \pmod 4) &= (1-i, 1+i, -1+i, -1-i, 1-i, ...),\\
(v_m \pmod 4) &= (2, 2, -2, -2, 2, 2, -2, -2, ...),\\
(x_n \pmod 4) &= (-i, -i, -i, -i, -i, -i, -i, ...,),\\
(x'_n \pmod 4) &= (-i, -i, -i, -i, -i, -i, -i, ...,),\\
(u_n \pmod 4) &= (i, i, i, i, i, i, i, i, i, , ...,),\\
(u'_n \pmod 4) &= (i, i, i, i, i, i, i, i, i, ...,),\\
(w_n \pmod 4) &= (2, 0, -2, 0, 2, 0, -2, 0, 2, ...).
\end{aligned}$$

By these relations, we deduce that the only possible equations are:

$$v_m = \pm w_n, \text{ or } v_m = \pm w'_n.$$

The equations $v_m = \pm w_n$ and $v_m = \pm w'_n$ can be solved similarly as in previous sections.

• $k = -4i$, $k = 1 - 2i$, $k = -2 - i$, $-2 - 2i$.

By conjugating, this cases becomes the same as the previous one.

• $k = -4$, $k = 2 - 2i$, $k = 2 + 2i$, $k = -2 + i$, $k = 2 + i$, $k = -1 - 2i$, $k = -1 + 2i$.

Lemma 6.1. *Let $k \in \mathbb{Z}[i]$. The set $\{-k-2, -k+2, -4k, d\}$ is a $D(4)$-quadruple in $\mathbb{Z}[i]$ if only if $\{k-2, k+2, 4k, -d\}$ is a $D(4)$-quadruple in $\mathbb{Z}[i]$.*

Proof. $\{-k-2, -k+2, -4k, d\}$ is a $D(4)$-quadruple in $\mathbb{Z}[i]$ if only if there exist x, y, z in $\mathbb{Z}[i]$ such that

$$d(-k-2) + 4 = x^2,\ d(-k+2) + 4 = y^2,\ d(-4k) + 4 = z^2.$$

This relation is equivalent to

$$-d(k+2) + 4 = x^2,\ -d(k-2) + 4 = y^2,\ -d(4k) + 4 = z^2.$$

Consequently, if $k \in \mathbb{Z}[i]$ the set $\{-k-2, -k+2, -4k, d\}$ is a $D(4)$-quadruple in $\mathbb{Z}[i]$ if only if $\{k-2, k+2, 4k, -d\}$ is a $D(4)$-quadruple in $\mathbb{Z}[i]$.

□

Using Lemma 6.1, the above cases become the same as the previous cases.

7 The case $1 \leq |k| \leq 2$

• $k = 1$.

In this case, system (1) is equivalent to the following system of Pell equations

$$3x^2 + y^2 = 16 \tag{27}$$

$$4x^2 + z^2 = 20. \tag{28}$$

We have $z^2 \equiv 0 \pmod 4$, so there exists $z' \in \mathbb{Z}[i]$ such that $z = 2z'$. Equations (27) and (28) are equivalent to

$$3x^2 + y^2 = 16 \tag{29}$$

$$x^2 + z'^2 = 5. \tag{30}$$

Equation (30) implies $|x - iz'|^2 |x + iz'|^2 = 25.$ So the solution of the system of Pellian equations (27) and (28) is $(x, y, z) = (2, 2, 2)$. We deduce that $d = 0$.

• $k = i$.

In this case, the fundamental solution of (2) is $(x_0, y_0) = (2, 2)$ and the fundamental solution of (3) is $(x_1, z_1) = (2, 2)$. By repeating the procedure described in Section 5, we conclude that the above equations have two solutions $v_0 = w_0 = w'0 = 2$ and $v_2 = w'_2 = -4 - 2i$. So, the nontrivial extension of the $D(4)$-triple $\{i-2, i+2, 4i\}$ is obtained for $d = -8i$, i.e. $4k^3 - 4k$, for $k = i$.

• $k = 2i$.

In this case, the fundamental solution of (2) is $(x_0, y_0) = (2, 2)$ and the fundamental solution of (3) is $(x_1, z_1) = (2, 2)$. By repeating the procedure described in Section 5, we conclude that the above equations have two solutions $v_0 = w_0 = w'0 = 2$ and $v_2 = w'_2 = -10 - 4i$. So, the nontrivial extension of the $D(4)$-triple $\{2i-2, 2i+2, 8i\}$ is obtained for $d = -40i$, i.e. $4k^3 - 4k$, for $k = 2i$.

• $k = 1 + i$.

In this case, the fundamental solution of (2) is $(x_0, y_0) = (2, 2)$ and the fundamental solution of (3) is $(x_1, z_1) = (2, 2)$. Thus, we get two solutions$v_0 = w_0 = w'0 = 2$ and $v_2 = w'_2 = -4+2i$. Therefore, the nontrivial extension of the $D(4)$-triple $\{-1+i, 3+i, 4+4i\}$ is obtained for $d = -12 + 4i$, i.e. $4k^3 - 4k$, for $k = 1 + i$.

• $k = 1 - i, k = -2i, k = i$.

By conjugating, these cases become the same as the previous cases.

• $k = -1 - i, k = -1 + i, k = -1$.

Using Lemma 6.1, we make similar conclusion for these cases.

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